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LINEAR OPERATORS AS MEASURE PRESERVING TRANSFORMATIONS

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Abstract. We examine conditions under which a bounded linear operator in a separable complex Banach space accepts an invariant probability Borel measure. Also we define a class of operators for which the measure preserving transformations they define have always complete point spectrum.

§1. We consider a complex separable Banach space B and we denote by T a continuous linear operator in B and by m a probability measure defined on the Borel σ-algebra of B . The support of m is the subset of B consisting of the elements whose every neighborhood has non-zero measure. We say that $T:B\to B$ accepts an invariant probability measure if there exists m as above whose support spans B and for which $m(T^{-1}(.))= m(.)$. We are interested in characterizing the operators that have this property and also in determining the type of measure preserving transformation (m.p.t.) arising in each case. The case where B is finite dimmensional can be solved completely and we consider it first. In §2 we solve the problem for a class of operators in infinite dimensions which includes the isometries and finally we apply the results to an example. The present work is a continuation of the results in [3] .

We consider first some notions from the theory of m.p.t. If h is a m.p.t in a probability space its eigenvalues are the complex numbers $\{c\}$ for which the equation $f(h(.))=cf(.)$ has non-trivial complex-valued measurable solutions $f(.)$. We note that the eigenvalues of a m.p.t. h form always a subgroup of the unit circle group and they coincide with the eigenvalues of the isometry V induced by h in each of the (complex) L_p spaces, $1 \leq p \leq \infty$, over the probability space, defined by $Vf(.)=f(h(.))$. We say that h has complete point spectrum, if L_2 is spanned by the eigenvectors of $V:L_2 \to L_2$. These are in a sense the simplest m.p.t. being completely characterized by the spectrum of $V:L_2 \to L_2$. We will need the following result whose proof we ommit as it is similar to a corresponding result in [4.p214].

Lemma 1. h is a m.p.t. in a probability space. If a collection of eigenfunctions of h generates the σ-algebra of the space then h has complete point spectrum. Also the eigenvalues of h are given by the subgroup of the unit circle group generated by the eigenvalues of the collection.

Considering now the case where B is finite dimmensional we have:

Theorem 1. B is a finite dimmensional complex Banach space and $T:B \to B$ a linear operator. Then

(i) T accepts an invariant m as above iff B

is spanned by eigenvectors of T having eigenvalues of norm 1.

(ii) If T accepts an ivariant m then the m.p.t. defined by T has complete point spectrum given by the subgroup of the unit circle group generated by the eigenvalues of T.

<u>Proof</u>: (i←) T preserves the Haar measure on the torus defined by the cartesian product of the unit cirle group in each one-dimmensional eigenspace. (i→) Assuming the existence of an invariant m whose support spans B we consider also the duals B^*, T^*. Any eigenvalue of T^* must have norm 1 because it is also an eigenvalue of the m.p.t. defined by T. Assume now that an eigenvalue c of T^* does not have index 1. Then there exist x^*,y^* in B^* such that $T^*x^*=cx^*$ and $T^*y^*=cy^*+x^*$. It follows that $T^{*n}y^*=c^ny^*+nc^{n-1}x^*$, or considering the elements of B^* as functions on B that $y^*(T^n(.))=c^ny^*(.)+nc^{n-1}x^*(.)$ and in particular that

$$(*)\quad n|x^*(.)|<|y^*(T^n(.))|+|y^*(.)|,\ n=1,2,\ldots$$

We can find $\varepsilon>0$ so that for $A'=\{z:|x^*(z)|\geqslant\varepsilon\}$ we have $m(A')>0$ because of the assumption on the support of m. We can also find $M>0$ so that for $A''=\{z:|y^*(z)|\leqslant M\}$ we have $m(A'\cap A'')>0$. Setting $A=A'\cap A''$ it follows from the measure preserving property of h that for some integer $n>2M/\varepsilon$ we have $m(T^{-n}A\cap A)>0$ and then the inequality (*) above is contradicted for each $z\varepsilon T^{-n}A\cap A$. Indeed for the expression on the left side we have $z\varepsilon A\rightarrow z\varepsilon A'\rightarrow n|x^*(z)|\geqslant n\varepsilon>2M$ while for that on the right side we have $z\varepsilon A\rightarrow\{z\varepsilon A_2$ and $T^nz\varepsilon A\}\rightarrow\{z\varepsilon A_2$ and

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$T^n z\varepsilon A_2\} \Rightarrow |y^*(z)| \leq M$ and $|y^*(T^n z)| \leq M$. It follows that T^* has a spanning set of eigenvectors having eigenvalues of norm 1. Hence so does T, which proves (i). As for part (ii) we note that the functions $\{x^*(.):x^*\varepsilon B^*\}$ generate the Borel σ-algebra of B . By above so do the functions $\{x^*(.):x^*$ an eigenvector of $T^*\}$ which are also eigenfunctions of the m.p.t. T having the same eigenvalues . The result then follows from Lemma 1. Q.E.D.

