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Abstract

In this paper, we prove the "relative connectedness" of graphs which satisfy a polynomial volume growth and a Poincaré-type inequality on balls. By "relative connectedness", we mean that every two vertices at distance R from a vertex x can be joined by a path within an annulus $A(x, \alpha^{-1}R, \alpha R)$. We apply this result first to control the behavior of harmonic functions outside a ball and then, in the case of Cayley graph of groups having polynomial volume growth, to obtain a Poincaré-type inequality on the annuli.

1 Introduction

Let Γ be an infinite undirected connected graph and note that we call Γ both the graph and its set of vertices when there is no ambiguity. Let two vertices x and y be neighbors (denoted $x \sim y$) when Γ has an edge between them. We also suppose the graph to be locally uniformly finite, which means:

$$\exists K > 0, \forall x \in \Gamma \quad \#\{y \in \Gamma : y \sim x\} \le K .$$

Let $d(x, y)$ be the natural distance on Γ, that is the minimal number of edges between x and y. Then we denote $\mathcal{S}(x, R) = \{y \in \Gamma : d(x, y) = R\}$ and $B(x, R) = \{y \in \Gamma : d(x, y) < R\}$ the sphere and the ball of radius R, centered at x.

We say that the spheres of Γ (with respect to d) are **relatively connected** if there exist constants $R_0 > 0$ and $\alpha > 1$ such that: for any $R > R_0$ and any vertex $x \in \Gamma$, every two vertices in the sphere $\mathcal{S}(x, R)$ can be joined by a path within the annulus $A(x, \alpha^{-1}R, \alpha R) = B(x, \alpha R) \backslash B(x, \alpha^{-1}R)$.

We suppose that Γ has **polynomial volume growth** of exponent D:

$$\exists C, D > 1, \forall x \in \Gamma, R > 0 \quad C^{-1}R^D \le \#B(x, R) \le CR^D . \qquad (1.1)$$

Our aim is first to prove the relative connectedness of the spheres of Γ (Proposition 2.1) when it satisfies (1.1) and a **D-Poincaré-type inequality** on balls: there is a constant $C(D)$ such that for any function u on Γ,

$$\forall x \in \Gamma, R > 0, \quad \sum_{y \in B(x,R)} |u(y) - u_{B(x,R)}|^D \leq C(D) R^D \sum_{y \in B(x,2R)} |\nabla u(y)|^D, \tag{1.2}$$

where $u_B = \frac{1}{\#B} \sum_{y \in B} u(y)$ for any set B, and

$$\nabla u(x) = \sum_{y \sim x} |u(y) - u(x)| .$$

The method is adapted from [5, Prop. 4.5] to our discrete setting. We will give the whole proof for the sake of completeness. Note that, for instance, the above assumptions are satisfied for Cayley graphs of groups with polynomial growth of exponent $D \geq 2$ [7, Th. 4.1].

Then, we extend to annuli (Theorem 3.1) the elliptic Harnack inequality on balls obtained by Delmotte [2] under (1.2) and the doubling of the volume (implied by (1.1)). From this inequality, we deduce a control on the behavior of harmonic functions outside a finite set (Theorem 3.2), by comparison with the behavior of the Green function. Finally, when Γ is the Cayley graph of a group having polynomial volume growth, we deduce Poincaré-type inequalities on annuli (Theorem 3.3).

2 Relative connectedness of the spheres

Proposition 2.1 *Let Γ be an infinite, locally uniformly finite, undirected connected graph which satisfies (1.1) and (1.2). Then, the spheres of Γ are relatively connected.*

PROOF: First, note that the definition (1.2) of the D-Poincaré-type inequality differs from the one in [5], which is, under the same conditions,

$$\frac{1}{\#B(x,R)} \sum_{y \in B(x,R)} |u(y) - u_{B(x,R)}| \leq C(D) \left(\sum_{y \in B(x,2R)} |\nabla u(y)|^D \right)^{1/D} . \tag{2.3}$$

Indeed, (2.3) is a consequence of (1.2) and Hölder inequality, and will be used below. Actually, (1.2) and (2.3) are equivalent (see [5]).

