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Abstract

In this paper, K denotes a complete, non-trivially valued, non-archimedean field. The entries of sequences, series and infinite matrices are in K. In the present paper, we prove the Silvermann-Toeplitz theorem for double sequences and series in K and apply it to Nörlund means for double sequences and series in K.

Throughout the present paper, K denotes a complete, non-trivially valued, non-archimedean field. The entries of sequences, series and infinite matrices are in K. In this paper, we prove the Silvermann-Toeplitz theorem for double sequences and series in K (see Theorem 2, proved in the sequel). We then introduce Nörlund means for double sequences and series in K and apply Silvermann-Toeplitz theorem for these means.

For analysis in the classical case a general reference is [2] while for analysis in non-archimedean fields a general reference is [1].

For a given infinite matrix $A = (a_{n,k})$ and a sequence $\{x_k\}$, the sequence $\{y_n\}$ is defined as follows:

$$y_n = \sum_{k=1}^{\infty} a_{n,k} x_k, \quad n = 1, 2, \cdots,$$

it being assumed that the series on the right converge. If $\lim_{n\to\infty} y_n = s$ whenever $\lim_{k\to\infty} x_k = s$, we say that A is regular. The criterion for A to be regular in terms of the entries of the matrix A are well-known (see [4], [6]).

Theorem 1. *$A = (a_{n,k})$ is regular if and only if*

(i) $\sup_{n,k} |a_{n,k}| < \infty$;

(ii) $\lim_{n\to\infty} a_{n,k} = 0,\ k = 1, 2, \cdots;$

and

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1.$

In the sequel, the following definitions are needed.

Definition 1. Let $\{x_{m,n}\}$ be a double sequence in K and $x \in K$. We say that $\lim_{m+n\to\infty} x_{m,n} = x$ if for each $\epsilon > 0$, the set $\{(m,n) \in \mathbb{N}^2 : |x - x_{m,n}| \geq \epsilon\}$ is finite. In such a case we say that x is the limit of $\{x_{m,n}\}$.

Definition 2. Let $\{x_{m,n}\}$ be a double sequence in K and $s \in K$. We say that

$$s = \sum_{m=1,n=1}^{\infty,\infty} x_{m,n},$$

if

$$s = \lim_{m+n\to\infty} s_{m,n},$$

where

$$s_{m,n} = \sum_{k=1,l=1}^{m,n} x_{k,l},\ m, n = 1, 2, \cdots.$$

Remark. If $\lim_{m+n\to\infty} x_{m,n} = x$, then the sequence $\{x_{m,n}\}$ is automatically bounded.

It is easy to prove the following results.

Lemma 1. $\lim_{m+n\to\infty} x_{m,n} = x$ if and only if

(i) $\lim_{n\to\infty} x_{m,n} = x,\ m = 1, 2, \cdots,$

(ii) $\lim_{m\to\infty} x_{m,n} = x,\ n = 1, 2, \cdots,$

and

(iii) for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x - x_{m,n}| < \epsilon$, for all $m, n \geq N$, which we write as $\lim_{m,n\to\infty} x_{m,n} = x$.

Lemma 2. $\lim_{m+n\to\infty} s_{m,n}$ exists if and only if

$$\lim_{m+n\to\infty} x_{m,n} = 0 . \tag{1}$$

Given the matrix $A = (a_{m,n,k,l})$, we define

$$y_{m,n} = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}, \quad m, n = 1, 2, \cdots, \tag{2}$$

assuming that the series on the right converge.

Necessary and sufficient conditions for $A = (a_{m,n,k,l})$ to be regular for the class of all double sequences and series in the classical case have been found by Kojima [3]. It has been found that convergence and boundedness play a vital role for double sequences and series, a role analogous to that of convergence for simple sequences and series. Robison [8] proved Silvermann-Toeplitz theorem for such a class of bounded and convergent double sequences in the classical case. We prove here its analogue in a complete, non-trivially valued, non-archimedean field.

