ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Mathématiques

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Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 60, série *Mathématiques*, n° 13 (1976), p. 129-134

<http://www.numdam.org/item?id=ASCFM_1976__60_13_129_0>

Ann. Sci. Univ. Clermont, Sér. Math., fasc. 13, 1976, pp. 129-133

THE THEORY OF BOOLEAN ALGEBRAS WITH A DISTINGUISHED SUBALGEBRA IS UNDECIDABLE

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§ 0. INTRODUCTION

We prove the following theorems :

Theorem 1** Let T_1 and T_2 be theories in the language $L = \{\cup, \cap, -, 0, 1\}$ such that there are infinite Boolean algebras (hereafter denoted by BA) B_1, B_2 such that $B_i \models T_i$ $i = 1,2$, let P be a unary predicate and $S = T_1 \cup T_2^{(P)}$, where $T_2^{(P)}$ is the relativization of T_2 to P, then S is undecidable.

Theorem 2 : The theory of 1-dimensional cylindric algebras (denoted by CA_1) is undecidable. Theorems 1 and 2 answer a question of Henkin and Monk in [2] Problem 7 ; there they also point out that the decidability problems of theorems 1 and 2 are closely related, this relation is formulated in the following proposition :

Proposition : (a) Let $\langle B,c\rangle$ be a CA_1 where B is a BA and c a unary operation on B then $A = \{ b \mid b \in B \text{ and } c(b) = b \}$ is a subalgebra of B, and for every $b \in B$ c(b) is the minimum of the set $\{ a \mid b \subseteq a \in A\}$.

(b) Let B be a BA and A be a subalgebra of B suppose that for every $b \in B$ $a_b = \min(\{ a \mid b \subseteq a \in A \})$ exists ; define $c(b) = a_b$, then $\langle B,c\rangle$ is a CA_1.

Let T_C be the theory of CA_1's and T_B be the theory of BA's with a distinguished subalgebra P, with the additional axiom that for every b there is a minimal a_b such that $P(a_b)$ and $b \subseteq a_b$, then certainly T_C and T_B are bi-interpretable.

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* This paper is part of the author's doctoral dissertation prepared at the Hebrew University under the supervision of Professor Saharon Shelah.

** R. McKenzie proved independently at about the same time, that the theory of Boolean algebras with a distinguished subalgebra is undecidable. The method of his proof is different from ours.

The classical result about the decidability of the theory of BA's appears in Tarski's [5], and in Ershov [1]. Ershov in [1] also proved that the theory of BA's with a distinguished maximal ideal is decidable, Rabin [4] proved the decidability of the theory of countable BA's with quantification over ideals.

Henkin proved that the equational theory of CA_2's is decidable and Tarski proved the undecidability of the equational theory of CA_n's for $n \geq 4$.

In our construction we interpret the theory of two equivalence relations in a model $\langle B, \cup, \cap, -, 0, 1, A \rangle$ but neither B nor A are complete BA's. We do not know the answer to the following question :

Let $K = \{\langle B, \cup, \cap, -, 0, 1, A\rangle \mid B$ is a BA, A is a subalgebra of B, A and B are complete$\}$ is Th(K) decidable ?

We also do not know whether an analogue of theorem 1 for T_B holds. For instance let S be T_B together with the axioms that say that both the universe and P are atomic BA's is S decidable ?

§ 1. THE CONSTRUCTION

$\cup$, $\cap$, $-$, 0, 1 denote the operations and constants of a BA and $\subseteq$ denotes its partial order. A,B,C denote BA's ; At(B), $A\ell(B)$, As(B) denote the set of atoms of B, the set of non-zero, non-maximal atomless elements of B and the set of non-zero, non-maximal atomic elements of B respectively. Let I(B) be the ideal generated by $A\ell(B) \cup As(B)$, $B^{(1)} = B/I(B)$ and if $b \in B$ $b^{(1)} = b/I(B)$. If $D \subseteq B$ $c\ell(D)$ denotes the subalgebra of B generated by D. $B \times C$ denotes the direct product of B and C. $\prod_{j \in J} B_j$ denotes the direct product of $\{B_j \mid j \in J\}$, and we assume that for every $j_1 \neq j_2$ $B_{j_1} \cap B_{j_2} = \{0\}$, so we can identify the element c of B_{j_0} with the element $f_c \in \prod_{j \in J} B_j$ where $f_c(j) = 0$ if $j \neq j_0$ and $f_c(j_0) = c$. We denote by 1_B the maximal element of B.

