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ANOTHER APPROACH TO INFINITE FORCING

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ABSTRACT - We interprete infinite forcing in terms of existential types. This yields a construction of generic models which resembles Henkin's construction and enables us to discuss conditions for the existence of prime generic models.

Let T be a theory. A maximal collection of existential formulas in the variables $x_1, ..., x_n$, which is consistent with T is called an existential n-type. We denote such a type by σ, by $\sigma(x_1, ..., x_n)$ or by $\sigma(\bar{x})$.

If $\sigma(x_1, ..., x_n)$ is an existential n-type and if $k \leq n$ then the collection σ' of the formulas in σ which are in the variables $x_1, ..., x_k$ is an existential k-type. We say that σ is an extension of σ'.

We define when does an existential type force a formula. We assume that all the free variables of the formula are among the free variables of the type.

1. Definition :

i) If Φ is atomic then $\sigma \Vdash \Phi$ if $\Phi \in \sigma$

ii) $\sigma \Vdash \Phi \vee \Psi$ if $\sigma \Vdash \Phi$ or $\sigma \Vdash \Psi$

iii) $\sigma \Vdash \Phi \wedge \Psi$ if $\sigma \Vdash \Phi$ and $\sigma \Vdash \Psi$

iv) $\sigma \Vdash \sim \Phi$ if not $\sigma \Vdash \Phi$

v) $\sigma \Vdash \exists x \, \Phi(x)$ if there is an extension σ' of σ and a variable x_k such that $\sigma' \Vdash \Phi(x_k)$.

The definition above looks different from Robinson's definition of infinite forcing [2]. Its main properties are the following :

2. Lemma :

i) If the free variables of Φ are among those of σ then $\sigma \Vdash \Phi$ or $\sigma \Vdash \sim \Phi$

ii) If $\sigma \Vdash \Phi$ and σ' extends σ then $\sigma' \Vdash \Phi$

iii) If $\sigma' \Vdash \Phi$ and σ' extends σ and if the free variables of Φ are among those of σ then $\sigma \Vdash \Phi$

iv) If Φ is an existential formula then $\sigma \Vdash \Phi$ iff $\Phi \in \sigma$.

To relate our definition to that of infinite forcing we recall the notion of weak forcing [2] : M weakly forces Φ ($M \Vdash_{\star} \Phi$) if $M \Vdash \sim\sim \Phi$. It is well known and easy to check that M weakly forces Φ if and only if every generic model which extends M forces (and satisfies) Φ. Forcing with types actually corresponds to weak forcing.

3. Theorem :

The following are equivalent :

a) $\sigma(x_1, ..., x_n) \Vdash \Phi(x_1, ..., x_n)$

b) If G is any generic model in which $a_1, ..., a_n$ realize the type σ then $G \models \Phi(a_1, ..., a_n)$.

c) If M is any model in which $a_1...a_n$ realize the type σ then $M \Vdash_{\star} \Phi(a_1, ..., a_n)$.

Proof :

(c) implies (b) since in generic models weak forcing, forcing and satisfaction coincide.
(b) implies (c) in view of the characterization above of weak forcing in terms of the generic models which extend M (and in which the type of $a_1...a_n$ remains σ).

We prove that (a) is equivalent to (b) by induction on the complexity of Φ. Let G be a generic model in which $a_1...a_n$ realize σ.

i) If Φ is atomic the equivalence follows from the definitions.

ii) If Φ is a conjunction or a disjunction the equivalence follows immediately from the induction assumption.

iii) If $\Phi = \sim \Psi$ then $\sigma \Vdash \Phi$ iff it is not true that $\sigma \Vdash \Psi$. By assumption this is the case iff it is not true that $G \models \Psi(a_1, ..., a_n)$ and this is the same as $G \models \Phi(a_1, ..., a_n)$.
iv) If $G \models \exists x \Phi(a_1...a_n x)$ then $G \models \Phi(a_1...a_n b)$ for some $b \in G$.

Let $\sigma'(x_1...x_n y)$ be the existential type which is realized by $a_1, ..., a_n$, b in G. By assumption $\sigma' \Vdash \Phi(x_1...x_n y)$. Since σ' extends σ we conclude that $\sigma \Vdash \exists y \Phi(x_1...x_n y)$.

Assume on the other hand that $\sigma \Vdash \exists y \Phi(x_1...x_n y)$. Then there is an extension $\sigma'(x_1...x_n y)$ of σ such that $\sigma' \Vdash \Phi(x_1...x_n y)$. Let d be a new constant. It is easy to see that $T \cup D(G) \cup \sigma'(a_1,..., a_n d)$ is consistent so that there is a generic model G' which extends G and by the induction assumption $G' \models \Phi(a_1...a_n, d)$. Since $G \prec G'$ we conclude that $G \models \exists y \Phi$.

