ANNALES MATHÉMATIQUES BLAISE PASCAL

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Annales mathématiques Blaise Pascal, tome 8, n° 2 (2001), p. 115-136
<http://www.numdam.org/item?id=AMBP_2001__8_2_115_0>

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A Mathematical model for Resin Transfer Molding

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Abstract

The known pseudo-concentration model is a generalization of the classical model of two immiscible fluids when the interface between the two fluids is not a sufficiently regular curve. Besides, it provides efficient and robust numerical methods. The aim of this article is to prove existence of solutions to a mathematical model, based on the pseudo-concentration function model, for the filling of shallow molds with polymers. Numerical methods and numerical simulations with comparison with experimental results have been presented in [6]; [7]. The proposed model is 2-D, the chemical reactivity of the fluid is accounted with the conversion rate satisfying a Kamal–Sourour model, and the temperature is not considered. We prove the existence of a renormalized solution to the mathematical model, and an analysis of time stability is carried out illustrating that the proposed model is suitable for describing the polymer state.

1 Introduction

Resin Transfer Molding (R.T.M.) is a very fast industrial process for direct production of thin components of complex shapes from low viscosity monomers or oligomers. It consists in low pressure injection of a reactive

[1]The first author is partially supported by by a *Action intégrée CNR-CNRS* grant AI 95/0845
[2]The second author is partially supported by *Action intégrée CNR-CNRS* grant AI 95/0845
[3]The third author is partially supported by *Action intégrée CNR-CNRS* grant AI 95/0845

compound in a hot shallow mold. The shallow mold will be represented by a bounded open space Ω of $\mathbb{R}^2$ which is its projection on a mean plan in the third direction, and T will be the time required for a complete filling, Π denoting the domain $]0,T[\times\Omega$. The reader is referred to [6] for a justification of the domain reduction. As time ellipse domain Ω will be split in sub-domains $\Omega_1(t)$ and $\Omega_2(t)$ separated by the interface $\Gamma_l(t)$ and such that $\overline{\Omega_1(t)}\cap\overline{\Omega_2(t)} = \Gamma_l(t)$ and $\Omega = \Omega_1(t)\sqcup\Omega_2(t)\sqcup\Gamma_l(t)$ which can be schematically represented by the following figure:

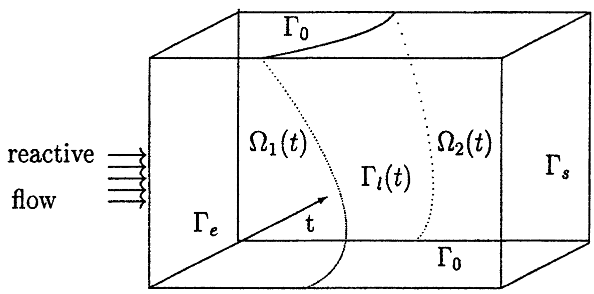

Figure 1

The domain $\Omega_1(t)$ represents the mold fraction filled with injected reactive fluid and $\Omega_2(t)$ is the mold fraction still full of air. It is experimentally well known that in mold process inertia terms are negligible compared to viscous terms in the equations describing fluid flow (see [1] or [6] for a formal justification issued from a perturbation development). In each domain $\Omega_k(t)$, for k=1,2 , the fluid flow speed u_k and the fluid pressure p_k satisfy Stokes law of viscous flows for viscosity η_k for any $t \in [0,T]$. Thus η_1 is the viscosity of the compound being injected and η_2 the viscosity of the air. In this simple mathematical model temperature variations are not taken into account and chemical reactions are represented by the conversion rate α (see [8] for a justification). Note that the function α is defined only in domain $\Omega_1(t)$, and a function of pseudo-concentration S (see [1]) such that $\{(t,y) \in \Pi \ / \ S(t,y) = 1\}$ will be used as characteristic of domain $\Omega_1(t)$ and $\{(t,y) \in \Pi \ / \ S(t,y) =$

0} of domain $\Omega_2(t)$. We apply the result of Nouri and Poupaud [10] to the mathematical model for proving existence. To determine the speed u and pressure p for the R.T.M. process, solutions of a Stokes problem, but also the pseudo-concentration function S, solution of a transport equation, and the conversion rate function α solution of a differential equation have to be considered. We will study firstly the existence and stability of the solution of this differential equation and, the existence of a fixed point solution of the coupled problem. In the case where domains $\Omega_1(t)$ and $\Omega_2(t)$ are smoothly variable (the interface $\Gamma_l(t)$ separating $\Omega_1(t)$ and $\Omega_2(t)$ is a curve), the model proposed becomes equivalent to the classical model with a free boundary. The cornerstone for proving the existence is to deal with renormalized solutions for the transport equation.
The outline of this work is the following. Firstly, we introduce the functional spaces and hypotheses needed. Then in section 2, we describe the proposed mathematical model to be investigated. Some results required for the study of the model are quoted. And in section 3, we show the result of existence of the proposed problem.

1.1 Functional spaces

Let Ω be an open bounded of $\mathbb{R}^2$ the boundary of which is piecewise C^1. The space of the continuous functions of compact support indefinitely differentiable on Ω is designated as $\mathcal{D}(\Omega)$ or $C_0^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$ is the distributions space. If

$$H^1(\Omega) = \{u \;/\; u \in L^2(\Omega) \text{ and } \frac{\partial u}{\partial x_i} \in L^2(\Omega); i = 1,2\}$$

with $\frac{\partial u}{\partial x_i}$ the derivative in distributional sense, the trace application from $H^1(\Omega)$ on $H^{\frac{1}{2}}(\partial\Omega)$ is denoted γ and the following spaces are defined

$$L_0^2(\Omega) = \left\{ p \in L^2(\Omega) / \int_\Omega p\,dx = 0 \right\};$$

$$H_0^1(\Omega) = \{ u \in H^1(\Omega)/(\gamma u)(x) = 0 \forall x \in \partial\Omega\}.$$

Then the space $\mathcal{V} = \{\varphi \in (\mathcal{D}(\Omega))^2 \;/\; div\varphi = 0\}$ is considered, and we write $H = \overline{\mathcal{V}}^{(L^2(\Omega))^2}$ and $V = \overline{\mathcal{V}}^{(H^1(\Omega))^2}$. Then $V = \{ \varphi \in$

$(H_0^1(\Omega))^2/div\varphi = 0\}$ and space H is equipped with the usual scalar product of $(L^2(\Omega))^2$ denoted $(\cdot,\cdot)$. The norm associated to this scalar product is denoted $|\,.\,|$. Space V is equipped with the following scalar product where $D_{ij}u = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ for any $u, v \in V$

$$((u,v)) = \sum_{i,j=1}^{2} \int_\Omega D_{ij}uD_{ij}v\,dx.$$

We denote $\epsilon(u) = (D_{ij}u)_{\substack{1\leq i\leq 2\\ 1\leq j\leq 2}}$ and we also write

$$((u,v)) = \int_\Omega \epsilon(u) : \epsilon(v)\,dx \ .$$

The associated norm of this scalar product is denoted $\|\,.\,\|$ and is equivalent to the norm of $(H^1(\Omega))^2$ as derived from Poincaré inequality

$$\exists\, c(\Omega) \text{ such that } \ |\, u \,|\leq c(\Omega)\ \|\, u\, \|_{H^1(\Omega)} \quad \forall u \in H_0^1(\Omega)\ .$$

