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## COMMENTS ON NONSTANDARD TOPOLOGY

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This paper is yet another variation on the general theme of «standard versus nonstandard topology».

Fenstad wrote a paper [5] to show that «nonstandard points and ultrafilters both describe all possible ways how a subfamily of a given family may converge».

Commenting on Fenstad's paper, Luxemburg [12] says : «The author shows the rather obvious fact that Robinson's [nonstandard] characterization of compactness follows from Cartan's [standard] criterion of compactness. We may add, however, that  ${}^*M$  [an enlargement of  $M$ ] contains far more information about the topological space  $M$  than the above criterion may indicate».

To be fair, one should also add that the space of ultrafilters  $\gamma(M)$  too contains far more information about  $M$  than Cartan's criterion may indicate.

Machover and Hirschfeld [13, p. 67] write : «Those who dislike non-standard analysis may develop an alternative and parallel theory, dealing with the set of ultrafilters on  $T$ , rather than with  $\hat{T}$  [the scope of  $T$  in an enlargement]. This can be done, but it has two disadvantages :

First : Considerations that are natural in non-standard analysis look unnatural and tricky when dealing with the set of ultrafilters.

Second : Some proofs become more difficult as one is not able to use the fact that logical statements hold for  $T$  if and only if they hold for  $\hat{T}$  (in the proper interpretation)».

Actually, as early as 1962, such a theory *has* been developed ([7], [8], [10]) dealing with the set of ultrafilters. It is true that some proofs might be easier by nonstandard methods, but others seem easier by standard methods.

No doubt, nonstandard methods can give a special insight in topological matters as they are mainly *a new way to look at old things*. But, as already noted by Fenstad [5] and Luxemburg [11], in the case of enlargements,  ${}^*E$  bears much resemblance to the Stone space  $\gamma(E)$  of ultrafilters on  $E$ . This will be made precise in the text, and there will be set a sort of «dictionary» that may serve to translate from the language of nonstandard topology into that of standard topology, and the other way round.

Now, the following relation holds in nonstandard models :

$$(1) \quad {}^*(E \times F) = {}^*E \times {}^*F,$$

whereas, if  $E$  and  $F$  are infinite, the space  $\gamma(E \times F)$  is not even homeomorphic to the product space  $\gamma(E) \times \gamma(F)$ . From there stems, in our opinion, the main technical superiority of nonstandard methods over standard methods *in topology*.

A main tool in nonstandard topology is the notion of *enlargement*. An enlargement is a certain kind of nonstandard model satisfying a sort of saturation property. Now, it is known that «saturation» is a property akin to compactness. This can be made precise when a suitable topology is introduced in a given model (viz. the S-topology of Luxemburg [11]). We would like to lay stress upon the fact that *compactness* is an essential feature of nonstandard methods in topology. It will be seen that it pervades the whole subject.

Note that the set  ${}^*(E \times F)$  can be equipped with two topologies : its own S-topology and the product topology on  ${}^*E \times {}^*F$  of the respective S-topologies on  ${}^*E$  and  ${}^*F$ . We will take advantage of this situation to introduce standard methods into nonstandard topology.

Standard and nonstandard methods will appear to be quite intermingled, and a proof using one of the methods can be almost automatically translated into the other. So far, all the known results proven by nonstandard methods in topology had previously been established by quite standard methods. It is to be expected, though, that the nonstandard point of view will help to discover new results because of the added intuition it can give.

## 0. Introduction.

The notion of an enlargement has been defined in several ways differing by technical details, the main idea being essentially the same, of course. The notion was introduced by Robinson [16] using a type-theoretical version of higher-order logic. Machover and Hirschfeld [13] dispense with the theory of types and use an instance of a first-order language with equality. Robinson and Zakon [17] describe a purely set-theoretical approach to the subject. Luxemburg [11] gives a «simplified» version of the theory developed by Robinson [16] using again higher-order structures and higher-order languages. We will use this last version for our purposes with *one slight modification* : We take into account the possibility for the same entity to have different types.

After some *preliminaries* in section 1, recalling certain notations and definitions, namely that of  $\gamma(E)$  the space of ultrafilters on  $E$ , we will study properties of *weakly Hausdorff spaces* and their reductions in section 2. Then, for future reference, we will proceed by recalling cursorily the theory of *enlargements* along the lines of Luxemburg in sections 3, 4, and 5. We do that in order to make the paper more self-contained, but also to add a few details to the theory here and there. The S-topology and the monads of filters will be discussed in higher-order nonstandard models. It turns out that every standard subset  ${}^*E$  is a compact weakly Hausdorff space in its S-topology if and only if the model is an enlargement, and then there is a canonical «reduction»  $\psi : {}^*E \longrightarrow \gamma(E)$ . This shows how close  ${}^*E$  is to  $\gamma(E)$ .

In section 6, under the name of *compact enlargements*, we will study those enlargements in which every internal set is compact in the S-topology. Section 7 is devoted to the *P-topology* on product sets in an enlargement, and to its relation with *maxed* filters and *grated* relations.

Our main sources on nonstandard topology are Robinson [16], Luxemburg [11], and Machover and Hirschfeld [13]. In the last four sections, we compare these nonstandard treatments to a standard treatment of the principal topological structures : topological spaces, uniform spaces, and proximity spaces. Section 8 is on *topological spaces*, with a few words on pretopologies of Choquet. It contains a characterization of the monads of topologies (8.9), and the fact that the monad of a topological space  $E$  is the preimage of the «nassa» of  $E$  under the canonical mapping  $\psi$  (8.6). In section 9, we give several new nonstandard characterizations of classical topological properties and

the property of «weak regularity». *Uniform spaces*, and their generalization, semi-uniform spaces, are studied in section 10 where a new nonstandard characterization of precompactness is given (10.12, 10.13). Finally section 11 is devoted to *proximity spaces* with a few words on topogenous orders of Császár.

The main results have been labelled theorems (8.6, 8.9, 10.6, 10.13, 11.3, 11.5, 11.7) except those in section 9 which are called propositions (namely 9.1 to 9.10, 9.12, 9.18, 9.19) because they are mere translations from standard topology in [7], [10]. Among other propositions, the most important are 4.10, 5.3, 5.4, 6.3, 7.4, and 7.12.

## 1. Preliminaries.

1.1. **Terminology and notations.** - Recall that a *graph* is a set of (ordered) couples. Because it will be important not to confuse an element  $x$  of a set  $X$  and the corresponding singleton  $\{x\}$  (which might also be an element of  $X$ ) we will adopt the following notation :

If  $G$  is a graph, and  $A$  a set, then  $G \langle A \rangle$  will denote the set  $\{y \mid \text{there exists } x \in A \text{ such that } (x,y) \in G\}$ . When  $A = \{a\}$  is a singleton, we will write  $G(a)$  instead of  $G \langle \{a\} \rangle$ .

Mappings can be assimilated to their graphs. However, when  $f : E \rightarrow F$  is a mapping, if  $y$  is the image of  $x \in E$  under  $f$ , then following the common usage, we have  $f(x) = y$ , and not  $f(x) = \{y\}$ . If  $A$  is a subset of  $E$ , then  $f \langle A \rangle = \{f(x) \mid x \in A\}$ .

Let  $G, H$  be graphs. The *composition*  $HG$  is the graph defined by  $(HG)(x) = H \langle G(x) \rangle$ . The *converse*  $\overset{-1}{G}$  of  $G$  is the graph defined by « $(x,y) \in \overset{-1}{G}$  if and only if  $(y,x) \in G$ ». Instead of  $GG$  and  $GGG$  we will write  $\overset{2}{G}$  and  $\overset{3}{G}$  respectively.

A graph  $G$  is called *reflexive* on a set  $E$  whenever  $G$  contains the diagonal of  $E \times E$ . It is called *symmetric* whenever  $\overset{-1}{G} = G$ . It is called *transitive* whenever  $\overset{2}{G} \subset G$ .

Recall that a *preorder* on a set  $E$  is a binary relation which is reflexive and transitive on  $E$ .

Dealing with topological spaces, we will follow the american usage. Thus, unless explicitly stated, a space need not be Hausdorff.

Finally, given a set  $E$ , we will denote by  $\mathcal{P}(E)$  the *set of subsets* of  $E$ .

1.2. **The space of ultrafilters.** - Given a set  $E$ , for every subset  $A$  of  $E$ , let  $\gamma_E(A)$  or simply  $\gamma(A)$  denote the set of those ultrafilters on  $E$  to which  $A$  belongs. Thus  $\gamma(E)$  is the set of all ultrafilters on  $E$ . The set  $\{\gamma(A) \mid A \subset E\}$  is a base for a topology on  $\gamma(E)$ . The topological space thus obtained is called the *space of ultrafilters* on  $E$ . It is known to be compact, Hausdorff, and extremally disconnected. It can be identified with the Stone-Ćech compactification of the discrete space  $E$ . See, for example, Gillman and Jerison [6].

Every element  $\gamma(A)$  of the base is both open and closed. An element in  $\gamma(E)$  is an isolated point if and only if it is a principal ultrafilter, that is a filter generated by a singleton on  $E$ . The set of isolated points is dense in  $\gamma(E)$ .

Notice that, when  $E$  and  $F$  are infinite sets, the product space  $\gamma(E) \times \gamma(F)$  is not extremally disconnected, and, hence is not homeomorphic to the space  $\gamma(E \times F)$ .

1.3. **Remark.** - The symbols  $\gamma, \mu, \varphi, \psi$  will constantly keep their meanings throughout the text. For the definitions of  $\mu$  and  $\varphi$ , see 4.4 for the definition of  $\psi$ , see 4.8.

Likewise, starting from the middle of subsection 4.2, the symbols  $\mathfrak{m}$  and  $^*\mathfrak{m}$  will retain their meanings. However, an additional condition will be imposed on  $^*\mathfrak{m}$ , from 5.5 onwards.

## 2. Weakly Hausdorff spaces.

2.1. **Definition.** - A topological space is said to be *weakly Hausdorff* whenever, given any two

points  $x, y$  such that  $\overline{\{x\}} \neq \overline{\{y\}}$ , there are two disjoint open sets  $U, V$  such that  $x \in U$  and  $y \in V$ .

A topological space is said to be *weakly  $T_1$*  whenever any neighbourhood of a point  $x$  contains  $\overline{\{x\}}$ .

Clearly, every Hausdorff space is weakly Hausdorff, and every  $T_1$  space is weakly  $T_1$ .

**2.2. Proposition.** - *Every weakly Hausdorff space is weakly  $T_1$ .*

**Proof.** Let  $U$  be a neighbourhood of  $x$ , and let  $y \notin U$ . Then  $x \notin \overline{\{y\}}$ . Therefore  $\overline{\{x\}} \neq \overline{\{y\}}$ . Now, if the space is weakly Hausdorff, there must be a neighbourhood of  $y$  to which  $x$  does not belong, so that  $y \notin \overline{\{x\}}$ . Hence  $\overline{\{x\}} \subset U$  and the space is weakly  $T_1$ . Qed

**2.3. The reduced space.** - Let  $X$  be a weakly Hausdorff space. Then the relation  $\overline{\{x\}} \cap \overline{\{y\}} \neq \emptyset$  is equivalent to  $\overline{\{x\}} = \overline{\{y\}}$  (use proposition 2.2). Therefore it is an equivalence relation. Let  $\rho$  be its graph, that is  $(x, y) \in \rho$  if and only if  $\overline{\{x\}} = \overline{\{y\}}$ . Given any subset  $A$  of  $X$ , the subset  $\rho < A >$  is called the *saturation* of  $A$ . The subset  $A$  is said to be *saturated* whenever  $\rho < A > = A$ . (Beware, these are the usual notions relative to equivalence relations; they have nothing to do with the model-theoretical notion of «saturation»).

The quotient space  $X/\rho$  will be called the *reduced space* associated with  $X$ . See below property  $P_8$ .

**2.4. Definition.** - Given a topological space  $X$ , a continuous mapping  $f : X \rightarrow Y$  is called a *reduction* of  $X$  whenever  $Y$  is a Hausdorff space, and, for every continuous mapping  $g : X \rightarrow Z$  of  $X$  into a Hausdorff space  $Z$ , there is a *unique* continuous mapping  $h : Y \rightarrow Z$  such that  $g = hf$ .

Of course, two reductions of the same space are always canonically «isomorphic».

