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THE PERIOD FUNCTION NEAR A POLYCYCLE WITH TWO SEMI-HYPERBOLIC VERTICES

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ABSTRACT. Let P be a polycycle of an analytic vector field on an open subset of the plane $\mathbb{R}^2$. Suppose that P is the union of two semi-hyperbolic singular points (*vertices* of P) connected by two trajectories (*sides* of P). Assume that one side is part of the center manifold of each vertex. Denote by L the other side. Assume also that P is a boundary component of an annulus of periodic orbits. Let Σ be a Poincaré section at the polycyle that intersects L. We show that the period function defined on Σ has a principal part of the form $kx^{-n}, k > 0, n \in \mathbb{N}$.

1. INTRODUCTION

Consider an analytic ordinary differential equation E on an open subset of the plane $\mathbb{R}^2$. Suppose that E has an annulus of periodic orbits, not necessarily bounded. It is known that a boundary component union of such an annulus is a *polycycle* P (cf. [P]); that is a finite connected union of singularities (*vertices* of P) and integral curves (*sides* of P) of E. A unique singular point may be considered as a polycycle. Let Σ be a small Poincaré section at the polycycle with a local coordinate s whose origin lies at the polycycle. The integral curve of E that passes through a point of Σ is a periodic orbit. The *period function* assigns to s the (minimum) period $T(s)$ of the corresponding periodic solution.

We are interested in the qualitative behavior of T, mainly in the asymptotic expansion of T and of its derivative, for small argument. The fact that a polycycle has a period function with an asymptotic expansion, and also the expansion itself, are problems of general interest. Also, the oscillatory character of T is of interest to us. We say that a function is *oscillatory* if the set of its critical points has accumulation points. The derivative of an oscillatory function either does not have an asymptotic expansion or it has an asymptotic expansion identically zero.

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The behavior of T and of its derivative, as its argument approach zero, depends on whether the polycycle is bounded or not, and also on the type of its vertices. We would like to know how the analytic local invariants of these singular points intervene in the behavior of such a function.

The period function T is analytic at every strictly positive coordinate s and it is analytic at the origin only if the polycycle is a nondegenerate center. Generally, both T and its derivative grow without bound when their arguments approach zero. Bounded polycycles have nonoscillatory period functions, [C-D]. On the other hand, in [S, Sa] it is proved that if the vertices of a polycycle (bounded or not) are formally linearizable after desingularization, then T and its derivative have asymptotic expansions in $\{s^{\mu}\}$ and $\{s^{\delta}\log s\}$, $\mu, \delta \in \mathbb{R}$. A consequence if such a polycycle is unbounded and has a finite vertex then it has a nonoscillatory period function.

In this work we are interested in a class of polycycles with semi-hyperbolic vertices. We determine the principal part of the period function for bounded polycycles in this class.

More precisely, we consider polycycles with two semi-hyperbolic singular points as vertices with a side that lies in the center manifold of both vertices. We call such a side *the center side* of the polycycle. The other side is called *hyperbolic side*. We prove that the period function defined on a transversal section through the hyperbolic side is of the form

$$T(s) = k^{-n}(1 + o(s)), \quad k > 0,\ n \in \mathbb{N}.$$

To prove this we decompose the period function in local time functions through the polycycle sides and the saddle sectors. The period function is the sum of each of these local time functions composed on the right with an appropriate transition map.

2. PRINCIPAL PART OF T

Consider the analytic differential equation E : $\frac{dx}{dt} = A(x,y)$, $\frac{dy}{dt} = B(x,y)$ on an open subset of the plane. A *polycycle* P of E is a connected union consisting of a finite number of singularities of E (*vertices* of P) and the integral curves of E (*sides* of P) such that a unilateral *return map* R exists, that is, there is an analytic curve

$$\gamma : [0,1] \to \Sigma \subset \mathbb{R}^2 \ , \quad \gamma(0) \in \mathrm{P}$$

transverse to P such that the integral curve through $\gamma(s)$ intersects Σ again for the first time at $\gamma(R(s))$ for each sufficiently small s.

The *period function* $T :]0,\epsilon[\rightarrow \mathbb{R}^+$, is defined when R is the identity map, given by $s \mapsto T(s)$ as the period of the periodic orbit through $\gamma(s)$.