Note that $V \subset H \subset V'$ with compact and continuous injections. Moreover for any $1 \leq p \leq \infty$ and for any Hilbert space $\mathcal{B}$, we set

$$L^p(0,T;\mathcal{B}) = \{v : (0,T) \rightarrow \mathcal{B};\ v \text{ is measurable and } \|v(.)\|_{\mathcal{B}} \in L^p(0,T)\}.$$

Then for any u_0 given in $(H^{\frac{1}{2}}(\partial\Omega))^2$ such that $\int_{\partial\Omega} u_0.n\,d\sigma = 0$ where n is the normal to $\partial\Omega$, we consider the following space $\mathcal{U}$

$$\mathcal{U} = \{u \in L^\infty(0,T;(H^1(\Omega))^2)\ /\ div(u) = 0\ ,\ \gamma(u) = u_0\}.$$

If moreover it is assumed that Ω has a Lipschitz continuous boundary and that this one is comprised of three connected components Γ_e, Γ_0 and Γ_s, then it is possible to define the following space $\mathcal{T}$:

$$\mathcal{T} = \{\varphi \in \mathcal{D}(\mathbb{R}^3)/\varphi(t,x) = 0, \forall x \in \Gamma_s, \forall t \in (0,T)\}.$$

The polymerization reaction is represented by a function $f : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ verifying the hypotheses:

H_1) $\exists\ 1 < p$ and $\exists\ 1 < k$, odd such that the function $z \longmapsto f(t,z) \in C^{k+p}(\mathbb{R}_+)$ $\forall t \in \mathbb{R}_+$. Moreover, $\exists\ 0 < \alpha_0 \leq 1$ such that $f(t,0) = f(t,\alpha_0) = 0\ \forall t \in \mathbb{R}_+$; $0 \leq f(t,z)$ if $0 \leq z \leq \alpha_0$ and $f(t,z) \leq 0$ if $\alpha_0 \leq z$ $\forall t \in \mathbb{R}_+$. Furthermore, $\partial_z^k f(.,\alpha_0) \in L^1(0,T)$ with $\partial_z^k f(t,\alpha_0) \leq \mu < 0$ on $[0,T]$ and $\partial_z^m f(t,\alpha_0) = 0\ \forall m \in \{1,2,...,k-1\}$.

H_2) f is continuous and $\partial_z f$ is bounded on bounded subsets of $\mathbb{R}_+ \times \mathbb{R}_+$

A function $U_0 \in L^\infty(0,+\infty;(C_0^1(\mathbb{R}^2))^2)$ verifying the following hypothesis will also be needed:

$$H_3) \begin{cases} divU_0 = 0; \\ U_0.n < 0 \text{ if } x \in \Gamma_e \text{ and } t \geq 0; \\ U_0.n = 0 \text{ if } x \in \Gamma_0 \text{ and } t \geq 0; \\ U_0.n > 0 \text{ if } x \in \Gamma_s \text{ and } t \geq 0. \end{cases}$$

Hypotheses $(H_1), (H_2)$ are related to the function f of the conversion rate and are limiting conditions but are satisfied by the Kamal–Sourour model for instance. The function U_0 is needed for extending the inside and outside speeds on Γ_e and on Γ_s which is crucial for defining renormalized solutions to the transport problem with a trace on Γ_s.

2 Mathematical Model

The mathematical model we propose for representing the process of filling the mold leads to the coupled following problem:

for u_0 , β_0 , S_0 and f given, find (u,p,S,α) verifying:

$$\begin{cases} div_x(\eta(t,x)\epsilon(u(t,x))) &= \nabla p \text{ in } \Omega; \\ div_x(u) &= 0 \text{ in } \Omega; \\ \gamma(u) &= u_0 \text{ on } \partial\Omega; \end{cases} \tag{2.1}$$

$$\begin{cases} \partial_t S(t,x) + (\vec{u}.\vec{\nabla}_x)S(t,x) &= 0\forall(t,x) \in \Pi; \\ S(0,x) &= S_0(x)\forall x \in \Omega; \\ S(t,x) &= 1\forall t \in]0,T[;\ \forall x \in \Gamma_e; \end{cases} \tag{2.2}$$

$$\begin{cases} \partial_t\alpha = f(t,\alpha)S(t,x) \quad t \in]0,T]; \\ \\ \alpha(0,x) = \beta_0. \end{cases} \tag{2.3}$$

Let $0 < m \leq M$ be two constants, and let $g \in C^0([0,1] \times \mathbb{R}_+; [m, M])$ be given such that

$$g(u,v) = \begin{cases} \eta_2 \text{ if } u = 0; \\ \eta_1 \text{ if } u = 1 \end{cases} \text{ for all } v \in \mathbb{R}_+. \tag{2.4}$$

The viscosity of the mixture in domain Π is described with the function $\eta : \Pi \longrightarrow [m, M]$ defined by $\eta(t,x) = g(S(t,x), \alpha(t))$. By setting $g_1(.) = g(1,.)$ we will assume that g_1 is a bounded increasing function. This last hypothesis expresses that during the polymerization process, η_1 the polymer viscosity is a continuous increasing function with respect to α such that $\eta_1(t) = g(1, \alpha(t))$ for any $t \in [0,T]$. The air viscosity η_2 is a constant. Please remark that similar problems to problem (1)-(2) settled in $\mathbb{R}^2$ are considered as quasi-stationary problems in [10]. The main result we obtained for solutions (see (3.32) for a definition) to Problem (1) – (3) is given by :

Theorem 1 *Let Ω be a Lipschitz continuous open bounded subset included in $\mathbb{R}^2$, the boundary of which consists in three connected parts Γ_e, Γ_0 and Γ_s such that there exist two open sets Ω_e and Ω_s verifying $\Gamma_e \subset \Omega_e$, $\Gamma_s \subset \Omega_s$ and $\Omega_e \cap \Omega_s = \emptyset$.*
Let $U_0 \in L^\infty(0, +\infty; C_0^1(\mathbb{R}^2))$ be a function satisfying (H_3). Let $S_0 \in L^\infty(\Omega; \{0,1\})$; $0 < \beta_0$ be given and let $f : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a function verifying (H_1) and (H_2). Then Problem (1) ; (2) and (3) has at least a solution (u,p,S,α) in $L^\infty(0,T; (H^1(\Omega))^2) \times L^\infty(0,T; (L_0^2(\Omega)) \times L^\infty[(0,T) \times \Omega] \times C^0(0,T; L^\infty(\Omega))$.

The proof will be given in section 3, and requires to solve independently Problem (1), Problem (2) and Problem (3) which is the issue of the following sections.

2.1 Stokes Problem (1)

In this subsection, we give a formulation of Stokes problem over the complete domain Ω. Assume η to be a continuous function with respect to time, positive bounded from above and from below, measurable with respect to x and let t to be fixed. While t is fixed we still denote the function $\eta(t, \cdot)$ by η.