**2.5. Properties.** - Here are some properties of weakly Hausdorff spaces. Their proof is routine topology.

$P_1$ . - *Every subspace of a weakly Hausdorff space is weakly Hausdorff.*

$P_2$ . - *The product of a family of weakly Hausdorff spaces is a weakly Hausdorff space.*

$P_3$ . - *In a weakly Hausdorff space, the saturation of a singleton  $\{x\}$  is its closure  $\overline{\{x\}}$ .*

$P_4$ . - *Closed subsets and open subsets in a weakly Hausdorff space are saturated.*

$P_5$ . - *A subset of a weakly Hausdorff space is closed (resp. open) if and only if a) it is saturated, and b) its image in the reduced space is closed (resp. open).*

$P_6$ . - *The topology of a weakly Hausdorff space is the preimage of the topology of its reduced space.*

$P_7$ . - *The reduced space associated with a weakly Hausdorff space is a Hausdorff space.*

$P_8$ . - *The canonical mapping of a weakly Hausdorff space into its reduced space is a reduction.*

Properties  $P_6$  and  $P_8$  show how close a weakly Hausdorff space is to its reduced space.

**2.6. Proposition.** - *Let  $X$  be a weakly Hausdorff space, and let  $A$  be a subset of  $X$ . Then the following statements are equivalent.*

a) *The subspace  $A$  is compact.*

b) *The image of  $A$  in the reduced space is compact.*

c) *The subspace  $\bar{A}$  is compact, and  $\bar{A}$  is the saturation of  $A$ .*

**Proof.** Let  $\rho$  be the canonical equivalence relation on  $X$ , and  $f : X \rightarrow X/\rho$  be the corresponding reduction.

a)  $\Rightarrow$  b). The space  $f \langle A \rangle$  is the continuous image of the compact space  $A$ , hence it is compact.

b)  $\Rightarrow$  c). Since  $\bar{A}$  is saturated by  $P_4$ , we have  $A \subset \rho \langle A \rangle \subset \bar{A}$ .

But  $\rho \langle A \rangle = \overline{f \langle A \rangle}$ , and  $f \langle A \rangle$  is closed since the reduced space is Hausdorff by  $P_7$ .

Therefore  $\rho \langle A \rangle$  is closed, and  $\rho \langle A \rangle = \bar{A}$ . The topology on  $\bar{A}$  is the preimage of the topology on  $f \langle A \rangle$  by  $P_6$ . Hence  $\bar{A}$  is compact.

c)  $\Rightarrow$  a). Let  $\mathcal{E}$  be a set of open subsets of  $E$  that covers  $A$ . Since  $\bar{A}$  is the saturation of  $A$ , and since the elements of  $\mathcal{E}$  are saturated by  $P_4$ ,  $\mathcal{E}$  also covers  $\bar{A}$ . Since  $\bar{A}$  is compact,  $\mathcal{E}$  contains a finite subcover of  $\bar{A}$  which also covers  $A$ , of course. Hence  $A$  is compact.

**Qed**

For the notion of *proper map* to be used now, see Bourbaki [1].

**2.7. Proposition.** - Let  $A$  be a compact space, and let  $B$  be a weakly Hausdorff space. Let  $f : A \rightarrow B$  be a continuous mapping. Then the following statements are equivalent.

a)  $f$  is proper.

b)  $f$  is closed.

c)  $f \langle \overline{\{a\}} \rangle = \overline{\{f(a)\}}$ , for every  $a \in A$ .

**Proof.** a)  $\Rightarrow$  b)  $\Rightarrow$  c) is immediate following Bourbaki [1, I, 10, 1. proposition 1, p. 113, and I, 5, 4, proposition 9, p. 61].

c)  $\Rightarrow$  a). We use a criterion of Bourbaki [1, I, 10, 2, théorème 1 d), p. 118]. Let  $\mathcal{U}$  be an ultrafilter on  $A$  such that  $f(\mathcal{U})$  converges to  $b \in B$ . Since  $A$  is compact,  $\mathcal{U}$  converges to at least one point  $a \in A$ . But then  $\mathcal{U}$  converges to all points in  $\overline{\{a\}}$ . Therefore  $f(\mathcal{U})$  converges to all points in  $f \langle \overline{\{a\}} \rangle = \overline{\{f(a)\}}$  because  $f$  is continuous. Since  $B$  is weakly Hausdorff,  $f(\mathcal{U})$  converges only to the points in  $\overline{\{f(a)\}}$ . Therefore  $b \in \overline{\{f(a)\}}$ . Then there is an  $x \in \overline{\{a\}}$  such that  $b = f(x)$  and  $\mathcal{U}$  converges to  $x$ . The criterion is satisfied.

**Qed**

The two preceding propositions can be considered as generalizations of results on Hausdorff spaces.

**2.8. Proposition.** - Let  $A, B$  be weakly Hausdorff spaces. Let  $f : A \rightarrow B$  be an open mapping such that  $\overline{f(b)}$  is compact, for every  $b \in B$ . Then  $\overline{\{f(a)\}} \subset f \langle \overline{\{a\}} \rangle$ , for every  $a \in A$ .

**Proof.** For every neighbourhood  $V$  of  $a$ ,  $f \langle V \rangle$  is a neighbourhood of  $f(a)$ , since  $f$  is open. Let  $b \in \overline{\{f(a)\}}$ , then  $f(a) \in \overline{\{b\}}$ , since  $B$  is weakly Hausdorff. Therefore  $b \in f \langle V \rangle$ , so that  $V \cap \overline{f(b)} \neq \emptyset$ . By hypothesis  $K = \overline{f(b)}$  is compact; and  $a \in \overline{K}$ . Therefore, by proposition 2.6, we have  $\overline{\{a\}} \cap K \neq \emptyset$  so that  $b \in f \langle \overline{\{a\}} \rangle$ .

**Qed**

### 3. Higher-order structures.

**3.1. Finite types.** - The class of (finite) *types* is defined as follows :

a)  $0$  is a type.

b) If  $t_1, \dots, t_n$  are types, then  $(t_1, \dots, t_n)$  is a type.

c) The class of types is the smallest class satisfying a) and b).

Let  $\mathcal{T}$  be the set of types, and let  $X$  be a set. Let  $X_0 = X$ , and, inductively, if  $t = (t_1, \dots, t_n)$ ,

let  $X_t = \mathcal{P}(X_{t_1} \times \dots \times X_{t_n})$ . Entities of type  $t$  on  $X$  are precisely elements of  $X_t$ . That is, entities of type  $t = 0$  are simply elements of  $X$  (also called *individuals*); when  $t = (t_1, \dots, t_n)$  (i.e.  $t \neq 0$ ) entities of type  $t$  are (graphs of)  $n$ -ary relations whose  $i$ -th arguments are entities of type  $t_i$ . Entities of a type of the form  $(t)$  are also called *subsets*.

**3.2. Higher-order structures.** - A *higher-order structure* over a set  $X$  is a family

$$\mathfrak{M} = (M_t)_t \in \mathcal{T} \text{ such that}$$

$$\text{a) } M_0 = X_0 = X.$$

$$\text{b) } M_t \subset X_t, \text{ for every } t \in \mathcal{T}$$

$$\text{c) If } t = (t_1, \dots, t_n) \text{ and } (A_1, \dots, A_n) \in R \in M_t, \text{ then } A_i \in M_{t_i}, \text{ for } 1 \leq i \leq n.$$

The *full* higher-order structure over a set  $X$  is, by definition, the family  $(X_t)_t \in \mathcal{T}$ .

**3.3. The formal language  $L$ .** - We use a formal language, called  $L$ , built on the first-order language by the adjunction of *type predicates* and a sequence of *basic predicates*. So the language  $L$  contains :

a) The usual *connectives*  $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$ . b) The usual *quantifiers*  $\forall, \exists$ . c) The *variables* (an infinite sequence of symbols). d) *Brackets*. e) *Constants* : a set  $K$  of symbols. f) *Type predicates* : a symbol  $T_t(\cdot)$ , for every type  $t \in \mathcal{T}$ . g) *Basic predicates* : a sequence  $\Phi_n(\cdot, \dots, \cdot)$  enclosing  $n + 1$  spaces,  $n = 1, 2, \dots$

*Well formed formulas* (wff's) and *sentences* (wff's in which every variable is under the scope of a quantifier) are defined in the usual way.

**3.4.  $L$ -structures.** - An  *$L$ -structure*  $(\mathfrak{M}, \theta)$  over a set  $X$  is a higher-order structure

$$\mathfrak{M} = (M_t)_t \in \mathcal{T} \text{ over } X \text{ together with an injective mapping } \theta : N \rightarrow K \text{ of the disjoint union}$$

$$N = \sum_{t \in \mathcal{T}} M_t \text{ (that is the set of couples } (R, t) \text{ where } R \in M_t \text{) into the set of constants } K \text{ of } L.$$

The «trick» of the disjoint union has to be used because, unfortunately, the same set-theoretical entity may have many different types in one and the same higher-order structure. In many instances, it will have to be understood from the context what type is under consideration, even though it is not explicitly stated. It is hoped that no special difficulties will arise for the reader from this procedure.

A sentence of  $L$  is *defined* in an  $L$ -structure whenever the constants contained in it are in the range of the injective mapping defining the structure.

A sentence of  $L$ , defined in an  $L$ -structure  $(\mathfrak{M}, \theta)$ , may be *true* or *false* in the  $L$ -structure according to the usual rules to which are added the following interpretations of the new predicates :

$$\text{a) } T_t(a) \text{ is true if and only if } A \in M_t \text{ and } a = \theta(A, t).$$

$$\text{b) } \Phi_n(a, a_1, \dots, a_n) \text{ is true if and only if } (A_1, \dots, A_n) \in A, \text{ and } a = \theta(A, t), a_i = \theta(A_i, t_i),$$

$$1 \leq i \leq n, \text{ and } t = (t_1, \dots, t_n).$$

#### 4. Nonstandard models.

**4.1. Definition.** - A *higher-order nonstandard model* of an  $L$ -structure  $(\mathfrak{M}, \theta)$  is an  $L$ -structure  $(^* \mathfrak{M}, ^* \theta)$  such that all sentences that are defined and hold in  $(\mathfrak{M}, \theta)$  are defined and also hold in  $(^* \mathfrak{M}, ^* \theta)$ .

**4.2.** Let  $A$  be an entity of type  $t$  in  $\mathfrak{M}$ , and let  $a = \theta(A, t)$ . Since  $T_t(a)$  holds in  $(\mathfrak{M}, \theta)$ , there must be an entity  $^*A$  of type  $t$  in  $^* \mathfrak{M}$  such that  $a = ^* \theta(^*A, t)$ . The mapping thus defined is unique

and injective over each  $M_t$ . (In some instances, the type  $t$  will not be explicitly stated, and it will be assumed implicitly that the star operation  $A \rightarrow {}^*A$  is taken for that type).

The entities of  ${}^*\mathfrak{M}$  that are of the form  ${}^*A$  are called *standard*. Now, suppose that the structure  $\mathfrak{M}$  is built over the set  $X$ . If one considers the *full* higher-order structure built over  ${}^*X$ , it has two sorts of entities : those that already belong to the structure  ${}^*\mathfrak{M}$  : these are called *internal* entities of  ${}^*\mathfrak{M}$  ; and those that do not belong to  ${}^*\mathfrak{M}$ , and these are called *external* entities of  ${}^*\mathfrak{M}$ . So that namely all individuals, and all standard entities of  ${}^*\mathfrak{M}$  are internal entities. Clearly, the structure  ${}^*\mathfrak{M}$  is full if and only if there are no external entities.

**In the sequel, we shall consider, once and for all, a given full L-structure  $\mathfrak{M} = (X_t)_{t \in \mathcal{T}}$  based over the set  $X$ , and a given nonstandard model  ${}^*\mathfrak{M}$  of  $\mathfrak{M}$ .**

The L-structure  ${}^*\mathfrak{M}$  need not be full as is well known.

If  $E_t$  is the identity relation on  $X_t$ , then, clearly,  ${}^*E_t$  is an equivalence relation on  ${}^*X_t$ . It is known (see Robinson [16, p. 39]) that, when  $t \neq 0$ ,  ${}^*E_t$  is precisely the identity relation on  ${}^*X_t$ . If  $t = 0$ , this needs no more be the case. However, *we will assume that our model  ${}^*\mathfrak{M}$  satisfies the condition that  ${}^*E_0$  is the identity relation for individuals in  ${}^*\mathfrak{M}$ .* (The existence of such models, satisfying also additional conditions, is guaranteed by ultrapower constructions.)

When a *notion* can be defined in the L-structure  $\mathfrak{M}$  using the language  $L$ , it can be transformed into a corresponding *star-notion* defined in the model  ${}^*\mathfrak{M}$ . Consider, for instance, the notion of *finiteness* : it can be defined in many different *but equivalent* ways in the L-structure  $\mathfrak{M}$ . For example : «The set  $E$  is finite if and only if no proper subset of  $E$  is isomorphic to  $E$ ». So that *\*-finiteness* has different (*but also equivalent*) definitions in  ${}^*\mathfrak{M}$ , one of which is «The internal set  $E$  is *\*-finite* if and only if no *\*-proper \*-subset* of  $E$  is *\*-isomorphic* to  $E$ ».

