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THE MODEL-COMPLETION OF STONE ALGEBRAS

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INTRODUCTION

The notion of model-completion was introduced and studied by A. Robinson in [5] . It leads to a general theory of «algebraically closed» structures. The model-completion of the theory of fields is the theory of algebraically closed fields and the theory of real closed fields is the model-completion of the theory of ordered fields. The model-completion of a theory need not exist but if it does it is unique. It is known that the model-completion of the theory of Boolean algebras is the theory of atomfree Boolean algebras. The model-completion of the theory of distributive lattices without endpoints is the theory of relatively-complemented distributive dense lattices without endpoints. These results are commonly known. In this paper the model-completion of the theory of Stone algebras is determined. Furthermore this theory is proved to be complete, substructure complete and $\aleph_0$- categorical.

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§ 1. MODEL THEORETICAL PRELIMINARIES

We assume familiarity with the basic concepts of model theory. We use capital Gothic letters $\mathfrak{A}$, $\mathfrak{B}$, ... to range over models; the corresponding capital Latin letters A, B ... always denote the corresponding universe. If $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ ($\mathfrak{A} \subseteq \mathfrak{B}$) and $a \in B$ then $\mathfrak{A}(a)$ denotes the substructure of $\mathfrak{B}$ generated by $A \cup \{a\}$. ${}^{X}Y$ is the set of all functions from X into Y and card A stands for the cardinality of A.

1.1. DEFINITION : A structure $\mathfrak{A}$ is called <u>$\aleph_0$-homogeneous</u> if for any two finitely generated substructures $\mathfrak{B}_1$, $\mathfrak{B}_2$ of $\mathfrak{A}$ any isomorphism f from $\mathfrak{B}_1$ onto $\mathfrak{B}_2$ and any $a \in A$ there is $b \in A$ such that f can be extended to an isomorphism $\bar{f}$ from $\mathfrak{B}_1(a)$ onto $\mathfrak{B}_2(b)$.

We shall employ the following two well-known theorems on $\aleph_0$-homogeneous structures.

1.2. THEOREM : Any two countable $\aleph_0$-homogeneous structures having upto isomorphism the same finitely generated substructures are isomorphic.

1.3. THEOREM : If $\mathfrak{A}$ is a countable $\aleph_0$-homogeneous structure then every isomorphism between two finitely generated substructures of $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$.

For proofs of these theorems see [6] lemma 20.1 and 20.4.

1.4. DEFINITION : A theory T is called <u>substructure complete</u> if for any two models $\mathfrak{A}_1$, $\mathfrak{A}_2$ of T and any common substructure $\mathfrak{B}$ holds $(\mathfrak{A}_1,b)_{b \in B} \equiv (\mathfrak{A}_2,b)_{b \in B}$.

1.5. DEFINITION : A theory T* is the <u>model-completion</u> of a theory T if

(i) $T \subseteq T^*$

(ii) every model of T can be embedded into a model of T*

(iii) for any two models $\mathfrak{A}_1$, $\mathfrak{A}_2$ of T^* and any common substructure $\mathfrak{B}$ which is a model of T holds

$$(\mathfrak{A}_1,b)_{b \in B} \equiv (\mathfrak{A}_2,b)_{b \in B}$$

These two definitions are related to the concept of $\aleph_0$-homogeneity by the following theorems.

1.6. THEOREM : Let T be a complete theory having only $\aleph_0$-homogeneous models then T is substructure complete.

PROOF : In showing that T satisfies the requirements of definition 1.4. we may w.l.o.g. assume that $\mathfrak{A}_1$, $\mathfrak{A}_2$ are countable models of T with $\mathfrak{B}$ as a finitely generated common substructure. Since T is complete there are a countable model $\mathfrak{C}$ of T and elementary embeddings g_i from $\mathfrak{A}_i$ into $\mathfrak{C}$. Since $\mathfrak{C}$ is $\aleph_0$-homogeneous by assumption the mapping $g_2 g_1^{-1}$ restricted to $g_1(B)$ can by theorem 1.3. be extended to an automorphism of $\mathfrak{C}$. Thus

$$(\mathfrak{C}, g_1(b))_{b \in B} \equiv (\mathfrak{C}, g_2(b))_{b \in B}$$

Since g_i is elementary this implies

$$(\mathfrak{A}_1, b)_{b \in B} \equiv (\mathfrak{A}_2, b)_{b \in B}$$

We note the following converse of theorem 1.6.

1.7. THEOREM : If T is $\aleph_0$-categorical and substructure complete then T has only $\aleph_0$-homogeneous models.

PROOF : Let $\mathfrak{A}$ be a model of T. Since T is $\aleph_0$-categorical $\mathfrak{A}$ is locally finite. Let $\mathfrak{B}_1 = \{a_0, ..., a_{n-1}\}$, $\mathfrak{B}_2 = \{b_0, ..., b_{n-1}\}$ be substructures of $\mathfrak{A}$, f an isomorphism between $\mathfrak{B}_1$ and $\mathfrak{B}_2$ such that $f(a_i) = b_i$. Substructure completeness yields

(1) $(\mathfrak{A}, a_0, ..., a_{n-1}) \equiv (\mathfrak{A}, b_0, ..., b_{n-1})$.

Let a_n be an arbitrary element of A. Since T is $\aleph_0$-categorical there is a formula $\varphi(v_0, ..., v_n)$ generating the type realized by $< a_0, ..., a_n >$ in $\mathfrak{A}$. Now using (1) we find $b_n \in A$ satisfying $\mathfrak{A} \models \varphi[b_0, ..., b_n]$. Thus $<a_0, ..., a_n>$ and $<b_0, ..., b_n>$ realize the same type in $\mathfrak{A}$ which implies that f can be extended to an isomorphism from $\mathfrak{B}_1(a_n)$ onto $\mathfrak{B}_2(b_n)$.

§ 2. ALGEBRAIC PRELIMINARIES

We briefly review the theory of Stone algebras as far as needed in the sequel. A thorough treatment may be found in [2].

2.1. DEFINITION : A structure $\mathfrak{A} = < A, \cap, \cup, *, 0, 1 >$ is called a pseudo complemented distributive lattice if $<A, \cap, \cup, 0, 1>$ is a distributive lattice with least and greatest element and the one-place operation * satisfies the following axioms

(SL1) $a \cap a^* = 0$

(SL2) $a \cap b = 0 \longrightarrow b \leq a^*$

A pseudo complemented distributive lattice is a Stone algebra if in addition holds

(SL3) $a^* \cup a^{**} = 1$

We denote the theory of Stone algebras by STA.

There are two interesting substructures of a Stone algebra $\mathfrak{A}$.

The skeleton of $\mathfrak{A} = Sk(\mathfrak{A}) = \{a \in A \mid a^{**} = a\} = \{a^* \mid a \in A\}$.

$Sk(\mathfrak{A}) = < Sk(\mathfrak{A}), \cap, \cup, *, 0, 1>$ is a Boolean algebra.

The set of dense elements $= D(\mathfrak{A}) = \{a \in A \mid a^* = 0\}$.

$D(\mathfrak{A}) = < D(\mathfrak{A}), \cap, \cup, 1 >$ is a distributive lattice with greatest element. Both substructures are linked together by the structure map $\sigma_{\mathfrak{A}}$, which is a homomorphism from

$Sk(\mathfrak{A})$ into the lattice of filters over $D(\mathfrak{A})$ preserving 0 and 1 defined by

$$\sigma_{\mathfrak{A}}(a) = \{x \in D(\mathfrak{A}) : x \geq a^*\}.$$

$\mathfrak{A}$ is upto isomorphism uniquely determined by the tripel $<Sk(\mathfrak{A}), D(\mathfrak{A}), \sigma_{\mathfrak{A}}>$.

Let $\mathfrak{B}$ be an arbitrary Boolean algebra, $\mathfrak{D}$ a distributive lattice with 1 and σ a homomorphism from $\mathfrak{B}$ into $F(\mathfrak{D})$, the lattice of filters over $\mathfrak{D}$, preserving 0 and 1. For any $b \in B$ we have $\sigma(b) \cup \sigma(b^*) = D$. Thus there are for any $x \in D$ uniquely determined elements x_1, x_2 such that $x_1 \in \sigma(b)$, $x_2 \in \sigma(b^*)$

$$x = x_1 \cap x_2$$

We use the notation $x_1 = \rho_b(x)$, $x_2 = \rho_{b^*}(x)$.

