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# INTERPRETING SET THEORY IN THE ENDOMORPHISM SEMI-GROUP OF A FREE ALGEBRA OR IN A CATEGORY

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ABSTRACT. - We prove that in the endomorphism semi-group of a free algebra $F_\lambda$ with $\lambda$-generators, we can interprete $\langle H(\lambda^+), \epsilon \rangle$ and get similar results for categories. The result was announced in [ S 3].

I would like to thank R. McKenzie, that seeing the proof, pointed out the holding of 3.3 (A) (by [ X ] ). I thank also M. Rubin for stimulating discussions on the problem during its solution, and for reading the previous version, and detecting errors. The readers should thank him for urging me to rewrite the paper, and in particular to state explicitely that $\underline{\underline{B}}$ is a Boolean algebra.

## § 0. INTRODUCTION.

This paper has two lines of thought as motivation : comparing category theory with set theory, and investigating the complexity of the theories of some natural structures.

Lawvere [ L ] proved that in the category of all maps between sets, we can interpret set theory (which is not surprising in view of Rabin [R] interpreting a general two-place relation by two one-place functions).

Eklof asked whether in Ab (the category of abelian group) we can define the free group of cardinality $\lambda$. He got a positive answer for $\lambda < \aleph_{(\omega^\omega)}$, $\lambda = 2^{\aleph_0}$, and $\lambda$ the first strongly compact cardinal. Feferman asked whether for some $\mu$ $Ab_\mu \prec Ab$ ($Ab_\mu$ - the category of abelian group of cardinality $< \mu$ ).

It is natural to replace the class of abelian group by any variety, and to concentrate on the free members. (In fact, our results hold for more general categories ; see § 5).

From another direction, there was interest in the first order theories of permutations groups. Mycielski [ My ] asked it.

This problem was dealt in Ershov [ E ],

McKenzie [M], Pinus [P], Shelah [ S1] , [S2 ] where it was totally solved, in fact.

The semi-group of endomorphism of a free algebra is a natural generalization.

For simplicity we shall restrict ourselves to $\lambda > |L|$, L the language of V.

Let for regular $\lambda$ , $H(\lambda)$ be the set of sets of hereditary power $< \lambda$, and for singular $\lambda$, $H(\lambda) = \bigcup_{\mu < \lambda} H(\mu^+)$. Our main result is that for a variety V, in $Cat_\lambda$ (the category of members of V of cardinality $< \lambda$ ), we can interpret uniformly a model $M_\lambda$ consisting of some copies of $(H(\lambda), \epsilon)$. For $\lambda = \infty$ this generalizes Lauvere theorem, and it also solves Feferman problem (i.e. reduce it to a problem of set theory). In categories in which the set of free members is definable, Eklof problem is answered too (e.g. for Ab).

There is an example showing that not always we can define the set of free members. However, we can characterize the algebras of cardinality $\leq \mu$ if e.g. $\mu^{|L|} = \mu$ (L-the language of V), and $\mu$ is definable in $H(\lambda)$ and we can characterize algebras which are free sums of subalgebras of cardinality $\leq 2^{|L|}$.

REMARKS : (1) By [S1], [S2], we can give a total analysis of the category of one-to-one maps ; by which we cannot interprete set theory in it.

(2) It is natural to ask what we can interpret in the automorphism group of a free algebra [ or the category of monomorphism ]. Clearly here the result depends on the variety. This converges with the question of M. Rubin who asked on the classificated of first-order theories, by biinterpretability of their saturated models automorphism groups. In [ Ru ] he solved the problem for Boolean algebras. If we allow quantification over elements, we can essentially solved the problem.

(3) It will be interesting, to change somewhat our main theorem 5.5 to get biinterpretability. Of course, if the set of identities of variety is definable in $H(\lambda^*)$ and there are no non-trivial beautiful terms, this holds for $M_{\lambda^*}$.

(4) Sabbagh and the author note that if in $Cat_\lambda$ , $F_1$ is definable <u>then</u> each automorphism of the skelton of $Cat_\lambda$ (i.e. the full subcategory of a set of representatives from each isomorphism class of objects) is induced by an automorphism of the (multi-sorted) algebras of terms.

NOTATION :

Let V be a fixed variety, and Cat : the category of all algebras in V with all homomorphisms. Let K be a fixed subcategory of Cat, usually we assume K is a full subcategory. Let $F_\lambda$ be the free algebra (in V) generated by $\lambda$ free generators $\{a_t : t \in I\}$. Let $G_\lambda$ be the endomorphism semi-group of $F_\lambda$. Elements of I will be denoted by t, s and $a_t$, I, t, s will appear in no other context. Several times, we deal with $K = G_\lambda$.

Two endomorphisms (or elements, or subalgebras) of $F_\lambda$ are called conjugate if some automorphism of $F_\lambda$ takes one to the other. We denote elements of algebras by a, b, c, d (a-usually a generator) and also by x, y, z which serve as individual constants too.

For any f let Rnf be its range, and for a set $B \subseteq A \in V$, Cl B is its closure in A, $\tilde{b} = \mathrm{Cl}\ \{b\}$. Let $\bar{x}, \bar{y}, \bar{z}$ denote finite sequences of variables, $\bar{x} = \langle x_0, \ldots, x_{n-1} \rangle$ usually. For a term $\tau$ we write $\tau = \tau(x_0, \ldots, x_{n-1}) = \tau(\bar{x})$, if every variable appearing in $\tau$ belong to $\{x_0, \ldots, x_{n-1}\}$. We can assume w.l.o.g. that if $\tau(\bar{x},\bar{y}) = \tau(\bar{x},\bar{z})$ is an identity (of V), $\bar{x}, \bar{y}, \bar{z}$ are pairwise disjoint, then for some term $\sigma(\bar{x})$, $\sigma(\bar{x}) = \tau(\bar{x},\bar{y})$ is an identity. A term $\tau(x_0, \ldots, x_{n-1})$ is called reduced if for no i and $\sigma$ is

$$\sigma(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}) = \tau(x_0, \ldots, x_{n-1})$$

an identity.

Clearly for every term $\tau(x_0, \ldots)$ there is a reduced term $\sigma(x_{i(0)}, \ldots)$ such that (i(0), i(1), ... are distinct and

$$\tau(x_0, x_1, \ldots) = \sigma(x_{i(0)}, x_{i(1)}, \ldots)$$

is an identity.

Clearly for every $b \in \mathrm{Cl}\ B$ ($B \subseteq A \in V$) there is a reduced $\tau$ and distinct $b_i \in B$ such that $b = \tau(b_0, b_1, \ldots)$. Also if $\tau(\bar{x})$ is reduced, $t_i \in I$ are distinct, $\tau(a_{t_1}, \ldots, a_{t_n}) = \sigma(a_{s_1}, \ldots, a_{s_m})$ (all in some $F_\lambda$ !) then $\{t_1, \ldots, t_n\} \subseteq \{s_1, \ldots, s_m\}$. We say $\tau(a_{t_0}, \ldots)$ is reduced if $\tau(x_0, \ldots)$ is reduced and the $t_i$ s are distinct. Notice that any function $h : \{a_t : t \in I\} \to A \in V$, has a unique extension $\hat{h} \in \mathrm{Hom}(F_\lambda, A)$. If $h : I \to I$, $\hat{h} \in G_\lambda$ is defined by $\hat{h}(a_t) = a_{h(t)}$.

We consider any K as a model : with two universes (the algebras and the morphisms, and the relations $f \in \mathrm{Hom}(A,B)$, $g = f \circ h$. Naturally a first-order language L is associated with it. For convenience, we can consider this language for $G_\lambda$ too.

## § 1. ON DEEPNESS.

DEFINITION 1.1. : Let h be a function from B into B. For every $x \in B$, we define its depth $Dp(x) = Dp(x,h)$ as an ordinal or $\infty$ by defining when it is $\geq \alpha$ :

$Dp(x) \geq 0 \Leftrightarrow x \in B$

$Dp(x) \geq \delta \Leftrightarrow Dp(x) \geq \alpha$ for every $\alpha < \delta$ (where $\delta$ denote a limit ordinal)

$Dp(x) \geq \alpha + 1 \Leftrightarrow$ for some $y \in B, f(y) = x$ and $Dp(y) \geq \alpha$ .

LEMMA 1.1. : Let $\{a_t : t \in I\}$ freely generate $F_\lambda$ , and h is a function from I into I. Define $\hat{h}$ by $\hat{h}(\tau(a_{t(1)}, ..., a_{t(n)})) = \tau(a_{h(t(1))}, ..., a_{h(t(n))})$. Then

(A) $\hat{h} \in G$ .

(B) $Dp[\tau(a_{t(1)}, ..., a_{t(n)}), \hat{h}] \geq \min_{e=1,n} Dp[t(e),h]$ .

(C) If in (B), $\tau(x_1, ..., x_n)$ is reduced and the t(e) are distinct, then equality holds.

PROOF.

(A) Immediate.

(B) We prove by induction on $\alpha$ that

$$\min_{e=1,n} Dp[t(e),h] \geq \alpha \Rightarrow Dp[\tau(a_{t(1)}, ..., a_{t(n)}] \geq \alpha$$

and this suffices. For $\alpha = 0$ or $\alpha$ limit, it is trivial. For $\alpha = \beta + 1$, by the assumption and definition of depth, there are $s(e) \in I$, $h(s(e)) = t(e)$ and $Dp(s(e),h) \geq \beta$ . Then $\min_e Dp[s(e),h] \geq \beta$, hence by the induction hypothesis $Dp[\tau(a_{s(1)}, ..., a_{s(n)}), \hat{h}] \geq \beta$
but

$$\hat{h}(\tau(a_{s(1)}, ..., a_{s(n)})) = \tau(a_{t(1)}, ..., a_{t(n)}), \text{ hence}$$

$$Dp[\tau(a_{t(1)}, ..., a_{t(n)}), \hat{h}] \geq \beta + 1 = \alpha .$$

(C) It suffices to prove by induction on $\alpha$ that

$$Dp[\tau(a_{t(1)}, ..., a_{t(n)}), \hat{h}] \geq \alpha \Rightarrow Dp(t(e), h) \geq \alpha$$

(for all e). For $\alpha = 0$ or $\alpha$ a limit ordinal, this is trivial. For $\alpha = \beta + 1$, by the definition of depth, there is a reduced term $\sigma(x_1, ..., x_m)$ and distinct s(e) ($e = 1,m$) such that

(i) $\hat{h}(\sigma(a_{s(1)}, ..., a_{s(m)}) = \tau(a_{t(1)}, ..., a_{t(n)})$ and

(ii) $Dp[\sigma(a_{s(1)}, ..., a_{s(m)}), \hat{h}] \geq \beta$ .