The function T is analytic on $]0,\epsilon[$. But T is not necessarily defined or analytic at $s = 0$, and $T(s)$ may converge to infinity as $s \mapsto 0^+$.

Next, consider a polycycle P with two vertices. Suppose that such vertices are semi-hyperbolic singular points of E (that is, if a is a vertex then the Jacobian matrix of (A, B) at a has one eigenvalue equal to zero and the other one different to zero). We suppose, moreover, that one side of P (center side) is contained in the center manifold of each vertex (see Fig. 1). Therefore

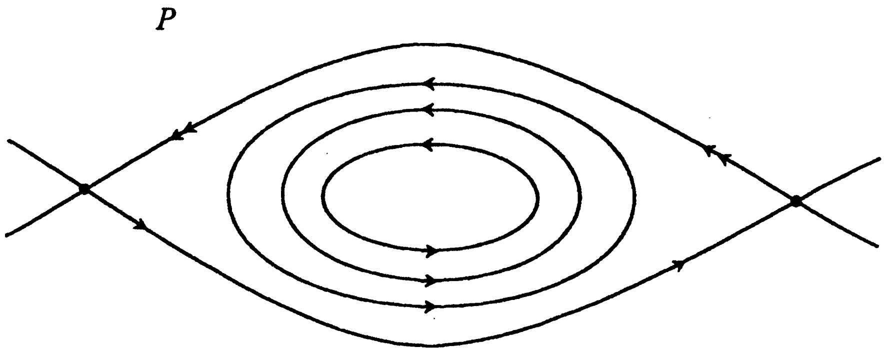

Fig. 1

Theorem. *Let Σ be a Poincaré section at P such that Σ intersects the hyperbolic side. The period function T defined on Σ satisfies*

$$T(s) = k^{-n}(1 + \varepsilon(s)),$$

where $k > 0$, $n \in \mathbb{N}$ with $\varepsilon(s) \mapsto 0$ if $s \mapsto 0^+$.

Proof. The orientation of the trajectories of E defines a sense of direction of P. Let a_1, a_2 be the vertices of P and let L_1 and L_2 be the hyperbolic side and the center side respectively, such that the end of L_1 and the start of L_2 is a_1 (see Fig. 2). We assume that a_1 is crossed by running first through L_1 and then through L_2. On a neighborhood of each vertex a_i $(i = 1, 2)$ of P, we choose two analytic semi-transversals

$$\gamma_i : [0,1] \rightarrow \Sigma_i \subset \mathbb{R}^2 \ , \ \ \gamma_i(0) = p_i \in P, \ \text{ and}$$

$$\pi_i : [0,1] \rightarrow \Pi_i \subset \mathbb{R}^2 \ , \ \ \pi_i(0) = q_i \in P,$$

where p_1, $p_2 \in L_1$ and q_1, $q_2 \in L_2$. We choose $\Sigma = \Sigma_1$.

Define the functions

$$g_1 :]0, \delta_1[\to \mathbb{R}^+ \text{ given by } s \mapsto g_1(s) \text{ and}$$

$$S :]0, \delta_1[\to \mathbb{R}^+ \text{ given by } s \mapsto S(s),$$

such that the positive semi-orbit of E through $\gamma_1(s)$ ($s \neq 0$) intersects $\Pi_1 - \{q_1\}$ at $\pi_1(g_1(s))$ and $\Sigma_2 - \{p_2\}$ at $\gamma_2(S(s))$.

Define also the functions

$$\sigma_1 :]0, \delta_1[\to \mathbb{R}^+ \text{ given by } s \mapsto \sigma_1(s) \ ,$$

$$\tau_1 :]0, \varepsilon_1[\to \mathbb{R}^+ \text{ given by } \kappa \mapsto \tau_1(\kappa)$$

and

$$\tau_2 :]0, \varepsilon_2[\to \mathbb{R}^+ \text{ given by } \xi \mapsto \tau_2(\xi) \ ,$$

where, $\sigma_1(s)$ is the time required for the integral curve starting at $\gamma_1(s)$ to intersect the transversal Π_1 for the first time at $\pi_1(g_1(s))$, the number $\tau_1(\kappa)$ is the time required for the integral curve starting at $\pi_1(\kappa) \in \Pi_1$ to intersect, for the first time the transversal Π_2, and $\tau_2(\xi)$ is the time required for the integral curve starting at $\gamma_2(\xi) \in \Sigma_2$ to intersect, for the first time, the transversal Σ_1 at $\gamma_1(s)$.

