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Creation and coupling of Rossby waves over topography and the breakdown of WKB theory

Rémi Tailleux [1]

Abstract

The standard oceanic Rossby wave modes are separable solutions of the primitive equations linearized around a resting, flat-bottom ocean whose stratification depends upon depth only. These waves have a modal structure in the vertical and a propagating horizontal part that satisfies the shallow water equations with an equivalent depth H_e that is different for each mode. These waves propagate independently of each other over a flat-bottom, but become everywhere coupled over topography. This coupling can be primarily interpreted as defining new dynamical wave modes, i.e. generalized topographic Rossby waves, which can be constructed explicitly by means of WKB theory under the usual scale separation assumption. WKB solutions may locally breakdown, however, when the wavenumbers and frequency of the mode considered approximately satisfy locally the dispersion relation of another wave mode supported by the system. In such a region, called a mode conversion point, the WKB solution becomes degenerate and can no longer be expressed in terms of a single wave mode. There, linear resonance occurs and energy can be exchanged between two different rays. This paper presents an application of mode conversion theory for Rossby waves in a two-layer ocean propagating over a mid-ocean Gaussian ridge varying with longitude only. This theory is shown to predict satisfactorily the location of mode conversion points, and the amount of energy exchanged between rays. In such a framework, wave creation thus occurs at points where WKB breaks down.

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1 Introduction.

There is strong observational evidence, e.g., [3], that mid-ocean ridges play a major role in creating/amplifying incident long baroclinic Rossby waves. Numerical ocean models have helped to provide evidence that new waves can be created over topographic ridges, e.g., [1], [15]. So far, however, a satisfactory theoretical description has been lacking. Perhaps the main attempt at interpretation has been the invocation to the concept of JEBAR (Joint Effect of Baroclinicity and Relief), e.g., [12]. Simply put, the JEBAR interpretation consists merely in stating that the standard barotropic and baroclinic modes, i.e., the modes defined for a flat-bottom reference ocean, are coupled over topography. Such a description, however, is not satisfactory because it is somehow largely tautological. Furthermore, stating that the standard modes are coupled does not imply wave creation. In fact, as explained in [15], the nature of this coupling can be twofold: a) it may result in new dynamical modes, which on may regard as generalized topographic Rossby wave modes; b) the new dynamical modes thus defined may themselves be coupled. The drawback of the JEBAR interpretation is that it does not distinguish between these two effects. To make progress, we therefore need a way to a) construct the generalized dynamical Rossby wave modes over topography; b) determine whether these modes are coupled or not.

When the medium variations are weakly non-uniform, the first issue can be addressed by means of WKB theory. Such an approach was pioneered in the present context by [11] for a constant buoyancy frequency and a topography varying in latitude only, and later generalized to more general topography and stratification by [2] and [13]. The latter studies focused essentially on how topography locally modifies the structure of the vertical normal modes. Examples of ray calculations were given only recently, e.g., [6], [9], and [14]. As long as it is valid, WKB theory provides a description of the system in terms of dynamically uncoupled wave modes. Direct numerical simulation, however, suggests that the generalized dynamical modes, regardless of the way they are defined, cannot remain uncoupled everywhere over topography. In the experiments realized by [15] for instance, two different wave modes are clearly observed in the western flat part of an ocean basin with a ridge at its center, starting with one type of mode in the eastern flat part, clearly indicating that wave creation occured over the central ridge. Within the framework of WKB theory, wave creation can only occur in regions where the WKB approximation breaks down because by construction a ray conserves its energy as long as WKB remains valid. To understand this issue, [6] investigated the validity of WKB theory in layered models of the ocean by comparing ray calculations with direct numerical simulations using a primitive equations model. He concluded that WKB theory was valid almost everywhere, except in localized area where bottom and surface intensified Rossby waves were found to exchange energy. [6] showed that these regions corresponded to places where the relation dispersion for the two modes osculated; the author failed, however, to provide a mathematical description of the phenomenon. Recently, [17] pointed out that the type of phenomenon described by Hallberg had been extensively studied in the context of plasma physics, which led to a pretty well developed theory called mode conversion theory, e.g., [4], [7]. This theory was recently applied in a geophysical context to the issue of the conversion of coastal Kelvin waves to equatorial Yanai waves by [8].

