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THE BANACH FIXED POINT METHOD FOR ITO STOCHASTIC DIFFERENTIAL EQUATIONS

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In this paper a simple proof is given for the existence and uniqueness of solutions of stochastic differential equations with respect to Brownian motion. The conditions imposed on the coefficients correspond to those introduced by CARATHÉODORY [2] for ordinary differential equations in 1918, except for requiring square integrability instead of integrability. The Banach fixed point method provides the result by means of a change of the norm.

There are several recent proofs of this fact under similar conditions, even in the more abstract setting of stochastic integrals with respect to semi-martingales, e.g. by DOLÉANS-DADE [3] 1976, and PROTTER [6] 1977. Relative to time dependence the Carathéodory conditions are slightly weaker than the ones used there. But the main interest of this note consists in the ease of carrying over methods of ordinary differential equations such as the point of view of solutions in the sense of Carathéodory and the method of change of the norm, as described e.g. by WALTER [7]. These methods also apply in the abstract setting.

Let $(W_t)_{t\geq 0}$ denote an m-dimensional Brownian motion on a probability space (Ω, F, P) and $(F_t)_{t\geq 0}$ an increasing family of sub-σ-fields of F such that $(W_t)_{t\geq 0}$ is an $(F_t)_{t\geq 0}$ - Brownian motion, i.e. $(W_t)_{t\geq 0}$ is adapted to $(F_t)_{t\geq 0}$ and $(W_{t+s} - W_t)_{s\geq 0}$ is independent of F_t for all $t\geq 0$. Consider on $\mathbb{R}_+ \times \Omega$ the σ-field P of predictable sets with respect to $(F_t)_{t\geq 0}$, and the vector spaces $L^2 = L^2(\mathbb{R}_+ \times \Omega, P, \lambda \otimes P)$ and H^2 of predictable processes $X = (X_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ such that

$$\|X\|_2 := (E(\int_0^\infty |X_t|^2 \, dt))^{1/2} < \infty$$

resp.

$$\|X\| := (\sup_{t\geq 0} E(|X_t|^2))^{1/2} < \infty$$

where $|\cdot|$ denotes Euclidean norm and λ Lebesgue measure.

Denote by L^2_{loc} (resp. H^2_{loc}) the space of processes X such that there is an increasing sequence of $(F_t)_{t\geq 0}$- stopping times $(T_n)_{n\in\mathbb{N}} \uparrow \infty$ P-a.s. such that $X 1_{[\![0,T_n]\!]}$ belongs to L^2 (resp. the process $X^{T_n^*} := X^{T_n} 1_{\{T_n > 0\}}$ belongs to H^2) for all $n \in \mathbb{N}$. The sequence $(T_n)_{n\in\mathbb{N}}$ is called a localizing sequence for X.

We have

$$L^2_{loc} = \{X \text{ predictable : for all } t \geq 0, \int_0^t |X_s|^2 ds < \infty \text{ P-a.s.}\}.$$

Further, every continuous adapted $\mathbb{R}^d$- valued process C belongs to H^2_{loc}, being localized by $T_n := \inf\{t \geq 0 : |C_t| \geq n\}$, $n \in \mathbb{N}$.

PROPOSITION 1: For any stopping time T, the quotient space H^2_T of H^2 by the kernel of the seminorm on H^2

$$\|X\|_T := \|X^T\|$$

is a Banach space with the norm $\|\cdot\|_T$.