In this context, the following definition is needed.

Definition 3. If whenever $\{x_{m,n}\}$ is a convergent sequence, $\{y_{m,n}\}$ converges to the same value, then the matrix $A = (a_{m,n,k,l})$ is said to be regular.

Theorem 2. *In order that whenever a sequence* $\{x_{m,n}\}$ *has a limit* x, $\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$ *shall converge and* $\lim_{m+n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} x_{k,l} = x$, *i.e., for* $A = (a_{m,n,k,l})$ *to be regular it is necessary and sufficient that*

(a) $\lim_{m+n\to\infty} a_{m,n,k,l} = 0, \quad k, l = 1, 2, \cdots;$

(b) $\lim_{m+n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} = 1;$

(c) $\lim_{m+n\to\infty} \sup_{k\geq 1} |a_{m,n,k,l}| = 0, \quad l = 1, 2, \cdots;$

(d) $\lim_{m+n\to\infty} \sup_{l\geq 1} |a_{m,n,k,l}| = 0, \quad k = 1, 2, \cdots;$

and

(e) $\sup_{m,n,k,l} |a_{m,n,k,l}| < \infty.$

Proof. Proof of necessity.

Define the sequence $\{x_{k,l}\}$ as follows: For any fixed p, q, let

$$x_{k,l} = \begin{cases} 1, & \text{when } k = p, \ l = q; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Then

$$y_{m,n} = a_{m,n,p,q}.$$

Since $\{x_{k,l}\}$ has limit 0, it follows that (a) is necessary.

Define the sequence $\{x_{k,l}\}$ where $x_{k,l} = 1, \ \ k, l = 1, 2, \cdots$.

Now,

$$y_{m,n} = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}, \quad m, n = 1, 2, \cdots.$$

$$\text{This shows that } \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} \text{converges for } m, n = 1, 2, \cdots. \tag{4}$$

Since $\{x_{k,l}\}$ has limit 1, it follows that

$$\lim_{m+n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} = 1,$$

so that (b) is necessary.

We now show that $\lim_{m+n\to\infty} \sup_{k\geq 1} |a_{m,n,k,l}| = 0$ for all $l \in \mathbb{N}$. Suppose not. Then there exists $l_0 \in \mathbb{N}$ such that $\lim_{m+n\to\infty} \sup_{k\geq 1} |a_{m,n,k,l_0}| = 0$ does not hold. So, there exists an $\epsilon > 0$, such that

$$\left\{(m, n) : \sup_{k\geq 1} |a_{m,n,k,l_0}| > \epsilon\right\} \text{ is infinite.} \tag{5}$$

Let us choose $m_1 = n_1 = r_1 = 1$. Choose $m_2, n_2 \in \mathbb{N}$ such that $m_2+n_2 > m_1 + n_1$ and

$$\sup_{1\leq k\leq r_1} |a_{m_2,n_2,k,l_0}| < \frac{\epsilon}{8}, \text{ using } (a);$$

and

$$\sup_{k\geq 1} |a_{m_2,n_2,k,l_0}| > \epsilon, \text{ using (5).}$$

Then choose $r_2 \in \mathbb{N}$ such that $r_2 > r_1$ and

$$\sup_{k>r_2} |a_{m_2,n_2,k,l_0}| < \frac{\epsilon}{8}, \text{ using (b).}$$

Inductively choose $m_p + n_p > m_{p-1} + n_{p-1}$ such that

$$\sup_{1\le k\le r_{p-1}} |a_{m_p,n_p,k,l_0}| < \frac{\epsilon}{8}; \tag{6}$$

$$\sup_{k\ge 1} |a_{m_p,n_p,k,l_0}| > \epsilon; \tag{7}$$

and then choose $r_p > r_{p-1}$ such that

$$\sup_{k>r_p} |a_{m_p,n_p,k,l_0}| < \frac{\epsilon}{8}. \tag{8}$$

In view of (6), (7), (8), we have

$$\sup_{r_{p-1}<k\le r_p} |a_{m_p,n_p,k,l_0}| > \epsilon - \frac{\epsilon}{8} - \frac{\epsilon}{8} = \frac{3\epsilon}{4}.$$