4. Corollary :

If M is existentially complete then M is generic if and only if the following condition holds : if $M \models \Phi(a_1, ..., a_n)$ and if σ is the existential type which is realized by $a_1...a_n$ in M then $\sigma \Vdash \Phi$.

Proof :

It is immediate from theorem 3 that the condition is necessary. It also follows from the theorem that if the condition holds then $M \Vdash_* \Phi$ whenever $M \models \Phi$. By [2; 6.4] this is a sufficient condition for M to be generic.

The following lemma supplies an interesting criterion for infinite genericity. It is also easy to see how to use it to characterize infinite generic models in terms of forcing with type without referring to the notion of satisfaction.

5. Lemma :

If M is existentially complete then the following is a sufficient condition for M to be generic : If $M \Vdash_* \exists y\, \Phi(a_1, ..., a_n, y)$ then $M \Vdash_* \Phi(a_1, ..., a_n, b)$ for some $b \in M$.

Proof :

We note first that by 2(i) and by 3 M weakly forces every formula in L(M) or its negation. Therefore it is enough to prove that if $M \Vdash_* \Phi$ then $M \Vdash \Phi$. This is easily shown by induction on the complexity of Φ. For negation forcing and weak forcing coincide, and an existential quantifier may be eliminated by the assumption of the lemma.

Using existential types we may construct the countable generic models similarly to Henkin's construction. At every step in the construction only the information about a finite number of elements is determined.

6. Theorem :

a) Let $\{\sigma_i(x_1, ..., x_{ni}) \mid i < \omega\}$ be a sequence of existential types such that

i) If $i < j$ then σ_j extends σ_i

ii) If $\sigma_i \Vdash \exists x\, \Phi(x)$ then for some $j \geq 1$ and for some $k \leq n_j$ $\sigma_j \Vdash \Phi(x_k)$.

Then the natural Henkin model M whose elements are (equivalence classes of) the variables is an infinite generic model for which $M \models \Phi(x_1, ..., x_n)$ iff $\sigma_n \Vdash \Phi(x_1, ..., x_n)$.

b) Every countable generic model can be constructed in this way.

Proof :

a) Let T be the theory which is composed of those formulas which are forced by some type in the sequence. (We treat now the x_i's as constant symbols). T is consistent by 2(ii) and 3. T is complete by 2(i) and T is a Henkin theory by assumption (ii) of the theorem. Let M be the corresponding Henkin model. Then $M \models \Phi(x_1 ... x_n)$ iff $T \vdash \Phi(x_1, ..., x_n)$ iff $\sigma_n \Vdash \Phi(x_1, ..., x_n)$. In particular $x_1, ..., x_{ni}$ realize in M the existential type σ_i (by 2(iv)) and M is existentially complete. By Lemma 5 M is also infinitely generic.

b) If G is a countable generic model we may order its elements in a sequence $< a_1, ..., a_n, ... >$. Choose now σ_i to be the existential type which is realized by $a_1, ..., a_i$ in G. It is easy to see that the sequence $< \sigma_i \mid i < \omega >$ satisfy the conditions of part (a) and that the

Henkin model obtained from this sequence is G.

Unfortunately theorem 6 does not yield a straightforward omitting type theorem. We may however, still discuss prime models for T^F using forcing and existential types.

7. Definition :

Given an existential type $\sigma\ (\bar{x})$ we denote by $\bar{\sigma}$ the collection $\bar{\sigma} = \{ \Phi(\bar{x}) \mid \sigma \Vdash \Phi\ (\bar{x})\}$.

$\bar{\sigma}$ is called the generic type generated by σ .

8. Theorem :

a) The generic types are exactly the complete types of T^F which are realized in generic models.

b) Every principal type in T^F is generic.

c) $\bar{\sigma}$ (x) is a principal type which is generated by the formula $\Phi\ (\bar{x})$ iff $\sigma\ (\bar{x})$ is the unique n-type which forces $\Phi\ (\bar{x})$.

Proof :

a) Let τ be the complete type which is realized by $a_1, ..., a_n$ in the generic model G and let σ be the set of existential formulas in τ . Since G is existentially complete σ is an existential type and by theorem 3 $\tau = \bar{\sigma}$ so that τ is a generic type.

Let now $\bar{\sigma}$ be any generic type and let G be a generic model in which $a_1, ..., a_n$ realize the existential type σ . Then again by theorem 3 $a_1, ..., a_n$ realize the type $\bar{\sigma}$.

b) Let τ be a principal type generated by the formula $\Phi\ (\bar{x})$. Since it is not the case that $T^F \vdash \sim \exists \bar{x}\ \Phi\ (\bar{x})$ there is a generic model (of T^F) in which the tuple $a_1...a_n$ satisfies $\Phi\ (\bar{a})$. It is easy to see that $a_1, ..., a_n$ realize the type τ , and τ is generic by (part (a).

c) Assume that σ is the only existential type that forces Φ . Then in every generic model every tuple $a_1...a_n$ which satisfies $\Phi\ (\bar{a})$ realizes the existential type σ , and the complete type $\bar{\sigma}$. Therefore for every $\Psi \in \bar{\Phi}$ and every generic model G we have $G \models \Phi \rightarrow \Psi$, and therefore $T^F \vdash \Phi \rightarrow \Psi$. Since also $\Phi \in \bar{\sigma}$ we conclude that $\bar{\sigma}$ is generated by Φ in T^F.