On the set of pairs $A = \{<x,b> \mid b \in B, x \in \sigma(b)\}$ we define a partial order by

$<x_1,b_1> \leq <x_2,b_2>$ iff $b_1 \leq b_2$ and $x_1 \leq \rho_{b_1}(x_2)$

$<A, \leq>$ induces a Stone algebra $\mathfrak{A}$. We furthermore have

$Sk(\mathfrak{A}) = \{<1,b> \mid b \in B\} \cong \mathfrak{B}$

$D(\mathfrak{A}) = \{<x,1> \mid x \in D\} \cong \mathfrak{D}$

$\sigma_{\mathfrak{A}}(<1,b>) = \{<x,1> \mid x \in \sigma(b)\}$

In the following we identify any Stone algebra $\mathfrak{A}$ with the algebra given by $<Sk(\mathfrak{A}), D(\mathfrak{A}), \sigma>$. We shall tacitly use the following rules of computation.

2.2. LEMMA :

$(a \cup b)^* = a^* \cap b^*$ $\qquad$ $(a \cap b)^* = a^* \cup b^*$

$a^{***} = a^*$ $\qquad$ $a \leq b \longrightarrow b^* \leq a^*$

$x \in \sigma(b)$ iff $\rho_b(x) = x$ iff $\rho_{b^*}(x) = 1$

$x \leq \rho_a(x)$ $\qquad$ $\rho_b(\rho_a(x)) = \rho_{a \cap b}(x)$

$x = \rho_a(x) \cap \rho_{a^*}(x)$ $\qquad$ $1 = \rho_a(x) \cup \rho_{a^*}(x)$

$\rho_a(x \cap y) = \rho_a(x) \cap \rho_a(x)$ $\qquad$ $\rho_a(x \cup y) = \rho_a(x) \cup \rho_a(y)$

$\rho_{a \cap b}(x) = \rho_a(x) \cup \rho_b(x)$ $\qquad$ $\rho_{a \cup b}(x) = \rho_a(x) \cap \rho_b(x)$

We conclude with two lemmas characterizing isomorphisms and subalgebras in terms of the corresponding tripels. The proofs may be found in [1].

2.3. LEMMA : Suppose $\mathfrak{A}_1$, $\mathfrak{A}_2$ are Stone algebras given by the tripels $<\mathfrak{B}_1, \mathfrak{D}_1, \sigma_1>$, $<\mathfrak{B}_2, \mathfrak{D}_2, \sigma_2>$ resp.

(i) Let F be an isomorphism from $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ then there are isomorphisms $f_1 : \mathfrak{D}_1 \longrightarrow \mathfrak{D}_2$ and $f_2 : \mathfrak{B}_1 \longrightarrow \mathfrak{B}_2$ such that

(α) $F(\langle x,a\rangle) = \langle f_1(x), f_2(a)\rangle$

(β) $\sigma_2(f_2(a)) = \{ f_1(x) \mid x \in \sigma_1(a) \}$

(ii) If $f_1 : \mathfrak{D}_1 \longrightarrow \mathfrak{D}_2$, $f_2 : \mathfrak{B}_1 \longrightarrow \mathfrak{B}_2$ are isomorphisms satisfying (β) then (α) defines an isomorphism from $\mathfrak{A}_1$ onto $\mathfrak{A}_2$.

(iii) Condition (β) is equivalent to

(γ) $f_1(\rho_b(x)) = \rho_{f_2(b)}(f_1(x))$.

2.4. LEMMA : Notation as in lemma 2.3.

(i) If $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$ then

(α) $\mathfrak{D}_1 \subseteq \mathfrak{D}_2$

(β) $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$

(γ) $\forall x \in D_1 \; \forall b \in B_1 \; (\rho_b(x) \in D_1)$

(ii) If (α) to (γ) hold then $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$

(iii) Condition (γ) is equivalent to

(δ) $\forall a \in B_1 \; (\sigma_1(a) = \sigma_2(a) \cap D_1)$

At some point in § 3 we shall make use of the following representation theorem

2.5. THEOREM : Every Stone algebra is a subdirect product of the three-element Stone algebra $\mathfrak{A}_3$. (where $A_3 = \{0, e, 1\}$ such that $0 < e < 1$ and $0^* = 1, e^* = 1^* = 0$).

PROOF : see [3].

§ 3. $\aleph_0$-HOMOGENEOUS STONE ALGEBRAS

Before we state and prove the main theorem we dispose of some trivial exeptions. A Stone algebra $\mathfrak{A}$ is called <u>trivial</u> if card $D(\mathfrak{A}) = 1$ or card $Sk(\mathfrak{A}) \leq 4$.

The proofs of the following statements are easy or variations of arguments used in the proof of the main theorem, so we omit them.

3.1. THEOREM : Let $\mathfrak{A}$ be a Stone algebra.

(i) If card $D(\mathfrak{A}) = 1$ then $Sk(\mathfrak{A}) \cong \mathfrak{A}$ and the problem is reduced to Boolean algebras.

A Boolean algebra is $\aleph_0$-homogeneous iff it has at most 4 elements or is atomfree.

(ii) If card $Sk(\mathfrak{A}) = 2$ then

$\mathfrak{A}$ is $\aleph_0$-homogeneous iff $D(\mathfrak{A})$ is $\aleph_0$-homogeneous

iff $D(\mathfrak{A})$ is relatively complemented, without antiatoms and without least element

(iii) If card $S(\mathfrak{A}) = 4$ then

$\mathfrak{A}$ is $\aleph_0$-homogeneous iff $D(\mathfrak{A}) = 1$.

3.2. MAIN THEOREM : Let $\mathfrak{A}$ be a nontrivial Stone algebra.

$\mathfrak{A}$ is $\aleph_0$-homogeneous iff the following conditions hold

(PI) Sk($\mathfrak{A}$) is atomfree

(PII) D($\mathfrak{A}$) is relatively complemented distributive lattice without antiatoms and without least element

(PIII) For all $b \in Sk(\mathfrak{A})$, $b \neq 0$, $\sigma(b)$ has no least element

(PIV) $\forall x, y \in D(\mathfrak{A})\ (x \sqcup y = 1 \longrightarrow \exists c \in Sk(\mathfrak{A})\ (x \in \sigma(c)\ \&\ y \in \sigma(c^*)))$

We first prove necessity of the conditions PI to PIV.

3.3. LEMMA : Suppose $\mathfrak{A}$ is an $\aleph_0$-homogeneous Stone algebra. $b_1, b_2 \in Sk(\mathfrak{A})$, $b_1 \sqcap b_2 = 0$ and $b_1, b_2 \neq 0,1$.

Then there is an automorphism g of D($\mathfrak{A}$) such that $\sigma(b_1) = \{g(x) \mid x \in \sigma(b_2)\}$.

PROOF : Consider the subalgebra $\mathfrak{A}_1 \subset \mathfrak{A}$ generated by $\{<1,b_1>, <1,b_1^*>, <1,b_2>, <1,b_2^*>\}$. It is easily checked that there is an embedding F from $\mathfrak{A}_1$ into $\mathfrak{A}$ taking $<1,b_2>$ to $<1,b_1>$. By $\aleph_0$-homogeneity F can be extended to an automorphism $\overline{F} = <g,f>$ of $\mathfrak{A}$. Now lemma 2.3. yields $\sigma(b_1) = \sigma(f(b_2)) = \{g(x) \mid x \in \sigma(b_2)\}$ and g is an automorphism of D($\mathfrak{A}$).

3.4. COROLLARY : If $\mathfrak{A}$ is a nontrivial $\aleph_0$-homogeneous Stone algebra then $\sigma_{\mathfrak{A}}$ is an embedding.

PROOF : It suffices to show that for $b \in Sk(\mathfrak{A})$ $b \neq 0$ implies $\sigma(b) \neq \{1\}$. Assume $b \neq 0$ and $\sigma(b) = \{1\}$. This implies $\sigma(b^*) = D(\mathfrak{A})$.

By lemma 3.3. $\sigma(b)$ and $\sigma(b^*)$ have the same cardinality contradicting the assumption card D($\mathfrak{A}$) > 1.