<u>Proof</u>: Let $(X^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in H^2 for the seminorm $\|\cdot\|_T$. By considering a subsequence one may always suppose

$$\sum_{n=1}^{\infty} \|X^n - X^{n-1}\|_T \le c < \infty$$

where $X^o := 0$. For every $t\ge 0$ we have

$$(E(\sum_{n=1}^{\infty} |X^n_{t\wedge T} - X^{n-1}_{t\wedge T}|)^2)^{1/2} \le \sum_{n=1}^{\infty} (E(|X^n_{t\wedge T} - X^{n-1}_{t\wedge T}|^2))^{1/2} \le c < \infty,$$

so $\sum_{n=1}^{\infty} |X^n_{t\wedge T} - X^{n-1}_{t\wedge T}|$ belongs to $L^2(\Omega,P)$ and is P-a.s. absolutely convergent in $\mathbb{R}$. Hence $(X^n_{t\wedge T})_{n\in\mathbb{N}}$ converges P-a.s. in $\mathbb{R}^d$. The process

$$X_t(\omega) := \begin{cases} \lim_{n\to\infty} X^n_{t\wedge T}(\omega) & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

is predictable, and by Lebesgue's theorem, $X^n_{t\wedge T} \to X_t$ in $L^2(\Omega,P)$ for every $t\ge 0$ since the $|X^m_{t\wedge T}| = |\sum_{n=1}^{m} X^n_{t\wedge T} - X^{n-1}_{t\wedge T}|$ are dominated by $\sum_{n=1}^{\infty} |X^n_{t\wedge T} - X^{n-1}_{t\wedge T}|$, hence so is $|X_t|$. Also we have $(E(|X_t|^2))^{1/2} \le c$ for all $t \ge 0$, hence $X \in H^2$. It is now easy to see that $X^n \to X$ considered as elements of H^2_T.

<u>THEOREM 2</u>: Consider on $\mathbb{R}_+ \times \mathbb{R}^d$ the stochastic differential equation

$$X_t = C_t + \int_0^t f(s,X_s)ds + \int_0^t G(s,X_s)dW_s$$

where the initial process C is continuous adapted with values in $\mathbb{R}^d$, and the coefficients are measurable functions $f:\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $G:\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{dm}$ ($d\times m$ - matrix valued) satisfying the following conditions:

1) <u>Lipschitz condition</u>: There is a function $\ell \in L^2_{loc}(\mathbb{R}_+,\lambda)$ such tha

$$|f(t,x) - f(t,y)| \le \ell(t)\ |x-y|$$

$$|G(t,x) - G(t,y)| \le \ell(t)\ |x-y|$$

for all $t \in \mathbb{R}_+$, $x,y \in \mathbb{R}^d$.

2) $f(\cdot,x)$, $G(\cdot,x) \in L^2_{loc}(\mathbb{R}_+,\lambda)$ for some $x \in \mathbb{R}^d$.

Then there exists a global solution $X \in H^2_{loc}$ with continuous paths which is unique up to stochastic equivalence.

REMARKS: 1) The corresponding theorem for Carathéodory solutions of ordinary differential equations is valid with $L^1_{loc}(\mathbb{R}_+,\lambda)$ instead of $L^2_{loc}(\mathbb{R}_+,\lambda)$ (cf. WALTER [7], p.82).

2) By the Lipschitz condition, the statement of condition (2) holds for all $x \in \mathbb{R}^d$ if it holds for some.

The properties of the stochastic integral used in the sequel can be found e.g. in ARNOLD [1] or FRIEDMAN [4]. For predictable sets and more general stochastic integration see KUSSMAUL [5].

LEMMA 3: Under the hypotheses of theorem 2, the process $Y_t := \int_0^t f(s,X_s)ds$, $t \geq 0$, is in H^2_{loc} for every $X \in H^2_{loc}$. For $X \in H^2$, the stopping times $T_n := n$, $n \in \mathbb{N}$, localize Y.