Of course every finite internal set in  ${}^*\mathfrak{M}$  is *\*-finite* but a *\*-finite* set need not be finite.

**4.3. The S-topology.** - For every  $t \in \mathcal{T}$ , we define a topology on  ${}^*X_t$  called the S-topology as follows : the set of standard entities of type  $(t)$  in  ${}^*\mathfrak{M}$  is a base for open sets of the S-topology on  ${}^*X_t$ . Notice that since

$$(2) \quad {}^*(X_t - E) = {}^*X_t - {}^*E,$$

every open set in the base is also closed. Moreover, every closed subset  $A$  in  ${}^*X_t$  is the intersection of the set of standard subsets of type  $(t)$  that contain  $A$ .

This topology coincides with the S-topology defined by Luxemburg [11, p. 47] in the case of enlargements, as will presently be seen (remark 5.9).

Every subset  $A$  of  ${}^*X_t$  (internal or external) is thus endowed with a topology (the relative topology) also to be called the S-topology on  $A$ .

**4.4. Monads.** - Given an entity  $\mathcal{F}$  of type  $((t))$  in  $\mathfrak{M}$ , the monad  $\mu(\mathcal{F})$  of  $\mathcal{F}$  is defined as the intersection of all standard *\*-members* of  ${}^*\mathcal{F}$ , that is

$$\mu(\mathcal{F}) = \bigcap \{ {}^*E \mid E \in \mathcal{F} \text{ and } E \text{ is of type } (t) \}.$$

The subset  $\mu(\mathcal{F})$  of  ${}^*X_t$  might well be empty, even if  $\mathcal{F}$  is a filter. We will see, though, that the property «  $\mu(\mathcal{F}) \neq \emptyset$  », for every filter  $\mathcal{F}$  » is characteristic of enlargements (see proposition 5.3).



On the other hand, given a subset  $A$  of  ${}^*X_t$ , internal or external, define

$$\varphi(A) = \{E \mid E \subset X_t \text{ and } A \subset {}^*E\} .$$

Then  $\varphi(A)$  is an entity of type  $((t))$  in  $M$ . If  $A = \emptyset$ , then  $\varphi(A) = X_{(t)} = \mathcal{P}(X_t)$ , the «improper filter» on  $X_t$ . If  $A \neq \emptyset$ , then  $\varphi(A)$  is a filter on  $X_t$ . If  $A = \{a\}$  is a singleton, then  $\varphi(\{a\})$  is an ultrafilter on  $X_t$ , since, given any subset  $E$  of  $X_t$ , either  $E \in \varphi(\{a\})$  or  $(X_t - E) \in \varphi(\{a\})$  using relation (2).

From the definitions, it is clear that

$$(3) \quad \mathcal{F} \subset \varphi(\mu(\mathcal{F})).$$

**4.5. Proposition.** - Given a subset  $A$  of  ${}^*X_t$ , let  $\bar{A}$  be its closure in the  $S$ -topology of  ${}^*X_t$ . Then  $\bar{A} = \mu(\varphi(A))$ .

**Proof.** By a previous remark,  $\bar{A}$  is the intersection of all standard subsets of  ${}^*X_t$  that contain  $A$ , hence the conclusion. Qed

**4.6. Corollary.** - A nonempty subset of  ${}^*X_t$  is closed if and only if it is the monad of a filter on  $X_t$ .

**4.7. Proposition.** - The space  ${}^*X_t$  in its  $S$ -topology is weakly Hausdorff.

**Proof.** Let  $x, y$  be points in  ${}^*X_t$  such that  $\overline{\{x\}} \neq \overline{\{y\}}$ . Then, according to the previous proposition  $\mu(\varphi(\{x\})) \neq \mu(\varphi(\{y\}))$ . Therefore the ultrafilters  $\varphi(\{x\})$  and  $\varphi(\{y\})$  are distinct. Therefore there exists a subset  $E$  of  $X_t$  such that  $E \in \varphi(\{x\})$  and  $F = X_t - E \in \varphi(\{y\})$ . Hence  ${}^*E$  and  ${}^*F$  are disjoint open sets such that  $x \in {}^*E$  and  $y \in {}^*F$ . Qed

**4.8. The mapping  $\psi$ .** - According to proposition 4.7 and to property  $P_1$  in 2.5, given a subset  $E$  in  $\mathfrak{M}$ , the space  ${}^*E$  is weakly Hausdorff in its  $S$ -topology.

Consider the mapping  $\psi: {}^*E \rightarrow \gamma(E)$  defined by  $\psi(x) = \varphi(\{x\})$ . Given any subset  $A$  of  $E$ , we have  $\psi^{-1}(\gamma(A)) = {}^*A$ . So  $\psi$  is a continuous mapping. But  $\psi$  need not be onto. In fact, as will be seen in corollary 5.4, the condition that  $\psi$  be onto is characteristic of enlargements. Proposition 4.10 will show that  $\psi({}^*E)$  can be identified with the reduced space of  ${}^*E$ .

**4.9. Lemma.** -  $\varphi(\mu(\psi(x))) = \psi(x)$ .

**Proof.** We have  $x \in \mu(\psi(x)) \neq \emptyset$ , so  $\varphi(\mu(\psi(x)))$  is a filter which contains  $\psi(x)$  by relation (3). But  $\psi(x)$  is an ultrafilter. Hence the conclusion. Qed

**4.10. Proposition.** - The mapping  $\psi: {}^*E \rightarrow \psi\langle {}^*E \rangle$  is a reduction.

**Proof.** Since  ${}^*E$  is weakly Hausdorff, it suffices to show that  $\psi\langle {}^*E \rangle$  is a quotient space for the canonical equivalence relation  $\rho$  on  $E$ . But  $(x, y) \in \rho$  means  $\overline{\{x\}} = \overline{\{y\}}$  which, by proposition 4.5, is equivalent to  $\mu(\psi(x)) = \mu(\psi(y))$  which, by lemma 4.9, is equivalent to  $\psi(x) = \psi(y)$ . Moreover, given a subset  $A$  of  $E$ , we have  $\psi\langle {}^*A \rangle = \gamma(A) \cap \psi\langle {}^*E \rangle$ . So that the mapping  $\psi: {}^*E \rightarrow \psi\langle {}^*E \rangle$  is open. So  $\psi\langle {}^*E \rangle$  can be identified with a quotient space  ${}^*E/\rho$ . Qed

Notice that  $\psi \langle {}^*E \rangle$  contains the set of isolated points in  $\gamma(E)$ . Therefore  $\psi \langle {}^*E \rangle$  is dense in  $\gamma(E)$ . (see 1.2).

**4.11. Infinitesimal members of filters.**- Given a filter  $\mathcal{F}$  on  $X_t$ , an *infinitesimal member* of  $\mathcal{F}$  is any (internal) element  $A$  of  ${}^*\mathcal{F}$  such that  $A \subset \mu(\mathcal{F})$ .

Since  $\mathcal{F}$  is a  ${}^*$ -filter, every element of  ${}^*\mathcal{F}$  is nonempty, so that any infinitesimal member of a filter is nonempty.

For every  $E \in \mathcal{F}$ , let  $\mathcal{F}_E = \{A \mid A \subset E \text{ and } A \in \mathcal{F}\}$ , then  $\{\mathcal{F}_E \mid E \in \mathcal{F}\}$  is a filter base, the base of the so-called filter of sections of  $\mathcal{F}$  which will be denoted by  $\mathcal{F}_s$ . The following result is immediate.

**4.12. Proposition.**- Let  $\mathcal{F}$  be a filter in  $\mathfrak{M}$ . Then  $\mu(\mathcal{F}_s)$  is the set of infinitesimal members of  $\mathcal{F}$ .

**4.13. Remark.**- Let  $G$  be a graph, and  $A$  be a subset in  $\mathfrak{M}$ . We will freely use in the sequel the following relations.

$$(4) \quad {}^*(G \langle A \rangle) = {}^*G \langle {}^*A \rangle .$$

$$(5) \quad \text{The converse of } {}^*G \text{ is equal to } {}^*(G)^{-1}.$$

## 5. Enlargements.

**5.1. Concurrent relations.**- A binary relation  $R$  of type  $(s,t)$  in  $\mathfrak{M}$  is called *concurrent* (or *finitely satisfiable*) whenever, given entities  $A_1, \dots, A_n$  of type  $s$  and entities  $B_1, \dots, B_n$  of type  $t$  in  $\mathfrak{M}$ , such that  $(A_i, B_i) \in R$ ,  $1 \leq i \leq n$ , there exists an entity  $B$  of type  $t$  in  $\mathfrak{M}$  such that  $(A_i, B) \in R$ ,  $1 \leq i \leq n$  (otherwise stated, whenever, if  $r = \theta(R, (s,t))$ , the set of «equations» in  $y$

$$\Phi_3(r, x_i, y), 1 \leq i \leq n$$

has a common solution in  $\mathfrak{M}$  if (and only if) each individual «equation» has a solution in  $\mathfrak{M}$ ).

**5.2. Definition.**- The model  ${}^*\mathfrak{M}$  is said to be an *enlargement* of the L-structure  $\mathfrak{M}$  whenever, for every concurrent relation  $R$  of type  $(s,t)$  in  $\mathfrak{M}$ , there exists an internal entity  $B$  of type  $t$  in  ${}^*\mathfrak{M}$  such that  $(A_1, B_1) \in R$  implies  $({}^*A_1, B) \in {}^*R$ .

We will say that a subset  $E$  in  $\mathfrak{M}$  is *coated* by a subset  $F$  in  ${}^*\mathfrak{M}$  whenever  $x \in E$  implies  ${}^*x \in F$ .

**5.3. Proposition.**- The following statements are equivalent.

- The model  ${}^*\mathfrak{M}$  is an enlargement of  $\mathfrak{M}$ .
- Every subset in  $\mathfrak{M}$  is coated by a  ${}^*$ -finite internal subset in  ${}^*\mathfrak{M}$ .
- Every filter in  $\mathfrak{M}$  has an infinitesimal member in  ${}^*\mathfrak{M}$ .
- The monad of every filter in  $\mathfrak{M}$  is nonempty in  ${}^*\mathfrak{M}$ .
- Every standard subset in  ${}^*\mathfrak{M}$  is compact in its  $S$ -topology.

**Proof.** The proof is  $a) \Rightarrow b) \Rightarrow c) \Rightarrow d) \Rightarrow e) \Rightarrow a)$ .

$a) \Rightarrow b)$ . Let  $E$  be a subset in  $\mathfrak{M}$ . Consider the relation « $x \in A \subset E$  and  $A$  is finite». This is a concurrent relation in  $(x, A)$ . If  ${}^*\mathfrak{M}$  is an enlargement of  $\mathfrak{M}$ , there is a  ${}^*$ -finite internal subset  $F$  in  ${}^*\mathfrak{M}$  such that  $x \in E$  implies  ${}^*x \in F$ . Hence  $E$  is coated by  $F$ .

$b) \Rightarrow c)$ . Let  $\mathcal{F}$  be a filter in  $\mathfrak{M}$ . If  $\mathcal{F}$  is coated by a  ${}^*$ -finite internal subset  $\mathcal{G}$  in  ${}^*\mathfrak{M}$ , consider  ${}^*\mathcal{F} \cap \mathcal{G}$ . This is a  ${}^*$ -finite internal subset of  ${}^*\mathcal{F}$ . But  ${}^*\mathcal{F}$  is a  ${}^*$ -filter. Hence  $A = \bigcap ({}^*\mathcal{F} \cap \mathcal{G})$  is an element of  ${}^*\mathcal{F}$  such that  $E \in \mathcal{F}$  implies  $A \subset {}^*E$ . Hence  $A$  is an infinitesimal member of  $\mathcal{F}$ .

c)  $\Rightarrow$  d). We have already noted that an infinitesimal member is always nonempty. Hence the conclusion.

d)  $\Rightarrow$  e). We show first that  ${}^*X_t$  is compact : Let  $\mathcal{G}$  be a filter on  ${}^*X_t$ . Consider

$$\mathcal{F} = \{A \mid A \subset X_t \text{ and there is } B \in \mathcal{G} \text{ such that } B \subset {}^*A\} .$$

Then  $\mathcal{F}$  is a filter on  $X_t$  and  $\mu(\mathcal{F}) \neq \emptyset$  is the set of cluster points of  $\mathcal{G}$ . Hence the conclusion.