By (ii) and the induction hypothesis $Dp[s(e),h] \geq \beta$ for $e = 1,m$.

By (i) and the definition of $\hat{h}$,

$$\sigma(a_{h(s(1))}, ..., a_{h(s(m))}) = \tau(a_{t(1)}, ..., a_{t(n)}).$$

As $\tau(x_1, ..., x_n)$ is reduced and the t(e) are distinct,
$\{t(1), ..., t(n)\} \subseteq \{h(s(1)), ..., h(s(m))\}$. So for each $e = 1,n$ there is $k_e$, $1 \leq k_e \leq m$ such that $t(e) = h(s(k_e))$ hence $Dp[t(e),h] \geq Dp[s(k_e),h] + 1 \geq \beta + 1 = \alpha$.

LEMMA 1.2. : Let $h_1, h_2$ be functions from B into B which commute i.e. $h_1 \circ h_2 = h_2 \circ h_1$. Then for any $x \in B$

$$Dp[x, h_1] \leq Dp[h_2(x), h_1].$$

PROOF : We prove by induction on $\alpha$ that

$$Dp[x, h_1] \geq \alpha \Rightarrow Dp[h_2(x), h_1] \geq \alpha.$$

For $\alpha = 0$, or $\alpha$ a limit ordinal, it is immediate.
For $\alpha = \beta + 1$.

If $Dp[x, h_1] \geq \beta+1$, then for some $y \in B$, $h_1(y) = x$ and $Dp[y, h_1] \geq \beta$.
So $h_1(h_2(y)) = h_1 \circ h_2(y) = h_2 \circ h_1(y) = h_2(h_1(y)) = h_2(x)$, and by the induction hypothesis $Dp[h_2(y), h_1] \geq \beta$ (as $Dp[y, h_1] \geq \beta$) hence
$Dp[h_2(x), h_1] \geq Dp[h_2(y), h_1] + 1 \geq \beta + 1 = \alpha$.

LEMMA 1.3. : Let $\{a_t : t \in I\}$ freely generates $F_\lambda$, $J \subseteq I$, $|I - J| = |I|$, $J = \bigcup_{\alpha < \alpha(0)} J_\alpha$ and let $B = Cl\{a_t : t \in J\}$. Then we can find $f \in G_\lambda$, so that

(A) if $t \in J_\alpha$ then $Dp[a_t, f] = \alpha$

(B) if $g \in G_\lambda$, g and f commute, and g maps B into B, then for every $\alpha < \alpha(0)$, $t \in J_\alpha$, $g(a_t) \in Cl\left\{a_s : s \in \bigcup_{\alpha \leq \beta < \alpha(0)} J_\beta\right\}$

(C) every function g from $\{a_t : t \in J\}$ into B satisfying the condition from (B), can be extended to an endomorphism of $G_\lambda$, commuting with f and mapping B into B

(D) f is the form mentioned in 1.1.

PROOF : By renaming we can assume $I\text{-}J = I_0 \cup \{\langle 0, t, \eta\rangle : t \in J_\alpha, \alpha < \alpha(0)$ $\ell(\eta) > 0$, $\eta$ a decreasing sequence ordinals, $\eta(0) < \alpha\} \cup \{\langle 1, t, n\rangle : 0 < n < \omega\}$ and we identify $\langle 0, t, \langle\rangle\rangle$ and $\langle 1, t, 0\rangle$ with t. Let us define a function h on I :

(i) for $s \in I_0$ $h(s) = s$

(ii) for $t \in J$, $\langle 0, t, \eta^\frown\langle\beta\rangle\rangle \in I$

$$h(\langle 0, t, \eta^\frown\langle\beta\rangle\rangle) = \langle 0, t, \eta\rangle$$

(iii) for $t \in J$, $h(\langle 1, t, n\rangle) = \langle 1, t, n+1\rangle$.

Let f be $\hat{h}$ as defined in lemma 1.1. It is easy to prove that if $t \in J$, $\ell(\eta) = n$, then $Dp[\langle 0, t, \eta \rangle, h] = \eta(n\text{-}1)$. Hence (A) follows immediately by 1.1 (C), and $f \in G_\lambda$ by 1.1 (A). As for part (B), if $g \in G_\lambda$ commutes with f, g maps B into B, $t \in J_\alpha$, $\alpha < \alpha(0)$, let $g(a_t) = \sigma(a_{s(1)}, \dots, a_{s(n)})$ be reduced. As g maps B into B, $g(a_t) \in B$, hence necessarily $s(e) \in J$ for $e = 1, n$; so let $s(e) \in J_{\alpha(e)}$. By 1.1 (C) and a remark above

$$Dp[g(a_t), f] = Dp[\sigma(a_{s(1)}, \dots, a_{s(e)}), f] = \min_e Dp[s(e), h] = \min_e \alpha(e).$$

On the other hand by lemma 1.2, as g commutes with f,

$$\alpha = Dp[a_t, f] \leq Dp[g(a_t), f].$$

Combining both we get $\alpha \leq \alpha(e)$ for $e = 1,n$. Hence $g(a_t) \in Cl\{a_t : t \in J_\beta, \alpha \leq \beta < \alpha(0)\}$, so we proved (B).

As for (C), extend g to a function $g_1$ from $\{a_t : t \in I\}$ into $F_\lambda$, by :

if $g(t) = \tau(a_{t(1)}, \dots, a_{t(n)})$

let $g_1(a_{\langle 0, t, \eta \rangle}) = \tau(a_{\langle 0, t(1), \eta \rangle}, a_{\langle 0, t(2), \eta \rangle}, \dots, a_{\langle 0, t(n), \eta \rangle})$

$g_1(a_{\langle 1, t, m \rangle}) = \tau(a_{\langle 1, t(1), m \rangle}, a_{\langle 1, t(2), m \rangle}, \dots, a_{\langle 1, t(n), m \rangle})$

and $g_1(a_s) = a_s$ for $s \in I_0$.

It is easy to check that $g_1$ is well defined (because $t \in J_\alpha$, $t(e) \in J_\beta \Rightarrow \alpha \leq \beta$) and it has a unique extension to $g_2 \in G_\lambda$. In order to check that $g_2$ and f commute, it suffices to prove that for every $s \in I$, $f \circ g_2(a_s) = g_2 \circ f(a_s)$ which is quite easy.

Now (D) holds by the definition of f.

## § 2. SIMPLE PROPERTIES EXPRESSIBLE IN FIRST-ORDER LOGIC.

LEMMA 2.1. : Each of the following properties (in a full subcategory K) is expressible by formulas (of L) :

(A) proj (f) - which means f is a projection

(B) f is an automorphism, aut(f) in short

(C) g is a projection, and $Rn(f) \subseteq Rn(g)$

(D) g is a projection, $Rn(f \restriction Rng) \subseteq Rn(g)$

(E) g is a projection, $Rn(f_e) \subseteq Rn(g)$ for $e = 1, 2, 3$ and

$$[f_3 \restriction Rn(g)] = [f_1 \restriction Rn(g)] \circ [f_2 \restriction Rn(g)]$$

(F) g, f are projections and $Rn(g) = Rn(f)$.

REMARK : Clearly for any $f \in Hom(A, B)$, Rn(f) is a subalgebra of B.

PROOF : We give the expression or an indication of it in each case

(A) $f \circ f = f$

(B) $(\exists g, A)(f \circ g = g \circ f = 1_A)$ [ $1_A$ - the identity of A define by

$$(\forall f,B)(f \in Hom(B,A) \rightarrow h \circ f = f)]$$

(C) proj (g) $\wedge$ $g \circ f = f$.

As g is a projection, $x \in Rn(g) \Leftrightarrow g(x) = x$. Hence when proj (g), and $f : B \rightarrow A$, $g : A \rightarrow A$.

$$g \circ f = f \Leftrightarrow (\forall x \in B)\ (g(f(x)) = f(x)) \Leftrightarrow (\forall x)\ [f(x) \in Rn(g)] \Leftrightarrow$$
$$\Leftrightarrow Rn(f) \subseteq Rn(g)$$

(D) proj (g) $\wedge$ $g \circ f \circ g = f \circ g$

(the proof similar to the previous one)

(E) proj (g) $\wedge \bigwedge_{i=1}^{3} g \circ f_i = f_i \wedge f_1 \circ f_2 \circ g = f_3 \circ g$

(F) Immediate by (C).

CLAIM 2.2.

(A) If B is the range of some projection $f \in G_\lambda$, then any homomorphism $h : B \rightarrow B$ (B considered as a subalgebra) can be extended to an $h' \in G_\lambda$

(B) If $B = Cl\{a_t : t \in J\}$ where $J \subseteq I$ and $\{a_t : t \in I\}$ freely generates $F_\lambda$, then a homomorphism $h : B \rightarrow B$ is onto B iff for some homomorphism $g : B \rightarrow B$ $h \circ g$ is the identity.

PROOF.

(A) Clearly $h \circ f \in G_\lambda$ extend h and its domain is $F_\lambda$.

(B) Clearly if $h \circ g$ is the identity (on B), then for any $y \in B$, $g(y) \in B$ and $h(g(y)) = y$, hence h is into B.

Let h be onto B, so for every $t \in J$, $a_t = h(\tau^t(a_{s(1,t)}, \ldots))$ for some term $\tau^t$ and $s(e,t) \in J$. There is a unique homomorphism $g : B \to B$, $g(a_t) = \tau^t(a_{s(1,t)}, \ldots)$. Clearly for any $t \in J$, $h \circ g(a_t) = a_t$ hence $h \circ g$ is the identity.

REMARK. Sabbagh had proved that «f is one-to-one» and «f is onto» are (first-order) definable in $Cat_\lambda$.

§ 3. BEAUTIFUL TERMS.

DEFINITION 3.1. : The term $\tau(x_1, ..., x_n)$ is called beautiful if

(A) For any term $\sigma(x_1, ..., x_m)$

$$\tau(\sigma(x^1_1, x^1_2, ..., x^1_m), \sigma(x^2_1, ..., x^2_m), ..., \sigma(x^n_1, ..., x^n_m)) =$$

$$\sigma(\tau(x^1_1, x^2_1, ..., x^n_1), \tau(x^1_2, ..., x^n_2), ..., \tau(x^1_m, ..., x^n_m))$$

is an identity (of $F_\lambda$)

(B) $\tau(\tau(x^1_1, x^1_2, ..., x^1_n), \tau(x^2_1, ..., x^2_n), ..., \tau(x^n_1, ..., x^n_n)) = \tau(x^1_1, x^2_2, ..., x^n_n)$

is an identity

(C) $\tau(x, ..., x) = x$ is an identity.