For given u_0 find $u \in (H^1(\Omega))^2$ and $p \in L_0^2(\Omega)$ verifying:

$$\begin{cases} div(\eta\epsilon(u)) &= \nabla p \text{ in } (\mathcal{D}'(\Omega))^2; \\ div(u) &= 0 \text{ a. e. in } \Omega; \\ \gamma(u) &= u_0 \text{ in } (H^{\frac{1}{2}}(\partial\Omega))^2 \end{cases} \tag{2.5}$$

To give a weak formulation of problem (2.5) we have to extend the boundary conditions in the domain Ω.

Lemma 1 *The following proposal are equivalent:*

i) $u_0 \in (H^{\frac{1}{2}}(\partial\Omega))^2$ *and* $\int_{\partial\Omega} u_0.n\, d\sigma = 0.$

ii) There exists $U_0 \in (H^1(\Omega))^2$ *such that* $\gamma(U_0) = u_0$ *and* $divU_0 = 0$.

Proposition 2.1 *Let* $\eta : \Omega \longrightarrow \mathbb{R}$ *be a measurable function such that there exist two* m *and* M *reals verifying* $0 < m \leq \eta \leq M$. *Let* $u_0 \in (H^{\frac{1}{2}}(\partial\Omega))^2$ *such that* $\int_{\partial\Omega} u_0.n\, d\sigma = 0$ *and* U_0 *as defined by lemma 1 Then problem (2.5) is equivalent to the following*

$$\begin{cases} \text{Find } v \in V \text{ such that } v = u - U_0 \text{ and for all } w \in V; \\ \int_\Omega \eta\epsilon(v) : \epsilon(w)\, dx \; + \; \int_\Omega \eta\epsilon(U_0) : \epsilon(w)\, dx = 0. \end{cases} \tag{2.6}$$

For the existence of solutions to problem (2.6) we have the classical following result:

Proposition 2.2 *Within the hypotheses of proposition 2.1, problem (2.6) has a unique solution* v *such that* $u = v + U_0$ *is the unique solution to (2.5). Moreover we have:*

$$||u|| \leq (\frac{2M}{m\, c(\Omega)} + 1)||U_0||$$

where $c(\Omega)$ *is the poincaré constant.*

Now we deal with a function η which is a measurable function defined in $\Pi = (0,T) \times \Omega$ such that $0 < m \leq \eta \leq M$ a. e. in Π. Then a weak solution to Problem (1) is a couple $(u,p) \in \mathcal{U} \times L^\infty(0,T; L_0^2(\Omega))$ such that $v = u - U_0$ verifying for all $w \in V$

$$\int_\Omega \eta\epsilon(v) : \epsilon(w)\, dx \; + \; \int_\Omega \eta\epsilon(U_0) : \epsilon(w)\, dx = 0 \quad \text{a. e.} t \in (0,T). \tag{2.7}$$

For $0 < n$, let ρ_n be the classical mollifier function, we defined the regularized function η^n by convolution

$$\forall t \in [0,T],\ \eta^n(t,x) = (\eta(\cdot,x) * \rho_n)(t) \quad \text{for a. e.} x \in \Omega.$$

We have, $\eta^n \to \eta$ in $L^1(\Pi)$ when n goes to infinity. We define u^n as the solution to problem (2.6) when η is replaced by η^n. Propositions (2.1), (2.2) apply, and for every $t \in [0,T]$ we get, the existence and uniqueness and a uniform bound with respect to n for u^n in $L^\infty(0,T;H^1(\Omega))$ verifying

$$\begin{aligned}&\int_0^T \int_\Omega \varphi(t)\eta^n(t,x)\epsilon(u^n(t,x)) : \epsilon(w(x))\,dtdx = \\ &\int_0^T \int_\Omega f(t,x)\varphi(t)w(x)\,dtdx,\ \forall\varphi \in \mathcal{D}((0,T)); \forall w \in \mathcal{V}.\end{aligned} \tag{2.8}$$

Since u^n is uniformly bounded with respect to n in $L^\infty(0,T;H^1(\Omega))$, we deduce the existence of $u \in L^\infty(0,T;H^1(\Omega))$ such that, up to a subsequence $u^n \to u$ in $L^\infty(0,T;H^1(\Omega))$ weak star. Since $\varphi\eta^n(t,x)\epsilon(w(x)) \to \varphi\eta(t,x)\epsilon(w(x))$ in $L^1(0,T;L^2(\Omega))$, we pass to the limit in the previous equation and we get

$$\int_0^T \varphi(t)\left(\int_\Omega \eta(t,x)\epsilon(u(t,x)) : \epsilon(w(x)) - fw(x)\,dx\right)dt = 0$$

for all $\varphi \in \mathcal{D}((0,T))$ and $w \in \mathcal{V}$. From this we get that (u,P) is a weak solution to Problem (1). The uniqueness of weak solution is straightforward from its definition. Thus we deduce the existence of $u \in L^\infty(0,T;H^1(\Omega))$ such that, up to a subsequence $u^n \to u$ in $L^\infty(0,T;H^1(\Omega))$ weak star.
We have proved that propositions (2.1), (2.2) are still valid when problem (2.6) is replaced by problem (2.7) for which $t \in (0,T)$ and equations (2.5) by equations (2.1).

Remark 2.1 *If the domains Ω_k have Lipschitz continuous boundaries, defining now the domains*

$$\Pi_1 = \{(t,x) \in \Pi/x \in \Omega_1(t)\};$$

$$\Pi_2 = \{(t,x) \in \Pi/x \in \Omega_2(t)\};$$

$$L = \{(t,x) \in \Pi/x \in \Gamma_l(t)\},$$

we have $\Pi = L \cup \Pi_1 \cup \Pi_2$. For $1 \leq k \leq 2$, we set:

$$V_0(\Pi_k) = \{p\,|_{\Pi_k} /; p \in L^2(0,T;L^2(\Omega))\};$$

$$V_1(\Pi_k) = \{u\,|_{\Pi_k} \in L^\infty(0,T;(H^1(\Omega))^2)\}.$$

The weak formulation of the Stokes problem, over the complete domain Ω is equivalent to the weak formulation of the Stokes problem in each sub-domains $\Omega_k(1 \leq k \leq 2)$ with matching conditions on the interface Γ_l(cf [10]). In each domain Π_k $((1 \leq k \leq 2))$, the speed $u_k \in V_1(\Pi_k)$ and pressure $p_k \in V_0(\Pi_k)$ verify for a. e. t in the interval $[0,T]$:

$$\left\{\begin{array}{lcl} div(\eta_k(t,x)\epsilon(u_k(t,x))) & = & \nabla p_k\ (t,x) \in \Omega_k(t); \\ div(u_k) & = & 0(t,x) \in \Omega_k(t); \\ \gamma(u_k) & = & u_0(t,x) \in \partial\Omega_k(t) \setminus \Gamma_l(t). \end{array}\right. \tag{2.9}$$

And we have also the linking conditions at the interface $\Gamma_l(t)$. The constraint tensor σ_k is defined by $\sigma_k = \eta_k\epsilon(u_k) - p_k Id$ and we set $U_k = (1,u_k)^T$ and $\Sigma_k = (1,\sigma_k)^T$ then the continuity conditions for the constraints tensor, the speed trace continuity and immiscibility over L are:

$$\Sigma_1.N = \Sigma_2.N \text{ over } L; \tag{2.10}$$

$$u_1 = u_2 \text{ over } \Gamma_l; \tag{2.11}$$

$$U_1.N = U_2.N = 0 \text{ over } L, \tag{2.12}$$

where N is normal to L defined by $N = N_1 = -N_2$ and N_k for $k = 1,2$, is the normal to $\partial\Pi_k \setminus \partial\Omega$.