Proof: Let $(T_n)_{n\in\mathbb{N}}$ be a localizing sequence for $X \in H^2_{loc}$. Replacing T_n by $T_n \wedge n$ if necessary, we may always assume $T_n \leq n$. Using Hölder's inequality, we get for all $t \geq 0$

$$E(|Y_t^{T_n}|^2) = E(|\int_0^{t\wedge T_n} f(s,X_s)ds|^2) \leq E((t\wedge T_n)\int_0^{t\wedge T_n}|f(s,X_s)|^2ds)$$

$$\leq 2nE(\int_0^{T_n}|f(s,X_s) - f(s,0)|^2 + |f(s,0)|^2ds)$$

$$\leq 2nE(\int_0^{T_n}|X_s|^2\, \ell^2(s)ds + \int_0^{T_n}|f(s,0)|^2ds)$$

$$\leq 2n\, \|X^{T_n^*}\|^2 \int_0^n \ell^2(s)ds + 2n\int_0^n |f(s,0)|^2ds < \infty.$$

Hence $Y \in H^2_{loc}$, and $(T_n)_{n\in\mathbb{N}}$ localizes Y.

LEMMA 4: Under the hypotheses of theorem 2, the process $Z_t := \int_0^t G(s,X_s)dW_s$, $t \geq 0$, is in H^2_{loc} for every $X \in H^2_{loc}$. For $X \in H^2$, the stopping times $T_n := n$, $n \in \mathbb{N}$, localize Z.

Proof: As the stochastic integral with respect to W is an isometry of L^2 into H^2, we have only to show that the process $GX_t := G(t,X_t)$, $t \geq 0$, is in L^2_{loc}. Let $(T_n)_{n\in\mathbb{N}}$ be a localizing sequence for $X \in H^2_{loc}$ such that $T_n \leq n$. We have

$$E(\int_0^\infty |GX_t 1_{[\![0,T_n]\!]}(t)|^2 dt) \leq 2E(\int_0^{T_n} |G(t,X_t) - G(t,0)|^2 + |G(t,0)|^2 \, dt)$$

$$\leq 2E(\int_0^{T_n} |X_t|^2 \, \ell^2(t) + |G(t,0)|^2 \, dt)$$

$$\leq 2 \, \|X^{T_n^*}\|^2 \int_0^n \ell^2(t)dt + 2\int_0^n |G(t,0)|^2 dt < \infty.$$

Hence $GX \in L^2_{loc}$, and $(T_n)_{n\in\mathbb{N}}$ localizes GX, hence Z.

Proof of theorem 2: Consider the, in general non - linear, operator $S: H^2_{loc} \to H^2_{loc}$ defined by

$$SX_t := C_t + \int_0^t f(s,X_s)ds + \int_0^t G(s,X_s)dW_s \,, \qquad t \geq 0.$$

The initial process $C \in H^2_{loc}$ is localized by the stopping times $T_n := \inf\{t \geq 0: |C_t| \geq n\} \wedge n$, $n \in \mathbb{N}$, and so is SX for all $X \in H^2$ by Lemma 3 and 4. The map $X \to (SX)^{T_n^*}$ therefore defines an operator $S_n: H^2 \to H^2$. For all $X,Y \in H^2$ and $t \geq 0$ we get

$$E(|S_n X_t - S_n Y_t|^2)$$

$$\leq 2E(T_n \int_0^{t\wedge T_n} |f(s,X_s) - f(s,Y_s)|^2 ds) + 2E(|\int_0^{t\wedge T_n} G(s,X_s) - G(s,Y_s)dW_s|^2)$$

$$\leq 2nE(\int_0^{t\wedge T_n} |f(s,X_s) - f(s,Y_s)|^2 ds) + 2E(\int_0^{t\wedge T_n} |G(s,X_s) - G(s,Y_s)|^2 ds)$$

$$\leq 2(n+1) \int_0^{t\wedge n} E(|X_s^{T_n} - Y_s^{T_n}|^2) \, \ell^2(s)ds$$

showing in particular that S_n can be considered as an operator on the quotient space $H^2_{T_n}$. Inserting $e^{-\alpha L(s)}e^{\alpha L(s)}$ with $\alpha \geq 0$ and $L(s) := \int_0^s \ell^2(r)dr$ for $s \geq 0$ yields