REMARK : The beautiful terms for $n > 1$ cause us much trouble. For the free abelian group, only x is beautiful and reduced. However, if F is a free algebra of the identities $T(T_1(x), T_2(x)) = x$, $T_1(T(x,y)) = x$ and $T_2(T(x,y)) = y$ and $T_3(x,y) = T(T_1(x), T_2(y))$, then the identities which hold in $(|F_\lambda|; T_3)$ define a variety for which $T_3(x,y)$ is a beautiful term.

LEMMA 3.1.

(A) The set of beautiful terms is closed under substitution, i.e. if $\sigma(x_1, ..., x_m)$, $\tau_i(x_1, ..., x_{n(i)})$ $(i = 1, ..., n)$ are beautiful terms, then so is

$$\sigma^*(y_1, ..., y_k) = \sigma(\tau_1(y_{j(1,1)}, y_{j(1,2)}, ..., y_{j(1,n(1))}), \tau_2(y_{j(2,1)}, ...), ...,$$
$$\tau_k(y_{j(k,1)}, y_{j(k,2)}, ..., y_{j(k,n(k))}))$$

(B) x is a beautiful term, and there is no other beautiful term $\tau(x)$ ; and $\tau(x_1, ..., x_n) = x_i$ is beautiful.

(C) The two-place beautiful terms ($\tau(x,y)$) generate by substitution all the beautiful terms.

PROOF.

(A) The checking has no problems.

(B) By the third demand in Def. 3.1 ; the second phrase - by checking.

(C) If $\tau = \tau(x_1, ..., x_n)$, $(n > 2)$ let $\tau_1(x,y) = \tau(x, ..., x, y)$

$\tau_2(x_1, ..., x_{n-1}) = \tau(x_1, ..., x_{n-1}, x_{n-1})$. So $\tau(x_1, ..., x_n) =$

$\tau_2(x_1, ..., x_{n-2}, \tau_1(x_{n-1}, x_n))$ is an identity as

$\tau_2(x_1, ..., x_{n-2}, \tau_1(x_{n-1}, x_n)) =$

$\tau(x_1, ..., x_{n-2}, \tau_1(x_{n-1}, x_n), \tau_1(x_{n-1}, x_n)) =$

$\tau(\tau(x_1, ..., x_1), ..., \tau(x_{n-2}, ..., x_{n-2}), \tau(x_{n-1}, ..., x_{n-1}, x_n), \tau(x_{n-1}, ..., x_{n-1}, x_n)) =$

$\tau(x_1, ..., x_{n-2}, x_{n-1}, x_n)$.

So we can prove our assertion by induction on n.

DEFINITION 3.2. : The beautiful Boolean algebra of a variety is the Boolean algebra $\underset{\sim}{B}$ such that

(1) Its elements are beautiful terms of the form $\tau(x,y)$ (x,y here are fixed).

(2) Its zero is y, its unit is x.

(3) The intersection is defined by

$$\tau_1(x,y) \cap \tau_2(x,y) = \tau_1(\tau_2(x,y), y) = \tau_2(\tau_1(x,y), y).$$

(4) The complement is define by $\tau(x,y)^c = \tau'(x,y)$ where $\tau(x,y) = \tau'(y,x)$.

DEFINITION 3.3. : For any filter T of $\underset{\sim}{B}$, let $\approx_T$ be the relation (defined on elements of members of Cat)

$$a \approx_T b \text{ iff } a = \tau(b,a) \text{ iff } b = \tau(a,b)$$

and it is similarly defined on each Hom(A,B), A, B $\in$ Cat (see Def. 3.4 (A)).

DEFINITION 3.4. :

(A) If $f_i : A \to B$ (in Cat), $\tau$ a beautiful term, $\tau(f_1, \ldots, f_n) : A \to B$ is defined by

$$\tau(f_1, \ldots, f_n)(a) = \tau(f_1(a), \ldots, f_n(a))$$

(B) If $A_i$ ($i = 1, \ldots, n$) belong to Cat, $\tau$ a beautiful term, the algebra $\tau(A_1, \ldots, A_n)$ is defined as follows :

its elements are $\langle a_1, \ldots, a_n \rangle$ where $a_i \in A_i$ and $\langle a_1, \ldots, a_n \rangle = \langle b_1, \ldots, b_n \rangle$ iff for each i, $a_i \approx_{T_i} b_i$ where $T_i$ is the filter generated by $\tau(y, \ldots, y, x, y, \ldots, y)$ (the x-in the ith place) ; for a term $\sigma$, $\sigma(\langle a^1_1, \ldots, a^1_n \rangle, \langle a^2_1, \ldots, a^2_n \rangle, \ldots) =$

$$\langle \sigma(a^1_1, a^2_1, \ldots), \sigma(a^1_2, a^2_2, \ldots), \ldots \rangle$$

(C) If $f_i : A_i \to B_i$ (in Cat) $\tau$ a beautiful term, then

$$\tau(f_1, \ldots, f_n) : \tau(A_1, \ldots, A_n) \to \tau(B_1, \ldots, B_n)$$

is defined naturally.

REMARK : There is an ambiguity in the definition of $\tau(f_1, \ldots, f_n)$ ; so when both definitions are meaningful we prefer (A) ; and even better : in (B) if $\bigwedge_{i=1}^{n} A_i = A$ or $\tau(x_1, \ldots, x_n) = x_i$ is an identity, we make $\tau(A_1, \ldots, A_n) = A_i$. In fact $\tau(A_1, \ldots)$ is essentially defined up to isomorphism.

NOTATION : If $\bar{x}^e = \langle x^e_0, \ldots, x^e_{m-1} \rangle$ then

$$\tau(\bar{x}^0, \ldots, \bar{x}^{n-1}) = \langle \tau(x^0_0, x^1_0, \ldots), \tau(x^0_1, x^1_1, \ldots), \ldots \rangle$$

LEMMA 3.2. :

(A) The beautiful Boolean algebra is really a Boolean algebra.

(B) For any filter T of $\underline{B}$ and $A \in$ Cat, $\approx_T$ is an equivalence relation over A, and even a congruence relation, so $A/\approx_T$ is defined and $\in$ Cat and $\tau(x,y) \in T$ implies the identity $\tau(x,y) = x$ holds in $A/\approx_T$.

(C) For any $f_i : A \to B$ (in Cat) and any beautiful $\tau$, $\tau(f_1, ..., f_n)$ is a homomorphism from A into B. If $A = B$, $f_i$ a projection then $\tau(f_1, ..., f_n)$ is a projection.

(D) If $A_i \in$ Cat, $\tau$ is beautiful then $\tau(A_1, ..., A_n) \in$ Cat.

(E) If $f_i : A_i \to B_i$ in Cat, $\tau$ beautiful then

$\tau(f_1, ..., f_n) : \tau(A_1, ..., A_n) \to \tau(B_1, ..., B_n)$ is in Cat.

(F) If $T \subseteq \underline{B}$ is an ultrafilter, for the category $\{A/\approx_T : A \in \text{Cat}\}$, there are no beautiful terms. It is the category of the variety, whose identities are those of V and $\{\tau(x,y) = x : \tau \in T\}$. This holds for filters T too.

PROOF : Easy.

LEMMA 3.3. :

(A) For every $\tau_1(x,y) = \tau_2(y,x)$, let $T_e$ be the filter generated by $\tau_e(x,y)$. Then any $A \in$ Cat is the direct product of $A/\approx_{T_1}$ and $A/\approx_{T_e}$. In $A/\approx_{T_e}$, $\tau_e(x,y) = x$ is an identity.

(B) If our variety V is the modules over a ring R, then the beautiful two-place terms are $ax + (1\text{-}a)y$ where a is in the center of R and is idempotent (i.e. $a^2 = a$).

PROOF : Easy.

LEMMA 3.4. : Let $\tau$ be a beautiful term.

(A) Let $K \subseteq$ Cat be a subcategory, such that for $f_i : A \to B$ in K, $\tau(f_1, ..., f_n)$ is in K. In the proper language $L^*$ (i.e. with variables for objects, and for morphisms, and a partial operation of composition, and the relation $f \in \text{Hom}(A,B)$ plus our $\tau$) for every first order formula $\phi_0(f_1, ... ; A_1, ...)$, there is a finite set $\Phi$ of formulas of $L^*$ with the same free variables such that :

(i) $\phi_0 \in \Phi$

(ii) if $g_i^j : B_{i(1)} \to B_{i(2)}$, $g_i = \tau(g_i^1, ..., g_i^n)$, then the sets

$$\Phi^j = \{\phi(f_1, ... ; A_1, ...) \in \Phi : K \models \phi[g_1^j, g_2^j, ... ; B_1, ... ]\}$$

totally determine $\Phi^* = \{\phi(f_1, \dots ; A_1, \dots) \in \Phi : K \models \phi[g_1, g_2, \dots ; B_1, \dots]\}$

hence in particular the truth of $\phi_0(g_1, \dots ; B_1, \dots)$

(iii) if in (ii) the $\Phi^j$'s are equal then $\Phi^* = \Phi^j$.

(B) We can expand $L^*$ by letting in variables for elements, and more terms of our variety ; and also let in (ii) $g^j_i : B^j_{i(1)} \to B^j_{i(2)}$ ; and the same conclusion holds, provided that we are careful for the question when

$$\tau(B_{i(1,1)}, B_{i(1,2)}, \dots) = \tau(B_{i(2,1)}, B_{i(2,2)}, \dots).$$

REMARK : This is just a varient of Feferman Vaught [ FV ] , as refined by Weinstein [W] and Galvin [G] .

PROOF :

(A) We define $\Phi = \Phi_{\phi_0}$ by induction on the structure of $\phi_0$.

Let $\tau^j(x,y) = \tau(y, \dots, y, x, y, \dots, y)$ (x - in the jth place) for $j = 1,n$.

If $\phi_0$ is $f_i \in \mathrm{Hom}(A_{i(1)}, A_{i(2)})$ or $A_i = A_j$ let $\Phi_{\phi_0} = \{\phi_0\}$

If $\phi_0$ is $f_{k(1)} = f_{k(2)} \circ f_{k(3)}$ let

$$\Phi_{\phi_0} = \{f_{k(1)} = \tau^j(f_{k(2)} \circ f_{k(3)}, f_{k(1)}) : j = 1,n\} \cup \{\phi_0\}$$

(if $\tau(f,g)$ is not defined, $h = \tau(f,g)$ is false)

If $\phi_0 = f_k = \tau(f_{k(1)}, \dots, f_{k(n)})$ let

$$\Phi_{\phi_0} = \{f_k = \tau^j(f_{k(1)}, \dots, f_{k(n)}) : j = 1,n\} \cup \{\phi_0\}$$

If $\phi_0 = \neg\psi$ or $\phi_0 = \psi_1 \wedge \psi_2$ then $\Phi_{\phi_0} = \Phi_\psi \cup \{\phi_0\}$, $\Phi_{\phi_0} = \Phi_{\psi_1} \cup \Phi_{\psi_2} \cup \{\phi_0\}$ resp.