2.2 The transport problem (2)

In the following, Ω is assumed to be an open domain bounded and that boundary is made of three connected Lipschitz continous components Γ_0 , Γ_e and Γ_s. Let $U_0 \in L^\infty(0,+\infty,(C_0^1(\mathbb{R}^2))^2)$ be a function verifying the hypotheses given by (H_3). The transport problem is defined by: for given $u \in \mathcal{U}$, $S_0 \in L^\infty(\Omega)$, find $S \in L^\infty(\Pi)$ verifying:

$$\left\{\begin{array}{lcl} \partial_t S(t,x) + (\vec{u}.\vec{\nabla}_x)S(t,x) & = & 0\forall(t,x) \in \Pi; \\ S(0,x) & = & S_0(x)\forall x \in \Omega; \\ S(t,x) & = & 1\forall t \in]0,T[;\ \forall x \in \Gamma_e. \end{array}\right. \tag{2.13}$$

Definition 1 If u $\in \mathcal{U}$, then a weak solution of (2.13) is a function $S \in L^\infty(\Pi)$ such that:

$$\int_0^T \int_\Omega S(t,x)\left(\partial_t\varphi + \vec{u}.\vec{\nabla}\varphi\right)(t,x)\,dx\,dt = \int_0^T \int_{\Gamma_e} \varphi(t,x)|U_0.n|\,d\sigma(x)dt -$$

$$\int_\Omega S_0(x)\varphi(0,x)\,dx, \text{ for any } \varphi \in \{\delta \in \mathcal{D}(\mathbb{R}^3)/\delta(t,x) = 0\ \forall x \in \Gamma_s;\ \forall t \in [0,T]\}.$$

Definition 1 If $u \in \mathcal{U}$ then a renormalized solution of (2.13) is a function $S \in L^\infty(\Pi)$ such that for any function $\beta \in C^1(\mathbb{R})$; $\beta(S)$ is a weak solution of (2.13) with $\beta(S_0)$ and $\beta(1)$ as initial and boundary conditions.

Theorem 2 *Let Ω be an open bounded domain with Lipschitz continuous boundary of $\mathbb{R}^2$ and that its boundary consists in three connected components Γ_0, Γ_e and Γ_s, and let $S_0 \in L^\infty(\Omega)$ and $u \in \mathcal{U}$. Then the transport problem (2.13) has a unique renormalized solution S with a trace S_{Γ_s} over Γ_s which belongs to $L^\infty((0,T) \times \Gamma_s)$ and verifying:*

$$\int_0^T \int_\Omega S(t,x)(\partial_t \varphi + (u.\nabla)\varphi)(t,x)\,dxdt + \int_0^T \int_{\Gamma_e} \varphi(t,x)|U_0.n|\,d\sigma(x)dt =$$

$$\int_0^T \int_{\Gamma_s} S_{\Gamma_s}(t,x)\varphi(t,x)|U_0.n|\,d\sigma(x)dt - \int_\Omega S_0(x)\varphi(0,x)\,dx. \tag{2.14}$$

for any $\varphi \in \mathcal{D}([0,T] \times \mathbb{R}^2)$. Moreover

$$\begin{aligned}\int_0^T \int_\Omega S^2(t,x)\,dx \;+\; \int_0^T (T-t) \int_{\Gamma_s} S^2_{\Gamma_s}(t,x)|U_0.n|\,d\sigma(x)dt = \\ \int_0^T (T-t) \int_{\Gamma_e} |U_0.n|\,d\sigma(x)dt \;+\; T\int_\Omega S_0^2(x)\,dx,\end{aligned} \tag{2.15}$$

and $S \in L^\infty(\Pi; \{0,1\})$ if $S_0 \in L^\infty(\Omega; \{0,1\})$. □

The method used for proving existence of solutions to Transport problem consists in introducing solutions to the problem extended to $\mathbb{R}^2$ and applying the results of Diperna–Lions to the transport equation. For a proof of theorem (2) the reader is referred to [10] or to [9].

Remark 2.2 *It is important to consider the renormalized solutions to get uniqueness for Equation (2.13) (which will be crucial to obtain strong convergence) and to get solutions with values in $\{0,1\}$. Renormalized solutions have a trace in $L^2(\Gamma_s; |U_0.n|d\sigma)$; which is not valid for the weak solutions of problem (2.13) (cf [3]).* □

Remark 2.3 *When the interface Γ_l is a curve, one can prove that Transport problem(2.13) and immiscibility condition $U.N = 0$ over Γ_l are equivalent (see [10]). When the boundaries of domains Ω_k are not Lipschitz continuous, Transport problem(2.13) is a generalization of the immiscibility condition.* □

2.3 The polymerization Problem (3)

It is assumed that the polymerization rate of the monomer is derived from empirical model such as Kamal–Sourour [8], i.e. there exists a function f : $\mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfying hypotheses (H_1) and (H_2) and such that for $\beta_0 \in \mathbb{R}_+^*$ given the function $\alpha(.) : [0,T] \subset \Pi \longrightarrow \mathbb{R}_+$ verifies

$$\begin{cases} \frac{d}{dt}\alpha &= f(t,\alpha);\ t \in]0,T]; \\ \alpha(0) &= \beta_0. \end{cases} \tag{2.16}$$

The function α does not depend on x. This means that the polymerization reaction starts at the beginning of the filling process. We study the stability and the asymptotic stability of the solutions to Problem (2.16). Stability of the solutions to mathematical model is important for asserting that this model represents correctly the physical process of RTM.

The results which are going to be derived for Problem (2.16) still remain valid for Problem (3), since function $S \in L^\infty$, thus the function $z \mapsto f(\cdot,z)S(\cdot,x)$ is continuous and locally Lipschitz continuous with respect to z independently of time and space. So the required hypotheses of Carathéodory Theorem are satisfied, the fact that $t,z,x \mapsto f(t,z)S(t,x)$ is not continuous with respect to time is not a restriction.