We can now find $k_p \in \mathbb{N}$, $r_{p-1} < k_p \le r_p$ such that

$$|a_{m_p,n_p,k_p,l_0}| > \frac{3\epsilon}{4}. \tag{9}$$

Define the sequence $\{x_{k,l}\}$ as follows:

$$x_{k,l} = \begin{cases} 0, & l \ne l_0; \\ 1, & \text{if } l = l_0,\ k = k_p,\ p = 1, 2, \cdots. \end{cases}$$

We note that $\lim_{k+l\to\infty} x_{k,l} = 0$. Now, in view of (6),

$$\left| \sum_{k=1}^{r_{p-1}} a_{m_p,n_p,k,l_0} x_{k,l_0} \right| \le \sup_{1\le k\le r_{p-1}} \left| a_{m_p,n_p,k,l_0} \right| < \frac{\epsilon}{8}; \tag{10}$$

Using (8), we have,

$$\left| \sum_{k=r_p+1}^{\infty} a_{m_p,n_p,k,l_0} x_{k,l_0} \right| \le \sup_{k>r_p} \left| a_{m_p,n_p,k,l_0} \right| < \frac{\epsilon}{8}; \tag{11}$$

and using (9), we get,

$$\left|\sum_{k=r_{p-1}+1}^{r_p} a_{m_p,n_p,k,l_0} x_{k,l_0}\right| = \left|a_{m_p,n_p,k_p,l_0}\right| > \frac{3\epsilon}{4}. \tag{12}$$

Thus

$$\begin{aligned}
|y_{m_p,n_p}| &= \left|\sum_{k=1}^{\infty} a_{m_p,n_p,k,l_0} x_{k,l_0}\right| \\
&\geq \left|\sum_{k=r_{p-1}+1}^{r_p} a_{m_p,n_p,k,l_0} x_{k,l_0}\right| - \left|\sum_{k=1}^{r_{p-1}} a_{m_p,n_p,k,l_0} x_{k,l_0}\right| - \left|\sum_{k=r_p+1}^{\infty} a_{m_p,n_p,k,l_0} x_{k,l_0}\right| \\
&\geq \left|a_{m_p,n_p,k_p,l_0}\right| - \sup_{1\leq k\leq r_{p-1}} \left|a_{m_p,n_p,k,l_0}\right| - \sup_{k>r_p} \left|a_{m_p,n_p,k,l_0}\right| \\
&> \frac{3\epsilon}{4} - \frac{\epsilon}{8} - \frac{\epsilon}{8}, \text{ using (10), (11) and (12)} \\
&= \frac{\epsilon}{2}, \quad p = 1, 2, \cdots.
\end{aligned}$$

Consequently $\lim_{m+n\to\infty} y_{m,n} = 0$ does not hold, which is a contradiction. Thus (c) is necessary. The necessity of (d) follows in a similar fashion.

To establish (e), we shall suppose that (e) does not hold and arrive at a contradiction. Since K is non-trivially valued, there exists $\pi \in K$ such that $0 < \rho = |\pi| < 1$. Choose $m_1 = n_1 = 1$. Using (a), (b), choose $m_2 + n_2 > m_1 + n_1$ such that

$$\sup_{1\leq k+l\leq m_1+n_1} |a_{m_2,n_2,k,l}| < 2, \text{ using (a);}$$

$$\sup_{k+l\geq 1} |a_{m_2,n_2,k,l}| > \left(\frac{2}{\rho}\right)^6;$$

and

$$\sup_{k+l>m_1+n_1} |a_{m_2,n_2,k,l}| < 2^2, \text{ using (b) and Lemma 1, Lemma 2.}$$