Now, given any subset  $E$  of  $X_t$ , the subspace  ${}^*E$  is closed in  ${}^*X_t$ , so it is also compact.

e)  $\Rightarrow$  a). Let  $R$  be a binary relation of type (s,t) in  $\mathfrak{M}$ , and consider

$\mathcal{G} = \{R(x) \mid x \in X_s \text{ and } R(x) \neq \emptyset\}$  . If  $R$  is concurrent, then  $\mathcal{G}$  has the finite intersection property.

Since  ${}^*X_t$  is compact, then  $\mu(\mathcal{G}) = \bigcap \{{}^*A \mid A \in \mathcal{G}\} \neq \emptyset$  . If  $y \in \mu(\mathcal{G})$ , then

$(x_1, y_1) \in R$  implies  $(x_1, y) \in R$ . Hence  ${}^*\mathfrak{M}$  is an enlargement of  $\mathfrak{M}$  .

**Qed**

**5.4. Corollary.-** *The model  ${}^*\mathfrak{M}$  is an enlargement of  $\mathfrak{M}$  if and only if, for every subset  $E$  in  $\mathfrak{M}$ , the canonical mapping  $\psi: {}^*E \rightarrow \gamma(E)$  is onto.*

**Proof.** Since  $\psi \langle {}^*E \rangle$  is dense in  $\gamma(E)$  (see 4.10) the canonical mapping

$\psi: {}^*E \rightarrow \gamma(E)$  is onto if and only if  $\psi \langle {}^*E \rangle$  is compact if and only if  ${}^*E$  is compact (see proposition 2.6).

**Qed**

From here on it will be assumed that  ${}^*\mathfrak{M}$  is an enlargement of  $\mathfrak{M}$  .

The existence of enlargements is guaranteed by ultrapower constructions.

**5.5. Proposition.-** *Let  $\mathcal{F}$  be a filter on  $X_t$  and  $\mathcal{G}$  be a set of subsets of  $X_t$  . Then the following statements are equivalent.*

a) *For every  $A \in \mathcal{F}$  there is a  $B \in \mathcal{G}$  such that  $B \subset A$*

b) *There is an infinitesimal member  $A$  of  $\mathcal{F}$  and  $B \in {}^*\mathcal{G}$  such that  $B \subset A$ .*

**Proof.** a)  $\Rightarrow$  b). If sentence a) holds in  $\mathfrak{M}$  then it holds in any model, and since  ${}^*\mathfrak{M}$  is an enlargement  $\mathcal{F}$  has an infinitesimal member. Hence the conclusion.

b)  $\Rightarrow$  a). For every  $E \in \mathcal{F}$ , the sentence «there exists  $B \in {}^*\mathcal{G}$  such that  $B \subset {}^*E$ » holds in  $\mathfrak{M}$ , since  $A \subset {}^*E$ . Therefore the sentence «there exists  $B \in \mathcal{G}$  such that  $B \subset E$ » holds in  $\mathfrak{M}$  .

**Qed**

**5.6. Corollary.-** *Let  $\mathcal{F}$  be a filter on  $X_t$  and  $\mathcal{G} \subset \mathcal{F}$  . Then  $\mathcal{G}$  is a basis for  $\mathcal{F}$  if and only if there is an infinitesimal member of  $\mathcal{F}$  that belongs to  ${}^*\mathcal{G}$  .*

**5.7. Corollary.-** *Let  $\mathcal{F}$  and  $\mathcal{G}$  be filters on  $X_t$  . Then  $\mathcal{G} \subset \mathcal{F}$  if and only if  $\mu(\mathcal{F}) \subset \mu(\mathcal{G})$  . In particular, if  $\mu(\mathcal{F}) = \mu(\mathcal{G})$  then  $\mathcal{F} = \mathcal{G}$  .*

**Proof.** Clearly  $\mathcal{G} \subset \mathcal{F}$  implies  $\mu(\mathcal{F}) \subset \mu(\mathcal{G})$  . Conversely, let  $\mu(\mathcal{F}) \subset \mu(\mathcal{G})$  and let  $A$  and  $B$  be infinitesimal members of  $\mathcal{F}$  and  $\mathcal{G}$  respectively. Then  $A \cup B$  is an infinitesimal member of  $\mathcal{G}$  which belongs to  ${}^*\mathcal{F}$ . The conclusion follows from proposition 5.5.

**Qed**

**5.8. Corollary.-** *Let  $\mathcal{F}$  be a filter on  $X_t$  . Then  $\varphi(\mu(\mathcal{F})) = \mathcal{F}$  .*

This is a strengthening of relation (3) in the case of enlargements.

**5.9. Remark.-** Using proposition 4.5 and corollary 4.6, it is readily seen that our S-topology

coincides with the S-topology introduced by Luxemburg [11, theorem 2.5.3, p. 47], and with the topology introduced by Machover and Hirschfeld [13, 5.1.8, p. 22] in the case of enlargements.

The notion of a *monad* was first introduced by Robinson [16, 4.1.1, p. 90] in the special case of neighbourhood filters, and later generalized by Luxemburg [11, p. 46]. Machover and Hirschfeld [13, p. 18] rediscovered the notion and studied it under the name of *nucleus*.

The monad  $\mu(\mathcal{F})$  need not be an internal subset in  ${}^*\mathfrak{M}$ . In fact, it has been shown by Luxemburg [11, theorem 2.2.6, p. 43] and by Machover and Hirschfeld [13, theorem 5.2.3, p. 25] that, in the case  ${}^*\mathfrak{M}$  is an enlargement and  $\mathcal{F}$  is a filter,  $\mu(\mathcal{F})$  is internal if and only if  $\mathcal{F}$  is a *principal filter*.

The definition of an *infinitesimal member* is due to Machover and Hirschfeld [13, p. 18] but the notion can be traced back to Robinson [16, 4.2, p. 97, on directed sets].

The notion of *concurrent relation* and the notion of *enlargement* are both due to Robinson [16].

Many of the results in proposition 5.3 are essentially known. They can be found interspersed in the literature : See Luxemburg [11, p. 37-38], and theorem 5.1.2, p. 18 of Machover and Hirschfeld [13]. Proposition 5.5 is implicit in the proof of that theorem. Corollaries 5.6, 5.7, 5.8 are not new : See Luxemburg [11, corollary 2.1.6, p. 38], and Machover and Hirschfeld [13, theorems 5.1.2 and 5.1.3, p. 18-19].

## 6. Compact enlargements.

As we have already seen in proposition 5.3, every *standard* subset in an enlargement is compact in its S-topology. However an *internal* subset might well be non compact (see Luxemburg [11, p. 54-55] for an example).

**6.1. Definition.-** The enlargement  ${}^*\mathfrak{M}$  will be called *compact* whenever every internal subset in  ${}^*\mathfrak{M}$  is compact in its S-topology.

**6.2. Remark.-** Theorem 2.7.10 of Luxemburg [11] can now be reworded as follows

*If  ${}^*\mathfrak{M}$  is a countable ultralimit of a sequence of successive ultrapower enlargements of  $\mathfrak{M}$ , or if  ${}^*\mathfrak{M}$  is  $\kappa$ -saturated where  $\kappa > \text{Card}(\cup (X_t))$ , then  ${}^*\mathfrak{M}$  is a compact enlargement of  $\mathfrak{M}$ .*

Many theorems in Luxemburg [11] are stated and proved under the implicit assumption of a compact enlargement. These theorems on internal subsets are in fact more generally true for any *compact* subset (internal or external) whether the enlargement is compact or not. See, namely, theorems 2.7.11, 2.8.1, 3.2.2, 3.2.3, 3.4.2, 3.5.1, and 3.6.1 in [11].

Since the space  ${}^*X_t$  is weakly Hausdorff, we can use proposition 2.6 to prove compactness of subsets in  ${}^*\mathfrak{M}$ . Here is a more elaborate criterion though.

**6.3. Proposition.-** *Given a subset A of type (t) in  ${}^*\mathfrak{M}$ , the following statements are equivalent.*

*a) The space A is compact in its S-topology.*

*b) For every filter  $\mathcal{F}$  on  $X_t$ , the condition  $A \cap \mu(\mathcal{F}) = \emptyset$  implies that there exists  $E \in \mathcal{F}$  such that  $A \cap {}^*E = \emptyset$ .*

**Proof.** a)  $\Rightarrow$  b). This is immediate since  $\{{}^*E \mid E \in \mathcal{F}\}$  is a filter base on  ${}^*X_t$  whose elements are closed subsets.

b)  $\Rightarrow$  a). We show that the saturation  $\rho \langle A \rangle$  of A is equal to its closure  $\bar{A}$  in  ${}^*X_t$  and we apply proposition 2.6. Let  $x \in {}^*X_t$  and  $x \notin \rho \langle A \rangle$ . Let  $\mathcal{F} = \psi(x)$ , then  $\mu(\mathcal{F}) = \{\bar{x}\}$  and  $A \cap \mu(\mathcal{F}) = \emptyset$ . So, by b),  $x \notin \bar{A}$ . Hence  $\bar{A} \subset \rho \langle A \rangle$ , therefore  $\bar{A} = \rho \langle A \rangle$ . Since  ${}^*X_t$  is compact, so is  $\bar{A}$ . Hence A is compact. Qed

Notice that this result applies to the case when  $A$  is a standard subset in  ${}^*\mathfrak{M}$ , so that corollary 5.8 can be viewed as a special case of proposition 6.3.

**6.4. Proposition.-** *The following statements are equivalent.*

a) *The enlargement  ${}^*\mathfrak{M}$  is compact.*

b) *Every internal subset in  ${}^*\mathfrak{M}$  which coats a filter in  $\mathfrak{M}$  includes an infinitesimal member of that filter.*

**Proof.** a)  $\Rightarrow$  b). Let  $\mathcal{F}$  be a filter in  $\mathfrak{M}$  coated by an internal subset  $\mathcal{A}$  in  ${}^*\mathfrak{M}$ . Consider again  $\mathcal{F}_S$ , the filter of sections of  $\mathcal{F}$ . Using proposition 6.3 we see that  $\mathcal{A} \cap \mu(\mathcal{F}_S) \neq \emptyset$ . Then, by proposition 4.12,  $\mathcal{A}$  includes an infinitesimal member of  $\mathcal{F}$ .

b)  $\Rightarrow$  a). Let  $A$  be an internal subset of type (t) in  ${}^*\mathfrak{M}$ . Let  $\mathcal{F}$  be a filter on  $X_t$  such that  $A \cap \mu(\mathcal{F}) = \emptyset$ . Consider  $\mathcal{B} = \{C \mid A \cap C \neq \emptyset \text{ and } C \text{ is internal of type (t)}\}$ . Since  $A$  is internal, so is  $\mathcal{B}$ . But  $\mathcal{B}$  does not include any infinitesimal member of  $\mathcal{F}$ . Therefore  $\mathcal{B}$  does not coat  $\mathcal{F}$ , and there exists  $E \in \mathcal{F}$  such that  ${}^*E \notin \mathcal{B}$ , that is  $A \cap {}^*E = \emptyset$ . By proposition 6.3,  $A$  is compact.

**Qed**

This is essentially theorem 2.7.3 and its proof in Luxemburg [11].

Let  $f : E \rightarrow F$  be a mapping in  $\mathfrak{M}$ . Then  ${}^*f : {}^*E \rightarrow {}^*F$  is an open and continuous mapping for the S-topologies. Indeed

$${}^*f \langle {}^*A \rangle = {}^*(f \langle A \rangle) \text{ and } ({}^*f)^{-1} \langle {}^*B \rangle = {}^*(f^{-1} \langle B \rangle).$$

**6.5. Proposition.-** *Given any mapping  $f : E \rightarrow F$  in  $\mathfrak{M}$ , when  ${}^*\mathfrak{M}$  is a compact enlargement, the mapping  ${}^*f : {}^*E \rightarrow {}^*F$  is proper and*

$${}^*f \langle \overline{\{x\}} \rangle = \overline{\{{}^*f(x)\}}, \text{ for every } x \in {}^*E.$$

**Proof.** The result follows from propositions 2.8 and 2.7, since  $({}^*f)^{-1}(y)$  is internal for every  $y \in {}^*F$ .

**Qed**

This is a generalization of 5.1.10 in Machover and Hirschfeld [13]. The result is to be used in the proof of proposition 9.19.

## 7. The P-topology.

**7.1. Definition.-** Given subsets  $E$  and  $F$  in  $\mathfrak{M}$ , the product topology on  ${}^*(E \times F) = {}^*E \times {}^*F$  of the respective S-topologies on  ${}^*E$  and  ${}^*F$  will be called the *P-topology*.