Mode conversion theory provides the mathematical framework to predict where mode conversion should take place. It also provides connection formula to link the incident wave action flux $J_{incident}$ to the outcoming converted and transmitted action fluxes $J_{converted}$ and $J_{transmitted}$ through connection formula

$$J_{transmitted} = T J_{incident}, \qquad J_{converted} = (1-T) J_{incident} \tag{1.1}$$

where the transmission coefficient T can be linked to the properties of the dispersion relation. The purpose of this paper is to describe a specific example of WKB mode coupling for two-layer long Rossby waves over a Gaussian ridge, by using linear mode conversion theory. The present discussion remains descriptive, and summarizes the detailed results which the interested reader may find in [16].

2 Model and notations

The dynamics of coupled Rossby waves over topography is studied with the same two-layer model as used in [15], whose main features are depicted in Fig. 1. The model equations are

$$f\hat{\mathbf{z}} \times \mathbf{U}_i + H_i \nabla p_i = 0, \qquad i = 1, 2, \tag{2.2}$$

$$\frac{\partial(\eta_1 - \eta_2)}{\partial t} + \mathrm{div}\mathbf{U}_1 = 0, \tag{2.3}$$

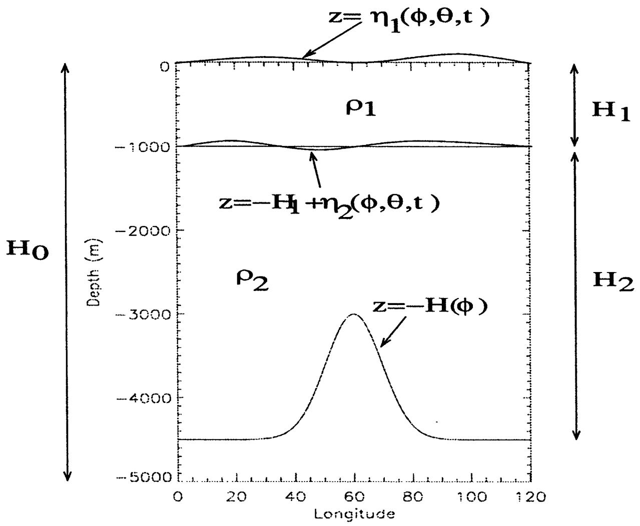

Figure 1: Model geometry and notation.

$$\frac{\partial \eta_2}{\partial t} + \text{div}\mathbf{U}_2 = 0. \tag{2.4}$$

Eq. (2.2) refer to the linearized momentum equations; it expresses geostrophic balance in each layer; Eqs (2.3) and (2.4) are linearized mass conservation equations for each layer. The geometry considered is a rectangular basin bounded by two meridians at $\phi = 0$ and $\phi = 120°$, but unbounded in latitude ($= \theta$). The topography is a Gaussian ridge that varies with longitude only, whose equation reads

$$H(\phi) = H_0 - \delta H \exp\left\{ -\frac{1}{2}\left(\frac{\phi - \phi_0}{\delta\phi}\right)^2 \right\}.$$

The notations are: H_1 and $H_2 = H(\phi) - H_1$ represent the unperturbed layer thicknesses; $\eta_1(\phi, \theta, t)$ and $\eta_2(\phi, \theta, t)$ are the surface and interface displacements; the total layer thicknesses are therefore $h_1 = H_1 + \eta_1 - \eta_2$ and $h_2 = H_2 + \eta_2$; $\epsilon = (\rho_2 - \rho_1)/\rho_0$ is a dimensionless parameter measuring the density difference across the interface; $\mathbf{U}_i = h_i\mathbf{u}_i \approx H_i\mathbf{u}_i$ is the horizontal transport in layer $i = 1, 2$; $f = 2\Omega \sin\theta$ is the Coriolis parameter, where