For Problem (3) we use the theorem of carathéodory (see remark p. 60 of [5] or [4]) and we get the existence. For the uniqueness we use Theorem 3.6 p. 64 of [5] which is still valid in the case where the function f, the right hand side of the ordinary differential equation is not a continuous function with respect to time (in the proof of the theorem 3.6 of [5], integrate the equation verified by φ). We have existence and uniqueness of $\alpha \in L^\infty(]0,T_x[\times\Omega, \mathbb{R}_+^*)$, continuous with respect to time, a weak solution to Problem (3) defined by

$$\alpha(t,x) = \beta_0 + \int_0^t f(s,\alpha(s,x))S(s,x)\,ds \quad \forall t \in [0,T_x] \text{ and a. e. } x \in \Omega. \tag{2.17}$$

Finally, accounting for the asymptotic stability results of the lemma 2.3, we get that α defined by (2.17) verifies $\alpha \in L^\infty(\Pi, \mathbb{R}_+^*)$.

In order to be more readable, the results concerning the stability are not given with the function $f(t,z)S(t,x)$ but only with the function $f(t,z)$. Nevertheless, these results are still valid for the function$f(t,z)S(t,x)$. The main result of this sub-section is given by:

Proposition 2.3 *Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a function verifying hypotheses $(H1)$ and (H_2) and let β_0 a non negative constant given. Then Problem (2.16) has a unique maximal solution α continuous with respect to time. The function α called a weak solution of Problem (2.16) is defined by*

$$\alpha(t) = \beta_0 + \int_0^t f(s, \alpha(s))\, ds \quad \text{for } t \in [0, T]. \tag{2.18}$$

Moreover, α_0 the solution at equilibrium of the differential equation (2.16) over $[0, +\infty[$, as defined by hypothesis (H_1), is asymptotically stable, whereas 0 the other solution at equilibrium of the equation (2.16) is not asymptotically stable.

Proof. The theorem (3.1) of [5] brings us to assert that Problem (2.16) has a unique continuous in time solution over a given interval $[0, t_1]$ for any initial condition β_0 at time 0.

In order to demonstrate that this solution is maximal and study its stability and asymptotic stability, we start by the verification of these properties for the solution of the following associated problem to (2.16).

$$\begin{cases} \frac{d}{dt}\alpha_l = \dfrac{\partial_z^k f(t, \alpha_0)}{k!}(\alpha_l - \alpha_0)^k; \\ \alpha_l(0) = \beta_0. \end{cases} \tag{2.19}$$

With $y = \alpha_l - \alpha_0$ and $y_0 = \beta_0 - \alpha_0$ problem (2.19) becomes:

$$\begin{cases} \frac{d}{dt}y = \dfrac{\partial_z^k f(t, \alpha_0)}{k!} y^k; \\ y(0) = y_0. \end{cases} \tag{2.20}$$

Lemma 2 *If $f : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a function verifying hypotheses $(H_1), (H_2)$ and $y_0 \in \mathbb{R}_+^*$ the initial condition, Problem (2.20) has a unique solution which is maximal y and the solution at equilibrium 0 is asymptotically stable. Moreover if $0 < y_0$, then $0 < y \leq y_0$ and if $y_0 < 0$ then $y_0 \leq y < 0$.*

Proof. If $0 < y_0$ the solution of (2.20) y is given by:

$$y(t) = \left[\frac{1}{[y_0]^{1-k} + (1-k) \displaystyle\int_0^t \frac{\partial_z^k f(s, \alpha_0)}{k!}\, ds} \right]^{\frac{1}{k-1}}$$

for any $t \geq 0$. And if $y_0 < 0$ the solution of (2.20) y is given by:

$$y(t) = -\left[\frac{1}{[y_0]^{1-k} + (1-k)\displaystyle\int_0^t \frac{\partial_z^k f(s,\alpha_0)}{k!}\,ds}\right]^{\frac{1}{k-1}}$$

for any $t \geq 0$.

□

Now it is possible to give a similar result for the non-linear problem(2.16).

Lemma 3 *If $f : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a function verifying hypotheses (H_1), (H_2) and if $\beta_0 \in \mathbb{R}_+^*$ are given then the Problem (2.16) has a unique maximal solution α over $[0, +\infty[$. Moreover if $\alpha_0 < \beta_0$ then $\alpha_0 < \alpha \leq \beta_0$ and if $\beta_0 < \alpha_0$ then $\beta_0 \leq \alpha < \alpha_0$.*

Proof. If α is the solution of Problem (2.16) that the proposition (2.3) confirms the existence and uniqueness over $[0,T]$, as α is continuous, that shows that there exists $T_1 \in]0,T[$ such that $\alpha_0 < \alpha$ and due to the sign of f i. e. (H2) $\alpha \leq \beta_0$ for any $t \in [0,T_1]$ if β_0 belongs to a neighborhood of α_0 with $\alpha_0 < \beta_0$ and $\alpha < \alpha_0$ for any $t \in [0,T_1]$ if β_0 belongs to a neighborhood of α_0 with $\beta_0 < \alpha_0$.

Now considering the stable point 0. If we take as initial condition β_0 close to 0 with $0 < \beta_0$ then $\alpha(t) \leq \alpha_0$ for any $t \in [0,t_1]$. Effectively supposing that there exists a real T_1 such that $T_1 \in]0,t_1[$ and $\alpha_0 < \alpha(T_1)$ then there exists a real δ such that $\alpha(T_1) > \delta > \alpha_0 > \beta_0$ and there exists a real $t_2 \in]0,T_1[$ such that $\alpha(t_2) = \delta$. Thus α is strictly increasing after a given time $t_3 \in [t_2,T_1[$. Considering the differential equation:

$$\begin{cases} \frac{d}{dt}z = f(t,z) & \text{over } (t_2,T_1]; \\ z(t_2) = \delta, \end{cases} \tag{2.21}$$

where α is strictly increasing solution over $[t_3,T_1]$. We have for all $\bar{t}$ and $\underline{t}$ such that $t_3 \leq \underline{t} \leq \bar{t}$

$$0 \leq z(\bar{t}) - z(\underline{t}) = \int_{\underline{t}}^{\bar{t}} f(s,z(s))\,ds \tag{2.22}$$

which is contradictory with (H2) $(f(\cdot,z) \leq 0 \;\alpha_0 \leq z)$. Since we have established that $\alpha \in [0, \max(\alpha_0,\beta_0)]$ we deduce that the solution can't blow up, so it is a maximal solution.

□

The stability result for Problem (2.16) can be written as:

Lemma 4 *Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a function verifying hypotheses (H_1), (H_2) and β_0 a non negative constant, then the solution at equilibrium α_0 of the differential equation (2.16) over $[0,+\infty[$ is asymptotically stable whereas the solution at equilibrium 0 of the differential equation (2.16) over $[0,+\infty[$ is not asymptotically stable.*

Proof. If $0 < \epsilon$ then for any initial condition β_0 such that $|\beta_0 - \alpha_0| < \epsilon$ the solution α of problem (2.16) given by lemma 3 with the initial condition β_0 verifies $|\alpha(t) - \alpha_0| < \epsilon$ for any $t \in [0,+\infty[$. Thus the equilibrium solution $\alpha = \alpha_0$ of (2.16) over $[0,+\infty[$ is stable. Moreover as $f(t,\alpha)$ in the neightbourhood of α_0 is written as:

$$f(t,\alpha) = [\frac{\partial_z^k f(t,\alpha_0)}{k!} + (\alpha-\alpha_0)^p R(t,\alpha,\alpha_0)](\alpha-\alpha_0)^k \; 1 \leq p,$$

if $0 < \nu$ is fixed it can be noted that if $\sup\limits_{\substack{0 \leq s \\ |\alpha-\alpha_0| \leq \nu}} |R(s,\alpha,\alpha_0)| = 0$

Problem (2.16) is reduced to problem (2.20).