Let $x \in \Gamma$ and $R \in \mathbb{N}^*$ be large enough. Let x_1, x_2 be two vertices on $\mathcal{S}(x,R)$ and take $\alpha > 21$. If $d(x_1,x_2) \leq \alpha^{-1}R$, then they can be joined within $A(x,\alpha^{-1}R,\alpha R)$. So we only need to consider $d(x_1,x_2) > \alpha^{-1}R$. Let F_1 (resp. F_2) be a path from x_1 (resp. x_2) to $\mathcal{S}(x,R/2)$ of length $R/2$. We suppose there is no path between F_1 and F_2 within $A(x,\alpha^{-1}R,\alpha R)$, and will prove that this becomes impossible for large α.

Let u be a function on $A(x,\alpha^{-1}R,\alpha R)$ such that $u \equiv 0$ on F_1 and $u(y) \geq 1$ for $y \in F_2$. Suppose $u_{B(x,R)} \leq 1/2$ and let $y \in F_2$, then $|u_{B(x,R)} - u(y)| \geq 1/2$.

Let $B_i = B(y, 2^{2-i}R)$ and C be a constant whose value may change from one line to another. Then by (1.1), as $B(x,R) \subset B_1$,

$$\begin{aligned}
|u_{B(x,R)} - u(y)| &\leq |u_{B(x,R)} - u_{B_1}| + |u_{B_1} - u(y)| \\
&\leq |u_{B(x,R)} - u_{B_1}| + \sum_{i=1}^{\infty} |u_{B_{i+1}} - u_{B_i}| \\
&\leq 2C \sum_{i=1}^{\infty} \frac{1}{\# B_i} \sum_{z \in B_i} |u(z) - u_{B_i}|\,.
\end{aligned}$$

So, using (2.3), we obtain

$$\begin{aligned}
|u_{B(x,R)} - u(y)| &\leq C \sum_{i=0}^{\infty} \left(\sum_{z \in B_i} |\nabla u(z)|^D \right)^{1/D} \\
&\leq C R^{1/D} \sup_{r<4R} \left(r^{-1} \sum_{z \in B(y,r)} |\nabla u(z)|^D \right)^{1/D} \sum_{i=0}^{\infty} 2^{(2-i)/D}.
\end{aligned}$$

So, there is a constant C depending on D and the constants from (1.1) and (1.2), such that for each $y \in F_2$, there is a radius $r(y) < 4R$ with

$$\sum_{z \in B(y,r(y))} |\nabla u(z)|^D \geq C r(y)/R\,.$$

By the covering Lemma [5, Th. 14.12], among all the $B(y,r(y))$, there is a sub-collection of balls $B(y_i, r(y_i))$ ($y_i \in F_2$), pointwise disjoint, such that $F_2 \subset \cup_i B(y_i, 5r(y_i))$. As a consequence, by definition of F_2, we must have $\sum_i r(y_i) \geq R/20$. So,

$$\sum_{z \in B(x,5R)} |\nabla u(z)|^D \geq C \sum_i \sum_{z \in B(y_i,r(y_i))} |\nabla u(z)|^D \geq C'$$

If $u_{B(x,R)} \geq 1/2$, then taking F_1 instead of F_2 leads to the same result. Finally

$$\sum_{z \in B(x,5R)} |\nabla u(z)|^D \geq C' . \tag{2.4}$$

As C' does not depend on α, the aim is to choose u such that the left hand side of (2.4) tends to 0 as α goes to infinity.