**7.2. Properties.-** a) Since the S-topologies are weakly Hausdorff, the P-topology is also weakly Hausdorff by property  $P_2$  in 2.5.

b) Since the S-topologies on  ${}^*E$  and  ${}^*F$  are compact, the P-topology on  ${}^*(E \times F)$  is compact.

c) For every subsets  $A \subset E$  and  $B \subset F$ , the subset  ${}^*(A \times B) = {}^*A \times {}^*B$  is open in the S-topology on  ${}^*(E \times F)$ . Therefore the P-topology on  ${}^*(E \times F)$  is coarser than the S-topology.

d) The reduced space associated with the P-topology on  ${}^*(E \times F)$  can be identified with  $\gamma(E) \times \gamma(F)$ .

In order to characterize closed subsets in P-topologies, we will introduce the following definitions.

7.3. **Definition.-** Given sets  $E$  and  $F$ , we will call a *macle* on  $E \times F$  any subset of  $E \times F$  which is a finite union of subsets of the form  $A \times B$ , where  $A \subset E$  and  $B \subset F$ .

Given a filter on  $E \times F$ , we will say it is *macled* whenever it has a base all of whose elements are macles on  $E \times F$ .

Notice that finite unions, finite intersections, and complements of macles are macles.

7.4. **Proposition.-** Let  $E, F$  be subsets in  $\mathbb{N}$ . A nonempty subset  $G$  of  ${}^*(E \times F)$  is closed in the  $P$ -topology if and only if  $G$  is the monad of a macled filter on  $E \times F$ .

**Proof.** Clearly if  $H$  is a macle on  $E \times F$ , then  ${}^*H$  is closed in the  $P$ -topology. So the monad of a macled filter is closed in the  $P$ -topology. Conversely, suppose that  $G \neq \emptyset$  is closed in the  $P$ -topology. Given  $x \in {}^*(E \times F)$  and  $x \notin G$ , there is a  $P$ -neighbourhood of  $x$  which does not meet  $G$ . So there are subsets  $A$  and  $B$  such that  $x \in {}^*(A \times B)$  and  ${}^*(A \times B) \cap G = \emptyset$ . Consider  $H = E \times F - A \times B$ . Then  $H$  is a macle on  $E \times F$ . Moreover  $x \notin {}^*H$  and  $G \subset {}^*H$ . If  $\mathcal{X}$  is the set of macles belonging to  $\varphi(G)$ , then  $\mathcal{X}$  is the base of a macled filter  $\mathcal{L}$  and  $G = \mu(\mathcal{L})$ . Hence the result.

**Qed**

Compare this result to corollary 4.6.

7.5. **Corollary.-** Let  $A$  be a nonempty subset of  ${}^*(E \times F)$ . The closure of  $A$  in the  $P$ -topology is the monad of the macled filter having for a base the set of macles belonging to  $\varphi(A)$ .

We will need the following result in the sequel.

7.6. **Lemma.-** Let  $A$  be a subset of  ${}^*(E \times F)$ ; and let  $B$  be the closure of  $A$  in the  $P$ -topology. Then, for every standard element  $x$  in  ${}^*E$ , the set  $B(x)$  is equal to the closure of the set  $A(x)$  in the  $S$ -topology on  ${}^*F$ .

**Proof.** Let  $C(x)$  be the closure of  $A(x)$  in the  $S$ -topology on  $F$ . Clearly  $C(x) \subset B(x)$  which is closed in that topology. On the other hand, given any subset  $D$  of  $F$  such that  $y \in {}^*D$ , the set  $\{x\} \times {}^*D$  is a neighbourhood of  $(x,y)$  in the  $P$ -topology. So that  $y \in B(x)$  implies  $(\{x\} \times {}^*D) \cap A \neq \emptyset$ , that is  ${}^*D \cap A(x) \neq \emptyset$ . Therefore  $y \in B(x)$  implies  $y \in C(x)$ . Hence the conclusion.

**Qed**

7.7. **Definition.-** A *grate* on a set  $E$  is a nonempty set  $\mathcal{G}$  of nonempty subsets of  $E$  satisfying the following condition

$$A \cup B \in \mathcal{G} \quad \text{if and only if} \quad A \in \mathcal{G} \quad \text{or} \quad B \in \mathcal{G} .$$

This notion is due to Choquet [ 3 ].

Recall the following. If  $\mathcal{F}$  is a grate on  $E$ , then  $S\mathcal{F} = \{ A \mid E - A \notin \mathcal{F} \}$  is a filter on  $E$ , and, conversely, if  $\mathcal{F}$  is a filter on  $E$ , then  $S\mathcal{F} = \{ A \mid E - A \notin \mathcal{F} \}$  is a grate on  $E$ . Ultrafilters are characterized by the fact that they are both filters and grates. For an ultrafilter  $\mathcal{U}$ , we have  $S\mathcal{U} = \mathcal{U}$ . Moreover, using operation  $S$ , it can be seen that a nonempty set  $\mathcal{G}$  of subsets of  $E$  is a grate on  $E$  if and only if, for every element  $A \in \mathcal{G}$ , there exists at least one ultrafilter  $\mathcal{U}$  on  $E$  such that  $A \in \mathcal{U} \subset \mathcal{G}$ .

7.8. **Definition.** - We will say that a binary relation  $R \subset \mathcal{P}(E) \times \mathcal{P}(F)$  is *grated* whenever a)  $(\emptyset, \emptyset) \notin R$ , and b) for every nonempty subsets  $A \subset E$  and  $B \subset F$ , the sets  $R(A)$  and  $R(B)$  are grates on  $F$  and  $E$  respectively.

**7.9. Lemma.-** Let  $R \subset \mathcal{P}(E) \times \mathcal{P}(F)$  be a graded binary relation, and let  $\mathcal{F}$  be a filter on  $E$ . Then  $R[\mathcal{F}] = \bigcap \{R(V) \mid V \in \mathcal{F}\}$  is empty or is a grate on  $F$ .

**Proof.** If  $A \in R[\mathcal{F}]$  or  $B \in R[\mathcal{F}]$ , then clearly  $A \cup B \in R[\mathcal{F}]$ . Conversely, let  $A \notin R[\mathcal{F}]$  and  $B \notin R[\mathcal{F}]$ , then there exist  $V \in \mathcal{F}$  and  $W \in \mathcal{F}$  such that  $A \notin R(V)$  and  $B \notin R(W)$ . But  $R(V \cap W) \subset R(V) \cap R(W)$ , so  $A \notin R(V \cap W)$  and  $B \notin R(V \cap W)$ . Therefore  $A \cup B \notin R(V \cap W)$ , and  $A \cup B \notin R[\mathcal{F}]$ .

**Qed**

**7.10. Monads and relations.-** Given sets  $E$  and  $F$ , let  $R \subset \mathcal{P}(E) \times \mathcal{P}(F)$  be a binary relation. Define the subset  $M(R) \subset {}^*E \times {}^*F$  as follows

$(a,b) \in M(R)$  if and only if  $a \in {}^*A$  and  $b \in {}^*B$  imply  $(A,B) \in R$ , for every  $A \subset E$  and  $B \subset F$ .

As is easily seen, we have

$$M(R) = \bigcap \{ {}^*E \times {}^*F - {}^*A \times {}^*B \mid (A,B) \notin R \}.$$

So that  $M(R)$  is closed in the  $P$ -topology on  ${}^*E \times {}^*F$ . Then, by a misuse of language, we will call  $M(R)$  the *monad* of the relation  $R$ .

Conversely, given a graph  $G \subset {}^*E \times {}^*F$ , define the binary relation  $P(G) \subset \mathcal{P}(E) \times \mathcal{P}(F)$  as follows

$(A,B) \in P(G)$  if and only if  $({}^*A \times {}^*B) \cap G \neq \emptyset$ .

It is easily seen that  $P(G)$  is a graded relation whenever  $G$  is nonempty.

The following result is an easy consequence of the definitions.

**7.11. Proposition.-** Given sets  $E$  and  $F$ , let  $G \subset {}^*E \times {}^*F$  be a graph. Then  $M(P(G))$  is the closure of  $G$  in the  $P$ -topology. In particular,  $M(P(G)) = G$  if and only if  $G$  is closed in the  $P$ -topology.

Clearly  $P(M(R)) \subset R$ . Moreover, we have the following result.

**7.12. Proposition.-** Given sets  $E$  and  $F$ , let  $R \subset \mathcal{P}(E) \times \mathcal{P}(F)$  be a binary relation. Then  $P(M(R)) = R$  if and only if  $R$  is empty or graded.

**Proof.** If  $M(R) \neq \emptyset$ , then  $P(M(R))$  is graded, as already noted. If  $M(R) = \emptyset$ , then  $P(M(R))$  is empty. All we have to show now is that  $R \subset P(M(R))$  if  $R$  is graded. Let then  $R$  be graded and  $(A,B) \in R$ . Then  $A$  and  $B$  are nonempty,  $R(B)$  is a grate on  $E$ , and  $A \in R(B)$ . Therefore there exists an ultrafilter  $\mathcal{U}$  on  $E$  such that  $A \in \mathcal{U} \subset \overset{-1}{R}(B)$ . By lemma 7.9, the set  $R[\mathcal{U}] = \bigcap \{R(V) \mid V \in \mathcal{U}\}$  is a grate on  $F$  since  $B \in R[\mathcal{U}]$ . Therefore there exists an ultrafilter  $\mathcal{V}$  on  $F$  such that  $B \in \mathcal{V} \subset R[\mathcal{U}]$ . Then  $C \in \mathcal{U}$  and  $D \in \mathcal{V}$  imply  $(C,D) \in R$ . The monads  $\mu(\mathcal{U})$  and  $\mu(\mathcal{V})$  being nonempty, consider  $a \in \mu(\mathcal{U})$  and  $b \in \mu(\mathcal{V})$ . Then  $a \in {}^*C$  and  $b \in {}^*D$  imply  $C \in \mathcal{U}$  and  $D \in \mathcal{V}$  which imply  $(C,D) \in R$ . That is,  $(a,b) \in M(R)$ . Since  $a \in {}^*A$  and  $b \in {}^*B$ , we have  $({}^*A \times {}^*B) \cap M(R) \neq \emptyset$ . So  $(A,B) \in P(M(R))$ .

**Qed**

**7.13. Remark.-** Given sets  $E$  and  $F$ , proposition 7.4 establishes a canonical bijection between the set  $\mathfrak{M}(E,F)$  of maced filters  $\mathcal{F}$  on  $E \times F$  and the set  $\mathcal{G}(E,F)$  of nonempty closed subsets  $G$  in the  $P$ -topology on  ${}^*E \times {}^*F$ . Propositions 7.11, 7.12 establish a canonical bijection between the set  $\mathcal{G}(E,F)$  and the set  $\mathcal{G}(E,F)$  of nonempty graded binary relations  $R \subset \mathcal{P}(E) \times \mathcal{P}(F)$ . Therefore there is a canonical bijection between the sets  $\mathfrak{M}(E,F)$  and  $\mathcal{G}(E,F)$ . This could also have been explicated in the following way :

Given any filter  $\mathcal{F}$  on  $E \times F$ , the relation  $\langle\langle (A \times B) \cap V \neq \emptyset, \text{ for every } V \in \mathcal{F} \rangle\rangle$  is a graded binary relation  $R \subset \mathcal{P}(E) \times \mathcal{P}(F)$ . Conversely, given any binary relation  $R \subset \mathcal{P}(E) \times \mathcal{P}(F)$ , the set  $\{E \times F - A \times B \mid (A, B) \notin R\}$  generates a maxed filter  $\mathcal{F}$  on  $E \times F$  or the «improper filter»  $\mathcal{P}(E \times F)$ .

Restricted to the sets  $\mathfrak{M}(E, F)$  and  $\mathcal{S}(E, F)$ , this gives the desired canonical bijection. One should try to prove that directly, without using the intermediate step of the P-topology.

### 8. Nonstandard topology

**8.1. The monad of a topology.-** Let  $E$  be a topological space in  $\mathfrak{M}$ . Consider the graph

$\tau \subset {}^*E \times {}^*E$  defined as follows

$(a, b) \in \tau$  if and only if, for every open subset  $A$  of  $E$ , the condition  $a \in {}^*A$  implies  $b \in {}^*A$ . This graph will be called the *monad* of the topological space  $E$  (relative to the enlargement  ${}^*\mathfrak{M}$ ). Given any subset  $A$  of  ${}^*E$ , define

$$\tau[A] = \cap \{ {}^*B \mid B \text{ is open in } E \text{ and } A \subset {}^*B \}$$

So  $\tau[A]$  is the monad of the set of open subsets in  $E$  which belong to  $\varphi(A)$ . When  $A = \{a\}$  is a singleton, we have

$$\tau[\{a\}] = \tau(a) = \{ b \mid (a, b) \in \tau \},$$

and when  $a = {}^*x$  is a standard point, then  $\tau({}^*x)$  is the monad of the neighbourhood filter of  $x$  in  $E$ .