§2. In infinite dimmensions the problem considered above with T assumed to be a linear contraction appears in [1] in the following form : "Solve the equation $X(h(.))=TX(.)$ where h is an ergodic m.p.t. in a probability space (S,Σ,μ), $T:B\to B$ is a linear contraction and $X:S\to B$ is Borel measurable." Using this setting we can construct m.p.t. (B,T,m) of various types as follows:

Example. T is taken to have the property that there exists a sequence $\{x_i: i=0,1,2,...\}$ with $Tx_o=0$, $Tx_{i+1}=x_i$ and $\Sigma|x_i|<\infty$. Let also h be any m.p.t. on a probability space (S,Σ,μ) and $f\varepsilon L_1(S,\Sigma,\mu)$. We define

$$X(.)=\Sigma f(h^2(.))x_i$$

Then $X(h(.))=TX(.)$ and therefore $m(.)=\mu(X^{-1}(.))$ defines a Borel probability measure on B , invariant under T . The m.p.t. so defined inherits many of the properties of h . In particular if does not have any eigenvalues if h does mot, so that Theorem 1 does not hold generally in infinite dimmensions . We do note however that all of the unit circle group belongs to the point spectrum of the operator T.

Next we extend Theorem 1 to a class of operators in infinite dimmensions, considering however only invariant probability measures for which the norm function on B is integrable, as is also the case in the example above. We mention that if B is a Banach space then a set of functionals $\{x^*\} \subset B^*$ is called total if $x^*(x)=0$ for every $x^* \varepsilon \{x^*\}$ implies $x=0$, or equivalently if $\{x^*\}$ spans B^* in its B-topology.

Theorem 2. B is a separable complex Banach space and $T:B \to B$ a continuous linear operator with the property that a total set of functionals has bounded orbits under T^*. Then:

(i) T accepts an invariant m of integrable norm iff B is spanned by eigenvectors of T having eigenvalues of norm 1.

(ii) If T leaves m of integrable norm invariant, then the m.p.t. defined by T has complete point spectrum given by the subgroup of the unit circle group generated by the eigenvalues of T. Concerning the structure of T we add:

(iii) If T leaves m of integrable norm invariant then the eigenvalues of the operators T, T^* are countable, they coincide and they all have norm 1. Also the eigenvectors of T^* span B^* in its B-topology.

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<u>Proof:</u> (i←) Let $\{x_i : i\varepsilon I\}$ be a countable collection of eigenvectors of T spanning B with eigenvalues $\{c_i : i\varepsilon I\}$ of norm 1. We consider the torus group $S=\Pi C_i$, $i\varepsilon I$, where C_i is the unit circle group, equipped with the Haar measure, and the m.p.t. $h:S\to S$ given by multiplication by the element $(c_i : i\varepsilon I)\varepsilon S$. The projection functions $\varphi_i : S\to C_i$ are eigenfunctions of h with eigenvalues c_i correspodingly. The function $X:S\to B$ defined by $X(.)=\Sigma\varphi_i(.)x_i/2^{|i|}$ is strongly integrable and satisfies $X(h(.))=TX(.)$. Also the essential range of X spans B . Clearly the measure induced in B by X satisfies all the requirements.

For the rest of the proof we use the following construction. Assuming that T accepts an invariant m of integrable norm we consider the bounded linear operator $K:L_\infty(B,m)\to B$ defined by the strong integral $Kf=\int zf(z)dm$. The adjoint $K^*:B^*\to L_1 \subset L_\infty^*$ is defined by $K^*x^*=x^*(.)\varepsilon L_1$. We have

(*) $K^*T^*=VK^*$

where $V:L_1\to L_1$ is the isometry induced by the m.p.t. T. We take now $x^*\varepsilon B^*$ for which the orbit $\{x^*, T^*x^*, T^{*2}x^*, \ldots\}$ under T^* is norm bounded. Considering B^* with its B-topology we have that the closure $C(x^*)$ of the orbit is compact and the restriction of K^* to $C(x^*)$ is injective and continuous. Indeed K^* is injective because the support of m spans B . Also from [5] we have that K is compact and therefore the restriction of K^* to the norm bounded subset $C(x^*)$ of B^* equipped with the B-topology is continuous [2,p.486]. Denoting by S the image of $C(x^*)$ we have : (a) S is compact in the norm