It now follows that

$$\sup_{k+l>m_2+n_2} |a_{m_2,n_2,k,l}| < 2^2.$$

Choose $m_3 + n_3 > m_2 + n_2$ such that

$$\sup_{1 \leq k+l \leq m_2+n_2} |a_{m_3,n_3,k,l}| < 2^2;$$

$$\sup_{k+l \geq 1} |a_{m_3,n_3,k,l}| > \left(\frac{2}{\rho}\right)^8;$$

and

$$\sup_{k+l > m_3+n_3} |a_{m_3,n_3,k,l}| < 2^4.$$

Inductively, choose $m_p + n_p > m_{p-1} + n_{p-1}$, such that

$$\sup_{1 \leq k+l \leq m_{p-1}+n_{p-1}} |a_{m_p,n_p,k,l}| < 2^{p-1}; \tag{13}$$

$$\sup_{k+l \geq 1} |a_{m_p,n_p,k,l}| > \left(\frac{2}{\rho}\right)^{2p+2} \tag{14}$$

and

$$\sup_{k+l > m_p+n_p} |a_{m_p,n_p,k,l}| < 2^{2p-2}. \tag{15}$$

Using (13), (14), (15), we have,

$$\sup_{m_{p-1}+n_{p-1} < k+l \leq m_p+n_p} |a_{m_p,n_p,k,l}| > \left(\frac{2}{\rho}\right)^{2p+2} - 2^{2p-2} - 2^{p-1}$$

$$\begin{aligned}
&\geq \left(\frac{2}{\rho}\right)^{2p+2} - \left(\frac{2}{\rho}\right)^{2p-2} - \left(\frac{2}{\rho}\right)^{p-1}, \text{ since } \frac{1}{\rho} > 1 \\
&= \left(\frac{2}{\rho}\right)^{p-1} \left[\left(\frac{2}{\rho}\right)^{p+3} - \left(\frac{2}{\rho}\right)^{p-1} - 1\right] \\
&\geq \left(\frac{2}{\rho}\right)^{p-1} \left[\left(\frac{2}{\rho}\right)^{p+3} - \left(\frac{2}{\rho}\right)^{p-1} - \left(\frac{2}{\rho}\right)^{p-1}\right], \text{ since } \left(\frac{2}{\rho}\right)^{p-1} \geq 1 \\
&= \left(\frac{2}{\rho}\right)^{p-1} \left[\left(\frac{2}{\rho}\right)^{4} \left(\frac{2}{\rho}\right)^{p-1} - 2\left(\frac{2}{\rho}\right)^{p-1}\right], \\
&> \left(\frac{2}{\rho}\right)^{p-1} \left[\left(\frac{2}{\rho}\right)^{4} \left(\frac{2}{\rho}\right)^{p-1} - \left(\frac{2}{\rho}\right)\left(\frac{2}{\rho}\right)^{p-1}\right], \text{ since } \frac{2}{\rho} > 2
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{2}{\rho}\right)^{2p-1}\left[\left(\frac{2}{\rho}\right)^{3}-1\right] \\
&> \left(\frac{2}{\rho}\right)^{2p-1}[2^3-1], \text{ since } \frac{2}{\rho}>2 \\
&= 7\left(\frac{2}{\rho}\right)^{2p-1} \\
&> 4\left(\frac{2}{\rho}\right)^{2p-1} \\
&= \frac{2^{2p+1}}{\rho^{2p-1}} \\
&> \frac{2^{2p+1}}{\rho^{p}}, \text{ since } \frac{1}{\rho}>1. \qquad (16)
\end{aligned}
$$