Let B_T be the BA of finite and cofinite subsets of ω and B_L the countable atomless BA. Let F_1 be the non-principal ultrafilter of B_T and F_2 be an ultrafilter in B_L ; let B_M be the following subalgebra of $B_T \times B_L$: $B_M = \{(a,b) \mid a \in F_1 \text{ iff } b \in F_2\}$; notice that $B_M^{(1)} \cong \{0,1\}$. For every i let $B_i \cong B_M$, $B^{>} = \prod_{i \in \omega} B_i$ and $B^{<} = c\ell\left(\bigcup_{i \in \omega} B_i\right)$. We denote 1_{B_i} by 1_i.

Lemma 3 : Let E_0 and E_1 be equivalence relations on ω then there is a model $M = \langle B, \cup, \cap, -, 0, 1, A\rangle \models T_B$ such that $\langle \omega, E_0, E_1\rangle$ is explicitly interpretable in M.

Proof : We denote by i/E_ε the E_ε-equivalence class of i and by ω/E_ε the set of E_ε-equivalence classes. For every $i \in \omega$ let

$\{b^i_{\varepsilon,\sigma,j} \mid \varepsilon \in \{0,1\}, \sigma \in \omega/E_\varepsilon, j \in \omega\} \subseteq A\ell(B_i)$ be such that

$\langle\varepsilon,\sigma,j\rangle \neq \langle\varepsilon',\sigma',j'\rangle \Rightarrow b^i_{\varepsilon,\sigma,j} \cap b^i_{\varepsilon',\sigma',j'} = 0$ and for every $b \in A\ell(B_i)$

$1 \leq |\{\langle\varepsilon,\sigma,j\rangle \mid b \cap b^i_{\varepsilon,\sigma,j} \neq 0\}| < \aleph_0$. For every $i \in \omega$ let

$\{\bar{a}^i_{\varepsilon,\sigma,j} \mid \varepsilon \in \{0,1\}, \sigma \in \omega/E_\varepsilon, j \in \omega\}$ be a set of pairwise disjoint subsets of

$At(B_i)$ such that $At(B_i) = \bigcup\{\bar{a}^i_{\varepsilon,\sigma,j} \mid \varepsilon \in \{0,1\}, \sigma \in \omega/E_\varepsilon, j \in \omega\}$ and

$$|\bar{a}^i_{\varepsilon,\sigma,j}| = \begin{cases} 1 & \varepsilon = 0 \text{ and } i \in \sigma \\ 2 & \varepsilon = 0 \text{ and } i \notin \sigma \\ 3 & \varepsilon = 1 \text{ and } i \in \sigma \\ 4 & \varepsilon = 1 \text{ and } i \notin \sigma \end{cases}$$

For every ε, σ and j as above let $c_{\varepsilon,\sigma,j} \in B^{>}$ be $c_{\varepsilon,\sigma,j} = \cup\{b^i_{\varepsilon,\sigma,j} \cup -a^i_{\varepsilon,\sigma,j} \mid i \in \omega\}$ where $\cup D$ denotes the supremum of D in $B^{>}$. Let $A = c\ell(\{c_{\varepsilon,\sigma,j} \mid \varepsilon \in \{0,1\}, \sigma \in \omega/E_\varepsilon, j \in \omega\}$, $B = c\ell(B^{<} \cup A)$ and $M = \langle B, \cup, \cap, -, 0, 1, A\rangle$. We show that $M \models T_B$. It suffices to show that $a_b = \min(\{a \mid b \subseteq a \in A\})$ exists for elements $b \in B$ of the following forms: $b \in At(B_i) \cup A\ell(B_i)$; $b \in B_i$ and $b^{(1)} = 1_i^{(1)}$; $b \in B^{<}$ and $1_i \subseteq b$ for almost all $i \in \omega$; this follows from the fact that every $b \in B$ can be represented in the form $\bigcup_{i=1}^{n} (b_i \cap a_i)$ where each b_i is of the above form and $a_i \in A$. In each of the above cases the existence of a_b is easily checked. Thus $M \models T_B$.

We now define formulas $\varphi_U(x)$, $\varphi_{Eq}(x,y)$, $\varphi_\varepsilon(x,y)$ $\varepsilon \in \{0,1\}$ such that $M \models \varphi_U[a]$ iff for some $i \in \omega$ $a^{(1)} = 1_i^{(1)}$, $M \models \varphi_{Eq}[a,b]$ iff $a^{(1)} = b^{(1)}$ and $M \models \varphi_\varepsilon[a_1, a_2]$ iff for some $i_j \in \omega$ $a_j^{(1)} = 1_{i_j}^{(1)}$ and $\langle i_1, i_2\rangle \in E_\varepsilon$.