3.5. COROLLARY : If $\mathfrak{A}$ is a nontrivial $\aleph_0$-homogeneous Stone algebra then (PIV) holds.

PROOF : Let x, y be elements of D($\mathfrak{A}$) satisfying $x \sqcup y = 1$. Since card Sk($\mathfrak{A}$) > 4 we find $a \in Sk(\mathfrak{A}) : 0 < a < 1$.

By corollary 3.4. this implies : $\{1\} < \sigma(a) < D(\mathfrak{A})$

This enables us to choose $u \in \sigma(a)$, $v \in \sigma(a^*)$ such that

$u = 1$ iff $x = 1$

$v = 1$ iff $y = 1$

In any case $u \sqcup v \in \sigma(a) \sqcap \sigma(a^*) = \{1\}$ yields $u \sqcup v = 1$.

Denote by $\mathfrak{D}_0$ the sublattice of $\mathfrak{D}$ generated by $\{x,y\}$. Denote by $\mathfrak{D}_1$ the sublattice of $\mathfrak{D}$ generated by $\{u,v\}$. Obviously there is an isomorphism f_1 from $\mathfrak{D}_0$ onto $\mathfrak{D}_1$ such that $f_1(x) = u$ and $f_1(y) = v$.

Define the subalgebras $\mathfrak{A}_j$ of $\mathfrak{A}$ for $j = 0,1$ by

$A_j = \{<d,1> \mid d \in D_j\} \cup \{<1,0>\}$

The mapping $F = < f_1, f_2 >$, where f_2 is the identity map on $\{0,1\} = Sk(\mathfrak{A}_j)$, is an isomorphism from $\mathfrak{A}_0$ onto $\mathfrak{A}_1$.

$\aleph_0$-homogeneity of $\mathfrak{A}$ provides us with an automorphism $\bar{F} = < \bar{f}_1, \bar{f}_2 >$ of $\mathfrak{A}$ extending F. Let c be the pre-image of a under $\bar{f}_2$, i.e. $\bar{f}_2(c) = a$.

Now $\sigma(a) = \sigma(\bar{f}_2(c)) = \{ \bar{f}_1(z) \mid z \in \sigma(c) \}$

$\sigma(a^*) = \sigma(\bar{f}_2(c^*)) = \{ \bar{f}_1(z) \mid z \in \sigma(c^*) \}$

implies

$x \in \sigma(c)$ and $y \in \sigma(c^*)$

proving PIV.

Conditions PI to PIII are derived in quite the same way so we omit the details.

It is easily seen that the class of all Stone algebras satisfying PI to PIV is a finitely axiometized elementary class, we denote its theory by STA*. We proceed to show that every model of STA* is $\aleph_0$-homogeneous. We start with the following simple though very useful lemma.

3.6. LEMMA : Let $\mathfrak{A}$ be an arbitrary Stone algebra $x,y \in D(\mathfrak{A})$, $x < y < 1$ and $a \in Sk(\mathfrak{A})$. Assume that the relative complement of y in $[x,1]$ exists and denote it by y'. Assume further that the relative complement of $\rho_a(y)$ in $[\rho_a(x), 1]$ exists and denote it likewise by $(\rho_a(y))'$. Then holds

$\rho_a(y') = (\rho_a(y))' \cup \rho_a(x)$.

PROOF : $y' \cap y = x$ and $y' \cup y = 1$ implies $\rho_a(y') \cap \rho_a(y) = \rho_a(x)$ and $\rho_a(y') \cup \rho_a(y) = 1$, i.e. $\rho_a(y')$ is the relative complement of $\rho_a(y)$ with respect to $[\rho_a(x), 1]$. It is easily checked that also $(\rho_a(y))' \cup \rho_a(x)$ is a relative complement of $\rho_a(y)$ with respect to $[\rho_a(x), 1]$. Uniqueness of relative complements in distributive lattices yields the claim.

We still need one preparatory lemma.

3.7. LEMMA : Let $\mathfrak{A}$ be a model of STA*. Then holds

(PV) $\sigma_{\mathfrak{A}}$ is an embedding

(PVI) $\forall b \in Sk(\mathfrak{A}) \, \forall x,y \in \sigma(b) \, [0 < b \,\&\, x \cup y = 1 \longrightarrow$

$\longrightarrow \exists c \in Sk(\mathfrak{A}) \, (0 < c < b \,\&\, x \in \sigma(c) \,\&\, y \in \sigma(c^*))]$

(PVII) Let $b \in Sk(\mathfrak{A})$, $b \neq 0$. Let $x, y, z_0, \ldots, z_{n-1} \in \sigma(b)$ satisfy

$\forall i < n \, (z_i \neq 1)$

$\forall i,j < n \, (i \neq j \longrightarrow z_i \cup z_j = 1)$ and

$\forall i < n \, (z_i \cup x = z_i \cup y = 1)$ and $x \cup y = 1$

then there is $c \in Sk(\mathfrak{A})$ such that $0 < c < b$

$x \in \sigma(c)$, $y \in \sigma(c^*)$ and

$\forall i < n \, (z_i \notin \sigma(c) \;\&\; z_i \notin \sigma(c^*))$

PROOF : (PV) follows easily from (PIII).

In proving (PVI) we distinguish the following three cases

case 1 : $x = y = 1$

Use (PI) to find $c \in Sk(\mathfrak{A})$ such that $0 < c < b$

case 2 : $x \neq 1, y \neq 1$

Using (PIV) we obtain $c_0 \in Sk(\mathfrak{A})$ such that $x \in \sigma(c_0)$, $y \in \sigma(c_0^*)$. Defining $c = c_0 \sqcap b$ we obviously get $x \in \sigma(c)$ and $y \in \sigma(c_0^*) \subseteq \sigma(c^*)$. $c = 0$ would imply $x \in \sigma(0) = \{1\}$; $c = b$ would imply $b^* \geq c_0^*$ thus $y \in \sigma(b \sqcap b^*) = \{1\}$. Both are contradictory to our assumptions. So $0 < c < b$ holds.

case 3 : $x \neq 1, y = 1$

By (PIII) there is $z \in \sigma(b)$, $z < x$. By (PII) there is $\overline{y} \in D(\mathfrak{A})$ satisfying $z = x \sqcap \overline{y}$ and $1 = x \sqcup \overline{y}$. Note that $\overline{y} \neq 1$ and $\overline{y} \in \sigma(b)$. Using case 2 we obtain $c \in Sk(\mathfrak{A})$: $0 < c < b$, $x \in \sigma(c)$, $\overline{y} \in \sigma(c^*)$.

Since $y = 1 \in \sigma(c^*)$ trivially holds we have proved (PVI).

To prove (PVII) let $x, y, z_0, \ldots, z_{n-1} \in \sigma(b)$ be given satisfying the assumptions. By PII it is possible to choose $u_i \in D(\mathfrak{A})$ such that for all $i < n : z_i < u_i < 1$.

By w_i we denote the relative complement of u_i with respect to $[z_i, 1]$ which exists by PII :

$$z_i = u_i \sqcap w_i \quad ; \quad 1 = u_i \sqcup w_i$$

Define $x_o = \bigcap_{i<n} u_i$, $\quad y_0 = \bigcap_{i<n} w_i$

Using distributivity we obtain the following equation

$$(x \sqcap x_o) \sqcup (y \sqcap y_0) = (x \sqcup y) \sqcap \bigcap\{x \sqcup u_i \mid i < n\} \sqcap \bigcap\{y \sqcup w_i \mid i < n\} \sqcap \bigcap\{u_i \sqcup w_j \mid i,j < n\}$$

Observing that

$$x \sqcup y = 1$$
$$x \sqcup u_i \geq x \sqcup z_i = 1$$
$$y \sqcup w_i \geq y \sqcup z_i = 1$$
$$u_i \sqcup w_j \geq z_i \sqcup z_j = 1 \quad \text{if } i \neq j$$
$$u_i \sqcup w_i = 1$$

we conclude

$$(x \sqcap x_0) \sqcup (y \sqcap y_0) = 1.$$

Since $\sigma(b)$ is a filter we still have $x \sqcap x_0$, $y \sqcap y_0 \in \sigma(b)$.