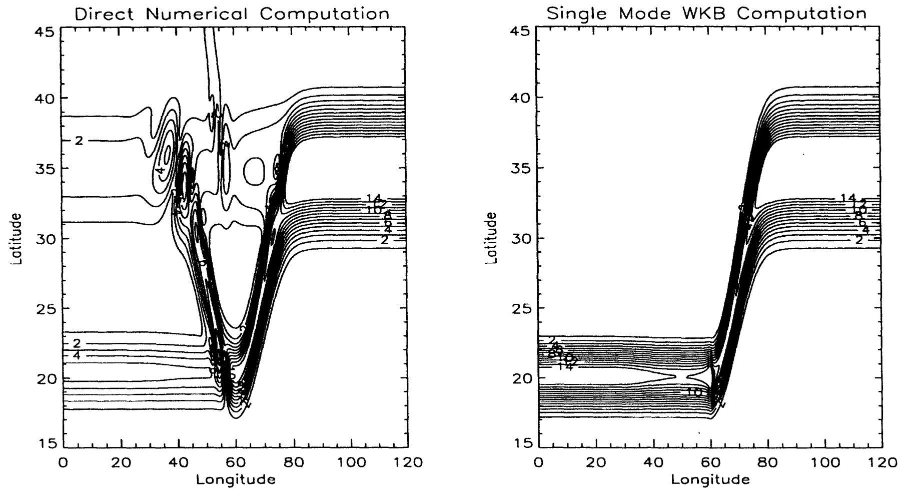

Figure 2: The root-mean square of the interface displacement as computed by: direct numerical simulation (left panel); from the wave action ray tube conservation equation of single mode WKB theory (right panel). Note that the contour lines represent isolines, not rays!

Ω is Earth's rotation rate, R is Earth's radius, g is Earth's gravitational acceleration, and $g' = g\epsilon$ is a reduced gravitational acceleration.

To illustrate the creation of Rossby waves over topography, we consider an eastern wavemaker experiment where waves are excited at the annual period along the eastern boundary. The response is therefore purely periodic, so that the solution of (2.2- 2.4) can be expressed as:

$$\eta_1(\phi,\theta,t) = \hat{\eta}_1(\phi,\theta)e^{-i\omega t}, \qquad \eta_2(\phi,\theta,t) = \hat{\eta}_2(\phi,\theta)e^{-i\omega t} \tag{2.5}$$

where $\hat{\eta}_1$ and $\hat{\eta}_2$ are therefore complex amplitudes. The result of direct the numerical simulation for the wave amplitude $|\hat{\eta}_2|(\phi,\theta)$ is depicted in the left panel of Fig. 2. The numerical values used are: $H_1 = 1000\,\mathrm{m}$, $H_0 = 4500\,\mathrm{m}$, $\delta = 1500\,\mathrm{m}$, $g' = 2.10^{-2}\,\mathrm{m.s^{-1}}$, $\omega = 2\pi/(1\mathrm{year})$, $\delta\phi = 9.6°$. In the eastern part, the bottom is flat and the wave energy propagate uniformly westward. Over the ridge, one part of the energy is seen to be deflected southward, approximately following the potential vorticity contours $H_2/f = cst$ of the lower layer, the other part continuing westward, following the contours H_1/f of the upper layer. Another splitting of the energy appears to occur near the top of the ridge, one part of the energy being deflected northward this time, the other part continuing westward. For comparion, the right panel of Fig. 2) shows the prediction of the classical single mode WKB theory. As said in the introduction, this calculation assumes the energy to be conserved along a ray; as a consequence, no energy splitting occurs, with a unique propagation path that contrasts sharply with that of the real solution. To describe the features of the real solution, WKB theory must be extended to include mode conversion points, which is addressed in the following section.

3 Mode conversion theory

Mode conversion points occur typically where two branches of the dispersion relation osculate. The standard features of mode conversion in one dimension are depicted in Fig. 3. The propagation is described here in the (k, x) space, where k is the wavenumber, and x the corresponding spatial coordinate. The thick lines represent two branches of the dispersion relation $k = k(\omega, x)$, where ω is the wave frequency which remains constant along a ray if the medium is time independent. The wavenumber varies with position as a result of the medium being inhomogeneous. Initially, only the upper branch is excited (the incident ray), the propagation taking place from right to left. The lower curve represents the dispersion relation for another wave mode supported by the system, but which is not excited initially. The two thin lines represent idealized dispersion curves that represent uncoupled propagation. In the present case, for instance, this idealized propagation would correspond to propagation taking place entirely in one layer or the other. The location where mode conversion occurs is associated physically with the branch crossing of the two uncoupled dispersion curves, and is indicated in the figure by the circle area. Outside this region, the energy splits between a converted and transmitted wave.