Let $g(z) = (\ln \alpha)^{-1} d(x,z)^{-1}$ when $z \in A(x, \alpha^{-1}R, \alpha R)$ and 0 elsewhere. We define the function

$$u(z) = \inf_{\gamma_z} \sum_{y \in \gamma_z} g(y) ,$$

where the γ_z's are the paths from z to F_1. Then $u \equiv 0$ on F_1 and, assuming that F_1 and F_2 cannot be joined within $A(x, \alpha^{-1}R, \alpha R)$, any path between them must leave the annulus. If it does so through its exterior boundary, the path should go from $\mathcal{S}(x,R)$ to $\mathcal{S}(x,\alpha R)$ and come back. Likewise, if the path exits the annulus through its interior boundary, it should go from $\mathcal{S}(x,R/2)$ to $\mathcal{S}(x,\alpha^{-1}R)$ and come back. Then, for $y \in F_2$, we obtain

$$u(y) \geq \min \left\{ (\ln \alpha)^{-1} 2 \sum_{k=R}^{\alpha R} k^{-1} , \, (\ln \alpha)^{-1} 2 \sum_{k=\alpha^{-1}R}^{R/2} k^{-1} \right\} \geq 1 .$$

For any $z \in A(x, \alpha^{-1}R, \alpha R)$ and $z' \sim z$, we easily see that

$$|u(z) - u(z')| \leq \max\{g(z), g(z')\} \leq 2g(z) .$$

So, $|\nabla u(z)|^D \leq K 2^D g(z)^D$, where K is a uniform bound for the number of neighbors.

Then, to prove that (2.4) leads to a contradiction it is sufficient to obtain

$$\sup_R \sum_{z \in B(x,5R)} g(z)^D \underset{\alpha \to \infty}{\longrightarrow} 0 . \tag{2.5}$$

Taking the supremum over R gives the uniformity in R of the constant α. Using the definition of g, we obtain

$$\begin{aligned}
\sum_{z \in A(x,\alpha^{-1}R,\alpha R)} g(z)^D &\leq (\ln \alpha)^{-D} \sum_{i=0}^{[2\log_2 \alpha]} \sum_{z \in A(x, 2^i \alpha^{-1} R, 2^{i+1} \alpha^{-1} R)} d(x,z)^{-D} \\
&\leq (\ln \alpha)^{-D} \left([2 \log_2 \alpha] + 1 \right) C 2^D .
\end{aligned}$$

The last inequality uses $\#A(x, 2^i\alpha^{-1}R, 2^{i+1}\alpha^{-1}R) \leq C(2^{i+1}\alpha^{-1}R)^D$. Since $g \equiv 0$ on $B(x, \alpha^{-1}R)$ and $D > 1$, we obtain (2.5) and then the result. □

Remark: In the hypothesis of Proposition 2.1, if we replace D by $p > D$ in the Poincaré-type inequality, the result fails. Indeed, take Γ the graph made of two copies of the two-dimensional lattice joined by a single edge. This graph, with polynomial volume growth of exponent 2, satisfies the above assumptions, but not a 2-Poincaré-type inequality, whereas the relative connectedness clearly fails. Conversely, the p-Poincaré-type inequality with $1 \leq p < D$ is stronger than the D-Poincaré-type inequality.

In the sequel, we will always denote α and R_0 the constants related to the relative connectedness of the spheres. Now, we give a control of the length of the path between two vertices of a sphere of Γ, within the annulus defined in Proposition 2.1.

Proposition 2.2 *Under the hypothesis of Proposition 2.1, there exists a positive constant λ such that, for all $R > R_0$ and all x_1, x_2 on the sphere $S(x, R)$ of Γ, there is a path from x_1 to x_2, within the annulus $A(x, (2\alpha)^{-1}R, 2\alpha R)$, of length at most λR.*

PROOF: By Proposition 2.1, we can pick one path between x_1 and x_2 within $A(x, \alpha^{-1}R, \alpha R)$. Let us take a sequence (v_i) of vertices on this path oriented from x_1 to x_2, by the following rules:

- $v_1 = x_1$,
- Given v_i, v_{i+1} is the last vertex along this path at distance $\lceil \alpha^{-1}R/2 \rceil$ (the lowest greater integer) from v_i,
- We stop at $i = I$ when x_2 belongs to the ball $B(v_I, \lceil \alpha^{-1}R/2 \rceil)$.