If $\phi_0 = (\exists g)\psi$ then $\Phi_{\phi_0} = \{(\exists g)\psi_1 : \psi_1$ a boolean combination of $\Phi_\psi\}$.

It is now easy to prove (ii) by induction and (iii) follows trivially by choosing $g^j_i = g^1_i$ for all i, j.

(B) Left to the reader.

## § 4. DEFINING ARBITRARY SETS WITH PARAMETERS.

THEOREM 4.1. : There is a formula $\phi^b_m$ such that the following holds. Let K be a full subcategory of Cat, and $F_\mu \in$ K. Suppose A, B $\in$ K, $|A| + |B| \leq \mu$ , and $\bar{f}_i$ $(i < i(0) < \mu^+)$ is an m-tuple of members of Hom (A,B). Then we can find $\bar{g}$ (from K) such that :

$$K \models \phi^b_m [\bar{f},\bar{g}] \quad \underline{\text{iff}} \quad \bar{f} = \tau(\bar{f}_{i(1)}, \ldots, \bar{f}_{i(n)})$$

for some beautiful term $\tau$ and $i(1) < \ldots < i(n)$.

REMARK : (1) The $f^*_i$ are added as individual constant during the proof, in order not to mention them explicitely during the proof. In the end, we include them in $\bar{g}$.

(2) Let $\phi^b_1 = \phi^b$.

PROOF : Let $\{a_t : t \in I^*\}$ freely generate $F_\mu$ , and let $J \subseteq I^*, \mu = |J| = |I^* - J|$ . For notational simplicity let $J = \{<\alpha,\beta> : \alpha,\beta < \mu\}$, $a_{<\alpha,\beta>} = a^\beta_\alpha$ , and $i(0) = \mu$ (as we can allow repeations).

MAIN LEMMA 4.2. : There is a formula $\phi(f)$, such that $G_\mu \models \phi[f]$ iff there is a beautiful $\tau = \tau(x_1, \ldots, x_n)$ and ordinals $\alpha(1), \ldots, \alpha(n)$, so that for every $\beta < \mu$ , $f(a^\beta_0) = \tau(a^\beta_{\alpha(1)}, \ldots, a^\beta_{\alpha(n)})$ .

PROOF OF 4.1 FROM 4.2 :

Let $f^*_0 : F_\mu \mapsto A$, be such that it maps $\{a^\alpha_0 : \alpha < \mu\}$ onto A, and $f^*_0(a^\beta_\alpha) = f^*_0(a^\beta_0)$. Let $\bar{f}_i = <f^0_i, \ldots, f^{m-1}_i>$ , and let $f^*_{0,e} : F_\mu \to B$, be such that $f^*_{0,e}(a^\beta_\alpha) = f^e_\alpha \circ f^*_0(a^\beta_\alpha)$, for $e < m$.

Let $f^*_1 : F_\mu \to F_\mu$ maps $\{a_t : t \in I^*\}$ onto $\{a^\beta_0 : \beta < \mu\}$

Let $\phi^b_m(f_0, \ldots, f_{m-1})$ says that there is $f : F_\mu \to F_\mu$, such that $\phi(f)$ and

$$\bigwedge_{e < m} f_e \circ f^*_0 \circ f^*_1 = f^*_{0,e} \circ f \circ f^*_1 .$$

PROOF OF 4.2 :

We partition our proof to three cases [ it suffice to prove each one separately, and then by comining the formulas and choosing the parameters for each case, we can easily find a unique formula ] .

We shall use § 2 freely : and restrict ourselves to $F_\mu$ and $G_\mu$ .

Case I : $\mu = \aleph_0$

Let $f_2^* \in G_\mu$ be such that $f_2^*(a_t) = a_0^0$ for each $t \in I^*$ and $f_i^* \in G_\mu$ $(i = 3, 7)$ be such that : $t \in I^* - J$ implies $f_i^*(a_t) = a_0^0$ , $f_3^*(a_n^m) = a_{n+1}^m$ , $f_4^*(a_n^m) = a_n^{m+1}$ , $f_5^*(a_n^m) = a_m^n$ , and $f_7^*(a_n^m) = a_n^0$; $f_6^*(a_n^m) = a_n^n$. Let $B_1 = Cl\{a_n^0 : n\}$, $B_2 = Cl\{a_n^m : n,m\}$, $B_0 = Cl\{a_0^0\}$.

Clearly there are first order formulas $\phi_e(X)$ such that :

(1) $\phi_1(f)$ says that $f \restriction B_1$ is into $B_1$ and it commutes with $f_3^* \restriction B_1$ (by 2.1 D, as $f_7^*$ is a projection, and $Rn\, f_7^* = B_1$)

(2) $\phi_2(f)$ says that $\phi_1(f)$ and $f \restriction B_2$ is into $B_2$ and it commutes with $f_4^* \restriction B_2$

(3) $\phi_3(f)$ says that $\phi_2(f)$ and for any g, $\phi_2(g)$ implies that $f \restriction B_2$ and $(f_5^* \circ g \circ f_5^*) \restriction B_2$ commute.

(4) $\phi_4(f)$ says that $\phi_3(f)$ and $(f_5 \circ f \circ f_5^* \circ f) \restriction B_0 = (f_6^* \circ f) \restriction B_0$

(5) $\phi_5(f)$ says that $\phi_4(f)$ and $(f_2^* \circ f) \restriction B_0 = f_2^* \restriction B_0$

(6) $\phi_6(f)$ says that for some f', $f' \restriction B_3 = f \restriction B_3$ and $\phi_5(f')$ where $B_3 = cl\{a_0^m : m < \omega\}$

Now we notice

<u>(1*)</u> $G_\mu \models \phi_1[f]$ iff $f(a_n^0) = \tau(a^0_{n+\ell(1)}, \dots, a^0_{n+\ell(k)})$ (for some $\tau, \ell(i)$, for every n)

[ Clearly the «if» part holds ; for the other direction, as $f(a_0^0) \in B_1$, $f(a_0^0) = \tau(a^0_{\ell(1)}, \dots, a^0_{\ell(k)})$ for some $\tau$ , $\ell(i)$ ; apply $f_3^*$ n times and we get our result ] .

<u>(2*)</u> $G_\mu \models \phi_2[f]$ iff $f(a_n^m) = \tau(a^m_{n+\ell(1)}, \dots, a^m_{n+\ell(k)})$

the «if» part is immediate, and for the «only if» use (1*), and apply $f_4^*$ m times on the right side

<u>(3*)</u> $G_\mu \models \phi_3[f]$ iff $f(a_n^m) = \tau(a^m_{n+\ell(1)}, \dots, a^m_{n+\ell(k)})$, (the $\ell(i)$ distinct) and $\tau(x_1, \dots, x_k)$ satisfies condition (A) for being beautiful.

[ Suppose $G_\mu \models \phi_3[f]$, then clearly $G_\mu \models \phi_2[f]$ hence by (2*) for some $\tau$ , $\ell(i)$, for every n,m $f(a^m_n) = \tau(a^m_{n+\ell(1)}, ..., a^m_{n+\ell(k)})$.

Now for every term $\sigma(x_1,..., x_p)$ let $g \in G_\lambda$ be such that $g(a^m_n) = \sigma(a^m_{n+1}, ..., a^m_{n+p})$.

So clearly $f^*_5 \circ g \circ f^*_5 (a^m_n) = \sigma(a^{m+1}_n, ..., a^{m+p}_n)$. By $(2^*)$, $G_\lambda \models \phi_2[g]$, hence

by $\phi_3$'s definition, $(f^*_5 \circ g \circ f^*_5) \circ f[a^0_0] = f \circ (f^*_5 \circ g \circ f^*_5)[a^0_0]$, computing we get

$$\tau(\sigma(a^1_{\ell(1)}, ..., a^p_{\ell(1)}), \sigma(a^1_{\ell(2)}, ..., a^p_{\ell(2)}), ..., \sigma(a^1_{\ell(k)}, ..., a^p_{\ell(k)}))$$

$$= \sigma(\tau(a^1_{\ell(1)}, ..., a^1_{\ell(k)}, \tau(a^2_{\ell(1)}, ..., a^2_{\ell(k)}), ..., \tau(a^p_{\ell(1)}, ..., a^p_{\ell(k)})).$$

As this holds for any $\sigma$, we prove that $\tau$ satisfies condition (A) from Definition 3.1. The other direction in $(3^*)$ should be easy now ].

<u>$(4^*)$</u> : $G_\mu \models \phi_4(f)$ iff $G_\lambda \models \phi_3[f]$, and the $\tau$ from $(3^*)$ satisfies condition (B) from Def. 3.1.

[ Assuming $G_\mu \models \phi_3[f]$, $\tau$ as in $(4^*)$, $f^*_5 \circ f \circ f^*_5 \circ f \restriction B_2 = f^*_6 \circ f \restriction B_2$ is equivalent to the equality of

$$f^*_6 \circ f(a^0_0) = f^*_6(\tau(a^0_{\ell(1)}, ..., a^0_{\ell(k)})) = \tau(a^{\ell(1)}_{\ell(1)}, ..., a^{\ell(k)}_{\ell(k)})$$

and $f^*_5 \circ f \circ f^*_5 \circ f(a^0_0) = f^*_5 \circ f \circ f^*_5 \quad (\tau(a^0_{\ell(1)}, ..., a^0_{\ell(k)}))$

$$= f^*_5 \circ f(\tau(a_0^{\ell(1)}, ..., a_0^{\ell(k)}))$$

$$= f^*_5(\tau(\tau(a^{\ell(1)}_{\ell(1)}, a^{\ell(1)}_{\ell(2)}, ...), \quad \tau(a^{\ell(2)}_{\ell(1)}, a^{\ell(2)}_{\ell(2)}, ...), ..., \tau(a^{\ell(k)}_{\ell(1)}, a^{\ell(k)}_{\ell(2)}, ...))$$

$$= \tau(\tau(a^{\ell(1)}_{\ell(1)}, a^{\ell(2)}_{\ell(1)}, ...), \quad \tau(a^{\ell(1)}_{\ell(2)}, a^{\ell(2)}_{\ell(2)}, ...,), \quad \tau(a^{\ell(1)}_{\ell(k)}, a^{\ell(2)}_{\ell(k)}, ...)$$

This clearly is equivalent to condition (B) from Def. 3.1 on $\tau$. The other direction is easy. ]

<u>$(5^*)$</u> $G_\mu \models \phi_5[f]$ iff $f(a^m_n) = \tau(a^m_{n+\ell(1)}, ..., a^m_{n+\ell(k)})$ (the $\ell(i)$ are distinct) where $\tau(x_1, ..., x_k)$ is beautiful.