$\varphi_U(x)$ says that $x^{(1)} \in At(B^{(1)})$ and for no $y \in At(A)$ $x^{(1)} = y^{(1)}$. $\varphi_{Eq}(x,y)$ says that $x^{(1)} = y^{(1)}$. $\varphi_0(x,y)$ says: $\varphi_U(x) \wedge \varphi_U(y)$ and there are x_1, y_1 such that $x^{(1)} = x_1^{(1)}$, $y^{(1)} = y_1^{(1)}$ and for every $z \in At(A)$ $|\{u \mid z \cap x_1 \supseteq u \in At(B)\}| = 1$ iff $|\{u \mid z \cap y_1 \supseteq u \in At(B)\}| = 1$. φ_1 is defined similarly. The desired properties of φ_U, φ_{Eq} and φ_ε are easily checked, and the lemma is proved.

Since the theory of two equivalence relations is undecidable T_B and T_C are undecidable and theorem 2 is proved.

Theorem 1 easily follows from the following lemma.

Lemma 4 : Let E_1, E_2 be equivalence relations on ω then there are models $M_i = \langle B_i, \cup, \cap, -, 0, 1, A_i\rangle$ $i = 1,\dots,4$ such that $\langle \omega, E_1, E_2\rangle$ is explicitly interpretable in M_i and B_1, A_1 are atomic, B_2, A_2 are atomless, B_3 is atomic A_3 is atomless, and B_4 is atomless A_4 is atomic.

Proof : Let B_0, A_0, M_0 denote B, A and M of lemma 3 respectively. For $i = 1,2$ M_i can easily be constructed so that $\langle B_i/H_i, \cup, \cap, -, 0, 1, A_i/H_i\rangle \cong M_0$ where $H_i = \{b \mid b \in B_i$ and for every $a \subseteq b$ $a \in A_i\}$. Since such an H_i is definable in M_i M_0 can be interpreted in M_i $i = 1,2$.

For i = 3 a similar construction works. Let B be an atomic saturated countable BA and I a maximal non-principal ideal of B. Let A be an atomless subalgebra of B such that :
(a) for every $b \in B$ which contains infinitely many atoms there is a non-zero $a \in A$ such that $a \subsetneq b$;
(b) for every $b \in A_s(B)$ there is an $a \in A$ such that $(a-b) \cup (b-a)$ contains only finitely many atoms of B. Let $J = I \cap A$. For every non-zero $a \in B_0$ let F_a be an ultrafilter in B which contains a, and $< B_a, A_a, I_a, J_a >$ a copy of $< B, A, I, J>$. Let $B^1 = \Pi \{B_a \mid 0 \neq a \in B_0\}$ and let B_3 be the following subalgebra of B^1 :
$B_3 = c\ell(\cup\{I_a \mid 0 \neq a \in B_0\} \cup \{g_a \mid 0 \neq a \in B_0\})$ where $g_a(b) = 1_{B_b}$ iff $a \in F_b$ and $g_a(b) = 0$ otherwise. Let $A_3 = c\ell(\cup\{J_a \mid 0 \neq a \in B_0\} \cup \{g_a \mid a \in A_0\})$.
Certainly B_3 is atomic and A_3 is atomless. Let $I = \{a \mid |\{b \mid a \supset b \in At(B_3)\}| < \aleph_0\}$.
I is an ideal in B_3, and I is definable in M_3 by the formula
$\varphi(x) \equiv \forall y(0 \neq y \subseteq x \to \sim P(y))$. Let $B_3^1 = B_3/I$ and $A_3^1 = \{a/I \mid a \in A_3\}$, then $< B_3^1, \cup, \cap, -, 0, 1, A_3^1 > \cong M_2$, so M_2 is interpretable in M_3 and thus $< \omega, E_1, E_2 >$ is interpretable in M_3 as desired.

In order to construct M_4 we assume that B_1 is a subalgebra of $P(\omega)$ and $At(B_1) = \{\{n\} \mid n \in \omega\}$. Let $B_L^i \cong B_L$ for every $i \in \omega$ B_4 is the following subalgebra of $\prod_{i \in \omega} B_L^i$: $B_4 = c\ell(\bigcup_{i \in \omega} B_L^i \cup \{f_a \mid a \in B_1\})$ where $f_a(n) = 1_{B_L^n}$ if $n \in a$ and $f_a(n) = 0$ otherwise. Let $A_4 = c\ell(\{f_a \mid a \in A_1\})$ and $M_4 = < B_4, \cup, \cap, -, 0, 1, A_4 >$.

Certainly B_4 is atomless and A_4 is atomic. Let $B_4^1 = \{b \mid b \in B_4$ and for every $a \in At(A_4)$ either $b \supset a$ or $-b \supset a\}$, then $< B_4^1, \cup, \cap, -, 0, 1, A_4 > \cong M_1$ and B_4^1 is certainly definable in M_4, thus $< \omega, E_1, E_2>$ is definable in M_4 and the lemma is proved.

We omit the proof of theorem 1 which follows easily from lemma 4, the fact that every countable BA can be embedded in e.g. B_L, and from [6] pp. 293-302.

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