Such a sequence exists and is finite. Note that all the balls $B(v_i, \alpha^{-1}R/4)$ are disjoint by construction. Recall the constant C in (1.1). All these balls are included in $B(x, (\alpha+\alpha^{-1}/2)R)$, whose volume is less than $C(\alpha+\alpha^{-1}/2)^D R^D$. On the other hand

$$\sum_{1 \leq i \leq I} \#B(v_i, \alpha^{-1}R/4) \geq IC^{-1}(4\alpha)^{-D}R^D.$$

Hence we must have $I \leq C^2(\alpha + \alpha^{-1}/2)^D(4\alpha)^D$. Note also that all the $B(v_i, \lceil \alpha^{-1}R/2 \rceil)$ are included in $A(x, (2\alpha)^{-1}R, 2\alpha R)$. Then, within this annulus, there are paths from v_i to v_{i+1} of length $\lceil \alpha^{-1}R/2 \rceil$. So, the annulus $A(x, (2\alpha)^{-1}R, 2\alpha R)$ contains a path between x and y of length less than λR with $\lambda = 4C^2(\alpha + \alpha^{-1}/2)^D(4\alpha)^{D-1}$. □

3 Behavior of harmonic functions outside a ball

We always suppose Γ to be an infinite, locally uniformly finite, undirected connected graph which satisfies (1.1) and (1.2). A function f on Γ is called **harmonic** on a set of vertices E if

$$\text{for all } x \in E, \ \ \Delta f(x) = \sum_{y \sim x} (f(x) - f(y)) = 0 \, .$$

Likewise we call a function sub-harmonic (resp. super-harmonic) if $\Delta f(x) \leq 0$ (resp. $\Delta f(x) \geq 0$).

Let u be a non-negative function defined on Γ, harmonic on an annulus $A(z, s, t) = B(z, t) \backslash B(z, s)$. We write $A(s, t)$ for $A(z, s, t)$. Recall the constant α from Proposition 2.1. First we extend to annuli an **elliptic Harnack inequality** on balls.

Theorem 3.1 *Assume that $s > R_0$ ($R_0 > 0$ large enough), and $t/\alpha > 4s\alpha$. Let u be a non-negative function defined on Γ, harmonic on $A(s,t)$. Then u satisfies an elliptic Harnack inequality on the annulus $A(2s\alpha, t/(2\alpha))$, namely:*

$$\max_{A(2s\alpha, t/(2\alpha))} u \leq c(t/s) \min_{A(2s\alpha, t/(2\alpha))} u \, ,$$

where $c(t/s)$ is a constant depending only on t/s and the graph.

PROOF: Let x, y be in $A(2s\alpha, t/(2\alpha))$. By Proposition 2.1, there is a path between x and y within $A(2s, t/2)$. We first take a sequence of vertices v_i ($i = 1$ to I) along this path, as in the proof of Proposition 2.2, with $\lceil s/3 \rceil$ instead of $\lceil \alpha R/2 \rceil$ for the distance between two successive vertices. Then, we obtain that $I \leq c(t/s)^D$ where D is the constant in (1.1).

With (1.1) and (1.2), Delmotte [2] has proved that an harmonic function on a ball $B(x, 2n)$ satisfies an elliptic Harnack inequality on $B(x, n)$ whose constant is independent of x and r:

$$\max_{B(x,n)} u \leq c \min_{B(x,n)} u \, .$$

We can apply this result for u on all the balls $B(v_i, s/6)$, because their doubles $B(v_i, s/3)$ are all included in $A(s,t)$. Moreover we can write the Harnack inequality on the union of all these balls, putting the constant to the power I depending only on t/s. The result follows. □

We have an immediate corollary.