In the present case, the propagation is two-dimensional, and thus must be represented in the four-dimensional phase space $(k_\phi, k_\theta, \phi, \theta)$, where k_ϕ and k_θ are the zonal and meridional wavenumbers respectively. To apply mode conversion theory, the problem must be reduced to the standard one-dimensional form. This is achieved by solving the standard canonical ray equations, e.g., [10]. The idea is to express k_θ and θ as functions of the longitude ϕ and ray index ξ, which denotes the starting latitude of the ray along the eastern boundary. When this is done, all quantities can be expressed in

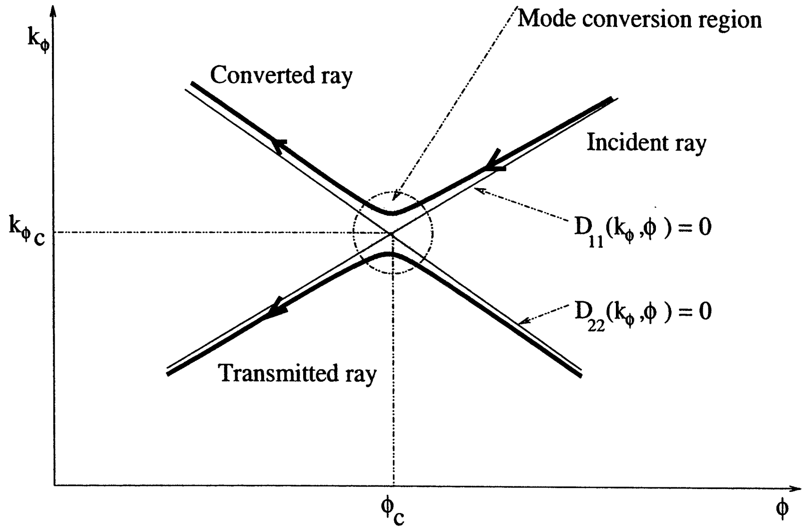

Figure 3: Idealized depiction of mode conversion. The two thick lines of equations $k_\phi = k_\phi^+(\phi)$ and $k_\phi = k_\phi^-(\phi)$ represent two branches of the dispersion relation in the (ϕ, k_ϕ) space. Mode conversion occurs in the shaded area where the incident ray originating from the right separates into a converted and transmitted ray. The amount of wave action splitting among the two depends on the nature and strength of the mode coupling. The thin lines represent the dispersion curves $D_{11}(k_\phi, \phi) = 0$ and $D_{22}(k_\phi, \phi) = 0$ of the wave modes in the absence of coupling. These curves intersect at the point $(\phi_c, k_{\phi,c})$, where mode conversion is expected to occur for the coupled wave modes.

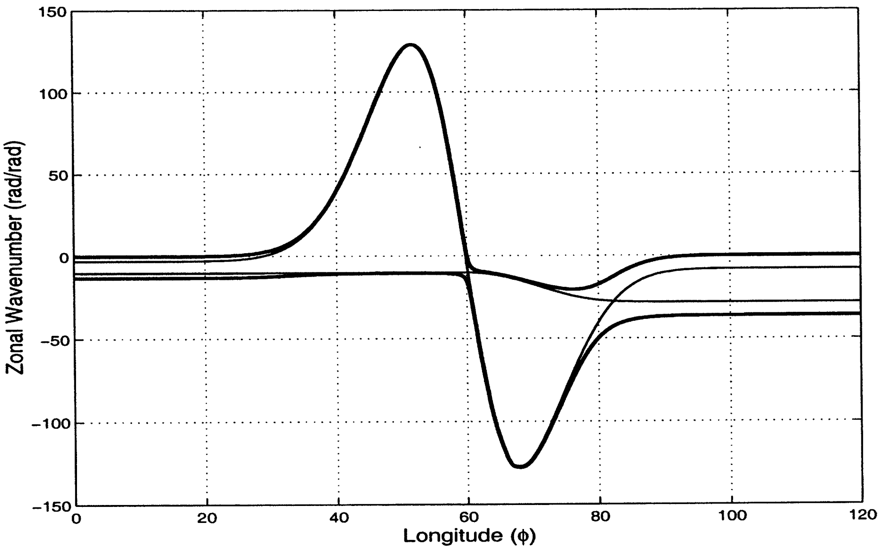

Figure 4: Evolution of the coupled (thick lines) and uncoupled (thin lines) wave modes represented in the (k_ϕ, ϕ) space, as in the previous figure, corresponding to a ray originating at $\xi = 35°N$. The uncoupled rays are the ones intersecting at two locations near the bottom and top of the ridge eastern flank respectively, where mode conversion must theoretically take place.