If we set $\epsilon_0 = \left[\dfrac{\mu}{2 \sup\limits_{\substack{0\leq s \\ |\alpha-\alpha_0|\leq\nu}} |R(s,\alpha,\alpha_0)|}\right]^{\frac{1}{p}}$ (where μ is defined at hypothesis $(H1)$ by $\partial_z^k f(t,\alpha_0) \leq \mu < 0$) then there exists a real $0 < \delta$ ($\delta = \min(\nu, \epsilon_0)$ from the proof of lemma 3) such that if $|\beta_0 - \alpha_0| < \delta$, then α verifies $|\alpha(t) - \alpha_0| < \delta$ for any $t \in [0,+\infty[$. And thus for any condition $0 < \beta_0$ such that $|\beta_0 - \alpha_0| < \delta$ we get:

$$\frac{\partial_z^k f(t,\alpha_0)}{k!} + (\alpha-\alpha_0)^p R(t,\alpha,\alpha_0) \leq \frac{\mu}{2}$$

for any $t \in [0,+\infty[$. For a given β_0 we set $y(t) = \alpha(t) - \alpha_0$, then (2.16) can be written:

$$\begin{cases} \frac{d}{dt}y = \Big[\dfrac{\partial_z^k f(t,\alpha_0)}{k!} + y^p R(t,\alpha,\alpha_0)\Big] y^k; \\ y(0) = \beta_0 - \alpha_0 = y_0. \end{cases} \tag{2.23}$$

The solution of (2.23) is given by:

$$y(t) = \left[\frac{1}{[y_0]^{1-k} + (1-k) \int_0^t [\frac{\partial_z^k f(s,\alpha_0)}{k!} + (y(s))^p R(s,\alpha,\alpha_0)]\, ds} \right]^{\frac{1}{k-1}}$$

$\forall t \in [0, +\infty[$.

As moreover $\frac{(1-k)\mu}{2}(t-0) \leq (1-k) \int_0^t [\frac{\partial_z^k f(s,\alpha_0)}{k!} + y(s))^p R(s,\alpha,\alpha_0)]\, ds$ for any $t \in [0, +\infty[$, the result is that $\lim_{t \to +\infty} \alpha(t) = \alpha_0$ and the solution at equilibrium $\alpha = \alpha_0$ is asymptotically stable for any initial condition $0 < \beta_0$ in a neighborhood of α_0.

If now we set $0 < \beta_0$ initial condition close to 0 then just as the solution α of (2.16) over $[0, +\infty[$ verifies $\alpha(t) \leq \alpha_0$; for any $t \in [0, +\infty[$ we have $\lim_{t \to +\infty} \alpha(t)$ exists and verifies $0 < \beta_0 \leq \lim_{t \to +\infty} \alpha(t) \leq \alpha_0$. Then $0 < \lim_{t \to +\infty} \alpha(t)$ and there is not asymptotic stability in this case. □

□

3 Fixed point and existence result

Two functions $S_0 \in L^\infty(\Omega; \{0,1\})$; $f : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ verifying hypotheses (H_1) and (H_2), and $\beta_0 \in \mathbb{R}_+^*$ are given. Let $U_0 \in L^\infty(0, T, (C_0^1(\Omega))^2)$ be such that $div_x(U_0) = 0$. We set $\gamma(U_0) = u_0$ over $(0,T) \times \partial\Omega$. Introduce the following set C

$$C = \{\varphi \in L^2((0,T) \times \Omega) \;/\; 0 \leq \varphi(t,x) \leq 1 \text{ a. e.}\}.$$

Obviously C is a close convex bounded of $L^2((0,T) \times \Omega)$.

Before the proof of Theorem 1, concerning existence of a solution (u, p, S, α) of the three coupled problems (1); (2); and (3) recalled here below:

$$\begin{cases} div_x(\eta(t,x)\epsilon(u(t,x))) &= \nabla p \text{ in } \Omega; \\ div_x(u) &= 0 \text{ in } \Omega; \\ \gamma(u) &= u_0 \text{ over } \partial\Omega; \end{cases} \tag{3.24}$$

$$\begin{cases} \partial_t S(t,x) + (\vec{u}.\vec{\nabla}_x) S(t,x) &= 0\ \forall (t,x) \in \Pi; \\ S(0,x) &= S_0(x)\ \forall x \in \Omega; \\ S(t,x) &= 1\ \forall t \in]0,T[;\ \forall x \in \Gamma_e; \end{cases} \tag{3.25}$$

$$\begin{cases} \partial_t \alpha = f(t,\alpha)S(t,x) \ t \in]0,T]; \\ \alpha(0,x) = \beta_0; \end{cases} \tag{3.26}$$

we give an intermediate result related to convergence of solutions to Problem (25). For any $0 < \epsilon$, let S^ϵ be a sequence of functions with $\|S^\epsilon\|_{L^\infty(\Pi)} \leq 1$. Using the results of subsection 2.3, for a. e. $x \in \Omega$ the existence and uniqueness of α^ϵ continuous with respect to time verifying

$$\begin{cases} \partial_t \alpha^\epsilon = f(t,\alpha^\epsilon)S^\epsilon(t,x) \ t \in]0,T]; \\ \alpha^\epsilon(0,x) = \beta_0, \end{cases} \tag{3.27}$$

is demonstrated. Moreover $\|\alpha^\epsilon\|_{L^\infty([0,T])} \leq \max(\alpha_0, |\beta_0|)$.

Lemma 5 *If S in $L^\infty(\Pi, \{0,1\})$, f verifying hypotheses (H_1) and (H_2), $\beta_0 \in \mathbb{R}_+^*$ are given and if S^ϵ is a sequence of functions such that $\|S^\epsilon\|_{L^\infty(\Pi)} \leq 1$ and verifying*

$$\underset{\epsilon \to 0}{S^\epsilon \longrightarrow S} \text{ in } L^2(\Pi).$$

Then the sequence α^ϵ of the solutions of problem (3.27) is converging in $L^2(\Pi; \mathbb{R}_+^*)$, towards

$$\alpha(t) = \beta_0 + \int_0^t f(r,\alpha(s))S(r,x)\,dr \tag{3.28}$$

a weak solution to problem (25).