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topology of L_1 as the continuous image of a compact set. (b) S is invariant under the isometry V by (*) and the fact that $C(x^*)$ is invariant under T^*, (c) $X=K^{*-1}: S \to B^*$ is norm bounded and continuous in the B-topology because $K^*: C(x^*) \to S$ is a homeomorphism. Also we have

$$(**) \qquad X(V(.))=T^*X(.)$$

Having a compact metric space S and an isometry $V:S\to S$ it is well known that there exists a Borel probability measure μ with support S which is invariant under V and such that the measure preserving transformation so defined has complete point spectrum. In fact it is also ergodic because $V:S\to S$ has a dense orbit by construction [1]. Let now $\{c_i\}$ be the collection of eigenvalues of the m.p.t. V and $\{f_i\}$ the corresponding collection of eigenfunctions with $|f_i|=1$ a.e. The weak integrals

$$x_i^* = \int \bar{f}\, X d\mu$$

are well defined as elements of B^* in the sense that $x_i^*(x) = \int \bar{f}_i X(x)\, d\mu$ for every $x \in B$. Also we have $T^* x_i^* = c_i x_i^*$ by (**) and finally we note that x^* lies in the subspace of B^* spanned by the collection $\{x_i^*\}$, in the B-topology. In particular we have, by the assumption on T^*, that B^* is spanned in the B-topology by the eigenvectors of T^*, which proves the last part of (iii). Let now $\{x_j^*\}$ be the collection of eigenvectors of T^*. It is a total set of functionals by above and therefore their images under K^* form a spanning set for $K^*B^* = \{x^*(.): x^* \in B^*\} \subset L_1$. From the separability of B it follows that the collection $K^*B^* \subset L_1$

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generates the Borel σ-algebra of B [6,p.74]. Therefore so does the collection $\{x_j(.):x_j$ eigenvector of $T^*\}$ which consists of eigenfunctions of the m.p.t. T. It fol-fows now from Lemma 1 that the m.p.t. T has complete point spectrum generated by the eigenvalues of T* which in particular must all have norm 1. This proves the first part of (ii).

(i→) Assuming that T accepts an invariant m of integrable norm we have shown above that the m.p.t. T has c.p.s. Let $\{\varphi_i(.)\}$ be the eigenfunctions of the m.p.t. T in $L_\infty(B,m)$. We define the strong integrals

$$x_i=\int\bar{\varphi}(z)\,zdm \quad (=K\bar{\varphi}_i)$$

We have that $\{x_i\}$ are eigenvectors of T having eigen-values of norm 1. Also they span B because the functions $\{\varphi_i(.)\}$ L_∞ form a total set of functionals for L_1 and $x^*(x_i)=0$ for all x_i implies $x^*(.)=0$ μ-a.e. which implies $x^*=0$ by the injectivity of K*.

(iii) Since the eigenvectors of T span B and tho-se of T* span B* in the B-topology if follows easily that the eigenvalues of T,T* must coincide, and also they are all of norm 1 because those of T* are so. This also proves the last part of (ii). Q.E.D.

<u>Corollary</u> If $T:B\to B$ has the property that all $x^*\epsilon B^*$ have bounded orbits under T* then Theorem 2 holds without the condition of integrability on the norm.

<u>Proof</u>: The assumption is equivalent to the assum-ption that the norms $\|T^i\|$, i=0,1,2,..., are uniform-ly bounded. In this case we can assume w.l.o.g. that

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T is a contraction by considering if necessary the equivalent norm $\| x\|'=\sup\{\| T^i x\| : i=0,1,2,\ldots\}$. If now T is a contraction and it accepts an invariant measure then the closed balls $B_a=\{x: \| x\| \leqslant a\}$ are m-invariant under T in the sense that $m(T^{-1}(B_a)\Delta B_a)=0$. Indeed assume that for some $A\subset B_a$ with positive measure we have $m(T^{-1}A\cap B)=0$. Then we have $m(T^{-i}A\cap A)=0$ for all $i=1,2,\ldots,$ which contradicts the measure preserving property of T. If follows that we can consider the restriction of m to the sets B_a which give measures of integrable morm in the subspaces spanned by B_a, and we can apply Theorem 2. Q.E.D.

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§3. In this section we apply Theorem 2 to an example and we pose a problem related to the condition of norm integrability.