[Because, assuming $G_\lambda \models \phi_4(f)$, $f^*_2 \circ f \restriction B_0 = f^*_2 \restriction B_0$ is equivalent to $a^0_0 = \tau(a^0_0, ..., a^0_0)$ ]

<u>$(6^*)$</u> $G_\mu \models \phi_6[f]$ iff $f(a^n_0) = \tau(a^n_{\ell(1)}, ..., a^n_{\ell(k)})$

for some beautiful $\tau$, and $\ell(i) < \omega$, for every $n < \omega$.

[Immediate, by (5), noticing $f^*_5 \circ f^*_7 \circ f^*_5$ is a projection onto $B_3$].

So we clearly finish case I.

<u>Case II</u>: $\mu = |J|$ is regular $> \aleph_0$.

Let $I^* - J = I_0 \cup \{<\alpha,\delta,n>: \alpha < \mu,\ \delta < \mu_1,\ \mathrm{cf}\,\delta = \aleph_0,\ n < \omega\}$.

$|I_0| = \mu$, and let us denote $a^{\beta,n}_\alpha = a_{<\alpha,\beta,n>}$; where $\mu_1 = \mu$. But for using in case III, we from now up to the end of the proof of claim 4.3, assume only $\mu_1 \leq \mu$, $\mu_1$ is regular. For each limit $\delta < \mu_1$, cf $\delta = \aleph_0$ choose an increasing sequence $\delta(n)$ $(n < \omega)$ of ordinals $< \delta$, whose limit is $\delta$; such that for each $\beta < \mu_1$, $n < \omega$, $\{\delta < \mu_1 : \beta = \delta(n)\}$ is a stationary subset of $\mu_1$ (see e.g. Solovay [So] ). Let us define some $f^*_e$'s, by defining $f^*_e(a_t)$ $(t \in I^*)$ understanding that when $f^*_e(a_t)$ is not explicitely defined, it is $a^0_0$. So let, for $\alpha < \mu$, $\beta < \mu_1$, $f^*_2(a^\beta_\alpha) = a^0_0$, $f^*_3(a^\beta_\alpha) = a^0_\alpha$, $f^*_4(a^\beta_\alpha) = a^\beta_0$, $f^*_5(a^\beta_\alpha) = a^\alpha_\beta$, $f^*_6(a^\beta_\alpha) = a^\alpha_\alpha$, and when $\delta < \mu_1$, cf $\delta = \aleph_0$, $f^*_7(a^\delta_\alpha) = a^{\delta,0}_\alpha$;

$\mathrm{cf}\,\beta \neq \aleph_0 \rightarrow f^*_7(a^\beta_\alpha) = a^\beta_\alpha$; $f^*_8(a^\beta_\alpha) = a^\beta_\alpha$; $f^*_8(a^{\delta,n}_\alpha) = a^{\delta,n+1}_\alpha$,

$f^*_9(a^\beta_\alpha) = a^\beta_\alpha$, $f^*_9(a^{\delta,n}_\alpha) = a^{\delta(n)}_\alpha$. Let $B_0 = \mathrm{Cl}\{a^0_0\}$, $B_1 = \mathrm{Cl}\{a^0_\alpha : \alpha < \mu\}$,

$B_2 = \mathrm{Cl}\{a^\beta_0 : \beta < \mu_1\}$, $B_3 = \mathrm{Cl}\{a^\beta_\alpha : \alpha < \mu,\ \beta < \mu_1\}$,

$B_4 = \mathrm{Cl}\{a^\beta_\alpha, a^{\beta,n}_\alpha : \alpha < \mu,\ \beta < \mu_1, n < \omega\}$ and $B_5 = \mathrm{Cl}\{a^\beta_0, a^{\beta,n}_0 : \beta < \mu_1, n < \omega\}$,

$B_6 = \mathrm{Cl}\{a^\beta_0 : \beta < \mu_1,\ \mathrm{cf}\beta = \aleph_0\}$, $B_7 = \mathrm{Cl}\{a^\beta_\alpha : \alpha < \mu,\ \beta < \mu_1,\ \mathrm{cf}\beta = \aleph_0\}$

Clearly $f^*_3, f^*_4, f^*_9$ are projections onto $B_1$, $B_2$, $B_3$ resp. and let $f^*_{10}, f^*_{11}, f^*_{12}$ be projections onto $B_4$, $B_5$, $B_6$ resp.

Now we apply lemma 1.3, with $\{<\alpha,\beta> : \alpha < \mu,\ \beta < \mu_1\}$ for J, and $\{<\alpha,\beta> : \alpha < \mu\}$ for $J_\beta$, and $I^*$ for I, and get $f^*_{13} \in G_\lambda$ as mentioned there.

Let the first order formula $\phi_1(f,g)$ says that f,g are conjugate to $f^*_2$, Rnf, Rng $\subseteq B_3$ and there is $h \in G_\mu$ commuting with $f^*_{13}$, and mapping $B_3$ into itself, such that $h \circ f = g$. We shall write $\phi_1(f,g)$ also in the form $f \leq g$. So by 1.3, if f,g are conjugate to $f^*_2$ and

$f(a^0_0) = a^\beta_\alpha$ , $g(a^0_0) = \tau(a^{\beta(1)}_{\alpha(1)}, \ldots, a^{\beta(k)}_{\alpha(k)})$ then $f \leq g$ iff $\beta \leq \beta(1) \wedge \ldots \wedge \beta \leq \beta(k)$.

Let the first order formula $\phi_2(f)$ says that $f \restriction B_2$ , $f \restriction B_5$ , $f \restriction B_0$, $f \restriction B_6$ are into $B_3, B_4, B_1, B_7$ resp. and for any g conjugate to $f^*_2$ , if Rng $\subsetneq B_2$ , then $g \leq f \circ g$ ; and $f \restriction B_5$ commute with $f^*_7, f^*_8, f^*_9$ and $f \restriction B_3$ commute with $f^*_3$ .

CLAIM 4.3 : $G_\lambda \models \phi_2[f]$ iff for each $\beta < \mu_1$

$f(a^0_0) = \tau(a^\beta_{\alpha(1)}, \ldots, a^\beta_{\alpha(k)})$, $f(a^{\beta,n}_0) = \tau(a^{\beta,n}_{\alpha(1)}, \ldots, a^{\beta,n}_{\alpha(k)})$

for some $\tau$ , $\alpha$ (e) (which do not depend on $\beta$ !).

PROOF : Assume $G_\mu \models \phi_2$ [ f ] , and let

$f(a^\beta_0) = \tau_\beta (a^{\gamma(\beta,1)}_{\alpha(\beta,1)}, a^{\gamma(\beta,2)}_{\alpha(\beta,2)}, \ldots, a^{\gamma(\beta,k(\beta))}_{\alpha(\beta,k(\beta))})$.

$\alpha$ ($\beta$,e) increase with e,w.l.o.g. By part of $\phi_2$ saying : g conjugate to $f^*_2$ , Rng $\subsetneq B_2$ implies $g \leq f \circ g$ ; it follows that $\beta \leq \gamma(\beta,e)$ (choose g such that $g(a_t) = a^\beta_0$ ).

As $\mu_1$ is regular, $\mu_1 > \aleph_0$ , for any $\beta_0 < \mu_1$

$\sup \{ \gamma(\beta,e) : e = 1, \ldots, k(\beta), \beta < \beta_0 \} < \mu_1$ , hence

$S = \{ \beta_0 : \beta_0 < \mu_1$ ; and $\beta < \beta_0, 1 \leq e \leq k(\beta)$ implies $\gamma(\beta,e) < \beta_0 \}$

is an unbounded subset of $\mu_1$ ; and by its definition it is closed.

Now we shall prove that for $\delta \in S$, cf $\delta = \aleph_0$ implies $\gamma(\delta,e) = \delta$ . For suppose $\gamma = \gamma(\delta, e_0) \neq \delta$ then as said above $\delta < \gamma(\delta, e_0)$. As $\delta_1(n)$ is increasing (as a function of n) and its limit is $\delta_1$ , (or $\delta_1(n)$ is not defined) for some $n < \omega$ big enough, $\delta < \gamma(\delta, e_0)(n)$, and $\gamma(\delta, e_1) \neq \gamma(\delta, e_2) \Rightarrow \gamma(\delta, e_1)(n) \neq \gamma(\delta, e_2)(n)$ (when they are defined).

As $f \restriction B_6$ is into $B_7$ necessarily $\gamma(\delta,e)$ has cofinality $\aleph_0$ , and as $f \restriction B_5$ commutes with $f^*_7$

$$f(a^{\delta,0}_0) = \tau_\delta (a^{\gamma(\delta,1),0}_{\alpha(\delta,1)}, a^{\gamma(\delta,2),0}_{\alpha(\delta,2)}, \ldots)$$

As $f \restriction B_5$ commutes with $f^*_8$

$$f(a^{\delta,n}_0) = \tau_\delta (a^{\gamma(\delta,1),n}_{\alpha(\delta,1)}, a^{\gamma(\delta,2),n}_{\alpha(\delta,2)}, \ldots)$$

As $f \restriction B_5$ commutes with $f_9^*$

$$f(a_0^{\delta(n)}) = \tau_\delta\,(a_{\alpha(\delta,1)}^{\gamma(\delta,1)(n)},\ a_{\alpha(\delta,1)}^{\gamma(\delta,2)(n)}, \ldots)$$

and $\tau_\delta\,(a_{\alpha(\delta,1)}^{\gamma(\delta,1)(n)}, \ldots)$ is reduced (by the choice of $\tau_\delta$ and of n).

But $f(a_0^{\delta(n)})$ is equal also to $\tau_{\delta(n)}\,(a_{\alpha(\delta(n),1)}^{\gamma(\delta(n),1)}, \ldots)$ which is reduced too.

Hence $\gamma(\delta,e_0)(n) \in \{\gamma(\delta(n),e) : 1 \leq e \leq k(\delta(n))\}$

but on the one hand $\delta < \gamma(\delta,e_0)(n)$ by the choice of $e_0$ and n,

and on the other hand $\delta(n) < \delta \Rightarrow \gamma(\delta(n),e) < \delta$ as $\delta \in S$, contradiction.