By (PVI) we obtain $c \in Sk(\mathfrak{A})$ satisfying

$$0 < c < b$$

and $x \cap x_0 \in \sigma(c), \quad y \cap y_0 \in \sigma(c^*)$

This implies $x \in \sigma(c), \quad y \in \sigma(c^*)$

$u_i \in \sigma(c), \quad w_i \in \sigma(c^*)$

again exploiting the filter property of $\sigma(c)$, resp. $\sigma(c^*)$.

This yields : $z_i \notin \sigma(c)$, $z_i \notin \sigma(c^*)$ for all $i < n$.

Since from $z_i \in \sigma(c)$ would follow $w_i \in \sigma(c)$ and $w_i \in \sigma(c) \cap \sigma(c^*) = \{1\}$ i.e. $w_i = 1$ contradicting the choice of u_i. Also $z_i \in \sigma(x^*)$ would entail in the same way $z_i = 1$ again contrary to assumption.

This completes the proof of lemma 3.7.

For the rest of this paragraph $\mathfrak{A} = \langle \mathfrak{B}, \mathfrak{D}, \sigma \rangle$ will denote a model of STA*. It is our goal to show that $\mathfrak{A}$ is $\aleph_0$-homogeneous. To this end we consider two finite subalgebras $\mathfrak{A}_i = \langle \mathfrak{B}_i, \mathfrak{D}_i, \sigma_i \rangle$, $i = 1,2$ and an isomorphism $F = \langle f_1, f_2 \rangle$ from $\mathfrak{A}_1$ onto $\mathfrak{A}_2$. Let $\langle x,a \rangle$ be an arbitrary element of A. $\bar{\mathfrak{A}}_1 = \langle \bar{\mathfrak{B}}_1, \bar{\mathfrak{D}}_1, \bar{\sigma}_1 \rangle$ denotes the subalgebra of $\mathfrak{A}$ generated by $A \cup \{\langle x,a \rangle\}$.

Problem : Extend F to an embedding $\bar{F} = \langle \bar{f}_1, \bar{f}_2 \rangle$ from $\bar{\mathfrak{A}}_1$ into $\mathfrak{A}$.

We shall proceed in the following steps

- Step 1 Form the closure of $\mathfrak{D}$ under complements
- Step 2 $\langle x,a \rangle = \langle x,1 \rangle$
- Step 3 $\langle x,a \rangle = \langle 1,a \rangle$
- Step 4 $\langle x,a \rangle$ arbitrary

If steps 2 and 3 will be accomplished, step 4 is trivial because $\langle x,a \rangle = \langle x,1 \rangle \cap \langle 1,a \rangle$.

STEP 1

Let d_{0i} denote the least element of $\mathfrak{D}_i$. For $x \in D_i$, $d_{oi} \leq x \leq 1$, x' denotes the relative complement of x with respect to $[d_{oi}, 1]$.

$\bar{D}_1 = \{ \bigcup \{ x_i \cap z_i' \mid i < n \} \mid n \in w, x_i, z_i \in D_1 \}$ defines a sublattice of $\mathfrak{D}$ closed under ' and containing D_1. Set $\bar{\sigma}_1(a) = \sigma(a) \cap \bar{D}_1$ for $a \in B_1$. We claim that $\bar{\mathfrak{A}}_1 = \langle \mathfrak{B}_1, \bar{\mathfrak{D}}_1, \bar{\sigma}_1 \rangle$ is a subalgebra of $\mathfrak{A}$. According to lemma 2.4. we have to show : for all $y \in \bar{D}_1$ and all $b \in B_1$ $\rho_a(y) \in \bar{D}_1$. To begin with take $z \in D_1$. By lemma 3.6. we have $\rho_a(z') = (\rho_a(z))' \cup \rho_a(d_{01})$. Since by assumption $\rho_a(z), \rho_a(d_{01}) \in D_1$ and D_1 is closed under ' we obtain $\rho_a(z') \in \bar{D}_1$.

For arbitrary $y = \bigcup \{ x_i \cap z'_i) \mid i < n \} \in \bar{D}_1$ we have

$\rho_a(y) = \bigcup \{ \rho_a(x_i) \cap \rho_a(z'_i) \mid i < n \} \in \bar{D}_1$.

Now we want to extend $F = < f_1, f_2 >$ to an embedding $\overline{F}$ from $\mathfrak{A}_1$ into $\mathfrak{A}$.
It is routine to check that there is an embedding $\overline{f}_1$ from $\overline{D}_1$ extending f_1 and satisfying
$\overline{f}_1 \left(\bigcup\{ x_i \cap z'_i \mid i < n\}\right) = \bigcup\{f_1(x_i) \cap f_1(z_i)' \mid i < n\}$.
In order to show that $\overline{F} = <\overline{f}_1, f_2>$ is an embedding from $\overline{\mathfrak{A}}_1$ into $\mathfrak{A}$ we need by lemma 2.3. only to know that for all $a \in B_1, y \in \overline{D}_1 : \overline{f}_1(\rho_a(y)) = \rho_{f_2(a)}(\overline{f}_1(y))$ holds. This is easily checked using lemma 2.2. and 3.6.

STEP 2

We adopt the following notation :
$D_1 = \{d_0, ..., d_{r-1}\}$ and agree that $\forall d \in D_1(d_0 \leq d)$
$B_1 = \{b_0, ..., b_{k-1}\}$
For $\pi \in {}^k2$ we use the abbreviation $b_\pi = \bigcap\{\pi(i)b_i \mid i < k\}$
where $0b = b$ and $1b = b^*$
Similarly for $\tau \in {}^r2$ $\quad d_\tau = \bigcup\{\tau(j)d_j \mid j < r\}$
where $0d = d$ and $1d = d'$ and d' is the relative complement of d in $[d_0, 1]$.
Furthermore ρ_π stands for ρ_{b_π}.
It is checked by straightforward computation that the following holds

3.8.
$$\pi_1 \neq \pi_2 \text{ implies } b_{\pi_1} \cap b_{\pi_2} = 0$$
$$\tau_2 \neq \tau_2 \text{ implies } d_{\tau_1} \cup d_{\tau_2} = 1$$
$$b_i = \bigcup\{b_\pi \mid \pi \in {}^k2 \text{ and } \pi(i) = 0\}$$
$$d_j = \bigcap\{d_\tau \mid \tau \in {}^r2 \text{ and } \tau(j) = 0\}$$
$$1 = \bigcup\{b_\pi \mid \pi \in {}^k2\}$$
$$d_0 = \bigcap\{d_\tau \mid \tau \in {}^r2\}$$

x is an arbitrary element of D. We may restrict to the case $x \in \sigma(b_\pi)$ for some $\pi \in {}^k2$.
If we have solved this restricted problem we might take up the general case by extending $\mathfrak{D}_1$ successively to $\overline{\mathfrak{D}}_1$ such that $\{\rho_\pi(x) \mid \pi \in {}^k2\} \subseteq \overline{D}_1$, since $\rho_\pi(x) \in \sigma(b_\pi)$.
Noticing that $x = \bigcap\{\rho_\pi(x) \mid \pi \in {}^k2\}$ holds we will have finished.
So we assume that $x \in \sigma(b_{\pi_0})$ holds for some $\pi_0 \in {}^k2$. Let $\overline{\mathfrak{D}}_1$ be the sublattice of $\mathfrak{D}$ generated by $D_1 \cup \{x\}$. Since for all $b \in B_1$ either $\rho_b(x) = x$ or $\rho_b(x) = 1$ holds $<\mathfrak{B}_1, \overline{\mathfrak{D}}_1, \overline{\sigma}_1>$ is a substructure of $\mathfrak{A}$ (of course $\overline{\sigma}_1(a) = \sigma(a) \cap \overline{D}_1$).
We shall distinguish the following three cases

Case 2.1. $d_0 \leq x \leq 1$

Case 2.2. $x < d_0$

Case 2.3. neither of the above

After case 2.1. and 2.2 have been accomplished case 2.3. will follow trivially : we first extend $\mathfrak{D}_1$ to contain $x \sqcap d_0$ using case 2.2. Now $x \sqcap d_0 < x$ and case 2.1. applies.

CASE 2.1. : Using step 1 we may suppose without loss of generality that $\mathfrak{D}_1$ is closed under complements. The embedding f_1 preserves complements in the sense that x' is mapped on the relative complement of $f_1(x)$ with respect to $[f_1(d_0), 1]$ which we also denote by $f_1(x)'$.