Corollary 3.1 *Let u be a non-negative function defined on Γ, harmonic outside a ball $B(z,N)$ ($z \in \Gamma$ and $N > 0$). There exists $N_0 \geq N$ such that u satisfies an elliptic Harnack inequality on all the dyadic annuli $A(z, 2^n, 2^{n+1})$ ($2^n > N_0$) with the same constant C_h.*

Now, we take the hypothesis of the previous corollary, and we study the asymptotic behavior of such a function u. We begin by an analysis made by Moser in [6] about its oscillations on the spheres. Since the center z is fixed, we denote $|x| = d(z,x)$. We define:

$$M(r) = \max_{|x|=r} u(x) \quad \text{and} \quad m(r) = \min_{|x|=r} u(x)\,.$$

Suppose $M(r)$ has two relative minima, say r_1 and r_2 ($r_2 > r_1$). Then, in the annulus $A(r_1, r_2)$, $M(r)$ attains his maximum inside the domain, so does u. This contradicts the maximum principle. So we are left with two cases: either $M(r)$ has one relative minimum at $\bar{r}$ and so $M(r)$ is increasing for $r > \bar{r}$. Or $M(r)$ has no relative minimum and so it is decreasing.

Likewise we see that $m(r)$ has at most one relative maximum $\underline{r}$. Therefore either $m(r)$ is decreasing for $r > \underline{r}$, or $m(r)$ is increasing. Finally, for r bigger than some r_0, $M(r)$ and $m(r)$ are both monotone, and we have four cases:

- Case 1: $M(r) \nearrow$ and $m(r) \searrow$
- Case 2: $M(r) \searrow$ and $m(r) \nearrow$
- Case 3: $M(r) \nearrow$ and $m(r) \nearrow$
- Case 4: $M(r) \searrow$ and $m(r) \searrow$

We denote $\mathrm{osc}(r) = M(r) - m(r)$, the oscillations of u on the sphere of radius r. The following proposition mimics [6, Th 4,5].

Proposition 3.1 *Let C_h be the constant of Corollary 3.1, then:*

(i) Case 1 *implies that osc(r) tends to infinity at least like a power p of r, and*

$$p = \log_2((C_h+1)/(C_h-1)).$$

(ii) Cases 2, 3, 4 *imply that osc(r) tends to zero. Hence:*

$$\lim_{|x|\to\infty} u(x) = u_\infty \text{ exists.}$$

Remark:

- As the four cases cover all possibilities, (i) and (ii) correspond respectively to u unbounded and u bounded.
- For case 2, the proof of Moser gives also that $\mathrm{osc}(r)$ tends to zero at most like a power $p' = \log_2((C_h-1)/(C_h+1))$ of r.

For cases 2,3,4, we want to bound $|u(x)-u_\infty|$ by an explicit power of $|x|$ which depends only on the constant D.

Theorem 3.2 *Suppose* Γ *has exponent* $D>2$*. Let* u *be a bounded (by say* U*) function on* Γ*, harmonic outside a ball* $B(z,N)$ *(*$z\in\Gamma$ *and* $N>0$*), then:*

$$\lim_{|x|\to\infty} u(x) = u_\infty \text{ exists,}$$

and

$$|u(x)-u_\infty| \le cN^{D-2}U|x|^{2-D},$$

where c *is a positive constant depending only on* Γ.

PROOF: First note that we can restrict ourselves to the non-constant functions. To apply Corollary 3.1, we need to consider only the x's such that $|x|>N_0$ with some N_0 large enough. Actually, we only need $N_0 = C'N$ with C' a constant depending on Γ. Then, when $|x|\le N_0$, we obtain $|u(x)-u_\infty| \le 2U \le 2U(C'N/|x|)^{D-2}$, and so the result still holds. As u is bounded below, we can look at the behavior of $u+U$ which has the same speed of convergence. Hence we can use the previous analysis for non-negative functions.

If u corresponds to case 4, then $-u+U$ corresponds to case 3 and has the same speed of convergence. So we just deal with cases 2, 3.