Hence $\gamma(\delta,e) = \delta$ for every $\delta \in S$.

Now for every $\beta < \mu_1$, n we know $\{\delta < \mu : \mathrm{cf}\,\delta = \aleph_0, \delta(n) = \beta\}$ is stationary, so there is $\delta \in S$, $\mathrm{cf}\,\delta = \aleph_0$ such that $\delta(n) = \beta$.

As before we can show that $f(a_0^{\delta(n)}) = \tau_\delta\,(a_{\alpha(\delta,1)}^{\delta(n)}, \ldots, a_{\alpha(\delta,k(\delta))}^{\delta(n)})$

$= \tau_{\delta(n)}(a_{\alpha(\delta(n),1)}^{\gamma(\delta(n),1)}, \ldots, a_{\alpha(\delta(n),k(\delta(n)))}^{\gamma(\delta(n),k(\delta(n)))})$ and the last is reduced.

Hence $\gamma(\delta(n),e) \in \{\delta(n)\}$ for each e, hence $\gamma(\delta(n),e) = \delta(n)$, that is $\gamma(\beta,e)$ for each $\beta$ and e. Hence, as $\tau_\beta\,(a_{\alpha(\beta,1)}^{\gamma(\beta,1)}, \ldots)$ is reduced, the ordinals $\alpha(\beta,e)$, $1 \leq e \leq k(\beta)$ are distinct.

As $f \restriction B_3$ commutes with $f_3^*$, for every $\beta$

$$\tau_\beta\,(a_{\alpha(\beta,1)}^{0}, \ldots, a_{\alpha(\beta,k(\beta))}^{0}) = \tau_0(a_{\alpha(0,1)}^{0}, \ldots, a_{\alpha(0,k(0))}^{0})$$

As the $\alpha(\beta,e)$ are distinct, necessarily $\{\alpha(\beta,e) : 1 \leq e \leq k(e)\} =$ $\{\alpha(0,e) : 1 \leq e \leq k(0)\}$, but as $\alpha(\beta,e)$ is increasing with e (for each $\beta$, by the choice of $\tau_\beta(\ , \ldots,))$ necessarily $\alpha(\beta,e) = \alpha(0,e)$, $k(\beta) = k(0)$ and we can assume that $\tau_\beta = \tau_0$.

Now clearly as before

$$f(a_0^{\beta,n}) = \tau_0\,(a_{\alpha(0,1)}^{\beta,n}, \ldots)$$

So we finish one direction of claim 3.3 where as the other is immediate.

* * *

Let $\phi_3$ (f) say that f maps $B_2$ into $B_3$ and if there is $f_1, f_1 \restriction B_2 \quad f \restriction B_2$ and $\phi_2(f_1)$.

It is easy to check that for every $\beta$ ; for some $\tau$ , $\gamma$ (e) :

$G_\mu \models \phi_3 [f] \Leftrightarrow f(a^\beta_0) = \tau(a^\beta_{\gamma(1)}, \ldots a^\beta_{\gamma(k)})$.

Let $\phi_4$(f) say that f maps $B_2$ into $B_3$ , $\phi_3$(f), and if $\phi_3$(g), then

$g \circ f_5 \circ f \restriction \tilde{a}^0_0 = f_5 \circ f \circ f_5 \circ g \restriction \tilde{a}^0_0$ and $f^*_5 \circ f \circ f^*_5 \circ f \restriction \tilde{a}^0_0 = f^*_6 \circ f \restriction \tilde{a}^0_0$ and

$f^*_2 \circ f \restriction \tilde{a}^0_0 = f^*_2 \restriction \tilde{a}^0_0$. (note that $\tilde{a}^0_0 = \mathrm{Rn}\, f^*_2$). As in case I we can check that

$G_\mu \models \phi_4 [f] \Leftrightarrow f(a^\beta_0) = \tau_0(a^\beta_{\gamma(1)}, \ldots, a^\beta_{\gamma(k)})$ (w.l.o.g. reduced)

and $\tau_0$ is beautiful.

<u>Case III</u> : $\mu$ a singular cardinal.

We give this case with less details.

We let $\mu_1 < \mu$ and $B_1, B_2, B_3$ be as in case II, $\mu_1$ regular , $\mu_1 > \aleph_0$ . By case II we can define $f^*_e$'s properly, so that, for some $\phi^0$ $G_\lambda \models \phi^0$ (f) <u>iff</u> there are $\tau$ and distinct $\alpha_1, \ldots, \alpha_n$ so that for each $\beta < \mu_1$ , $f(a^\beta_0) = \tau(a^\beta_{\alpha_1}, \ldots, a^\beta_{\alpha_n})$.

Let $\phi^1$ (f) say that $f(\tilde{a}^0_0) \subseteq B_2$ and for every g, $\phi^0$ (g) implies that

$(f \circ g) \restriction \tilde{a}^0_0 = (g \circ f) \restriction \tilde{a}^0_0$. It is easy to check that $G_\lambda \models \phi^1$ [f] <u>iff</u> there are $\sigma$ and distinct $\beta$ (1), ..., $\beta$ (m) such that for every $\alpha$ , $f(a^0_\alpha) = \sigma(a^{\beta(1)}_\alpha, \ldots, a^{\beta(m)}_\alpha)$ ,

$\sigma$ satisfying (A) of Def. 3.1.

As $\mu_1$ is regular, we can use case II. So there is $\phi^2$ such that $G_\lambda \models \phi^2$ [ f ] <u>iff</u> there are a beautiful term $\sigma$ and distinct $\beta$ (i) such that for every $\alpha$ , $f(a^0_\alpha) = \sigma(a^{\beta(1)}_\alpha, \ldots, a^{\beta(m)}_\alpha)$.

Let $\mu = \sum_{i < \mathrm{cf}\mu} \mu(i)$, $\mu(i) < \mu$ , $\mu$ (i) increasing.

We just prove that for every $\gamma < \mathrm{cf}\mu$ , there is $\bar{f}^*_\gamma$ as constructed in case II and above, such that

(i) $G_\lambda \models \phi^2[f, \bar{f}^*_\gamma]$ <u>iff</u> there are a beautiful term $\sigma$ and distinct $\beta$ (i) $< \mu(\gamma)^+$ , such that for every $\alpha < \mu$ , $f(a^0_\alpha) = \sigma(a^{\beta(1)}_\alpha, \ldots, a^{\beta(m)}_\alpha)$

(ii) $\bar{f}^*_\gamma$ (0) is a projection onto Cl { $a^\beta_\alpha$ : $\alpha < \mu$, $\beta < \mu(\gamma)^+$ }

Checking the construction carefully, we see that :

if $\tau$, $\sigma$ are beautiful terms, $\beta(i) < \mu\ (\gamma_\ell)^+$ $(i = 1, \ldots, m,\ \ell = 1, \ldots, n)$ and for every $\alpha < \mu$, $f(a^0_\alpha) = \sigma(a^{\beta(1)}_\alpha, \ldots, a^{\beta(m)}_\alpha)$, then $G_\lambda \models \phi^2(f, \tau(\bar f^*_{\gamma_1}, \ldots, \bar f^*_{\gamma_n}))$.

Now checking the proof of 4.1 from 4.2, the existence of the $\bar f^*_\gamma$ mentioned above, is sufficient to prove 4.1 when $i(0) < \mu$.

Hence there are $\phi^3$ and $g^*$ such that

$G_\mu \models \phi^3[\bar f, g^*]$ $\underline{\text{iff}}$ $\bar f = \tau(\bar f^*_{\gamma(1)}, \ldots, \bar f^*_{\gamma(n)})$ for some beautiful $\tau$. So let the formula $\phi^4[f, g^*]$ say «there is $\bar f_1$, such that $\phi^3[\bar f_1, \bar g^*]$ and for every $\bar f_2$ which satisfies $\phi^3[\bar f_2, \bar g^*] \wedge \text{Rn}\ \bar f_1(0) \subseteq \text{Rn}\ \bar f_2(0)$ satisfies also $\phi^2[f, \bar f_2]$ ».

Clearly $G_\mu \models \phi^4[f, \bar g^*]$ iff for some beautiful $\tau$ and $\beta(i) < \mu$, for every $\alpha < \mu$, $f(a^0_\alpha) = \tau(a^{\beta(1)}_\alpha, \ldots)$ and $\phi^4(f^*_5 \text{ o f o } f^*_5, g^*)$ is our desired formula.

THEOREM 4.4 : We can find formulas $\phi^b$, $\phi^{eq}$, $\phi^r$, $\phi^x$ (in the language of $G_\lambda$) such that for any ultrafilter $T \subseteq \underline{B}$, $\lambda$, and free set of generators $\{a_t : t \in I^*\} \subseteq F_\lambda$ and $f_t \in F_\lambda$ defined by $f_t(a_s) = a_t$ ; there is $\bar f^* \in G_\lambda$ such that $f^*_0 = f_t$ for some t and

(A) $G_\lambda \models \phi^b[f, \bar f^*]$ iff $f = \tau(f_{t_1}, \ldots)$ for some beautiful $\tau$.

(B) For any f,g, (see Def. 3.3)

$G_\lambda \models \phi^{eq}[f, g ; \bar f^*]$ $\underline{\text{iff}}$ $\phi^b[f, \bar f^*] \wedge \phi^b[g, \bar f^*]$ and $f \approx_T g$

(C) For any $h : I^* \to I^*$ and $f, g \in G_\lambda$,

$\phi^{eq}[f, g : \bar f^*]$ implies $\phi^{eq}[\hat h \circ f, \hat h \circ g ; \bar f^*]$

(D) For any two-place relation R on the class of $\phi^{eq}$-equivalence classes, there is $\bar g \in G$ such that for any $f_1, f_2 \in G_\lambda$

$G_\lambda \models \phi^r[f_1, f_2 ; \bar g]$ iff $\phi^b[f_1 ; \bar f^*] \wedge \phi^b[f_2 ; \bar f^*]$ $\langle f_1/\phi^{eq}, f_2/\phi^{eq}\rangle \in R$

(E) $G_\lambda \models \phi^x[f]$ iff $f \in G_\lambda$ is a projection onto some $\text{Cl}\{\tau_e(a_{t(e,1)}, \ldots) : e < n\}$ (for some $n < \omega$, beautiful $\tau_e$, and $t(e,1), \ldots \in I^*$.