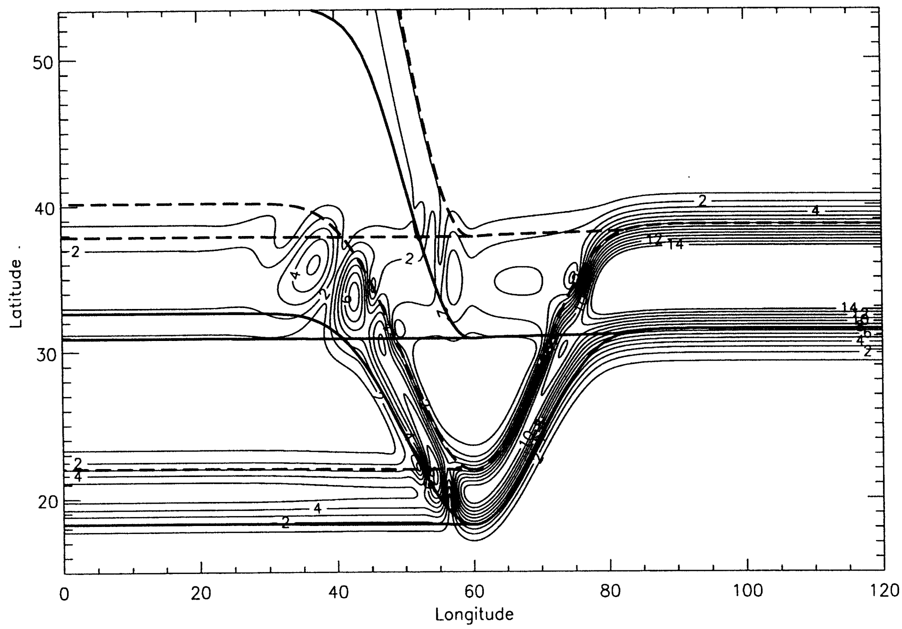

Figure 5: A pair of rays (denoted by the heavy solid and dashed lines) originating from the eastern boundary and their successive bifurcations predicted by mode conversion theory superimposed on the direct numerical computation of the r.m.s. interface displacement (reproduced from left panel of Fig. 2).

terms of k_ϕ, ϕ and ξ. Fig. 4 shows the evolution of the zonal wavenumber k_ϕ for the present case, for a ray initiated at $\xi = 35°$. The lower thick curve represents the zonal evolution of the zonal wavenumber for the initially excited baroclinic ray. The upper thick curve corresponds to the zonal wavenumber for the second wave mode. The near coincidence of the thin and thick lines over the topography indicates the quasi perfect decoupling of the two layers over the topography, as previously noted by [6]. The thick and thin lines strongly differ near the mode conversion points, and thus indicate a strong coupling of the layers there. According to the geometric interpretation of [5], transmission is expected to be the strongest: a) the closer the dispersion curves; b) the sharpest the angle between the dispersion curves before and after the mode conversion point. From Fig. 5, we therefore expect weak transmission near the bottom ridge, because the two thick lines are quite far away from each other, while remaining also quite smooth. In contrast, the thick lines appear very close near the hilltop, while at the same time they vary abruptly; here, transmission is therefore expected to be the strongest. This qualitative prediction is confirmed in the next section.

4 Test of mode conversion theory

The first test of mode conversion theory one can make is with regard to its predictions about where mode conversion should take place. To that end, we started a new ray each time a mode conversion point was encountered. As shown in Fig. 4, two such mode conversion points where found here, one occuring near the bottom of the ridge, the other one near the hilltop, in both cases on the eastern flank. The result is illustrated in Fig. 5 where we plotted two rays and their successive bifurcations superimposed on the left panel of Fig. 2. Wave activity appears clearly to lie within the regions delineated by the the various rays, which thus indicates that mode conversion theory correctly predicts the regions where to expect energy.