Let f be a non-negative function on Γ, sub-harmonic outside the ball $B(z,N)$ and vanishing at infinity. We denote $G_z(x)$ the Green's function rooted at z, i.e. the unique solution of $\Delta u = \delta_z$ which vanishes at infinity. Under (1.2) and (1.1) with exponent $D > 2$, there exists a constant c such that (see [3])

$$\text{for every } x \neq z, \quad G_z(x) \leq c|x|^{2-D}.$$

With $N_0 = C'N \in \mathbb{N}$, we denote

$$a = \min_{\mathcal{S}(z,N_0)} G_z(x) \quad \text{and} \quad b = \max_{\mathcal{S}(z,N_0)} f.$$

Then $(b/a)G_z \geq f$ on $\mathcal{S}(z,N_0)$. $(b/a)G_z - f$ is a super-harmonic function therefore we can use the minimum principle. Since $(b/a)G_z - f$ vanishes at infinity and is non-negative on $\mathcal{S}(z,N_0)$, this function remains non-negative out of $B(z,N_0)$. Otherwise it should have a local minimum, which would contradict the minimum principle. Hence for all $x \notin B(z,N_0)$,

$$f(x) \leq (b/a)G_z(x) \leq cU|x|^{2-D}. \tag{3.6}$$

For case 3, $f = u_\infty - u$ is a non-negative function on Γ, sub-harmonic outside the ball $B(z,N)$ and vanishing at infinity. So, by the above argument, the claim follows. For case 2, the sign of $f = u - u_\infty$ may change since $M(r)$ is decreasing and $m(r)$ is increasing, both tending to u_∞ at infinity. We denote $f_1 = \max(0,f)$ and $f_2 = \max(0,-f)$. These are non-negative functions and we easily see that they are sub-harmonic outside $B(z,N)$. As they vanish at infinity, we obtain (3.6) for f_1 and f_2 and the result is also true. □

Example: Let $S^x(k)$ be the simple random walk on Γ started at x, and $\tau_{B(z,N)}$ be the hitting time of $B(z,N)$. Then, the function $u(x) = \mathbf{P}^x\{\tau_{B(z,N)} < \infty\}$ is harmonic outside $B(z,N)$, bounded by 1 and tends to 0 when $|x|$ goes to infinity. Then Theorem 3.2 gives

$$\mathbf{P}^x\{\tau_{B(z,N)} < \infty\} \leq c(|x|/N)^{2-D}.$$

3.1 Poincaré-type inequality on annuli

Here, Γ is the Cayley graph of a finitely generated group G, associated with a symmetric finite generating set S: its vertices are the elements of G, and there is an edge between x and y when $yx^{-1} \in S$. We assume the polynomial volume growth (1.1), then (see [1, 7]) Γ satisfies (1.2), and so the relative connectedness of the spheres. On this kind of graph, the proof of the Poincaré-type inequality on balls relies on the construction of a particular path $\gamma_{x,y}$ between each pair of vertices x, y in a ball $B(z, R)$ ($z \in \Gamma$, $R > 0$). The set of these paths should have the property to pass "not too often" through any edge in $B(z, 2R)$. To obtain the same type of inequality on annuli we will need to define the path $\gamma_{x,y}$ in a way adapted to our setting. Recall the constants α and R_0 from Theorem 2.1. The Poincaré-type inequalities on annuli are the following.

Theorem 3.3 *Let Γ be the Cayley graph of a group with polynomial volume growth of exponent $D > 1$. Then, for all $p \geq 1$, there exists a constant $C(p)$ such that for all $z \in \Gamma$, $s, t \in \mathbb{N}$ ($t \geq s \geq R_0$), and any function f on Γ, we have*

$$\sum_{y \in A(z,s,t)} |f(y) - f_{A(z,s,t)}|^p \leq C(p)(t/s)^{Dp} s^p \sum_{y \in A(z, \alpha^{-1}s/3, \alpha t + 2\alpha^{-1}s/3)} |\nabla f(y)|^p.$$