PROOF: Writing α^ϵ solution of problem (3.27) in the following way for all $t \in [0,T]$ and for a.e. $x \in \Omega$:

$$\alpha^\epsilon(t) = \beta_0 + \int_0^t f(r,\alpha^\epsilon(s))S^\epsilon(r,x)\,dr. \tag{3.29}$$

Let us show that α^ϵ is converging towards α in $L^2(\Pi; \mathbb{R}_+^*)$. Using hypothesis (H_2) and L^∞-estimate for S^ϵ and denoting by

$$K_1 = \sup_{\substack{t\in[0,T] \\ z\in[0,\max(\alpha_0,\beta_0)]}} |\partial_z f(t,z)|; \quad \overline{m} = \sup_{\substack{t\in[0,T] \\ z\in[0,\max(\alpha_0,\beta_0)]}} |f(t,z)|$$

we get for all $t \in [0,T]$ and for a.e. $x \in \Omega$

$$|\alpha^{\epsilon_1}(t) - \alpha^{\epsilon_2}(t)| \le K_1 \int_0^t |\alpha^{\epsilon_1}(r) - \alpha^{\epsilon_2}(r)|\, dr + \overline{m} \int_0^t |S^{\epsilon_1}(r,x) - S^{\epsilon_2}(r,x)|\, dr\ .$$

Gronwall's lemma implies that for all $t \in [0,T]$ and for a.e. $x \in \Omega$:

$$|\alpha^{\epsilon_1}(t) - \alpha^{\epsilon_2}(t)| \le \overline{m} \int_0^t \int_0^r e^{-K_1(\theta - r)} |S^{\epsilon_1}(\theta,x) - S^{\epsilon_2}(\theta,x)|\, d\theta dr\ .$$

The conclusion of this inequality is that α^ϵ is a Cauchy sequence in $L^2(\Pi;\mathbb{R}_+^*)$ since S^ϵ is a Cauchy sequence.

In the same way for almost the complete $(t,x) \in \Pi$ there is:

$$|\alpha^\epsilon(t) - (\beta_0 + \int_0^t f(r,\alpha(s))S(r,x)\, dr)| \le$$
$$\overline{m} \int_0^t \int_0^t e^{-K_1(\theta - r)} |S^\epsilon(\theta,x) - S(\theta,x)|\, d\theta dr\ .$$

and it follows that

$$|\alpha^\epsilon(t) - (\beta_0 + \int_0^t f(r,\alpha(s))S(r,x)\, dr)| \underset{\epsilon \to 0}{\longrightarrow} 0 \ \text{ in } L^2(\Pi). \tag{3.30}$$

Lemma 5 is demonstrated. □

In what following an operator F defined over C with values in C is built such that the solution of problem (1)-(2)-(3) is a fixed point of F in C. To prove that F admits a fixed point, the sequential compactness of F as an operator from $(C, \|\cdot\|_{L^2(\Pi)})$ onto itself is used.

Building an operator F over C:

The operator Λ_{polym} can be defined by use of the results of subsection 2.3

$$\begin{array}{lll} \Lambda_{polym} : L^2(\Pi;\{0,1\}) & \longrightarrow L^2(\Pi) & \\ S & \longmapsto \Lambda_{polym}(S) = \alpha & \text{defined by (2.17).} \end{array}$$

With proposition 2.2 it is possible to define :

$$\begin{array}{ll} \Lambda_{stokes} : L^\infty(\Pi;\mathbb{R}_+ \cap \{m \le z \le M\}) & \longrightarrow \mathcal{U} \\ \eta & \longmapsto \Lambda_{stokes}(\eta) = u \ \text{ a} \\ & \text{solution to problem (1).} \end{array}$$

With theorem 2 it is possible to introduce the operator:

$$\begin{array}{lll} \Lambda_{trans} : \mathcal{U} & \longrightarrow L^2(\Pi; \{0,1\}); & \\ u & \longmapsto \Lambda_{trans}(u) = S & \text{solution to problem (2)defined in Theorem 2.} \end{array}$$

For a given $S \in C$, then α; η; u and $\tilde{S}$ are defined by:

$$\left\{ \begin{array}{l} \alpha = \Lambda_{polym}(S)\ \eta = g(S,\alpha) \quad \text{(where g is defined by (4))} \\ u = \Lambda_{stokes}(\eta) \ \text{ and finally } \ \tilde{S} = \Lambda_{trans}(u) \end{array} \right. \tag{3.31}$$

The operator F is defined by

$$F : C \longrightarrow C \text{ associates } \tilde{S} \text{ to } S.$$

A solution of problem (1)-(2)-(3) will be a quadruplet (u,p,S,α) with $u \in L^2\Big[0,T;(H^1(\Omega))^2\Big], p \in L^2(0,T;L^2_0(\Omega)), S \in L^2((0,T)\times\Omega), \alpha \in C^0(0,T;L^\infty(\Omega))$ such that

$$\alpha = \Lambda_{polym}(S); \quad \eta = g(S,\alpha); \quad u = \Lambda_{stokes}(\eta); \quad F(S) = S, \tag{3.32}$$

and p is defined by proposition 2.1.
The existence of a solution to problem (1)-(2)-(3) is then derived of the existence of a fixed point of F in C. Let us begin with a technical result concerning the sequence of renormalized solutions of problem (3).

Lemma 6 *If $(u^n)_{n\in\mathbb{N}}$ is a sequence in $L^2(0,T;(H^1(\Omega))^2)$ such that $u^n \rightharpoonup u$ weakly in $L^2(0,T;(H^1(\Omega))^2)$ with $\gamma(u^n) = \gamma(U_0)$ and $divu^n = 0$ for any n, and such that $(S^n = \Lambda_{tans}(u^n)_{n\in\mathbb{N}})$ is the sequence of solutions to transport problems associated to u^n with $S_0 \in L^\infty(\Omega;\{0,1\})$ defined in Theorem 2. Then the sequence $(S^n)_{n\in\mathbb{N}}$ converges in $L^2((0,T)\times\Omega)$ towards the function $S = \Lambda_{tans}(u)$ a renormalized solution of the transport problem associated to u and given by Theorem 2.*

For a proof the reader is referred to [10] corollary 5.1. The key point for the strong convergence of $(S^n)_{n\in\mathbb{N}}$ in L^2 is to work with the space

$$X = L^2((0,T)\times\Omega) \times L^2(0,T;L^2(\Gamma_s, |U_0.n|d\sigma(x)))$$

and providing X of the following norm

$$|||(\delta,\mu)||| = \left[\|\delta\|^2_{L^2((0,T)\times\Omega)} + \int_0^T (T-t)\int_{\Gamma_s} \mu^2 |U_0.n|\, d\sigma(x)dt\right]^{\frac{1}{2}}.$$

PROOF: [Proof of theorem 1]
Let us first establish that the function F is continuous. Let S^n be strongly convergent in $L^2(\Pi)$ towards $S \in L^\infty(\Pi, \{0,1\})$. We have to prove that $(\xi^n = F(S^n))_{n\in\mathbb{N}}$ converges in C towards $F(S)$. The sequences (α^n, η^n, u^n) are defined by

$$\alpha^n = \Lambda_{polym}(S^n); \quad \eta^n = g(S^n, \alpha^n); \; u^n = \Lambda_{stokes}(\eta^n).$$