To illustrate the splitting process, we decomposed the real solution depicted in the left panel of Fig. 2 on the two WKB wave modes. Fig. 6 (middle panel) shows the projection on the "baroclinic" WKB mode initially excited, while the bottom panel shows the projection on the "barotropic" WKB mode. For comparison, we also depicted the prediction of single WKB theory accounting for the loss of energy taking successively place at the two mode conversion points. The latter is therefore directly comparable with the

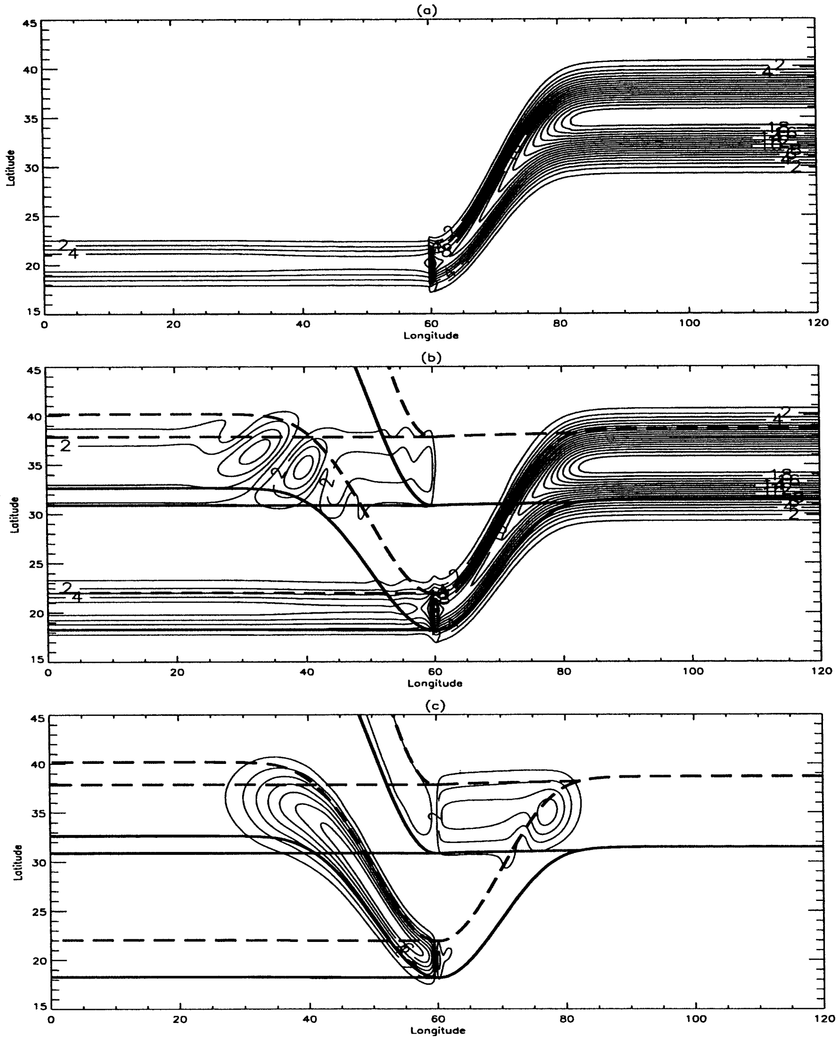

Figure 6: (a) Root-means square of the layer interface as computed by single-WKB mode theory modified to account for the two mode conversion events described in the text. (b) The root-mean square of the layer interface of the true solution projected on the WKB normal mode. (c) The root-mean square of the residual of the true layer interface minus the part depicted in (b). A pair of rays originating from the eastern boundary and their successive bifurcations predicted by mode conversion theory were superimposed on the middle and bottom panels.

part of the wave activity that originates from the eastern boundary, where the two signals appear to be in good agreement. The bottom panel makes it clear that generalized "barotropic" waves are created at the bottom and top of the ridge's eastern flank. The middle panel also indicates the creation of a new generalized "baroclinic" wave issued from the barotropic wave created at the bottom of the ridge.

One can test the theory more quantitatively by comparing the amplitudes of the real solution and that predicted by single mode WKB theory, accounting or not for mode conversion, for several latitudinal sections taken at selected longitudes, as shown in Fig. 7. The thin line represents the real solution, while the thick line represents the prediction of mode conversion theory accounting for the mode conversion events. The comparison of the two curves show a very good agreement. The dashed line, on the other hand, represents the prediction of standard single mode WKB theory, which is seen to greatly overestimate the amplitude of the wave in the western part of the basin.