**8.2. Remark.-** The set  $\tau(a)$  was first introduced by Robinson [16, p. 90] in the case  $a$  is a standard point (under the name of «monad of  $a$ »), and later generalized by Luxemburg [11] to all points, standard or not.

Now, to any subset  $A$  of  ${}^*E$  can be associated two subsets of  ${}^*E$ , namely  $\tau[A]$  and  $\tau \langle A \rangle = \cup \{ \tau(a) \mid a \in A \}$ . Clearly, we have  $\tau \langle A \rangle \subset \tau[A]$ .

**8.3. Proposition.-** Let  $A$  be a subset of  ${}^*E$ . The following statements are equivalent.

- a) The subset  $A$  is compact in the S-topology.
- b) We have  $\tau \langle A \rangle = \tau[A]$ , for every topology  $\tau$  on  $E$ .

**Proof.** a)  $\Rightarrow$  b). Let  $x \notin \tau \langle A \rangle$ . Then, for every  $a \in A$ , there exists an open subset  $B_a$  in  $E$  such that  $a \in {}^*B_a$  and  $x \notin {}^*B_a$ . The set  $\mathcal{G} = \{ {}^*B_a \mid a \in A \}$  is an open covering of the compact subspace  $A$  in the S-topology. Then  $\mathcal{G}$  contains a finite subcovering  $\mathcal{S}$ . Consider  $B = \cup \mathcal{S} = {}^*C$ , where  $C$  is an open subset in  $E$ . Then  $A \subset {}^*C$  and  $x \notin {}^*C$ . Therefore  $x \notin \tau[A]$ . So  $\tau[A] \subset \tau \langle A \rangle$ . Hence the conclusion.

b)  $\Rightarrow$  a). Take for  $\tau$  the monad of the discrete topology on  $E$ . Then  $\tau[A] = \mu(\varphi(A))$  is the closure of  $A$ , by proposition 4.5. Since  $\tau \langle A \rangle$  is the saturation of  $A$ , the conclusion follows from proposition 2.6.

**Qed**

The implication a)  $\Rightarrow$  b) is a generalization of theorem 3.2.2 of Luxemburg [11], and its proof is essentially the same.

**8.4. Proposition.-** Let  $E$  be a topological space in  $\mathfrak{M}$ , and let  $\tau$  be its monad in  ${}^*\mathfrak{M}$ .

a) We have  $\tau = \cap \{ {}^*E \times {}^*E - ({}^*A \times {}^*(E - A)) \mid A \text{ is open in } E \}$ .

b) We have  $\tau^{-1} \langle {}^*A \rangle = {}^*(\bar{A})$ , for every subset  $A$  of  $E$ , where  $\bar{A}$  is the closure of  $A$  in  $E$ .



**Proof.** a) follows from the definition of the monad  $\tau$ .

b), We have  $x \notin \tau^{-1} \langle *A \rangle$  if and only if  $*A \cap \tau(x) = \emptyset$  if and only if there exists an open subset  $B$  of  $E$  such that  $x \in *B$  and  $*A \cap *B = \emptyset$ , by proposition 6.3. So

$$*E - \tau^{-1} \langle *A \rangle = \bigcup \{ *B \mid B \text{ is open in } E \text{ and } A \cap B = \emptyset \} = *(E - \bar{A}).$$

**Qed**

**8.5. The nassa of a topology.**- Let  $E$  be a topological space. Consider the graph  $T \subset \gamma(E) \times \gamma(E)$  defined as follows

$(a, b) \in T$  if and only if every open set belonging to  $a$  also belongs to  $b$ .

This graph has been introduced in [ 7 ] under the name of «nasse» of the topological space  $E$ . We will call it the *nassa* of  $E$  (the latin equivalent of the french word «nasse»).

Comparing the definitions of the monad and the nassa, we readily have the following result.

**8.6. First Main Theorem.**- *Let  $E$  be a topological space in  $\mathfrak{M}$ . Let  $\tau$  be its monad, and  $T$  its nassa. If  $\psi : *E \rightarrow \gamma(E)$  is the canonical mapping, then*

$(a, b) \in \tau$  if and only if  $(\psi(a), \psi(b)) \in T$ .

This theorem may serve to translate from the language of nassas into that of nonstandard topology, and the other way round. As an exercise, one could translate theorems from Luxemburg [ 11 ], stated in terms of monads, into corresponding results on nassas. One would then find that some of these results had already been stated in [ 10 ].

On the other hand, here are some results translated from nassas to monads. The first is a characterization of the monads of topological spaces. The others are characterizations of topological properties in terms of monads. *No proofs are given since the proofs in [ 10 ] and the main theorem are sufficient to yield the results.* Nevertheless, the reader might want to try to establish these results directly by nonstandard methods. This will be left as an exercise.

In order to characterize the monads of topological spaces, we will need the following definition due to Choquet [ 2 ].

**8.7. Definition.**- Given topological spaces  $E$  and  $F$ , a graph  $G \subset E \times F$  is said to be *half-open* in  $E \times F$  whenever, for every open subset  $B$  of  $F$ , the subset  $\overset{-1}{G} \langle B \rangle$  is open in  $E$ .

**8.8. Remark.**- It is clear that a mapping  $f : E \rightarrow F$  is continuous if and only if its graph is half-open in  $E \times F$ . So that half-open graphs are one form of the generalization of continuity to «multivalued functions».

If  $\tau$  is the monad of a topology on  $E$ , then, by proposition 8.4, for every standard subset  $B$  of  $*E$ , the subset  $\overset{-1}{\tau} \langle B \rangle$  is also standard. Therefore  $\tau$  is half-open in  $*E \times *E$  (where  $*E$  is considered with its  $S$ -topology). Moreover, by the same proposition 8.4, the graph  $\tau$  is closed in the  $P$ -topology on  $*E \times *E$ . Finally, as is easily seen from the definition,  $\tau$  is the graph of a preorder on  $*E$ .

**8.9. Theorem.**- *Let  $G \subset *E \times *E$  be a graph. Then  $G$  is the monad of a topology on  $E$  if and only if a)  $G$  is the graph of a preorder on  $*E$ , b)  $G$  is closed in the  $P$ -topology on  $*E \times *E$ , and c)  $G$  is half-open in  $*E \times *E$ .*

See [ 10, p. 29 ].

**8.10. Remark.**- The so-called *pretopologies* of Choquet [ 2 ] are a generalization of topologies. They could be treated in a similar way. Thus, if condition a) in theorem 8.9 is relaxed to « $G$  is

reflexive on  ${}^*E$ , then one would obtain a characterization of the monads of pretopologies. See [7], [10, p. 26-27] for the corresponding treatment by nassas.

**9. Characterizations of topological properties**

**Let  $\tau$  be the monad of a topological space  $E$ .**

**9.1. Proposition.-** *Let  $A$  be a subset of  $E$ .*

*a)  $A$  is open in  $E$  if and only if  $\tau \langle {}^*A \rangle = {}^*A$ .*

*b)  $A$  is closed in  $E$  if and only if  $\tau^{-1} \langle {}^*A \rangle = {}^*A$ .*

See [10, 1.5 proposition 3, p.28] . Compare with Robinson [16, 4.1.4 and 4.1.5, p. 90-91] .

Let  ${}^\circ E = \{ {}^*x \mid x \in E \}$  denote the set of standard elements in  ${}^*E$ . The elements of

$\tau \langle {}^\circ E \rangle$  are called *near-standard points* in the literature (see [16], [11], [13]). By

$\psi : {}^*E \rightarrow \gamma(E)$ , they correspond to the ultrafilters that converge on  $E$ .

**9.2. Proposition.-** *The monad  $\tau$  is equal to the closure of  $({}^\circ E \times {}^*E) \cap \tau$  in the  $P$ -topology on  ${}^*E \times {}^*E$ .*

See [10, 1.1 caractérisations, p. 23-24] .

**9.3. Proposition.-**

*a) The space  $E$  is a  $T_0$ -space if and only if, for every  $x, y$  in  ${}^\circ E$ ,  $x \in \tau(y)$  and  $y \in \tau(x)$  imply  $x = y$ .*

*b) The space  $E$  is weakly  $T_1$  if and only if, for every  $x, y$  in  ${}^\circ E$ ,  $\tau^{-1}(x) \cap \tau^{-1}(y) \neq \emptyset$  implies  $\tau^{-1}(x) = \tau^{-1}(y)$ .*

*c) The space  $E$  is a  $T_1$ -space if and only if, for every  $x, y$  in  ${}^\circ E$ ,  $\tau^{-1}(x) \cap \tau^{-1}(y) \neq \emptyset$  implies  $x = y$ .*

*d) The space  $E$  is weakly Hausdorff if and only if, for every  $x, y$  in  ${}^\circ E$ ,  $\tau(x) \cap \tau(y) \neq \emptyset$  implies  $\tau(x) = \tau(y)$ .*

*e) The space  $E$  is Hausdorff if and only if, for every  $x, y$  in  ${}^\circ E$ ,  $\tau(x) \cap \tau(y) \neq \emptyset$  implies  $x = y$ .*

See [10, 2.2 propositions 5 to 9, p. 37-39] . Results a) and e) have been established by Robinson [16, 4.1.8 and 4.1.9, p. 92] .

The space  $E$  is said to be *completely Hausdorff* (or a Urysohn space) whenever any distinct points  $x, y$  in  $E$  have disjoint closed neighbourhoods  $U, V$ .

**9.4. Proposition.-** *The space  $E$  is completely Hausdorff if and only if, for every  $x, y$  in  ${}^\circ E$ ,  $\tau^{-1}(x) \cap \tau^{-1}(y) \neq \emptyset$  implies  $x = y$ .*

See [10, 2.2 proposition 10, p. 39] .

Recall that the space  $E$  is said to be *semi-regular* whenever the set of open subsets  $A$  such that

$\overset{\circ}{A} = A$  is a base for its topology.

**9.5. Proposition.-**

*a) The space  $E$  is regular if and only if, for every  $x$  in  ${}^\circ E$ ,*

$$\tau^{-1} \tau(x) = \tau(x).$$

b) The space  $E$  is semi-regular if and only if, for every  $x$  in  ${}^\circ E$ ,

$$\tau^{-1} \langle *E - \tau^{-1} \tau(x) \rangle = *E - \tau(x).$$

See [10, 2.2 propositions 11 and 12, p. 39].

Compare a) with Robinson [16, 4.1.11, p. 93].

The space  $E$  is said to be *generalized H-closed* whenever every filter having an open base has an adherent point in  $E$ . An *H-closed* space is a generalized H-closed space which is also Hausdorff.

#### 9.6. Proposition.-

a) The space  $E$  is generalized H-closed if and only if

$$\tau^{-1} \langle \tau < {}^\circ E \rangle = *E.$$

b) The space  $E$  is compact if and only if

$$\tau < {}^\circ E \rangle = *E.$$

See [10, 2.2 propositions 13 and 14, p. 40]. Result b) is the compactness criterion established by Robinson [16, 4.1.13, p. 93].

#### 9.7. Proposition.-

a) The space  $E$  is normal if and only if

$$\tau^{-1} \tau \subset \tau^{-1} \tau$$

b) The space  $E$  is completely normal if and only if

$$\tau^{-1} \tau = \tau \cup \tau^{-1}$$

See [10, 2.2 propositions 15 and 17, p. 40-41].

Compare a) with Robinson [16, 4.1.12, p. 93].

9.8. Proposition.- The space  $E$  is extremally disconnected if and only if

$$\tau^{-1} \tau \subset \tau^{-1} \tau$$

See [10, 2.2 proposition 18, p. 41].

9.9. Proposition.- The space  $E$  is countably compact if and only if the closure of  $(*E - {}^\circ E) \cap \tau < {}^\circ E \rangle$  in the  $S$ -topology on  $*E$  is equal to  $*E - {}^\circ E$ .

See [10, 2.2 proposition 20, p. 41-42].

The space  $E$  is said to be *irreducible* whenever every nonempty open subset is dense in  $E$ .

9.10. Proposition.- The space  $E$  is irreducible if and only if

$$\tau^{-1} \tau = *E \times *E.$$

See [10, 2.2 proposition 22, p. 42].

9.11. Definition.- A set  $\mathfrak{X}$  of subsets of a given set is said to be *directed* whenever every finite union of elements of  $\mathfrak{X}$  is contained in an element of  $\mathfrak{X}$ . The space  $E$  is said to be *weakly regular* whenever, for every directed open covering  $\mathcal{G}$  of  $E$ , there exists an open covering  $\mathcal{U}$  of  $E$  such that  $\bar{\mathcal{U}} = \{\bar{V} \mid V \in \mathcal{U}\}$  is a refinement of  $\mathcal{G}$  (see [9]).

Every regular space is weakly regular, of course.