Thus there exist k_p and l_p, $m_{p-1}+n_{p-1}<k_p+l_p\leq m_p+n_p$ such that

$$
|a_{m_p,n_p,k_p,l_p}| > \frac{2^{2p+1}}{\rho^p}. \qquad (17)
$$

Now, define the sequence $\{x_{k,l}\}$ as follows:

$$
x_{k,l} = \begin{cases} \pi^p, & \text{if } k=k_p, l=l_p, p=1,2,\cdots; \\ 0, & \text{otherwise.} \end{cases}
$$

We note that $\lim\limits_{k+l\to\infty} x_{k,l}=0$. Now,

$$
\begin{aligned}
|y_{m_p,n_p}| &= \left|\sum_{k=1,l=1}^{\infty,\infty} a_{m_p,n_p,k,l}x_{k,l}\right| \\
&\geq \left|\sum_{k+l=(m_{p-1}+n_{p-1})+1}^{m_p+n_p} a_{m_p,n_p,k,l}x_{k,l}\right| \\
&\quad - \left|\sum_{k+l=1}^{m_{p-1}+n_{p-1}} a_{m_p,n_p,k,l}x_{k,l}\right| \\
&\quad - \left|\sum_{k+l=(m_p+n_p)+1}^{\infty} a_{m_p,n_p,k,l}x_{k,l}\right|
\end{aligned}
$$

$$\begin{aligned}
&\geq |a_{m_p,n_p,k_p,l_p}| \times |x_{k_p,l_p}| - \\
&\quad \sup_{1\leq k+l\leq m_{p-1}+n_{p-1}} |a_{m_p,n_p,k,l}| - \sup_{m_p+n_p<k+l<\infty} |a_{m_p,n_p,k,l}| \\
&> \frac{2^{2p+1}}{\rho^p}\rho^p - 2^{2p-2} - 2^{p-1}, \text{ using (13), (15) and (17)} \\
&= 2^{2p+1} - 2^{2p-2} - 2^{p-1} \\
&= 2^{2p-2}(2^3-1) - 2^{p-1} \\
&= 2^{2p-2}(7) - 2^{p-1} \\
&= 2^{p-1}[7\cdot 2^{p-1} - 1] \\
&\geq 2^{p-1}[7\cdot 2^{p-1} - 2^{p-2}] \\
&= 2^{p-1}[2^{p-2}(14-1)] \\
&= 2^{p-1}[13\cdot 2^{p-2}] \\
&= 13\cdot 2^{2p-3}
\end{aligned}$$

i.e., $|y_{m_p,n_p}| > 13\cdot 2^{2p-3}$, $p = 1, 2, \cdots,$
i.e., $\lim_{m+n\to\infty} y_{mn} = 0$ does not hold, which is a contradiction. Thus (e) is necessary.

Proof of Sufficiency.

Let $\lim_{m+n\to\infty} x_{m,n} = x$. Then

$$y_{m,n} - x = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}x_{k,l} - x.$$

From (b) we have

$$\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} + r_{m,n} = 1,$$

where

$$\lim_{m+n\to\infty} r_{m,n} = 0. \tag{18}$$

Hence,

$$y_{m,n} - x = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - x) + r_{m,n}x.$$

Given $\epsilon > 0$, we can choose sufficiently large p and q such that

$$\sup_{k+l>p+q} |x_{k,l} - x| < \frac{\epsilon}{5H}, \tag{19}$$

where $H = \sup_{m,n,k,l \geq 1} |a_{m,n,k,l}|$. Observe that $H > 0$ (from (b)).
Let $L = \sup_{k+l \geq 1} |x_{k,l} - x|$. We now choose $N \in \mathbb{N}$ such that whenever $m+n \geq N$, the following are satisfied:

$$\text{(i)} \quad \sup_{1 \leq k+l \leq p+q} |a_{m,n,k,l}| < \frac{\epsilon}{5pqL}, \text{ using (a);} \tag{20}$$

$$\text{(ii)} \quad \sup_{k \geq 1} |a_{m,n,k,l}| < \frac{\epsilon}{5qL}, \ l = 1, 2, \cdots, q, \text{ using (c);} \tag{21}$$

$$\text{(iii)} \quad \sup_{l \geq 1} |a_{m,n,k,l}| < \frac{\epsilon}{5pL}, \ k = 1, 2, \cdots, p, \text{ using (d);} \tag{22}$$

and

$$\text{(iv)} \quad |r_{m,n}| < \frac{\epsilon}{5|x|}, \text{ from the equation (18).} \tag{23}$$