PROOF: As before, we omit the reference to the center z of the annuli. We cover the annulus $A(\alpha^{-1}s, \alpha t)$ by a minimal number I of balls $B_i = B(v_i, \alpha^{-1}s/6)$ such that all the $B(v_i, \alpha^{-1}s/12)$ are disjoint. We denote $\mathcal{B} = \{B_i : 1 \leq i \leq I\}$ this covering. Then,

$$\sum_{i=1}^{I} \#B(v_i, \alpha^{-1}s/12) \leq \#A(3\alpha^{-1}s/4, \alpha t + \alpha^{-1}s/4),$$

which leads, as in Proposition 2.2, to $I \leq c(t/s)^D$. Let N_i be the number of balls intersected by B_i. Let us denote them $B(v_{i_k}, \alpha^{-1}s/6)$ ($k = 1$ to N_i). Remark that

$$\bigcup_{k=1}^{N_i} B(v_{i_k}, \alpha^{-1}s/6) \subset B(v_i, \alpha^{-1}s/2),$$

and, as all the $B(v_{i_k}, \alpha^{-1}s/12)$ are disjoint,

$$\sum_{k=1}^{N_i} \#B(v_{i_k}, \alpha^{-1}s/12) \leq \#B(v_i, \alpha^{-1}s/2).$$

So the N_i's are bounded by say N, which does not depend on s nor t.

Let $x, y \in A(s,t)$. We construct a path from x to y in the spirit of [4, Ex. 2.3]. By Theorem 2.1, there is a path γ from x to y within $A(\alpha^{-1}s, \alpha t)$. We define an index $i(x)$ associated to x:

$$i(x) = \inf\left\{ i : \ d(v_i, x) = \min_j d(v_j, x) \right\}$$

We remark that $x \in B_{i(x)}$. Likewise, we define $i(y)$.

The path γ intersects a sequence of B_i's denoted $B_{i(k)}$ ($k = 0$ to K) and we can take $i(0) = i(x)$ and $i(K) = i(y)$ (adding $B_{i(x)}$ and $B_{i(y)}$ to the sequence if necessary). We first construct a sequence of vertices (z_n) ($n = 0$ to $2K+1$), from x to y as follows

$$z_{2k} = v_{i(k)} v_{i(x)}^{-1} x \ \text{ and } \ z_{2k+1} = v_{i(k)} v_{i(y)}^{-1} y. \tag{3.7}$$

We denote $3B_{i(k)} = B(v_{i(k)}, \alpha^{-1}s/2)$ and $5B_{i(k)} = B(v_{i(k)}, 5\alpha^{-1}s/6)$. For all k, z_{2k} and z_{2k+1} belong to $B_{i(k)}$ and z_{2k+2} belongs to $3B_{i(k)}$. So, for all n, $d(z_n, z_{n+1}) \le 2\alpha^{-1}s/3$. Let $g(z_n^{-1}z_{n+1})$ be a minimal path from e to $z_n^{-1}z_{n+1}$. We join z_n and z_{n+1} by the translated path $z_n g(z_n^{-1}z_{n+1})$ which stays within $5B_{i([n/2])}$ (where $[\cdot]$ denotes the integer part). Finally, we obtain a path $\gamma_{x,y}$ from x to y within $A' = A(\alpha^{-1}s/6, \alpha t + 5\alpha^{-1}s/6)$, whose length is bounded by $Cs(t/s)^D$ for some non-negative constant C. Indeed,

$$(\text{length of } \gamma_{x,y}) \le \sum_{n=0}^{2K} d(z_n, z_{n+1}) \le 4I\alpha^{-1}s/3 \le Cs(t/s)^D .$$

Now, we prove our Poincaré-type inequality using the same technique as the one on balls (see [1, 7]). We denote $H(a,b) = \#\{\{x,y\} : \ (a,b) \in \gamma_{x,y}\}$, where (a,b) denotes the edge between two neighbors a and b.