9.12. Proposition.- The space  $E$  is weakly regular if and only if

$$\tau^{-1} \tau < {}^\circ E \rangle = \tau < {}^\circ E \rangle.$$

See [10, 2.2 proposition 25, p. 45].

**9.13. Lemma.-** *Let  $A$  be a compact subset of  ${}^*E$ . Then  $F = \{x \mid {}^*x \in \tau^{-1}\langle A \rangle\}$  is closed in  $E$ .*

**Proof.** Let  $x \notin F$ , then  ${}^*x \notin \tau^{-1}\langle A \rangle$ , that is  $A \cap \tau({}^*x) = \emptyset$ . By proposition 6.3, there exists an open neighbourhood  $V$  of  $x$  such that  $A \cap {}^*V = \emptyset$ , which implies  $F \cap V = \emptyset$ . Therefore  $x \notin \overline{F}$ . So  $\overline{F} \subset F$ , hence the conclusion.

**Qed**

As explained above in 6.2, this is a generalized version of part of theorem 3.4.2 of Luxemburg [11].

**9.14. Proposition.-** *Let  $A$  be a compact subset of  ${}^*E$ . If the space  $E$  is weakly regular, and if  $A \subset \tau\langle E \rangle$ , then  $F = \{x \mid {}^*x \in \tau^{-1}\langle A \rangle\}$  is compact in  $E$ .*

**Proof.** Let  $\mathcal{G}$  be a directed set of open subsets in  $E$  which covers  $F$ . By the preceding lemma,  $F$  is closed, so that  $G = E - F$  is open. Therefore  $\mathcal{D} = \{C \cup G \mid C \in \mathcal{G}\}$  is a directed open covering of  $E$ . Since  $E$  is weakly regular, there exists an open covering  $\mathcal{U}$  of  $E$  such that  $\overline{\mathcal{U}} = \{\overline{V} \mid V \in \mathcal{U}\}$  is a refinement of  $\mathcal{D}$ . The set  $\{V \mid V \in \mathcal{U}\}$  covers  $\tau\langle E \rangle$  and therefore also covers  $A$ . Since  $A$  is compact, there is a finite subset  $\{V_1, \dots, V_n\}$  of  $\mathcal{U}$  such that  $\{V_1, \dots, V_n\}$  covers  $A$ . By proposition 8.4 b),  $\{\overline{V}_1, \dots, \overline{V}_n\}$  covers  $F$ . Since  $\overline{\mathcal{U}}$  is a refinement of  $\mathcal{D}$ , there exists  $\{C_1, \dots, C_n\} \subset \mathcal{G}$  such that  $\overline{V}_i \subset C_i \cup G$ ,  $1 \leq i \leq n$ . Therefore  $\{C_1, \dots, C_n\}$  covers  $F$ . So that every directed open covering of  $F$  contains a finite subcovering. Therefore every open covering of  $F$  contains a finite subcovering. Hence  $F$  is compact.

**Qed**

**9.15. Corollary.-** *If the space  $E$  is weakly regular, then a subset  $A$  of  $E$  is relatively compact if and only if  ${}^*A \subset \tau\langle E \rangle$ .*

See also [10, 2.2 proposition 25, p. 45].

**9.16. Remark.-** Proposition 9.14 and corollary 9.15 are generalizations of theorem 3.6.1 of Luxemburg [11].

As the following example shows, these statements are no longer true if the words «weakly regular» are replaced by the word «Hausdorff» :

Take a nonempty compact Hausdorff space  $E$  and a subset  $A$  of  $E$  such that  $\overline{A} = \overline{E - A} = E$  (for example,  $E =$  unit interval,  $A =$  set of rational numbers in  $E$ ). Then retopologize  $E$  by adding  $A$  to the open sets. The new space thus obtained is Hausdorff, every ultrafilter on  $A$  converges in  $E$  so that  ${}^*A \subset \tau\langle E \rangle$ , and yet  $E = \{x \mid {}^*x \in \tau^{-1}\langle {}^*A \rangle\} = \overline{A}$  is not compact.

A more sophisticated proof of proposition 9.14 can be given using proposition 9.12 and the fact that  $\tau^{-1}\langle A \rangle$  is compact if  $A$  is compact in the  $S$ -topology.

For further details on weakly regular spaces see [9], [10].

**9.17. Characterizations of mappings.-**

Let  $E$  and  $F$  be topological spaces in  $\mathfrak{M}$ , and let  $\tau$  and  $\nu$  be their monads in  ${}^*\mathfrak{M}$  respectively.

**9.18. Proposition.-** *Let  $f : E \rightarrow F$  be a mapping. The following statements are equivalent.*

- a)  $f$  is continuous.
- b)  ${}^*f \tau \subset \nu {}^*f$ .

$$c) *f \overset{-1}{\tau} \subset \overset{-1}{\nu} *f.$$

d)  $(a,b) \in \tau$  implies  $(*f(a), *f(b)) \in \nu$ , for every  $a, b$  in  $*E$ .

See [10, 2.5 théorème 6, I, p. 55-56].

Let  $D_E$  (resp.  $D_F$ ) denote the diagonal of  ${}^\circ E \times {}^\circ E$  (resp.  ${}^\circ F \times {}^\circ F$ ).

**9.19. Proposition.-** Suppose  $*\mathfrak{m}$  is a compact enlargement. Let  $f : E \rightarrow F$  be a mapping.

a)  $f$  is an open mapping if and only if  $*f \tau \supset \nu *f$ .

b)  $f$  is a closed mapping if and only if  $*f \overset{-1}{\tau} \supset \overset{-1}{\nu} *f$ .

c)  $f$  is a proper mapping if and only if  $*f D_E \overset{-1}{\tau} = D_F \overset{-1}{\nu} *f$ .

See [10, 2.5 théorème 6, II to IV, p. 55-56] and use proposition 6.5 in the proof.

## 10. Uniform spaces

**10.1. Definition.-** Given a set  $E$ , we will say that a set  $\mathcal{F}$  of subsets of  $E \times E$  is *transitive* whenever, for every  $V \in \mathcal{F}$ , there exists  $W \in \mathcal{F}$  such that  $\overset{2}{W} \subset V$ .

Recall that a graph  $G \subset E \times E$  is said to be transitive whenever  $\overset{2}{G} \subset G$ . So that the graph  $G$  is transitive if and only if the singleton  $\{G\}$  is transitive.

**10.2. Proposition.-** A filter  $\mathcal{F}$  on  $E \times E$  is transitive if and only if its monad  $G = \mu(\mathcal{F})$  is transitive.

**Proof.** If  $\mathcal{F}$  is transitive, then clearly  $\overset{2}{G} \subset G$ . Conversely, let  $\overset{2}{G} \subset G$ . Consider an infinitesimal member  $A$  of  $\mathcal{F}$ . Then  $A \subset G \subset \overset{2}{G}$ . So that, given any  $V \in \mathcal{F}$ , the sentence «there exists  $A \in * \mathcal{F}$  such that  $\overset{2}{A} \subset *V$ » is true in  $*\mathfrak{m}$ . Therefore the sentence «there exists  $W \in \mathcal{F}$  such that  $\overset{2}{W} \subset V$ » is true in  $\mathfrak{m}$ . Hence  $\mathcal{F}$  is transitive.

**Qed**

**10.3. Lemma.-** Let  $G$  be a subset of  $*E \times *E$  which is closed in the  $S$ -topology, and let  $H$  be its closure in the  $P$ -topology on  $*E \times *E$ . If  $G$  is transitive, then  $H$  is also transitive.

**Proof.** If  $G = \emptyset$ , then  $H = \emptyset$  is clearly transitive. If  $G \neq \emptyset$ , then  $\mathcal{F} = \varphi(G)$  is a transitive filter on  $E \times E$  by proposition 10.2. Let  $\rho$  be the graph of the equivalence relation  $\overline{\{x\}} = \overline{\{y\}}$  on  $*E$ . Then  $(a,b) \in \rho G \rho$  if and only if  $(\overline{\{a\}} \times \overline{\{b\}}) \cap G \neq \emptyset$ . That is,  $\rho G \rho$  is the saturation of  $G$  in the weakly Hausdorff  $P$ -topology. Since  $G$  is closed in the  $S$ -topology, it is compact in that topology. Therefore  $G$  is compact in the  $P$ -topology which is coarser. Hence its closure  $H$  is equal to its saturation  $\rho G \rho$ , by proposition 2.6.

We first show that  $G \rho G \subset \rho G \rho$ . Let  $(a,b) \notin \rho G \rho = H$ . There exists, then, subsets  $A$  and  $B$  of  $E$  such that  $a \in *A$ ,  $b \in *B$ , and  $(*A \times *B) \cap G = \emptyset$ . Let  $V = E \times E - A \times B$ . Then  $G \subset *V$ , so that  $V \in \mathcal{F}$ . Therefore there exists  $W \in \mathcal{F}$  such that  $\overset{2}{W} \subset V$ . Let  $M = W \langle A \rangle$  and  $N = \overset{-1}{W} \langle B \rangle$ . We have  $(A \times B) \cap \overset{2}{W} = \emptyset$ , so that  $M \cap N = \emptyset$ . Since  $G \subset *W$ , we have  $G \langle *A \rangle \subset *W \langle *A \rangle = *M$  and  $\overset{-1}{G} \langle *B \rangle \subset *(\overset{-1}{W}) \langle *B \rangle = *N$ .

Since  $*M$  and  $*N$  are saturated, we have  $\rho G \langle *A \rangle \subset *M$  and  $\overset{-1}{\rho G} \langle *B \rangle \subset *N$ . Therefore  $\rho G \langle *A \rangle \cap \overset{-1}{\rho G} \langle *B \rangle = \emptyset$ , so that  $(*A \times *B) \cap G \overset{2}{\rho G} = \emptyset$ . So  $(a,b) \notin G \overset{2}{\rho G} = G \rho G$ .

Therefore  $G \rho G \subset \rho G \rho$ . Now,  $\overset{2}{H} = (\rho G \rho)(\rho G \rho) = \rho(G \rho G) \rho \subset \rho(\rho G \rho) \rho = \rho G \rho = H$ .  
So that  $\overset{2}{H} \subset H$ . Qed

**10.4. Remark.-** The lemma says that if a filter  $\mathcal{F}$  on  $E \times E$  is transitive, then the filter generated by the macles in  $\mathcal{F}$  is also transitive. One should try to give a straightforward proof of this fact.

We will need this lemma in the sequel to characterize the monad of the proximity associated to a uniform space.

**10.5. Semi-uniform spaces.-** (See Nachbin [14]). Recall that a *semi-uniform* (or quasi-uniform) space is a set  $E$  together with a set  $\mathcal{U}$  of subsets of  $E \times E$  such that

- a) every element in  $\mathcal{U}$  is reflexive on  $E$  ;
- b)  $V \in \mathcal{U}$  and  $V \subset W$  imply  $W \in \mathcal{U}$  ;
- c)  $V \in \mathcal{U}$  and  $W \in \mathcal{U}$  imply  $V \cap W \in \mathcal{U}$  ;
- d)  $\mathcal{U}$  is transitive.

Notice that, when  $E$  is nonempty,  $\mathcal{U}$  is a filter on  $E \times E$ .

The semi-uniform space  $(E, \mathcal{U})$  is said to be a *uniform space* whenever  $\mathcal{U}$  satisfies the additional symmetry condition

- e)  $V \in \mathcal{U}$  implies  $\overset{-1}{V} \in \mathcal{U}$  .

The monad  $\mu(\mathcal{U})$  will be called the *monad* of the semi-uniform (resp. uniform) space  $(E, \mathcal{U})$ . This notion has been introduced by Luxemburg [11] and Machover and Hirschfeld [13] in the case of uniform spaces. Clearly  $\mu(\mathcal{U})$  is the graph of a preorder on  ${}^*E$  which is closed in the S-topology on  ${}^*E \times {}^*E$ . Moreover, in the case of a uniform space, the preorder is an equivalence relation.

Using proposition 10.2, the following result is easily established. It should be compared to theorem 8.9.

**10.6. Theorem.-** Let  $G \subset {}^*E \times {}^*E$  be a graph. Then  $G$  is the monad of a semi-uniform (resp. uniform) space  $(E, \mathcal{U})$  if and only if a)  $G$  is the graph of a preorder (resp. equivalence relation) on  ${}^*E$ , and b)  $G$  is closed in the S-topology on  ${}^*E \times {}^*E$ .

This is a slight generalization of theorem 3.9.1 of Luxemburg [11, p. 72] and of a theorem in Machover and Hirschfeld [13, 7.2.1, p. 50].