Whenever $m + n \geq N$, we thus have,

$$\begin{aligned}
|y_{m,n} - x| &= \left| \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - x) + r_{m,n}x \right| \\
&\leq \left| \sum_{k=1,l=1}^{p,q} a_{m,n,k,l}(x_{k,l} - x) \right| + \left| \sum_{k=1,l=q+1}^{p,\infty} a_{m,n,k,l}(x_{k,l} - x) \right| \\
&\quad + \left| \sum_{k=p+1,l=1}^{\infty,q} a_{m,n,k,l}(x_{k,l} - x) \right| + \left| \sum_{k=p+1,l=q+1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - x) \right| \\
&\quad + |r_{m,n}| \, |x| \\
&< \frac{\epsilon}{5pqL} Lpq + \frac{\epsilon}{5pL} Lp + \frac{\epsilon}{5qL} Lq + \frac{\epsilon}{5H} H + \frac{\epsilon}{5|x|} |x| \\
&= \epsilon, \quad \text{using } (19), (20), (21), (22) \text{ and } (23).
\end{aligned}$$

Thus

$$\lim_{m+n \to \infty} y_{m,n} = x,$$

which completes the proof of the theorem.

Nörlund means for simple sequences and series in complete, non-trivially valued, non-archimedean fields were introduced by Srinivasan [9] and studied later in detail by Natarajan (for instance, see [7]). Nörlund means for double sequences and series in classical analysis were introduced by Moore [5]. We now define Nörlund means for double sequences and series in complete, non-trivially valued, non-archimedean fields and apply Theorem 2 for these means.

Definition 4. Given a doubly infinite set of elements $p_{m,n} \in K$, $m,n = 0,1,2,\cdots$, where $p_{0,0} \neq 0$, $|p_{i,j}| < |p_{0,0}|$, $(i,j) \neq (0,0)$, $i,j = 0,1,2,\cdots$, let

$$P_{m,n} = \sum_{i,j=0}^{m,n} p_{i,j}, \qquad m,n = 0,1,2,\cdots.$$

Given any double sequence $\{s_{m,n}\}$ we define

$$\sigma_{m,n} = (N, p_{m,n})(s_{m,n}) = \frac{S_{m,n}}{P_{m,n}} = \frac{\sum_{i,j=0}^{m,n} p_{m-i,n-j} s_{i,j}}{P_{m,n}}, \; m,n = 0,1,2,\cdots.$$

If $\lim_{m+n\to\infty} \sigma_{m,n} = \sigma$, we say that the double sequence $\{s_{m,n}\}$ is summable $(N, p_{m,n})$ to the value σ, written as

$$s_{m,n} \to \sigma(N, p_{m,n}).$$

Any double series $\sum_{m,n} u_{m,n}$ is said to be summable $(N, p_{m,n})$ to the value σ if the double sequence $\{s_{m,n}\}$, where

$$s_{m,n} = \sum_{i,j=0}^{m,n} u_{i,j}, \; m,n = 0,1,2,\cdots,$$

is summable $(N, p_{m,n})$ to the value σ.