Lemma 5 applies and we get that α^n converges towards $\alpha = \Lambda_{polym}(S)$ a weak solution of problem (25). Since (S^n, α^n) is strongly convergent towards (S, α) in $L^2(\Pi, \mathbb{R}_+^*)$, neglecting the extraction of a sub-sequence again noted (S^n, α^n) Theorem 4.9 in [2] claims that the sequence (S^n, α^n) is convergent towards (S, α) almost everywhere in Π. The continuity of g implies that $\eta^n = g(S^n, \alpha^n)$ is convergent almost everywhere towards $g(\alpha, S)$. The function g defined by (2.4) is bounded, it follows that the sequence η^n is uniformly bounded. It is deduced of proposition 2.2 that:

$$||u^n|| \leq (\frac{2M}{mc(\Omega)} + 1)||U_0||.$$

The weak sequential compactness of a reflexive Banach space unit ball leads to the existence of $u \in \mathcal{U}$ such that $u^{n_p} \rightharpoonup u$ in $L^2(0,T;(H^1(\Omega))^2)$ weak. Let us show that u is the solution to problem (25) associated to $\eta = g(S, \alpha)$, $u = \Lambda_{stokes}(\eta)$.
Let now the subsequences $(S^{n_p})_{p\in\mathbb{N}}$ of $(S^n)_{n\in\mathbb{N}}$ and $(\alpha^{n_p})_{p\in\mathbb{N}}$ of $(\alpha^n)_n$ be such that for any $p \in \mathbb{N}$, $u^{n_p} = \Lambda_{stokes}(\eta^{n_p})$ is the solution of Problem (25) associated to $\eta^{n_p} = g(S^{n_p}; \alpha^{n_p})$. Set t $v^{n_p} = u^{n_p} - U_0$, which is weakly convergent towards $v - U_0$ in $L^2(0,T;V)$. The function g is continuous and bounded, so we have for all $w \in \mathcal{D}(0,T;V)$

$$\eta^{n_p}\epsilon(w) \to \eta\epsilon(w) \text{ a.e. in } \Pi \text{ and } ||\eta^{n_p}\epsilon(w)||_{L^2(\Pi} \leq C_1 M ||w||_{L^2(0,T;(H^1(\Omega))^2)}. \tag{3.33}$$

The dominated convergence theorem provides for all $w \in \mathcal{D}(0,T;V)$

$$\eta^{n_p}\epsilon(w) \to \eta\epsilon(w) \text{ in } L^2(\Pi) \tag{3.34}$$

when p goes to ∞. Then, we deduce for all $w \in \mathcal{D}(0,T;V)$:

$$\lim_{p\to+\infty} \int_0^T \int_\Omega \eta^{n_p}(t,x)\epsilon(v^{n_p}(t,x)) : \epsilon(w)\,dx\,dt + \int_0^T \int_\Omega \eta^{n_p}(t,x)\epsilon(U_0(t,x)) : \epsilon(w)\,dx\,dt =$$
$$\int_0^T \int_\Omega \eta(t,x)\epsilon(v(t,x)) : \epsilon(w)\,dx\,dt + \int_0^T \int_\Omega \eta(t,x)\epsilon(U_0(t,x)) : \epsilon(w)\,dx\,dt.$$

But for any $p \in \mathbb{N}$, and any $w \in V$ and a. e. $t \in (0,T)$ we have:

$$\int_\Omega \eta^{n_p}(t,x)\epsilon(v^{n_p}(t,x)) : \epsilon(w)\,dx + \int_\Omega \eta^{n_p}(t,x)\epsilon(U_0(t,x)) : \epsilon(w)\,dx = 0\ .$$

Then for any $w \in V$ and for almost every $t \in (0,T)$:

$$\int_\Omega \eta(t,x)\epsilon(v(t,x)) : \epsilon(w)\,dx + \int_\Omega \eta(t,x)\epsilon(U_0(t,x)) : \epsilon(w)\,dx = 0\ .$$

So v is solution of Problem (2.6) and thus $u = v + U_0$ is a solution of (2.5) then a solution of (1) associated to η. Moreover u is unique, so it is all the sequence which converges, and we have $u = \Lambda_{stokes}(\eta = g(S,\alpha))$.
On the other hand, by using Corollary 5.1 of [10] we have that

$$S = \lim_{n\to\infty} F(S^n) = \Lambda_{trans}(u)$$

since u^n weakly converges to u in $L^2(0,T;(H^1(\Omega))^2)$. Thus we have $S = F(S)$.
Finally let us establish the compactness of F. We start with $(S^n)_{n\in\mathbb{N}}$ an arbitrary bounded sequence in C. The weak sequential compactness of the unit ball of $L^2(\Pi)$ implies the existence of $S \in L^2(\Pi)$ and of a sub-sequence $S^{n_p} \rightharpoonup S$ in $L^2(\Pi)$ weak. Let $\alpha^n = \Lambda_{polym}(S^n)$; $u^n = \Lambda_{stokes}(\eta^n = g(S^n,\alpha^n))$ which is uniformly bounded in $L^2(0,T;(H^1(\Omega))^2)$. The weak sequential compacity of the unit ball of the Banach space $\mathcal{U}$ implies the existence of $u \in \mathcal{U}$ and a sub-sequence $u^{n_p} \rightharpoonup u$ in $L^2(0,T;(H^1(\Omega))^2)$ weak. Corollary 5.1 of [10] applies, and we have $S^{n_p} \to S$ in $L^2(\Pi)$. Arguing as before we prove that S^{n_p} converges strongly in $L^2(\Pi)$ toward $S = \Lambda_{stokes}(\eta = g(S,\alpha))$.
Theorem 1 is proven.

□

Acknowledgement

We would like to thank the referee for the helpful remarks.

References

[1] J.F. Agassant, Y. Demay, and A. Fortin. Prediction of stationary interfaces in coextrusion flows. *Polymer engineering and science*, 34:121–134, 1994. N 14.

[2] H. Brézis. *Analyse fonctionnelle.* Masson, Paris, 1983.

[3] Bardos C. Problèmes aux limites pour les équations aux dérivées partielles du premier ordre à coefficients réels; théorèmes d'approximation. application à l'équation de transport. *Ann. Sci. École Normale Supérieure*, 4^{eme} série, t. 3:185–233, 1970.

[4] E. A. Coddington and L. Norman. *Ordinary differential equations.* Mcgran-hill book compagny, New-York ; Toronto ; London, 1955.

[5] M. Crouzeix and A. Mignot. *Approximation des équations différentielles ordinaires.* Masson, Paris, 1992.

[6] O. Diallo. Modélisation et simulation numérique de résines réactives dans un milieu poreux. PhD Thesis, Claude Bernard University Lyon 1, 2000.

[7] O. Diallo, J. Pousin, and T. Sassi. A posteriori error estimates for the transport equation applied to resin transfert molding problems. In R. Owens M. Deville, editor, *16th IMACS world congres Proceeding.* Pitman Reasearch Notes in Mathematics series 345 Longman, 2000.

[8] M. R. Kamal and S. Sourour. Kinetics and thermal characterization of thermoset cure. *Polymer Engineering and science*, 13:59–64, 1973.

[9] E. Maitre and P. Witomski. Transport equation with boundary conditions for free surface localization. *Numer. Math.*, 84:275–303, 1999. No. 2.

[10] A. Nouri and F. Poupaud. An existence theorem for the multifluid stokes problem. *Quart. Appl. Math.*, 55:421–435, 1997. N 3.

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