**10.7. Topology of a semi-uniform space.-** To every semi-uniform space  $(E, \mathcal{U})$  is associated a topology on  $E$  such that, for every  $x \in E$ , the set  $\mathcal{U}(x) = \{V(x) \mid V \in \mathcal{U}\}$  is the neighbourhood filter of  $x$  in that topology.

**10.8. Proposition.-** Given a semi-uniform space  $(E, \mathcal{U})$  let  $U = \mu(\mathcal{U})$  be its monad and  $\tau$  be the monad of its topology. Then, for every  $x \in {}^\circ E$ , we have  $\tau(x) = U(x)$ .

**Proof.** Both  $\tau({}^*a)$  and  $U({}^*a)$  are equal to the monad of the neighbourhood filter  $\mathcal{U}(a)$  of  $a$ .

Qed

See also Machover and Hirschfeld [13, 7.2.2, p. 51].

**10.9. Corollary.-** Every topological space is semi-uniformizable.

(That is, given any topology on a set  $E$ , there exists  $\mathcal{U}$  such that  $(E, \mathcal{U})$  is a semi-uniform space whose topology is precisely the given topology.)

**Proof.** Let  $\tau$  be the monad of a topology on  $E$ . By theorem 8.9,  $\tau$  is the graph of a preorder

on  ${}^*E$  which is closed in the P-topology, and hence in the S-topology, on  ${}^*E \times {}^*E$ . By theorem 10.6,  $\tau$  is then the monad of a semi-uniform space  $(E, \mathcal{U})$ . By proposition 10.8, the topology of  $(E, \mathcal{U})$  is precisely the original topology on  $E$ .

**Qed**

This result is not new. See Császár [4, p. 171], Pervin [15], and [10, 3.2 proposition 2, p. 58].

**10.10. Remark.**- It will be noticed that the theory of monads of uniform spaces, as developed by Luxemburg [11] and Machover and Hirschfeld [13], looks very much like the theory of *tramels* of such spaces (see [8], [10]). But of course, here, the monads are much simpler. In our opinion, this is essentially due to the fact that relation

$$(1) \quad {}^*(E \times F) = {}^*E \times {}^*F$$

holds, whereas the corresponding relation for ultrafilter spaces fails.

**10.11. Precompactness.**- Recall that a uniform space  $(E, \mathcal{U})$  is *precompact* if and only if, for every  $V \in \mathcal{U}$ , there exists a *finite* covering  $\mathcal{U}$  of  $E$  such that  $A \times A \subset V$ , for every  $A \in \mathcal{U}$ .

**10.12. Proposition.**- *Given a nonempty uniform space  $(E, \mathcal{U})$ , let  $U$  be its monad. Then the following statements are equivalent.*

- a) *The uniform space  $(E, \mathcal{U})$  is precompact.*
- b) *The monad  $U$  is closed in the P-topology on  ${}^*E \times {}^*E$ .*
- c) *The filter  $\mathcal{U}$  is maced.*

**Proof.** a)  $\Rightarrow$  b). Given  $V \in \mathcal{U}$ , there exists  $W \in \mathcal{U}$  such that  $\overset{-1}{W} = W$  and  $\overset{3}{W} \subset V$ . Since  $(E, \mathcal{U})$  is precompact, there exists a finite covering  $\mathcal{U}$  of  $E$  such that  $A \times A \subset W$ , for every  $A \in \mathcal{U}$ . Consider  $T = \bigcup \{ W(A) \times W(A) \mid A \in \mathcal{U} \}$ . Then  $W \subset T \subset \overset{3}{W} \subset V$  and  $T$  is a macle on  $E \times E$ . So  $U \subset {}^*W \subset {}^*T \subset {}^*V$  and  ${}^*T$  is closed in the P-topology. Since  $U$  is the intersection of all  ${}^*V$  such that  $V \in \mathcal{U}$ , then  $U$  is closed in the P-topology.

b)  $\Rightarrow$  c). Follows immediately from proposition 7.4.

c)  $\Rightarrow$  a). Since  $\mathcal{U}$  is maced, vien any  $V \in \mathcal{U}$ , there exists a macle  $W \in \mathcal{U}$  such that  $W \subset V$ . Let  $W = \{ A_i \times B_i \mid 1 \leq i \leq n \}$ , and  $C_i = A_i \cap B_i$ . Then  $\{ C_i \mid 1 \leq i \leq n \}$  is a finite covering of  $E$ , and  $C_i \times C_i \subset V$ , for  $1 \leq i \leq n$ . So  $(E, \mathcal{U})$  is precompact.

**Qed**

**10.13. Theorem.**- *Let  $U$  be the monad of a uniform space  $(E, \mathcal{U})$ , and let  $V$  be the closure of  $U$  in the P-topology on  ${}^*E \times {}^*E$ . Then*

- a)  *$V$  is the monad of a precompact uniform space  $(E, \mathcal{U})$*
- b) *Given any precompact uniform space  $(E, \mathcal{U})$ , then  $\mathcal{U} \subset \mathcal{U}$  if and only if  $\mathcal{U} \subset \mathcal{U}$*
- c) *The topologies of  $(E, \mathcal{U})$  and  $(E, \mathcal{U})$  are the same.*

**Proof.** a) follows from lemma 10.3, theorem 10.6, and proposition 10.12.

b) Let  $W$  be the monad of a uniform space  $(E, \mathcal{U})$ . Then  $\mathcal{U} \subset \mathcal{U}$  if and only if  $U \subset W$ . On the other hand,  $\mathcal{U} \subset \mathcal{U}$  if and only if  $V \subset W$ . If  $(E, \mathcal{U})$  is precompact, then  $W$  is closed in the P-topology, by proposition 10.12. Since  $V$  is the closure of  $U$  in that topology, we have  $U \subset W$  if and only if  $V \subset W$ . Hence the conclusion.

c) By lemma 7.6, we have  $U(x) = V(x)$ , for every  $x \in {}^\circ E$ . The conclusion follows by proposition 10.8.

**Qed**

Proposition 10.12 is a special case of theorem 10.13.

### 11. Proximity spaces

Precompact uniform spaces have been identified with the so-called *proximity spaces*. A proximity space is a set  $E$  together with a *proximity relation* on  $E$ .

11.1. **Definition.**- A *proximity relation* on a set  $E$  is a graded symmetric binary relation  $R \subset \mathcal{P}(E) \times \mathcal{P}(E)$  which satisfies the following two conditions

- r)  $A \cap B \neq \emptyset$  implies  $(A,B) \in R$ .
- t)  $(A,B) \notin R$  implies that there exists a subset  $C \subset E$  such that  $(A,E-C) \notin R$  and  $(C,B) \notin R$ .

This notion is due to Efremovič (see Császár [4, p. 66] for an equivalent formulation).

11.2. **Monad of a proximity relation.**- In 7.10, we have defined monads of binary relations between subsets. Given a proximity relation  $R$  on a set  $E$ , consider its *monad*  $M(R)$ . Due to relation r), the monad  $M(R)$  is reflexive on  ${}^*E$ . Since  $R$  is symmetric,  $M(R)$  is also symmetric. Due to condition t), the monad  $M(R)$  is transitive, as is easily seen. So  $M(R)$  is the graph of an equivalence relation on  ${}^*E$ . This has already been noted by Machover and Hirschfeld [13, 8.2.4, p. 63]. But, which is more, the monad  $M(R)$  is closed in the  $P$ -topology on  ${}^*E \times {}^*E$ . This turns out to be characteristic of the monads of proximity relations. Remember that, given any subset  $G$  of  ${}^*E \times {}^*E$ , we defined, in 7.10, a binary relation  $P(G) \subset \mathcal{P}(E) \times \mathcal{P}(E)$ .

11.3. **Theorem.**- Let  $G \subset {}^*E \times {}^*E$  be a graph. Then  $G$  is the monad of a proximity relation  $R$  on  $E$  such that  $R = P(G)$  if and only if a)  $G$  is the graph of an equivalence relation on  ${}^*E$ , and b)  $G$  is closed in the  $P$ -topology on  ${}^*E \times {}^*E$ .

**Proof.** The only thing which really needs a proof is that conditions a) and b) imply condition t) on  $R = P(G)$ . If  $G = \emptyset$ , this is clear. Suppose  $G \neq \emptyset$  and let  $(A,B) \notin R$ , that is  $({}^*A \times {}^*B) \cap G = \emptyset$ . Consider  $\mathcal{U} = \varphi(G)$ . Then  $\mathcal{U}$  is a filter whose monad is  $\mu(\mathcal{U}) = G$ . By proposition 6.3, there exists  $V \in \mathcal{U}$  such that  $({}^*A \times {}^*B) \cap {}^*V = \emptyset$ . By proposition 10.2,  $\mathcal{U}$  is transitive. So there exists  $W \in \mathcal{U}$  such that  $\overset{2}{W} \subset V$ . Then  $(A \times B) \cap \overset{2}{W} = \emptyset$ , that is,  $W(A) \cap \overset{-1}{W}(B) = \emptyset$ . Let  $C = W(A)$ . Then  $A \subset W(A) = C$  and  $B \subset \overset{-1}{W}(B) \subset E-C$ . Therefore  $W(A) \cap \overset{-1}{W}(E-C) = \emptyset$ , that is  $(A \times (E-C)) \cap W = \emptyset$ , hence  $(A,E-C) \notin R$ . On the other hand,  $C \cap \overset{-1}{W}(B) = \emptyset$ , that is  $(C \times B) \cap W = \emptyset$ , hence  $(C,B) \notin R$ . Qed

The comparison of proposition 10.12 and theorem 11.3 renders the identification between precompact uniform spaces and proximity spaces almost trivial.

11.4. **The nassa of a proximity space.**- Let  $(E,R)$  be a proximity space. Consider the graph  $P \subset \gamma(E) \times \gamma(E)$  defined as follows

- (a,b)  $\in P$  if and only if  $A \in a$  and  $B \in b$  imply  $(A,B) \in R$ .

This graph has been introduced in [8] under the name of «cotte» of the proximity space  $(E,R)$  and has been renamed «nasse» in [10]. We will call it the *nassa* of  $(E,R)$ .

Comparing the definitions of the monad and the nassa of a proximity space, we readily have the following result.



**11.5. Second Main Theorem.-** Let  $(E, R)$  be a proximity space. Let  $M(R)$  be its monad and  $P$  be its nassa. If  $\psi : {}^*E \rightarrow \gamma(E)$  is the canonical mapping, then

$(a, b) \in M(R)$  if and only if  $(\psi(a), \psi(b)) \in P$ .

Again, this theorem may serve to translate from the language of nassas into that of nonstandard topology, and the other way round.

Namely, theorem 11.3 is a translation of a result on nassas [10, p. 33], and this would have been sufficient proof for the theorem.

**11.6. Proximity of a uniform space.-** We will add just one more result on proximity spaces.

Given a uniform space  $(E, \mathcal{U})$ , recall that a binary relation  $R \subset \mathcal{P}(E) \times \mathcal{P}(E)$  is defined as follows

$(A, B) \in R$  if and only if  $(A \times B) \cap V \neq \emptyset$ , for every  $V \in \mathcal{U}$ .

This is known to be a proximity relation on  $E$ .

**11.7. Theorem.-** Given a uniform space  $(E, \mathcal{U})$ , let  $U$  be its monad and  $R$  be its proximity relation. Then the monad  $M(R)$  of  $R$  is the closure of  $U$  in the  $P$ -topology on  ${}^*E \times {}^*E$ .

**Proof.** By proposition 6.3,  $(A \times B) \cap V \neq \emptyset$ , for every  $V \in \mathcal{U}$ , if and only if  $({}^*A \times {}^*B) \cap U \neq \emptyset$ . Therefore  $(a, b) \in M(R)$  if and only if  $a \in {}^*A$  and  $b \in {}^*B$  imply  $(A, B) \in R$ , if and only if  $a \in {}^*A$  and  $b \in {}^*B$  imply  $({}^*A \times {}^*B) \cap U \neq \emptyset$ , if and only if  $(a, b)$  belongs to the closure of  $U$  in the  $P$ -topology.

**Qed**

**11.8. Topogenous orders.-** By way of a generalization, the so-called *topogenous orders* of Császár [4, p. 25] could be treated in a similar way. See [8], [10, p. 33-34] for the corresponding treatment by nassas. It would be seen, then, that there is a canonical bijection between the set of all topogenous orders on a set  $E$  and the set of those graphs  $G \subset {}^*E \times {}^*E$  which are reflexive on  ${}^*E$  and closed in the  $P$ -topology.

**Concluding remark.-** Somewhat schematically, one could say that nonstandard topology seems to be mainly concerned with certain closed subsets in two *standard* topologies, the  $S$ -topology and the  $P$ -topology, whereas standard topological structures on a given set can be viewed as certain preorders, equivalence relations or simply reflexive relations, on an *enlargement* of that set.

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