Definition 5. Given the Nörlund means $(N, p_{m,n}), (N, q_{m,n})$, we say that they are consistent if

$$s_{m,n} \to \sigma(N, p_{m,n}) \text{ and } s_{m,n} \to \sigma'(N, q_{m,n}) \Rightarrow \sigma = \sigma'.$$

We say that $(N, p_{m,n})$ is included in $(N, q_{m,n})$, written as

$$(N, p_{m,n}) \subseteq (N, q_{m,n}),$$

if

$$s_{m,n} \to \sigma(N, p_{m,n}) \Rightarrow s_{m,n} \to \sigma(N, q_{m,n}).$$

The two methods $(N, p_{m,n}), (N, q_{m,n})$ are said to be equivalent if

$$(N, p_{m,n}) \subseteq (N, q_{m,n}) \text{ and } (N, q_{m,n}) \subseteq (N, p_{m,n}).$$

In view of Theorem 2, it is easy to prove the following result.
Theorem 3. *The necessary and sufficient conditions for the regularity of the Nörlund means* $(N, p_{m,n})$ *are:*

$$\lim_{m+n\to\infty} \sup_{0\le j\le n} |p_{m-i,n-j}| = 0, \quad 0 \le i \le m; \tag{24}$$

$$\lim_{m+n\to\infty} \sup_{0\le i\le m} |p_{m-i,n-j}| = 0, \quad 0 \le j \le n. \tag{25}$$

In the sequel let $(N, p_{m,n}), (N, q_{m,n})$ be two regular Nörlund methods such that each row and each column of the infinite matrices $(p_{m,n}), (q_{m,n})$ is a regular Nörlund mean for simple sequences.
Theorem 4. *Any two such regular Nörlund methods are consistent.*
Proof. Given two Nörlund methods $(N, p_{m,n})$ and $(N, q_{m,n})$, where each row and each column of the infinite matrices $(p_{m,n}), (q_{m,n})$ is a regular Nörlund mean for simple sequences, we define a third method $(N, r_{m,n})$ by the equation

$$r_{m,n} = \sum_{i,j=0}^{m,n} p_{i,j} q_{m-i,n-j}, \ m, n = 0, 1, 2, \cdots.$$

We then readily obtain, for $s = \{s_{m,n}\}$,

$$(N, r_{m,n})(s) = \sum_{i,j=0}^{m,n} \gamma_{m,n,i,j}(N, q_{i,j})(s),$$

where

$$\gamma_{m,n,i,j} = p_{m-i,n-j} Q_{i,j} / \sum_{\mu,\nu=0}^{m,n} p_{m-\mu,n-\nu} Q_{\mu,\nu}.$$

Since $(N, p_{m,n})$ and $(N, q_{m,n})$ are regular, we have,

$$\lim_{m+n\to\infty} \sup_{0\le j\le n} |p_{m-i,n-j}| = 0 = \lim_{m+n\to\infty} \sup_{0\le i\le m} |p_{m-i,n-j}|.$$

It now follows that

$$\lim_{m+n\to\infty} \sup_{0\le j\le n} \gamma_{m,n,i,j} = 0 = \lim_{m+n\to\infty} \sup_{0\le i\le m} \gamma_{m,n,i,j}.$$

Consequently $(N, r_{m,n})$ is regular. The regularity of this transformation enables us to infer that

$$s_{m,n} \to \sigma'(N, q_{m,n}) \Rightarrow s_{m,n} \to \sigma'(N, r_{m,n}).$$

Similarly we can show that

$$s_{m,n} \to \sigma(N, p_{m,n}) \Rightarrow s_{m,n} \to \sigma(N, r_{m,n}).$$

These imply that the two Nörlund methods $(N, p_{m,n})$ and $(N, q_{m,n})$ are consistent, completing the proof of the theorem.