We shall make use of the following

3.9. CRITERION : There exists an embedding $\overline{F} : \overline{\mathfrak{A}}_1 \longrightarrow \mathfrak{A}$ extending F iff for every $\tau \in {}^{\Gamma}2$ there is $y_\tau \in D$ such that the following four conditions are satisfied :

(3.9.0) $y_\tau \geq f_1(d_\tau)$

(3.9.1) $d_\tau \sqcup x = 1$ iff $y_\tau = 1$

(3.9.2) $d_\tau \sqcup x' = 1$ iff $y_\tau' \sqcup f_1(d_\tau) = 1$

(3.9.3) $y_\tau \in \sigma(f_2(b_{\pi_0}))$

PROOF OF 3.9. : Necessity is clear by taking $y_\tau = \overline{f}_1(x \sqcup d_\tau)$.

To prove suffiency set $y = \bigcap\{y_\tau \mid \tau \in {}^{\Gamma}2\}$. Now 3.9.0 to 3.9.2 imply (using 3.8)

$$\forall d \in D_1(d \sqcup x = 1 \text{ iff } f_1(d) \sqcup y = 1)$$

$$\forall d \in D_1(d \sqcup x' = 1 \text{ iff } f_1(d) \sqcup y' = 1)$$

By a well-known theorem on extending isomorphisms between Boolean algebras (see e.g. [7] p. 37) applied to $\mathfrak{D}_1$ and the interval $[d_0, 1]$ there is an embedding $\overline{f}_1$ from $\overline{\mathfrak{D}}_1$ into $\mathfrak{D}$ extending f_1 such that $\overline{f}_1(x) = y$. It remains to show that $\langle \overline{f}_1, f_2 \rangle$ is an embedding from $\overline{\mathfrak{A}}_1$ into $\mathfrak{A}$.

First we note that for all $b \in B_1$ and $x \in D_1$ holds

3.10. : $\overline{f}_1(\rho_b(x)) = \rho_{f_2(b)}(\overline{f}_1(x))$

For either $b \sqcap b_{\pi_0} = 0$ or $b \sqcap b_{\pi_0} = b_{\pi_0}$ holds. In the first case $\overline{f}_1(\rho_b(x)) = \overline{f}_1(1) = 1$ and $\rho_{f_2(b)}(\overline{f}_1(x)) = \bigcap_\tau \rho_{f_2(b)}(y_\tau) = 1$, since for all $\tau \in {}^{\Gamma}2$ $f_2(b_{\pi_0}) = f_2(b) \sqcap f_2(b_{\pi_0})$ yields $\rho_{f_2(b)}(y_\tau) = 1$.

In the second case $\overline{f}_1(\rho_b(x)) = \overline{f}_1(x) = \bigcap_\tau y_\tau$ and

$$\rho_{f_2(b)}(\overline{f}_1(x)) = \bigcap_\tau \rho_{f_2(b)}(y_\tau) = \bigcap_\tau \rho_{f_2(b_{\pi_0})}(y_\tau).$$

Using 3.9.3. this yields $\rho_{f_2(b)}(\bar{f}_1(x)) = \bigcap \{y_\tau \mid \tau \in {}^r2\}$

This completes the proof of 3.10.

Every $y \in \bar{D}_1$ can be represented in the form

$$y = \bigcup_{i<n} (z_i \cap x) \qquad n < \omega,\ z_i \in D_1$$

$$\begin{aligned} \text{Thus } \bar{f}_1(\rho_b(y)) &= \bigcup_{i<n} [f_1(\rho_b(z_i)) \cap \bar{f}_1(\rho_b(x))] \\ &= \bigcup_{i<n} \left[\rho_{f_2(b)}(f_1(z_i)) \cap \rho_{f_2(b)}(\bar{f}_1(x))\right] \\ &= \rho_{f_2(b)}(\bar{f}_1(y)). \end{aligned}$$

This completes the proof of 3.9.

Now let $\tau \in {}^r2$ be given. We shall find y_τ satisfying 3.9.0 to 3.9.3.

In the trivial cases $d_\tau \cup x = 1$ and $d_\tau \cup x' = 1$ we choose $y_\tau = 1$, $y_\tau = f_1(d_\tau)$ respectively.

So we arrive at the non-trivial case : $d_\tau < d_\tau \cup x < 1$.

We claim that this implies

3.11. : $\rho_{\pi_0}(d_\tau \cup x) < 1$.

This can be seen as follows. $x \in \sigma(b_{\pi_0})$ yields for all $\pi \in {}^k2$, $\pi \neq \pi_0$: $x \cup d_\tau \in \sigma(b^*_\pi)$ i.e. $\rho_{\pi_0}(x \cup d_\tau) = 1$.

If contrary to 3.11 $\rho_{\pi_0}(d_\tau \cup x) = 1$ would also be true, we obtained $(d_\tau \cup x) = \bigcap_\pi \rho_\pi (x \cup d_\tau) = 1$. Contradiction.

From 3.11 and $\rho_{\pi_0}(d_\tau) \leq \rho_{\pi_0}(d_\tau \cup x)$ now follows :

$\rho_{\pi_0}(d_\tau) < 1$.

Using the assumption on $< f_1, f_2 >$ we infer :

$\rho_{f_2(b_{\pi_0})}(f_1(d_\tau)) < 1$.

By (PII) and the fact that $\sigma(f_2(b_\pi))$ is a filter on $\mathfrak{T}$ we may choose $y_\tau \in \sigma(f_2(b_{\pi_0}))$ such that :

$$\rho_{f_2(b_{\pi_0})}(f_1(d_\tau)) < y_\tau < 1$$

This implies :

$f_1(d_\tau) < y_\tau < 1$

and y_τ satisfies 3.9.0 to 3.9.3.

This completes case 2.1.

CASE 2.2. : $x < d_0$

In this case we have $\bar{D}_1 = D_1 \cup \{x\}$. Since $\sigma(f_2(b_{\pi_0}))$ contains by (PIII) no least element, we may choose $y \in \sigma(f_2(b_{\pi_0}))$ such that $y < f_1(d_0)$.

The mapping $\bar{f}_1$ defined by

$$\bar{f}_1(d) = \begin{cases} f_1(d) & \text{if } d \in D_1 \\ y & \phantom{\text{if }} d = x \end{cases}$$

is certainly an embedding from $\bar{\mathfrak{D}}_1$ into $\mathfrak{D}$.

Furthermore holds

$$\bar{f}_1(\rho_b(x)) = \begin{cases} 1 \\ y \end{cases} \text{ if } \begin{cases} b \cap b_{\pi_0} = 0 \\ b \cap b_{\pi_0} = b_{\pi_0} \end{cases} \quad \text{and}$$

$$\rho_{f_2(b)}(\bar{f}_1(x)) = \rho_{f_2(b)}(y) = \begin{cases} 1 \\ y \end{cases} \text{ if } \begin{cases} f_2(b) \cap f_2(b_{\pi_0}) = 0 \\ f_2(b) \cap f_2(b_{\pi_0}) = f_2(b_{\pi_0}). \end{cases}$$

This shows that $<\bar{f}_1, f_2> : \bar{\mathfrak{A}}_1 \longrightarrow \mathfrak{A}$ is an embedding.

This completes step 2.

STEP 3 : Let $\bar{\mathfrak{B}}_1$ be the subalgebra of $\mathfrak{B}$ generated by $B_1 \cup \{a\}$. Let $\bar{\mathfrak{D}}_1$ be the sublattice of $\mathfrak{D}$ generated by $D_1 \cup \{\rho_a(x) \mid x \in D_1\} \cup \rho_{a^*}(x) \mid x \in D_1\}$.

Finally $\bar{\sigma}_1(a) = \sigma(a) \cap \bar{D}_1$. It is not hard to see that $\bar{\mathfrak{A}}_1 = <\bar{\mathfrak{B}}_1, \bar{\mathfrak{D}}_1, \bar{\sigma}_1>$ is a subalgebra of $\mathfrak{A}$.

Denote by d_0 the least element of $\mathfrak{D}_1$. Obviously d_0 is also the least element of $\bar{\mathfrak{D}}_1$. For $x \in D, d_0 \leq x \leq 1$, x' denotes the relative complement of x with respect to $[d_0, 1]$. Step 1 allows us to assume w.l.o.g. that $\mathfrak{D}_1$ is closed under '.