PROOF : The set

$R = \{\langle f, g\rangle : f, g \in \{\tau(f_{t(1)}, \ldots) : \tau \text{ beautiful}, t(e) \in I^*\}$
and for some $\tau(x,y) \in T$, $f = \tau(g, f)\}$

is closed under beautiful terms, hence by 4.1 is definable by some $\phi$.
This observation suffice for (A), (B). Now (C) is a statement on $\approx_T$, in fact, which is easy to verify.
Now (D) follows by (C), as by (C) we can define arbitrary one-place functions from the set of $\phi^{eq}(X,Y\,;\,\bar{f}^*)$-equivalence classes into itself, so by Rabin [ Ra] , we can define arbitrary relations. Lastly, (E) follows by 4.1.

Now any such subalgebra is the range of a projection, because by Def. 3.1 and 3.1 (A) we can assume the subalgebra is

$$B = cl\,\{\tau(a_{t(i,0)}, a_{t(i,1)}, \ldots, a_{t(i,m-1)}) : i < n\}\ .$$

Choose $b \in B$ and let

$$h(a_t) = \tau(c_{s(t,0)}, \ldots, c_{s(t,m-1)})$$

where $c_{s(t,j)} = a_t$ if for some i, $t(i,j) = t$ and $c_{s(t,j)} = b$ otherwise.

### § 5. INTERPRATING SET THEORY IN A CATEGORY.

Here K will be a full subcategory of Cat, $A \in K \to |A| < \lambda^*$, and $\lambda < \lambda^* \to F_\lambda \in K$, and $|L(V)| < \lambda^*$ (in fact $(\forall \lambda < \lambda^*)(\exists \mu \geq \lambda)(F_\mu \in K)$ suffice).

In the formulas from 4.4 we now put $F_\mu$ explicitely as a parameter.

DEFINITION 5.1 : Let $\bar{f}$ be of the length of $\bar{f}^*$ (of 4.4) and let $\psi^a(\bar{f},A)$ be the formula saying (in K) :

(0) $f_e \in \text{End } A$ ( = Hom(A,A))

(1) $f_0$ is a projection

(2) $\phi^b(g;\bar{f},A)$ implies g is conjugate to $f_0$ (so $g \in \text{End } A$)

and $\phi^b(g_1,\bar{f},A) \wedge \phi^b(g_2,\bar{f},A) \to g_2 = g_2 \circ f_1$

(3) $\phi^{eq}(g_1,g_2;\bar{f},A)$ is an equivalence relation $E_{\bar{f}}$ over

$W_A^{\bar{f}} = \{g : \phi^b(g;\bar{f},A)\}$ with $>1$ equivalence classes

(4) $\phi^b(g;\bar{f},A)$ implies $\phi^x(g;\bar{f},A)$ which implies $g \in \text{End } A \wedge \text{proj } g$

(5) Suppose $\phi^x(g_1;\bar{f},A) \wedge \phi^x(g_2;\bar{f},A)$, then there is a $g_3$ such that :

(i) $\phi^x(g_3;\bar{f},A) \wedge \text{Rn } g_1 \subseteq \text{Rn } g_3 \wedge \text{Rn } g_2 \subseteq \text{Rn } g_3$

(ii) if $\phi^b(g_4;\bar{f},A) \wedge \text{Rn } g_4 \subseteq \text{Rn } g_3$

then $\bigvee_{e=1}^{2} (\exists g)\ [\phi^b(g;\bar{f},A) \wedge \text{Rn } g \subseteq \text{Rn } g_e \wedge \phi^{eq}(g_4, g;\bar{f},A)]$

(6) $\phi^P(g_1,g_2,g_3;\bar{f},A)$ represent a pairing function on the set of $E_{\bar{f}}$-equivalence classes.

(7) If (i) $\phi^b(g_e;\bar{f},A)$ (e = 1,6), $h \in \text{End } A$,

(ii) $(\forall g)\ [\phi^x(g;\bar{f},A) \wedge \text{Rn } g_1 \subseteq \text{Rn } g \wedge \text{Rn } g_2 \subseteq \text{Rn } g \to \text{Rn } g_3 \subseteq \text{Rn } g]$

(iii) $h \circ g_e = g_{e+3}$ for e = 1,2,3

then for e = 1,2 $\phi^{eq}(g_3,g_e;\bar{f},A) \equiv \phi^{eq}(g_6,g_{e+3};\bar{f},A)$

(at least one of them holds by (5))

(8) If $\phi^x(g^*,\bar{f},A)$ then there is $g_2^*$, such that $\phi^b(g_2^*,\bar{f},A)$ and for each $g_1^*$

satisfying $\phi^b(g_1^*,\bar{f},A) \wedge \text{Rn } g_1^* \subseteq \text{Rn } g^*$ the following holds : $\neg\phi^{eq}(g_1^*,g_2^*;\bar{f},A)$ and for every $g_3, g_4$ satisfying $\bigwedge_{e=3}^{4} \phi^b(g_e, \bar{f}, A)$ there is $h \in$ End A such that

$h \circ g_1^* = g_3$, $h \circ g_2^* = g_4$ and $(\forall g)\ [\phi^b(g,\bar{f},A) \to \phi^b(h \circ g, \bar{f}, A)]$

(9) If $\phi^b(g_e, \bar{f}, A)$ $(e = 1,2)$ then for some $h \in$ End A, $h \circ g_e = g_{1-e}$ and $(\forall g)\ (\phi^b(g,\bar{f},A) \equiv \phi^b(h \circ g, \bar{f}, A))$

CLAIM 5.1 :

(A) For each $\lambda < \lambda^*$, let $\bar{f}^*_\lambda \in$ End $F_\lambda$ be as constructed in 4.4.
Then $K \models \psi^a[\bar{f}^*_\lambda, F_\lambda]$

(B) Suppose $K \models \psi^a[\bar{f}, A]$. Then

(1) If $\bigwedge_{e=1}^{n+1} \phi^b(g_e, \bar{f}, A)$ and Rn $g_{n+1} \subseteq$ Cl $\bigcup_{e=1}^{n}$ Rn $g_e$
then $\bigvee_{e=1}^{n} \phi^{eq}(g_{n+1}, g_e; \bar{f}, A)$

(2) $\{g : \phi^b(g, \bar{f}, A)\}$ is closed under beautiful terms

(3) There is an ultrafilter $T = T(\bar{f}, A)$ such that
$\bigwedge_{e=1}^{2} \phi^b(g_e, \bar{f}, A)$ implies $\phi^{eq}(g_e, \tau(g_1, g_2), \bar{f}, A)$ where $e = 1 \Leftrightarrow \tau(x,y) \in T \Leftrightarrow e \neq 2$.

(4) $E_{\bar{f}}$ has infinitely many equivalence classes

PROOF :

(A) Immediate.

(B) (1) Easy, by part (5) (and (4)) of Def. 5.1 (for $n = 1$, use (2)).

(B) (2), (3). By part (8) of Def. 5.1 we can define inductively $g_i^*$ $(i < \omega)$ such that $\phi^b(g_i^*, \bar{f}, A)$ and if $i < j$, $\phi^b(g^1, \bar{f}, A) \wedge \phi^b(g^2, \bar{f}, A)$ <u>then</u> for some $h \in$ End A $h \circ g_i^* = g^1$, $h \circ g_j^* = g^2$ and $(\forall g)\ (\phi^b(g, \bar{f}, A) \to \phi^b(h \circ g, \bar{f}, A))$.

Let $\tau(x,y)$ be a beautiful term .
Let us apply 3.4 to $\Phi^b$, and get $\Phi = \{\psi_e : e < k < \omega\}$
So there are $i < j \leq 2^k$ such that $\bigwedge_{e<k} \psi_e(g_i, \bar{f}, A) \equiv \phi_e(g_j, \bar{f}, A)$ hence $\phi^b(\tau(g_i, g_j), \bar{f}, A)$. Now for any $g^1, g^2$ satisfying $\phi^b(g^1, \bar{f}, A) \wedge \phi^b(g^2, \bar{f}, A)$,

there is h such that $h \circ g_i = g^1, h \circ g_j = g^2$ and $(\forall g)(\phi^b(g,\bar f, A) \to \phi^b(h \circ g, \bar f, A))$. Hence $\tau(g^1, g^2) = h(\tau(g_i, g_j))$ satisfy $\phi^b(x,\bar f,A)$; hence (2) is proved. For (3) note that by (B) $\phi^{eq}(\tau(g_i, g_j), g_p, f, A)$, hence by (7) for any $g^1, g^2$ as above $\phi^{eq}(\tau(g^1,g^2), g^q, \bar f, A)$ where $p = i \leftrightarrow q = 1$.

(B) (4). By Def. 5.1, (7) and (8).

LEMMA 5.2 :

(A) There is a formula $\psi^s$ such that

(1) $K \models \psi^s [\bar f_\lambda, F_\lambda] \wedge (\forall \bar f, A) [\psi^s(\bar f,A) \to \psi^a(\bar f,A)]$

(2) If $K \models \psi^s[\bar f, A]$ then for any subset R of $W^{\bar f}_A / E_{\bar f}$, $|R| < \lambda^*$, there is $g^* \in K$ such that $g/E_{\bar f} \in R \Leftrightarrow K \models \phi^r [g, g^*, \bar f, A]$

(3) Like (2), for a two-place relation R.

(B) There is a formula $\psi^t$ such that if $K \models \psi^s [\bar f^e, A^e]$ $(e = 1,2)$ then :
$K \models \psi^t [\bar f^1, A^1, \bar f^2, A^2]$ iff $T(\bar f^1, A^1) = T(\bar f^2, A^2)$

(C) There are formulas $\psi^e$, $\theta$ such that

(1) $K \models \psi^e [\bar f_\lambda, F_\lambda] \wedge (\forall f,A) [\psi^e(\bar f,A) \to \psi^s[\bar f,A)]$

(2) $K \models \psi^e[\bar f^1, A^1] \wedge \psi^e [\bar f^2, A^2] \wedge \psi^t [\bar f^1, A^1, \bar f^2; A^2]$

implies : for every $g^1_i \in W^{\bar f^1}_{A^1}$ pairwise non $E_{\bar f^1}$-equivalent, $(i < \lambda < \lambda^*)$ and $g^2_i \in W^{\bar f^2}_{A^2}$ there is $\bar h$ such that

$\theta[g^1, g^2, \bar h] \Leftrightarrow (\exists i) [g^1 E_{\bar f^1} g^1_i \wedge g^2 E_{\bar f^2} g^2_i]$

(D) There is a formula $\psi^m$ such that

(1) $K \models \psi^m [\bar f_\lambda, F_\lambda] \wedge (\forall \bar f, A) [\psi^m(\bar f,A) \to \psi^e(\bar f,A)]$

(2) $K \models \psi^m [f,A]$ implies $E_{\bar f}$ has $< \lambda^*$ equivalence classes.