Assume on the other hand that both σ and σ' force Φ and let $e(\bar{x})$ be a formula in σ but not in σ'. Then in a generic model which realizes σ and σ' we do not have $\Phi \rightarrow e$ or $\Phi \rightarrow \sim e$. Hence also neither of these follow from T^F so that $\Phi\ (\bar{x})$ is not atomic in T^F.

9. Theorem :

The following condition is necessary and sufficient for T^F to be atomic :

If $\Phi\ (\bar{x})$ is forced by some type then there is a type σ which forces Φ and a formula $\Psi(\bar{x})$ such that σ is the only type which forces $\Psi\ (\bar{x})$.

Proof :

It is easy to see that $\Phi(\bar{x})$ is consistent with T^F if and only if it is forced by some existential type.

Now assume that T^F is atomic and that $\Phi(\bar{x})$ is forced by some type and is therefore consistent. Then $\Phi(\bar{x})$ is included in some principal type which is generated by a formula $\Psi(\bar{x})$. By theorem 8 this is a generic type $\bar{\sigma}$ where σ is the unique existential type which forces $\Psi(\bar{x})$. It is also easy to see that σ forces $\Phi(\bar{x})$.

Next assume that the condition of the theorem holds and that $\Phi(\bar{x})$ is consistent with T^F. Then there is an existential type σ and a formula $\Psi(\bar{x})$ such that $\sigma \Vdash \Phi$ and σ is the unique type which forces $\Psi(\bar{x})$. By theorem 8 $\bar{\sigma}$ is a principal type generated by $\Psi(\bar{x})$.

Clearly $\Phi(\bar{x}) \in \sigma$. Thus T^F is atomic.

10. Corollary :

If T has the joint embedding property then the condition of theorem 9 is a sufficient condition for T^F to have a prime model which is generic itself.

Proof :

If T has the joint embedding property then T^F is complete [2 ; 4]. Hence T^F has a prime model G_o[1 ; 2. 3. 2]. Since every generic model is a model of T^F and contains an elementary substructure which is isomorphic to G_o we conclude that G_o is generic.

11. Theorem :

If T has only a countable number of existential types then T^F is atomic.

Proof :

Assume that the condition of theorem 9 is not satisfied. Then there is a formula $\Phi(\bar{x})$ which is satisfied by different existential types σ_1 and σ_2 and such that for no formula $\Psi(\bar{x})$ $\Phi \wedge \Psi$ is forced by a unique type. Let $e_1(\bar{x})$ be a formula in $\sigma_1 - \sigma_2$ and $e_2(\bar{x})$ a formula in $\sigma_2 - \sigma_1$. Then σ_1 forces $\Phi \wedge e_1$ and σ_2 forces $\Phi \wedge e_2$. Neither of these types is the unique type with this property and we may split similarly e_1 and e_2. Continuing with this process we get $2^{\aleph_0}$ collections of existential formulas which may be extended (if necessary) to $2^{\aleph_0}$ existential types contradicting the assumption of the theorem.

12. Remark :

The proof above points out one of the difficulties that arise dealing with forcing. Beginning with a formula $\Phi(\bar{x})$ we use the existence of many types that force $\Phi(\bar{x})$ to conclude that there are uncountably many types. We do not know (although it seems reasonable) that there are uncountably many types that force Φ .

Theorem 11 relates us to the subject of stable theories. We deal only with the countable case.

13. Theorem :

Assume that T is ω-stable [1 ; 7. 1. 2.]

a) If M is a countable submodel of a model of T and if G is a generic model that contains M then $Th(G,a)_{a \in m}$ has a prime (generic) model.

b) If E is existentially complete then it has a prime generic extension (which is elementarily embeddable over E in every generic extension of E).

Proof :

a) It is easy to check that $Th(G,a)_{a \in m}$ has the joint embedding property and has a countable number of existential types. Hence corollary 9 applies. Since the prime model of this theory is an elementary submodel of G, it is also generic.

b) If we replace M in (a) by an existentially complete model E then the new generic models are <u>all</u> the old generic models which contain E, since any of them can be extended to a model of $Th(G,a)_{a \in E}$. The result now follows immediately.

As remarked above dealing with forcing is more difficult than with satisfaction. We were unable to come up with a natural condition to omit a type in a generic model. We also do not know whether a prime generic model (which can be embedded in every generic model) is necessarily a prime model for T^F.

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