We begin with three easy observations.

3.12. : $\bar{D}_1 = \{\rho_a(x) \cap \rho_{a^*}(z) \mid x,z \in D_1\}$

The right hand side of 3.12 contains

$D_1 \cup \{\rho_a(x) \mid x \in D_1\} \cup \{\rho_{a^*}(x) \mid x \in D_1\}$ and is closed under $\cap$ and $\cup$.

This is clear for $\cap$. For $\cup$ we obtain

$[\rho_a(x) \cap \rho_{a^*}(z)] \cup [\rho_a(u) \cap \rho_{a^*}(v)] = \rho_a(x \cup u) \cap \rho_{a^*}(z \cup v)$

since $\rho_a(x) \cup \rho_{a^*}(v) = \rho_{a^*}(z) \cup \rho_a(u) = 1.$

3.13. : For all $x \in D_1$: $[\rho_a(x)]' = x' \cap \rho_{a^*}(x)$

This is checked by direct computation.

3.14. : $\bar{D}_1$ is closed under complements.

Using 3.12 and 3.13 we obtain :

$$\left[\rho_a(x) \cap \rho_{a^*}(z)\right]' = \left[x' \cap \rho_{a^*}(x)\right] \cup \left[z' \cap \rho_a(z)\right]$$

By assumption the last term is an element of $\bar{D}_1$.

3.15. CRITERION : The embedding $F : \mathfrak{A}_1 \longrightarrow \mathfrak{A}$ can be extended to an embedding $\bar{F} : \bar{\mathfrak{A}}_1 \longrightarrow \mathfrak{A}$ iff there is $c \in B$ satisfying

3.15.1 $\forall b \in B_1(b \cap a = 0$ iff $f_2(b) \cap c = 0)$

3.15.2 $\forall b \in B_1(b \cap a^* = 0$ iff $f_2(b) \cap c^* = 0)$

3.15.3 $\forall y \in D_1(y \in \sigma(a)$ iff $f_1(y) \in \sigma(c))$

3.15.4 $\forall y \in D_1(y \in \sigma(a^*)$ iff $f_1(y) \in \sigma(c^*))$

PROOF OF 3.15 : To prove necessity take $c = \bar{f}_2(a)$. On the other hand conditions 3.15.1/ 2 ensure the existence of an embedding $\bar{f}_2$ from $\mathfrak{B}_1$ into $\mathfrak{B}$ extending f_2 such that $\bar{f}_2(a) = c$ (see [7] p. 37).

For $y = \rho_a(x) \cap \rho_{a^*}(z) \in \bar{D}_1$ define $\bar{f}_1(y) = \rho_c(f_1(x)) \cap \rho_{c^*}(f_1(z))$.

We assert that $\bar{f}_1$ is an embedding from $\bar{\mathfrak{D}}_1$ into $\mathfrak{D}$. To this end we have to verify that

3.16. : $\rho_a(x) \cap \rho_{a^*}(z) = \rho_a(u) \cap \rho_{a^*}(v)$ iff

$$\rho_c(f_1(x)) \cap \rho_{c^*}(f_1(z)) \doteq \rho_c(f_1(u)) \cap \rho_{c^*}(f_1(v))$$

$\rho_a(x) \cap \rho_{a^*}(z) = \rho_a(u) \cap \rho_{a^*}(v)$ is equivalent to the conjunction of the following two equations :

$$(\rho_a(x) \cap \rho_{a^*}(z)) \cup (\rho_a(u))' \cup (\rho_{a^*}(v))' = 1$$

$$(\rho_a(x))' \cup (\rho_{a^*}(z))' \cup (\rho_a(u) \cap \rho_{a^*}(z)) = 1$$

We continue with a list of equivalent rearrangements of the first of these two equations. Using 3.13. yields

$(\rho_a(x) \cap \rho_{a^*}(z)) \cup (u' \cap \rho_{a^*}(u)) \cup (v' \cap \rho_a(v)) = 1.$

Employing the fact that $u' \cap \rho_{a^*}(u) = \rho_a(u') \cap \rho_{a^*}(u') \cap \rho_{a^*}(u)$

we obtain $(\rho_a(x) \cap \rho_{a^*}(z)) \cup (\rho_a(u') \cap \rho_{a^*}(d_0)) \cup (\rho_{a^*}(v') \cap \rho_a(d_0)) = 1.$

Bringing the right hand side into disjunctive normal form shows that this equation is equivalent to the conjunction of the following eight equations :

(E1)	$\rho_a(x \sqcup u') \sqcup \rho_{a^*}(v') = 1$	(E5)	$\rho_{a^*}(z \sqcup v') \sqcup \rho_a(u') = 1$
(E2)	$\rho_a(x \sqcup u' \sqcup d_0) = 1$	(E6)	$\rho_{a^*}(z) \sqcup \rho_a(u' \sqcup d_0) = 1$
(E3)	$\rho_a(x) \sqcup \rho_{a^*}(d_0 \sqcup v') = 1$	(E7)	$\rho_{a^*}(z \sqcup d_0 \sqcup v') = 1$
(E4)	$\rho_a(x \sqcup d_0) \sqcup \rho_{a^*}(d_0) = 1$	(E8)	$\rho_{a^*}(z \sqcup d_0) \sqcup \rho_a(d_0) = 1$

Let (Fn) stand for the equation obtained from (En) by replacing x, u, u' ... by $f_1(x), f_1(u), (f_1(u))'$ ... and ρ_a, ρ_{a^*} by ρ_c, ρ_{c^*} respectively.

We claim that for all $1 \leq n \leq 8$ (Fn) is equivalent to (En).
Except for n = 2,7 this is trivial.
n = 2 :

$$\rho_a(x \sqcup u' \sqcup d_0) = 1 \quad \text{iff} \quad x \sqcup u' \sqcup d_0 \in \sigma(a^*)$$

by (3.15.4)

$$\text{iff} \quad f_1(x) \sqcup f_1(u)' \sqcup f_1(d_0) \in \sigma(c^*)$$

$$\text{iff} \quad \rho_c\,[f_1(x) \sqcup f_1(u)' \sqcup f_1(d_0)] = 1.$$

The case n = 7 is proved similarly now using (3.15.3).
Performing the same rearrangements as we did above in the reverse direction we see that the system of equations (F1) - (F8) is equivalent to

$$[\rho_c(f_1(x)) \sqcap \rho_{c^*}(f_1(z))] \sqcup (\rho_c(f_1(u)))' \sqcup (\rho_{c^*}(f_1(v)))' = 1$$

Similarly $(\rho_a(x))' \sqcup (\rho_{a^*}(z))' \sqcup (\rho_a(u) \sqcap \rho_{a^*}(z)) = 1$ is equivalent to

$$(\rho_c(f_1(x)))' \sqcup (\rho_{c^*}(f_1(z)))' \sqcup (\rho_c(f_1(u)) \sqcap \rho_{c^*}(f_1(z))) = 1$$

This proves 3.16.
It remains to show that $< \bar{f}_1, \bar{f}_2 >$ is an embedding from $\bar{\mathfrak{A}}_1$ into $\mathfrak{A}$.
Take $y \in \bar{D}_1$, $y = \rho_a(x) \sqcap \rho_{a^*}(z)$ for some $x, z \in D_1$ and $b \in \bar{B}_1$,
$b = \bigcup\{ b_i \sqcap \varepsilon_i a \mid i < n \}$ for $n \in \omega$, $b_i \in B_1$, $\varepsilon_i = 0, 1$.