PROOF OF 5.2 :

(A) Immediate by 4.1 and claim 5.1 (B) (2), (3) (for (3), use (2) and $\phi^P$).

(B) Suppose $K \models \psi^s\ [\bar{f}^e, A^e]$ (e = 1,2) and $T = T(\bar{f}^1, A^1) = T(\bar{f}^2, A^2)$.

For e = 1,2 let $g^e_n \in W^{\bar{f}^e}_{A^e}$ ($n < \omega$) be pairwise non-$E_{\bar{f}^e}$-equivalent.

Let $|A^1| + |A^2| \leq \lambda < \lambda^*$, let $\{a_i : i < \lambda\ \omega\}$ freely generate $F_\lambda$, and $f_n \in \text{End } F_\lambda$ be a projection onto $\{a_i : \lambda\ n \leq i < \lambda\ (n+1)\}$.

Let $\bar{f}^*$ be such that

$\underline{\phi}^b (g, \bar{f}^*, F_\lambda) \Leftrightarrow g = \tau (f_{\beta(1)}, \ldots), \tau$ beautiful

$\phi^{eq} (g_1, g_2, \bar{f}^*, F_\lambda) \Leftrightarrow \phi^b (g_1, \bar{f}^*, F_\lambda) \wedge \phi^b (g_2, \bar{f}^*, F_\lambda)$

and for some $\tau(x,y) \in T$

$g_1 = \tau (g_2, g_1)$

Let $h_e$ maps $\{a_i : \lambda\ n \leq i < \lambda\ (n+1)\}$ onto $\text{Rn } g^e_n$.

Let $\theta^2 (g^1, g^2) = \theta^2 (g^1, g^2 ; h_1, h_2, f, \bar{f}^1, \bar{f}^2, A^1, A^2, \bar{f}^*, F_\lambda)$ says :

$\phi^b (g^e, \bar{f}^e, A^e)$ (e = 1,2) and for some $g^e_1$ (e = 1,2)

$$g^e\ E_{\bar{f}^e}\ g^e_1$$

and there is f, $\phi^b (f, \bar{f}^*, F_\lambda)$ such that

$\text{Rn } h_e \circ f \subsetneq \text{Rn } g^e_1$ and for every $g^e_0$ (e = 1,2)

$g^e_0 \in \text{End } A^e \wedge \text{proj } g^e_0 \wedge \text{Rn } h_e \circ f \subseteq g^e_0 \rightarrow \text{Rn } g^e_0 \subsetneq \text{Rn } g^e_0$

Then $\theta^2 (g^1, g^2)$ <u>iff</u> there are beautiful $\tau$, and

n(1), ... such that

$g^e\ E_{\bar{f}^e}\ \tau (g^e_{n(1)}, \ldots)$ for e = 1, 2

Let $\psi^t\ [\bar{f}^1, A^1, \bar{f}^2, A^2]$ say that $\psi^s\ [f^e, A^e]$, e = 1,2 and for some $h_1, h_2, f^*, B$, $\theta^2(g^1, g^2) =$

$= \theta^2(g^1, g^2, h_1, h_2, \bar{f}^1, \bar{f}^2, A^1, A^2, \bar{f}^*, B)$.

define a one-to-one map from a subset of $W^{\bar{f}^1}_{A^1}$ into $W^{\bar{f}^2}_{A^2}$ ; and this set is infinite (i.e. is ordered in an order of type $\omega$, see (A)).

We have just proved that

$\psi^s\ [\bar f^e, A^e]\ \ (e = 1,2)$ and $T(\bar f^1, A^1) = T(\bar f^2, A^2)$ implies $\psi^t\ [\bar f^1, A^1; \bar f^2, A^2]$.

Suppose the conclusion holds, then clearly $\psi^s\ [\bar f^e, A^e]$ ; if $\tau(x,y) \in T(\bar f^1, A^1)$, $\tau(x,y) \notin T(\bar f^2, A^2)$ we get contradiction by 3.4.

(C) The same proof as (B), essentially.

(D) Let $\psi^m(\bar f, A)$ says that $\psi^e(\bar f, A)$ and there is $A^1$ such that for every $\bar f^1$, if $\psi^t\ [\bar f, A; \bar f^1, A^1]$ (and there is at least one such $\bar f^1$) then for some $\bar f^2$ and automorphism h of $A^1$ :

(0) $\psi^e(\bar f^2, A^1)$

(1) for some $\bar h$, $\theta(g^1, g^2, \bar h)$ define a map from the $E_{\bar f^2}$-equivalence classes onto the $E_{\bar f}$-equivalence classes

(2) if $\phi^b(g^1, \bar f^2, A^1) \wedge \phi^b(g^2, \bar f^2, A^1) \wedge \neg\phi^{eq}(g^1, g^2, \bar f^2, A^1)$ then $g^1 \circ h \circ f^1_0 \neq g^2 \circ h \circ f^1_0$.

For proving D(1) choose $A^1 = F_\lambda$, and $\bar f^2$ is chosen like $\bar f_\lambda$, but the range of the projections, is freely generated by $\aleph_0$ elements. For D(2), if $E_{\bar f}$ has $\geq \lambda^*$ equivalence classes, choose $\bar f^1$ such that Rn $f^1_0$ is finitely generated.

DEFINITION 5.2 : We say $< g, \bar f, A >$ represent the model (N,R) if :

$g \in$ End A, and let $\{ g_i : i < \alpha^* \}$ be representatives of the $E_{\bar f}$-equivalence classes, and let

$J = \{ i < \alpha^* : (\exists f)\ [\phi^r(f,g,\bar f,A) \wedge (\exists f_2)\ \ \phi^p(f, f_2, g_i, \bar f, A)\ ]\ \}$

and there is H, a one-to one function from J onto R, and :

$\langle H(i), H(j)\rangle \in R$ iff $(\exists f\ f_2\ g)\ (\phi^r(f,g,\bar f,A) \wedge \phi^p(f,f_2, g, \bar f, A) \wedge \phi^p(f_2, g_i, g_j, \bar f, \bar A))$

LEMMA 5.4 : There are formulas $\psi^f$, $\psi^h$ such that

(A) $\psi^f(\bar g, \bar f, A)$ iff $\psi^m(\bar f, A)$ and $< g, \bar f, A >$ represent a well founded model satisfying extensionality.

(B) $\psi^h(g^1, \bar f^1, A^1 ; g^2, \bar f^2, A^2)$ iff $\psi^f(g^e, \bar f^e, A^e)$ (e = 1, 2) and $T(\bar f^1, A^1) = T(\bar f^2, A^2)$ and $< g^e, \bar f^e, A^e >$ (e = 1,2) represent isomorphic models.

PROOF : Easy by 5.3.

THEOREM 5.5 : In K we can interpret a model $M_{\lambda^*}$ consisting of $\mu$ (V) disjoint copies of $H(\lambda^*)$ ( $\mu(V)$ - the number of ultrafilters of $\underline{B}$).

PROOF : Let the elements of the model be triples $< \bar{g}, \bar{f}, A >$ satisfying $\psi^h$. Equality to define by $\psi^h$, and $\epsilon$ is defined naturally.

DEFINITION 5.3 :

(A) The model $M = (A_1, ..., A_m ; R_1, ..., R_k)$ ($A_i$-universes, $R_i$-relations) is an explicitely interpratable expansion of the model $N = (A_1, ..., A_n ; R_1, ..., R_\ell)$ if there are formulas $\phi^0_i(\bar{x})$, $\phi^1_i(\bar{x},\bar{y})$, $\psi_j$ in L(N) ($n < i \leq m, 1 < j \leq k$) and function $F_i$ ($n < i \leq m$) such that

(1) $F_i$ is a function from $\{ \bar{a} : N \models \phi^0_i [\bar{a}] \}$ onto $A_i$

(2) $F_i(\bar{a}_1) = F_i(\bar{a}_2)$ iff $N \models \phi^1_i [\bar{a}_1, a_2]$

(3) $M \models R_j [F_{i(1)}, (\bar{a}_1), ... ]$ iff $N \models \psi_j [\bar{a}_1, ... ]$.

(B) The scheme of the explicite interpratable expansion is all the syntactical information involved.

DEFINITION 5.4 :

If $M^e$ is an explicite interpratable expansion of $N^e$ ($e = 1,2$) by the same scheme, and by the function $F^e_j$, and $N^1$ is a submodel of $N^2$, then the function G, $G \restriction A^1_i =$ the identity ($1 \leq i \leq n$), $G(F^1_j(\bar{a})) = F^2_j(\bar{a})$ when $N^1 \models \phi^0_j [\bar{a}]$ , is called the natural embedding of $M^1$ into $M^2$.

CLAIM 5.6 : If in Def. 5.4, $N^1$ is an elementary submodel of $N_2$ <u>then</u> the natural embedding of $M_1$ into $M_2$ is an elementary embedding.

PROOF : Trivial.

DEFINITION 5.5. : Let M(K) be the model with the universes and relations listed below : (functions and partial functions will be encode by relations).

(A) Those of K.

(B) $A_1 = \{ < a, T > : a \in H(\lambda^*), T \in S(\underline{B}) \}$ where $S(\underline{B})$ is the set of ultrafilters of $\underline{B}$. E will be an equivalence relation on $A_1$ defined by : $< a_1, T_1 > \; E \; < a_2, T_2 >$ iff $T_1 = T_2$. $R_1$ is defined by : $< a_1 , T_1 > \; R_1 \; < a_2 , T_2 >$ iff $a_1 \in a_2$ and $T_1 = T_2$.

(C) $A_2 = \{ \langle < a_T, T > : T \in S(\underline{B}) \rangle : a_T \in H(\lambda^*)\}$ and

$< a, T_0 > \; R_2 \; \langle < a_T, T > : T \in S(\underline{B}) \rangle$ iff $a = a_{T_0}$

(D) $A_3 = S(B)$

$F_1(< a, T >) = T$

$F_2(B, T) = B \approx_T$ for an algebra $B \in K$.

THEOREM 5.7 : M(K) is an explicitely interpratable expansion of K, assuming $2^{|L(V)|} < \lambda^*$.

REMARK : All the information we know on Eklof and Feferman problems mentioned in the introduction can be easily extracted from this theorem. We assume $2^{|L(V)|} < \lambda^*$ in order to simplify the definition of M(K).

PROOF : We go through Def. 5.5.

(A) No problem.

(B) Essentially, this was proved in 5.5.

(C) (D) Left to the reader.

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