$$\begin{aligned}
&E(|S_nX_t - S_nY_t|^2) \\
&\leq 2(n+1) \int_0^{t\wedge n} E(|X_s^{T_n} - Y_s^{T_n}|^2)\, e^{-\alpha L(s)}e^{\alpha L(s)}\, \ell^2(s)ds \\
&\leq 2(n+1) \sup_{s\geq 0} E(|X_s^{T_n} - Y_s^{T_n}|^2\, e^{-\alpha L(s\wedge n)}) \int_0^{t\wedge n} e^{\alpha L(s)}\, \ell^2(s)ds \\
&\leq 2(n+1)\, |||\, X - Y\, |||^2_{T_n}\, \frac{1}{\alpha}\, e^{\alpha L(t\wedge n)}
\end{aligned}$$

where $|||\cdot|||_{T_n}$ denotes the seminorm on H^2 defined by

$$|||\, X\, |||_{T_n} := \big(\sup_{t\geq 0} E(|X_t^{T_n}|^2\, e^{-\alpha L(t\wedge n)})\big)^{1/2}.$$

Since $e^{-\alpha L(n)/2}\|X\|_{T_n} \leq |||\, X\, |||_{T_n} \leq \|X\|_{T_n}$, it is equivalent to $\|\cdot\|_{T_n}$ and so it defines an equivalent norm on $H^2_{T_n}$. The above estimate now reads

$$|||\, S_nX - S_nY\, |||^2_{T_n} \leq \frac{2(n+1)}{\alpha}\, |||\, X - Y\, |||^2_{T_n} \quad .$$

For a suitable choice of α , e.g. $\alpha = 8(n+1)$, the mapping S_n is a contraction on $H^2_{T_n}$. By the Banach fixed point theorem there is a unique fixed point ${}^nX \in H^2_{T_n}$. As for $X \in H^2$, $|||\, X\, |||_{T_n} = 0$ is equivalent to $E(|X_t^{T_n}|) = 0$ for all $t \geq 0$, this fixed point corresponds to a process ${}^nX \in H^2$ such that for all $t \geq 0$, ${}^nX_t^{T_n}$ is uniquely determined up to $L(\Omega,P)$ - equivalence.

For $m > n$ and the corresponding fixed points ${}^mX \in H^2_{T_m}$ and ${}^nX \in H^2_{T_n}$ we have

$${}^mX^{T_n^*} = (S_m{}^mX)^{T_n^*} = S_n({}^mX^{T_n}) \qquad\qquad \text{in } H^2_{T_n},$$

hence by the uniqueness property ${}^mX_t^{T_n^*} = {}^nX_t^{T_n}$ P-a.s. for all $t \geq 0$ and any representatives ${}^mX, {}^nX$ in H^2, and a global solution $X \in H^2_{loc}$ is determined uniquely up to $L(\Omega,P)$ - equivalence.

By the continuity properties of the Lebesgue integral and of the Ito integral there exist continuous versions of X, and these versions are unique up to stochastic equivalence since it is possible to find a common null set outside of which the trajectories coincide.

REMARK: The reason for the use of $X^{T_n^*}$ instead of X^{T_n} in the definition of H^2_{loc} and in the proof of theorem 2 is to admit initial processes C without integrability restrictions. In the case of $C \in H^2$, one could use stopping times $(T_n)_{n \in \mathbb{N}}$ converging uniformely to ∞, e.g. $T_n := n$, and define S_n by $(SX)^{T_n}$ in the proof of theorem 2.

By the usual estimate of the successive approximation procedure starting with $X^0 := 0$ one gets the following

COROLLARY 5: Under the hypotheses of theorem 2 and with $\alpha = 8(n + 1)$, T_n, L defined as above, the solution $X \in H^2_{loc}$ satisfies

$$\|X^{T_n^*}\|^2 \leq 12(\|C^{T_n^*}\|^2 + (n + 1)k_n)\ e^{8(n + 1)L(n)}$$

for all $n \in \mathbb{N}$ where $k_n := \max\left(\int_0^n |f(t,0)|^2 dt, \int_0^n |G(t,0)|^2 dt\right)$.

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