Define $E_0 = \{ i < n \mid \varepsilon_i = 0 \}$ $\qquad E_1 = \{ i < n \mid \varepsilon_i = 1 \}$.

$$\begin{aligned}
\bar{f}_1(\rho_b(y)) &= \bar{f}_1(\rho_a \rho_b(x) \sqcap \rho_{a^*} \rho_b(z)) \\
&= \bar{f}_1\left(\bigcap_{i \in E_0} \rho_a \rho_{b_i}(x) \sqcap \bigcap_{i \in E_1} \rho_{a^*} \rho_{b_i}(z) \right) \\
&= \rho_c\left[f_1\left(\bigcap_{i \in E_0} \rho_{b_i}(x) \right) \right] \sqcap \rho_{c^*}\left[f_1\left(\bigcap_{i \in E_1} \rho_{b_i}(z) \right) \right] \\
&= \rho_c\left[\bigcap_{i \in E_0} \rho_{f_2(b_i)}(f_1(x)) \right] \sqcap \rho_{c^*}\left[\bigcap_{i \in E_1} \rho_{f_2(b_i)}(f_1(z)) \right] \\
&= \rho_u(f_1(x)) \sqcap \rho_w(f_1(z)) \text{ where } u = \bigcup_{i \in E_0} f_2(b_i) \sqcap c,\ w = \bigcup_{i \in E_1} f_2(b_i) \sqcap c^* \\
&= \rho_{\bar{f}_2(b)}\, \rho_c(f_1(x)) \sqcap \rho_{\bar{f}_2(b)}\, \rho_{c^*}(f_1(z)) \\
&= \rho_{\bar{f}_2(b)}(\bar{f}_1(y)).
\end{aligned}$$

This completes the proof of 3.15.

Enumerate $B_1 = \{b_0, ..., b_{k-1}\}$. For $\tau \in {}^k2$ b_τ is defined as in step 2.

3.17. CRITERION : F can be extended to an embedding $\overline{F}$ from $\mathfrak{A}_1$ into $\mathfrak{A}$ iff for every $\tau \in {}^k2$ there is $c_\tau \in B$ such that

(3.17.0) $c_\tau \leqq f_2(b_\tau)$

(3.17.1) $b_\tau \cap a = 0$ iff $c_\tau = 0$

(3.17.2) $b_\tau \cap a^* = 0$ iff $f_2(b_\tau) \cap c_\tau^* = 0$

(3.17.3) $\forall y \in \sigma_1(b_\tau)$ [$y \in \sigma(a)$ iff $f_1(y) \in \sigma(c_\tau)$]

(3.17.4) $\forall y \in \sigma_1(b_\tau)$ [$y \in \sigma(a^*)$ iff $f_1(y) \in \sigma(c^*)$]

PROOF OF 3.17. : To prove necessity take $c_\tau = \overline{f}_2(b_\tau \cap a)$.

Now assume c_τ exists for every $\tau \in {}^k2$ satisfying (3.17.0) - (3.17.4).

Set $c = \bigcup \{c_\tau \mid \tau \in {}^k2\}$. We shall show that c satisfies (3.15.1) - (3.15.4).

(3.15.1) For any $b_i \in B_1$ holds

$b_i \cap a = 0$ iff $\forall \tau \in {}^k2(\tau(i) = 0 \longrightarrow b_\tau \cap a = 0)$

(by 3.17.0, 3.17.1) iff $\forall \tau \in {}^k2(\tau(i) = 0 \longrightarrow f_2(b_\tau) \cap c_\tau = 0)$

(by 3.8.) iff $\tau \in {}^k2(\tau(i) = 0 \longrightarrow f_2(b_\tau) \cap c = 0$

iff $f_2(b_i) \cap c = 0$

(3.15.2) is proved analogously.

(3.15.3) Since $\bigcup \{\sigma_1(b_\tau) \mid \tau \in {}^k2\} = D_1$ there are for each $y \in D_1$ uniquely determined $y_\tau \in \sigma_1(b_\tau)$ such that $\bigcap \{y_\tau \mid \tau \in {}^k2\} = y$.

Now $y \in \sigma(a)$ implies by (3.17.3) : $\forall \tau \in {}^k2(y_\tau \in \sigma(a))$.

Thus $\forall \tau \in {}^k2(f_1(y_\tau) \in \sigma(c_\tau))$ which in turn yields

$f_1(y) = \bigcap \{f_1(y_\tau) \mid \tau \in {}^k2\} \in \sigma(c_\tau) \subseteq \sigma(c)$

This proves one part of 3.15.3

$f_1(y) \in \sigma(c) \longrightarrow \forall \tau \in {}^k2(f_1(y_\tau) \in \sigma(c))$

$\longrightarrow \forall \tau \in {}^k2(f_1(y_\tau) \in \sigma(f_2(b_\tau) \cap c))$

$\longrightarrow \forall \tau \in {}^k2(f_1(y_\tau) \in \sigma(c_\tau))$

(by 3.17.3) $\longrightarrow \forall \tau \in {}^k2(y_\tau \in \sigma(a))$

$\longrightarrow y \in \sigma(a)$

This proves the other part of (3.15.3).

(3.15.4) Take an arbitrary $y \in D_1$. Let y_τ be as above.

$y \in \sigma(a^*) \longrightarrow \forall \tau \in {}^k2(y_\tau \in \sigma(a^*))$

(by (3.17.4)) $\longrightarrow \forall \tau \in {}^k2(f_1(y_\tau) \in \sigma(c^*_\tau))$

Notice that for $\tau_1 \neq \tau_2$ $\quad y_{\tau_2} \in \sigma(b_{\tau_2}) \subseteq \sigma(b^*_{\tau_1})$ holds. Thus

$f_1(y) \in \sigma(f_2(b^*_{\tau_1})) \subseteq \sigma(c^*_{\tau_1})$.

So we obtain

$y \in \sigma(a^*) \longrightarrow \forall \tau \in {}^k2(f_1(y_\tau) \in \sigma(\bigcap \{c^*_\pi \mid \pi \in {}^k2\}) = \sigma(c^*))$

$\qquad \longrightarrow f_1(y) \in \sigma(c^*)$

Now assume $f_1(y) \in \sigma(c^*)$. This implies

$\forall \tau \in {}^k2(f_1(y) \in \sigma(c^*_\tau))$

$\rightarrow \forall \tau \in {}^k2(f_1(y_\tau) \in \sigma(c^*_\tau))$

(by (3.17.4)) $\quad \forall \tau \in {}^k2(y_\tau \in \sigma(a^*))$

Thus $y = \bigcap \{y_\tau \mid \tau \in {}^k2\} \in \sigma(a^*))$.

This completes the proof of 3.17.

We now show that the axioms (PI) - (PIV) suffice to find the elements $c_\tau \in B$ required in (3.17).

Take an arbitrary $\tau \in {}^k2$. We first dispose of the trivial cases.

If $b_\tau = 0$ or $a \cap b_\tau = 0$, we choose $c_\tau = 0$ and (3.17.0) - (3.17.4) are trivially satisfied.

If $b_\tau \cap a^* = 0$ we choose $c_\tau = f_2(b_\tau)$ and are through again.

So we are left with the only non-trivial case : $0 < a \cap b_\tau < b_\tau$.

We enumerate $\sigma_1(b_\tau) = \{y_0, ..., y_{r-1}\}$; we agree that y_0 is the least element of $\sigma_1(b_\tau)$.

For $\pi \in {}^r2$ define $y_\pi = \bigcup \{\pi(j)y_j \mid j < r\}$ as we have done in step 2.

Define $Y_1 = \{y_\pi \mid \pi \in {}^r2 \text{ and } y_\pi \in \sigma(a)\}$

$Y_2 = \{y_\pi \mid \pi \in {}^r2 \text{ and } y_\pi \in \sigma(a^*)\}$

$Y_3 = \{y_\pi \mid \pi \in {}^r2 \text{ and } y_\pi \notin \sigma(a) \text{ and } y \notin \sigma(a^*)\}$

Furthermore $X_i = f_1(Y_i)$ for $i = 1,2,3$.

By property P VI there is $c_\tau \in Sk(\mathfrak{A})$ such that

$0 < c_\tau < f_1(b_\tau)$

$X_1 \subseteq \sigma(c_\tau)$

$X_2 \subseteq \sigma(c^*_\tau)$

$\forall x \in X_3(x \notin \sigma(c_\tau) \text{ and } x \notin \sigma(c^*_\tau))$.

This implies immediately conditions (3.17.0) - (3.17.2) and for all $\pi \in {}^r2$

$$y_\pi \in \sigma(a) \quad \text{iff} \quad f_1(y_\pi) \in \sigma(c_\tau)$$

$$y_\pi \in \sigma(a^*) \quad \text{iff} \quad f_1(y_\pi) \in \sigma(c_\tau^*)$$

Since every $y_j \in \sigma_1(b_\tau)$ can be represented as

$$y_j = \bigcap \{ y_\pi \mid \pi \in {}^r2 \text{ and } \pi(j) = 0 \}$$

and for any $b \in B$ holds

$$y_j = \bigcap \{ y_\pi \mid \pi \in {}^r2 \text{ and } \pi(j) = 0 \} \in \sigma(b) \quad \text{iff}$$

$$\forall \pi \in {}^r2 (\pi(j) = 0 \longrightarrow y_\pi \in \sigma(b))$$

we infer that also (3.17.3) - (3.17.4) hold.