5 Conclusion

The coupling of the standard baroclinic and baroclinic modes can be interpreted most of the time as defining two dynamically independent wave modes, which can be constructed explicitly using WKB theory when the medium (here the topography) is weakly non-uniform. Unlike the standard Rossby wave modes, which are coupled everywhere over topography, the WKB Rossby wave modes are coupled only in special locations where they can exchange energy. These points correspond to local resonance induced by the topography. In the WKB description, energy exchange with another wave mode can only occur in places where WKB breaks down, as otherwise energy is conserved along the rays by assumption. Mode conversion theory describes the resonant energy exchange taking place between rays at points where two branches of the dispersion relation locally osculate, and where the WKB approximation breaks down. Such a theory permits to describe and quantify wave creation in a weakly nonuniform medium. It was shown to be qualitatively and quantitatively accurate to describe the coupling of Rossby waves over a Gaussian ridge in a two-layer model. The mechanism thus described is universal, and thus is potentially applicable to a wide variety of waves in geophysical fluids.

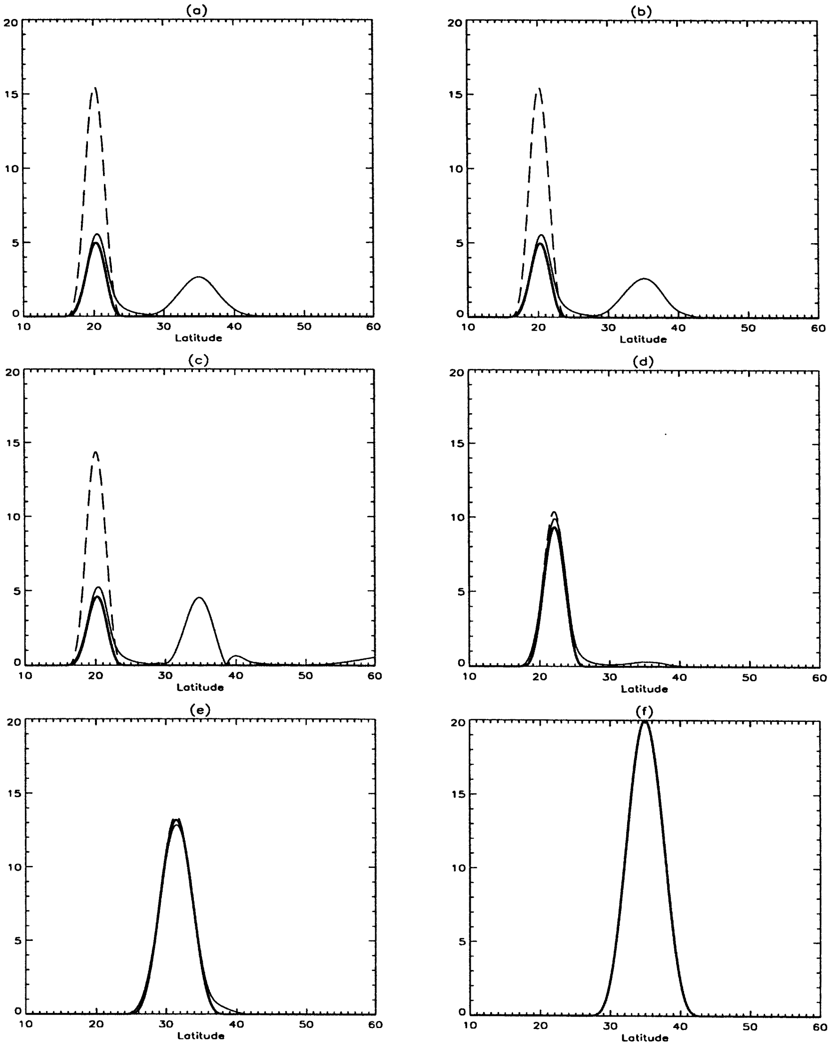

Figure 7: Latitudinal sections of $|\hat{\eta}_2|$ computed by: single WKB mode theory (dashed line); single WKB mode theory accounting for mode conversion (thick line); direct numerical simulation projected on WKB normal mode (thin line) for various longitudes: a) $\phi = 5°$; b) $\phi = 20°$; c) $\phi = 40°$; d) $\phi = 65°$; e) $\phi = 75°$; f) $\phi = 90°$. In panels d), e), and f) the three curves are almost indistinguishable from each other. In panels a), b), and c) the additional peak on the right appears only in the numerical solution, so that the comparison is to be made only for the peak on the left.

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