$$\begin{aligned}
\sum_{y\in A(s,t)} |f(y) - f_{A(s,t)}|^p &= \sum_{y\in A(s,t)} \left| f(y) - \frac{1}{\#A(s,t)} \sum_{x\in A(s,t)} f(x) \right|^p \\
&\le \sum_{y\in A(s,t)} \left(\frac{1}{\#A(s,t)} \sum_{x\in A(s,t)} |f(y) - f(x)| \right)^p \\
&\le \frac{1}{\#A(s,t)} \sum_{x,y\in A(s,t)} |f(y) - f(x)|^p
\end{aligned}$$

$$
\begin{aligned}
&\leq Ct^{-D} \sum_{x,y \in A(s,t)} (\text{length of } \gamma_{x,y})^{p-1} \sum_{(a,b)\in\gamma_{x,y}} |f(a)-f(b)|^p \\
&\leq Ct^{-D}s^{p-1}(t/s)^{D(p-1)} \sum_{(a,b)\in A'} |f(a)-f(b)|^p H(a,b) \\
&\leq Ct^{-D}s^{p-1}(t/s)^{D(p-1)} \sup_{(a,b)\in A'} H(a,b) \sum_{a\in A'} |\nabla f(a)|^p.
\end{aligned}
$$

To estimate $H(a,b)$, let first write

$$
H(a,b) = \sum_{i=1}^{I}\sum_{j=1}^{I} \#\left\{\{x,y\} \in A(s,t) :\ i(x)=i, i(y)=j, (a,b)\in\gamma_{x,y}\right\}
$$

Now, suppose i and j fixed. We want to bound the number of pair of vertices $\{x,y\}$ such that $i(x)=i$, $i(y)=j$ and $(a,b)\in\gamma_{x,y}$. Saying that the edge (a,b) belongs to $\gamma_{x,y}$ means that (a,b) belongs to one of the paths $z_n g(z_n^{-1}z_{n+1})$ defined above.

We need to obtain a bound K_1 for the number of balls in $\mathcal{B}$ that could contain such z_n, and likewise, a bound K_2 for the number of balls in $\mathcal{B}$ that could contain such z_{n+1}. Then, we want a bound K_3 for the number of elements $h\in\Gamma$ such that $h=z_n^{-1}z_{n+1}$. Finally, we need a bound K_4 for the number of possible z_n. By (3.7), $v_{i([n/2])}, v_{i([(n+1)/2])}, z_n, z_{n+1}, i=i(x)$ and $j=i(y)$ fully determine the pair $\{x,y\}$, so

$$
H(a,b) \leq I^2K_1K_2K_3K_4\,. \tag{3.8}
$$

Since $g(z_n^{-1}z_{n+1})$ has length less than $2\alpha^{-1}s/3$, the vertices a and b should be at distance less than $5\alpha^{-1}s/6$ from the center of any ball in $\mathcal{B}$ that contains such z_n. By definition of the overlapping bound N, for a fixed edge (a,b), there are at most $(N+1)^5$ such balls. So, $K_1 \leq (N+1)^5$ and likewise $K_2 \leq (N+1)^5$. As $z_n^{-1}z_{n+1} \in B(e,\alpha^{-1}s/3)$, there are, at most, $\#B(e,\alpha^{-1}s/3)$ choices for $z_n^{-1}z_{n+1}$, so $K_3 \leq Cs^D$. Once $z_n^{-1}z_{n+1}$ is fixed, knowing that (a,b) is one of the edges of $z_ng(z_n^{-1}z_{n+1})$ leaves $d(z_n,z_{n+1}) \leq 2\alpha^{-1}s/3$ choices for the starting point z_n, so $K_4 \leq 2\alpha^{-1}s/3$.

Finally, plugging these bounds and $I \leq c(t/s)^D$ into (3.8), we obtain

$$
H(a,b) \leq CI^2N^{10}s^{D+1} \leq C(t/s)^{2D}s^{D+1}\,.
$$

Therefore, the result follows. □

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