This completes step 3 and we proved the main theorem.

3.8. EXAMPLE : We shall explicitely construct a countable model of STA*.

We consider subsystems of the power set algebra on $\mathbb{Q} \times \mathbb{Q}$ (the cartesian product of the set of rational numbers with itself).

Let $\mathfrak{B}^0$ be the Boolean subalgebra generated by all subsets of the form $(a,b] \times (c,d]$ where $(a,b] = \{ x \in \mathbb{Q} \mid a < x \leq b \}$ and $a,b,c,d \in \mathbb{Q} \cup \{ -\infty, +\infty \}$. $\mathfrak{B}^0$ is an atomfree countable Boolean algebra.

A subset $X \subseteq \mathbb{Q} \times \mathbb{Q}$ is called thick if there are $n \in \omega$, $p_i, q_i \in \mathbb{Q}$ such that the complement of $X = \bigcup \{ \{p_i\} \times (a_i, b_i] \mid i < n \}$.

The thick subsets form a distributive lattice denoted by $\mathfrak{D}^0$. $\mathfrak{D}^0$ is relatively complemented without antiatoms and without least element.

For $b \in B^0$ define $\sigma(b) = \{ x \in D^0 \mid x \geq b^* \}$ then $\langle \mathfrak{B}^0, \mathfrak{D}^0, \sigma \rangle$ is a Stone algebra satisfying (PI), (PII). To prove (PIII) let $b \in B^0$ and $x \in D^0$ be such that $x \not\geq b^*$. Let the set theoretical complement of x be $\bigcup \{ \{p_i\} \times c_i \mid i < n \}$ and $b^* = \bigcup \{ b_{0i} \times b_{1i} \mid i < m \}$ such that $i \neq k$ implies $b_{0i} \cap b_{1k} = \emptyset$. There is $i < m$ such that $\{p_0\} \times c_0 \subset b_{0j} \times b_{1j}$. Since b_{0j} is infinite there is $p_n \in b_{0j}$, $p_n \notin \{p_0, \ldots, p_{n-1}\}$. Define $c_n = c_0$ and y to be the complement of $\bigcup \{ \{p_i\} \times c_i \mid i \leq n \}$ then $y \in D^0$ and $x \geq y \not\geq b^*$.

To prove (PIV) let x, y be set theoretical complements of elements in D^0 such that $x \cap y = \emptyset$. We have to find an element $b \in B^0$ satisfying $x \subset b$ and $y \subset b^*$. Taking complements we then arrive at (PIV).

We may represent x, y in the following way

$$x = \bigcup \{ \{p_i\} \times c_i \mid i < r \} \cup \bigcup \{ \{p_i\} \times c_i \mid r \leq i < n \}$$

$$y = \bigcup \{ \{p_i\} \times d_i \mid i < r \} \cup \bigcup \{ \{p_i\} \times d_i \mid n \leq i < k \}$$

where $i \neq j$ implies $p_i \neq p_j$. For every $i < k$ there is an interval $b_i \subset \mathbb{Q}$ satisfying $p_i \in b_i$ and $\forall\, j < k\, (i \neq j \longrightarrow p_j \notin b_i)$.

Set $b = \bigcup \{b_i \times c_i \mid i < n\}$ then $b \in B^0$, $x \subset b$ and $y \cap b = \emptyset$, i.e. $y \subset b^*$.

3.19. THEOREM : (i) STA* is $\aleph_0$-categorical

(ii) STA* is complete

(iii) STA* is substructure complete

(iv) STA* is the model completion of STA.

PROOF :

(i) Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be two countable models of STA*. We shall show that $\mathfrak{A}_1$, $\mathfrak{A}_2$ have upto isomorphism the same finite substructures. Theorem 3.2. and 1.2 will then yield $\mathfrak{A}_1 \cong \mathfrak{A}_2$.

Let $\mathfrak{C} = \langle \mathfrak{B}, \mathfrak{T}, \sigma \rangle$ be a finite substructure of $\mathfrak{A}_1$. We may w.l.o.g. suppose that $\mathfrak{L}$ is in fact a Boolean lattice. Since $D(\mathfrak{A}_2)$ is relatively complemented without antiatoms we find an embedding f_1 from $\mathfrak{T}$ into $D(\mathfrak{A}_2)$. Now $\langle f_1, id \rangle$ is an embedding from $\langle \{0, 1\}, \mathfrak{L}, \sigma \rangle$ into $\mathfrak{A}_2$ which using the methods employed in step 3 of the proof of theorem 3.2. can be extended to an embedding $\overline{F}$ from $\mathfrak{C}$ into $\mathfrak{A}_2$.

(ii) Follows from (i) by Vaught's test.

(iii) Follows immediately from theorem 1.6. (ii), and theorem 3.2.

(iv) Since STA $\subseteq$ STA* holds and STA* is substructure complete it remains to show that every model of STA can be embedded into a model of STA*. If $\mathfrak{A}$ is a model of STA we shall construct an increasing sequence $(\mathfrak{B}_n)_{n<\omega}$ of models of STA such that

$\mathfrak{A} \subseteq \mathfrak{B}_0$ and

if $n \equiv 0 \pmod 3$ then $Sk(\mathfrak{B}_n)$ is atomfree and $D(\mathfrak{B}_n)$ has no antiatoms and no least element

if $n \equiv 1 \pmod 3$ then $\mathfrak{B}_n$ satisfies axiom (PIII)

if $n \equiv 2 \pmod 3$ then $\mathfrak{B}_n$ satisfies axiom (PIV) and $D(\mathfrak{B}_n)$ is relatively complemented.

Let $\mathfrak{B}$ be the union of $(\mathfrak{B}_n)_{n<\omega}$. Since the axioms (PI) to (PIV) are $\forall\exists$-sentences and STA is even an equational theory $\mathfrak{B}$ will be a model of STA* extending STA.

If $n \equiv 0 \pmod 3$ take $\mathfrak{B}_n$ to be the free product of $\mathfrak{B}_{n-1}$ ω-times with itself. (see [2], section 17, [4]).

For the next two constructions we shall use theorem 2.5. Assume $n \equiv 1 \pmod 3$ and $\mathfrak{B}_{n-1} \subseteq \mathfrak{A}_3^I$. Take $J = I \times \mathbb{Q} \times \mathbb{Q}$ and let $\mathfrak{B}^0$, $\mathfrak{D}^0$ be as in example 3.18.

The mapping F from $\mathfrak{B}_{n-1}$ into $\mathfrak{A}_3^J$ defined by F(f) (i,p,q) = f(i) is an embedding. Consider the set $C = \{g \in A_3^J \mid \forall i \in I(\{\langle p,q\rangle \mid g(i,p,q) = c\} \in D^0$ & $\{\langle p,q\rangle \mid g(i,p,q) = 0\} \in B^0)$. It is easily seen that C is the universe of a subalgebra $\mathfrak{C} \subseteq \mathfrak{A}_3^J$ and $F(\mathfrak{B}_{n-1}) \subseteq \mathfrak{C}$. Using the same argument as in example 3.18. one shows that $\mathfrak{C}$ satisfies axiom (PIII).

If $n \equiv 2 \pmod 3$ and $\mathfrak{B}_{n-1} \subseteq \mathfrak{A}_3^I$ take $\mathfrak{B}_n = \mathfrak{A}_3^I$. Then $\mathfrak{B}_n$ is obviously relatively complemented and if $f, g \in D(\mathfrak{B}_n)$ are given satisfying $f \sqcup g = 1$ define h by

$$h(i) = \begin{cases} 0 & \\ & \text{if} \\ 1 & \end{cases} \quad \begin{matrix} f(i) \not\leq e \\ \\ f(i) = 1 \end{matrix}$$

Then $h \in Sk(\mathfrak{B}_n)$ and $h \subseteq f$, $h^* \subseteq g$. Thus $\mathfrak{B}_n$ satisfies axiom (PIV).
This completes the proof of theorem 3.19.

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