We consider a probability space (S,Σ,μ) and a non-singular transition probability $P(.,E)$ with the property $P\mu(E)\leqslant c\mu(E)$ for some $c>0$ and all $E\varepsilon\Sigma$, where $P\mu(E)=$ $=\int P(.,E)d\mu(.)$. We note that we always have a measure equivalent to μ for which this happens, e.g. the measure $\mu'=\Sigma_0^\infty P^n\mu/2^n$ where $P^n\mu=P(P^{n-1}\mu)$. If this condition is satisfied then P induces in each complex $L_p(S,\Sigma,\mu)$, $1\leqslant p\leqslant\infty$, a bounded linear operator denoted also by P, where $Pf(.)=\int f(s)dP(.,ds)$. Then every essentially bounded $f\varepsilon L_p$ has bounded orbit under P in L_p, because P is a contraction in L_∞. We can therefore apply Theorem 2 to the adjoint $T=P^*:L_q\to L_q$, for $1/p+1/q=1$. For simplicity we will apply Theorem 2 to the case where the transition probability is induced by a nonsingular point transformation $h:S\to S$ in the sense that $P(.,E)=1_E(h(.))$ where 1_E is the characteristic function. We note however that solutions obtained would be the same for the general case of a transition probability. For convenience we assume also that h is invertible as a m.p.t. and ergodic. The operator $T=P^*$ is now given by $Tf=f(h^{-1}(.))\varphi(.)$ where the R-N derivative $\varphi(.)=d\mu h^{-1}/d\mu$ is essentially bounded by assumption. Assume now that $T=P^*$ accepts an invariant m of integrable norm whose support spans $B\subset L_q$ for a fixed q, $1\leqslant q\leqslant\infty$ where B is also assumed separable if $q=\infty$. Then B is spanned by $\{f_i\}\subset L_q$, where

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$$f_i(h^{-1}(.))\varphi(.)=c_i f_i(.)\ ,\quad |c_i|=1,\ i=0,1,2,\ldots.$$

In particular we have $|f_i|(h^{-1}(.))\varphi(.)=|f_i|(.)$. We set $\chi(.)=|f_o|(.)$ and $d\mu'=\chi(.)d\mu$. By nonsingularity of h we have $\varphi(.)>0$ μ-a.e. and then by the ergodicity of h we have $\chi(.)>0$ μ-a.e. Assuming also $\chi(.)$ properly normalized we have that μ' is a probability measure equivalent to μ and clearly invariant under h. We also have that $\bar{f}_i/\chi$ are eigenfunctions of h having eigenvalues c_i. It is now clear how we can construct all solutions (B,T) for our example . We start with an ergodic invertible m.p.t. $h:(S,\Sigma,\mu')\to(S,\Sigma,\mu')$ and we choose a positive function $\chi(.)$ such that

(*) $\int\chi^{-1}(.)d\mu'=1$ and $\chi(h(.))/\chi(.)=\varphi(.)$ is ess. bd..

We then set $d\mu=\chi^{-1}d\mu'$. If $\{g_i\}$ are the eigenfunctions of h we set $f_i=g_i\chi$. These are the eigenfunctions of $T=P^*$. The solutions (B,T) are now given by the invariant subspaces of the space spanned by $\{f_i\}$ in L_q. Naturally even if the collection $\{g_i\}$ is nonempty, solutions exist iff $\{f_i\}\subset L_q$, which since $|g_i|(.)$=constant a.e. is equivalent to the condition $\chi\in L_q(\mu)$, where $\chi(.)$ satisfies (*). This condition is always satisfied if q=1 because of (*) while for $q>1$ it becomes

(**) $\int\chi^{q-1}d\mu'<\infty$ for $q>1$

χ is ess. bd. for $q=\infty$

Thus the solution depends on the eigenvalues of h and also on the integrability properties (**) of functions $\chi(.)$ satisfying (*).

Remark. The interest in the conditions above stems mainly from the following observation. Calling an ergodic m.p.t. h of type A if the condition $\chi(h(.))/\chi(.)$ ess. bd. implies $\chi \, d\mu' < \infty$, where $\chi(.) > 0$, we have: "If h is of type A then any solution $X(.) : S \to B$ of the eigenoperator equation $X(h(.)) = TX(.)$ is integrable" . While we can construct ergodic m.p.t. that are not of type A, e.g. cartesian product of a Bernoulli shift with irrational rotation on the circle, we do not know if any type A transformations exist. An equivalent condition is the following: "For $A \varepsilon \Sigma$ with $\mu'(A) > 0$ we construct the sets $A_n = \{s : s, \ldots, h^{n-1}(s) \notin A, \ h^n(s) \varepsilon A\}$. Then h is of type A iff $m'(A_n) \to 0$ faster than any power" . We do not know whether this is true even for irrational rotations on the unit circle. At any rate for ergodic m.p.t. of type A that are defined by linear operators we can use the constructions in the proof of Theorem 2, because the norm function is necessarily integrable.

R E F E R E N C E S

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