If $(N, p_{m,n})$, $(N, q_{m,n})$ are regular, in view of conditions (24), (25), we have,

$$\begin{aligned} P(x,y) &= \sum P_{m,n} x^m y^n, \\ Q(x,y) &= \sum Q_{m,n} x^m y^n, \\ p(x,y) &= \sum p_{m,n} x^m y^n, \\ q(x,y) &= \sum q_{m,n} x^m y^n, \end{aligned}$$

all converge for $|x|$, $|y| < 1$. The series

$$\begin{aligned} k(x,y) &= \sum k_{m,n} x^m y^n = \frac{q(x,y)}{p(x,y)} = \frac{Q(x,y)}{P(x,y)}, \\ l(x,y) &= \sum l_{m,n} x^m y^n = \frac{p(x,y)}{q(x,y)} = \frac{P(x,y)}{Q(x,y)} \end{aligned}$$

are convergent for $|x|, |y| < 1$ and further

$$\sum_{i,j=0}^{m,n} k_{i,j} p_{m-i,n-j} = q_{m,n}; \quad \sum_{i,j=0}^{m,n} k_{i,j} P_{m-i,n-j} = Q_{m,n}, \tag{26}$$

$$\sum_{i,j=0}^{m,n} l_{i,j} q_{m-i,n-j} = p_{m,n}; \quad \sum_{i,j=0}^{m,n} l_{i,j} Q_{m-i,n-j} = P_{m,n}. \tag{27}$$

Theorem 5. *If* $(N,p_{m,n}),(N,q_{m,n})$ *are regular, then* $(N,p_{m,n}) \subseteq (N,q_{m,n})$ *if and only if* $\lim_{m+n\to\infty} k_{m,n} = 0$.

Proof. Let $s(x,y) = \sum s_{m,n}x^m y^n$. Then for $|x|, |y| < 1$, we have,

$$\begin{aligned}\sum Q_{m,n}(N,q_{m,n})(s)x^m y^n &= \sum \left(\sum_{i,j=0}^{m,n} q_{m-i,n-j}s_{i,j} \right) x^m y^n \\ &= q(x,y)s(x,y);\end{aligned}$$

similarly

$$\sum P_{m,n}(N,p_{m,n})(s)x^m y^n = p(x,y)s(x,y).$$

Thus

$$\sum Q_{m,n}(N,q_{m,n})(s)x^m y^n = \sum k_{m,n}x^m y^n \sum P_{m,n}(N,p_{m,n})(s)x^m y^n$$

which implies that

$$Q_{m,n}(N,q_{m,n})(s) = \sum_{i,j=0}^{m,n} k_{m-i,n-j}P_{i,j}(N,p_{i,j})(s).$$

Hence,

$$(N,q_{m,n})(s) = \sum_{i,j=0}^{m,n} c_{m,n,i,j}(N,p_{i,j})(s), \tag{28}$$

where

$$c_{m,n,i,j} = k_{m-i,n-j}P_{i,j}/Q_{m,n}.$$

If $(N,p_{m,n}) \subseteq (N,q_{m,n})$, $(c_{m,n,i,j})$ is regular and so, by Theorem 2 (a), $\lim_{m+n\to\infty} c_{m,n,0,0} = 0$,

i.e., $$\lim_{m+n\to\infty} \frac{|k_{m,n}|\,|p_{0,0}|}{|q_{0,0}|} = 0,$$

which implies that $\lim_{m+n\to\infty} k_{m,n} = 0$.

Conversely, if $\lim_{m+n\to\infty} k_{m,n} = 0$, we can easily verify that $(c_{m,n,i,j})$ is regular. Consequently, using (28), it follows that $(N,p_{m,n}) \subseteq (N,q_{m,n})$. This completes the proof of the theorem.

Theorem 6, stated below, is an immediate consequence of Theorem 5.

Theorem 6. *If* $(N, p_{m,n})$ *and* $(N, q_{m,n})$ *are regular Nörlund methods, then they are equivalent if and only if* $\lim_{m+n\to\infty} k_{m,n} = 0$ *and* $\lim_{m+n\to\infty} l_{m,n} = 0$.

Remark. For the analogue of Theorem 6 in the classical case, see [5], Theorem III. Theorem 5, Theorem 6, in the case of regular Nörlund means for simple sequences, were established earlier by Natarajan (see[7], Theorem 3, Theorem 4).

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