Next, consider the equation

$$E^* : \frac{dx}{dt} = -A(x,y) \ ; \ \frac{dy}{dt} = -B(x,y)$$

and define the function $\sigma_2 :]0, \varepsilon_2[\to \mathbb{R}^+ \ \xi \mapsto \sigma_2(\xi)$,where $\sigma_2(\xi)$ is the time required for the integral curve of E^* starting at $\gamma_2(\xi) \in \Sigma_2$ to intersect the transversal Π_2 for the first time at $\pi_2(g_2(\xi))$. The functions σ_i ($i = 1, 2$) are called *the corner passage time functions relative to* Σ_i and Π_i. Therefore, the function T is given by

$$T(s) = \sigma_1(s) + \tau_1(g_1(s)) + \sigma_2(S(s)) + \tau_2(S(s)) \qquad (*)$$

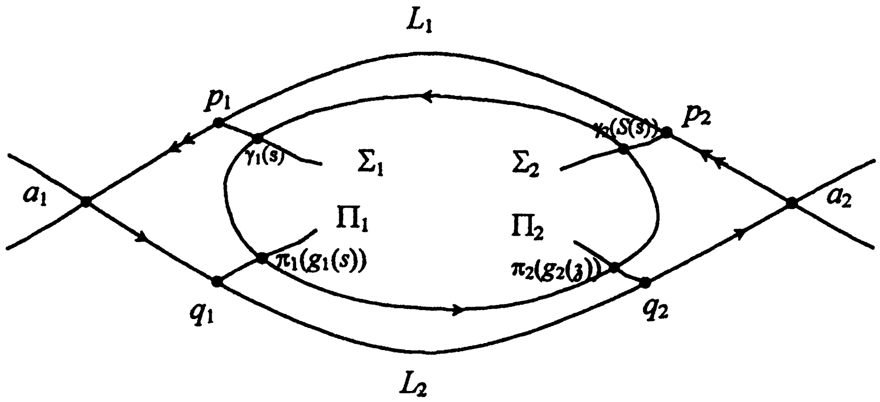

Fig. 2

The functions τ_1 and τ_2 are analytic at zero with $\tau_1(0) \neq 0$ and $\tau_2(0) \neq 0$. In effect, consider the function τ_1. From the Flow Box Theorem, analytic coordinates (u, v) exist on a neighborhood of the side L_2 of P such that the axis $v = 0$ is the side L_2, while the semitransversals Π_1 and Π_2 are graphs of the analytic functions, k and l, on $v \geq 0$ (the functions k and l are defined and analytic on a neighborhood of zero, and the points $(k(0), 0)$ and $(l(0), 0)$ correspond to q_1 and q_2 respectively). In these coordinates, X (the vector field associated with the differential equation E) becomes

$$X = A_1(u, v)\frac{\partial}{\partial u},$$

where A_1 is a strictly positive analytic function. We obtain that the time required for the integral curve starting at $(k(v), v)$ to intersect Π_2 at $(l(v), v)$ is the integral

$$\tau(v) = \int_{k(v)}^{l(v)} \frac{1}{A_1(u, v)}\, du\ .$$

Thus τ is an analytic function with $\tau(0) \neq 0$. Since the coordinate change $\kappa \mapsto v$ is analytic and fixes zero, we obtain that τ_1 is an analytic function on a neighborhood of zero with $\tau_1(0) \neq 0$. In the same way τ_2 is an analytic function with $\tau_2(0) \neq 0$.

Since $g_1(0) = 0$ and $S(0) = 0$ (with g_1 and S defined, at zero, by continuity), the principal parts of $\tau_1(g_1(s))$ and $\tau_2(S(s))$ are $\tau_1(0)$ and $\tau_2(0)$, respectively.

Now, from $(*)$, it remains to evaluate the principal part of $\sigma_1(s) + \sigma_2(S(s))$. For that purpose we shall need the following propositions (to be proved in the appendix).

Proposition 1. *There exist a positive integer n, such that*

$$\sigma_1(s) = k_1 s^{-n} + o(s^{-n})$$

and

$$\sigma_2(\xi) = k_2 \xi^{-n} + o(\xi^{-n}),$$

where k_1 and k_2 are strictly positive numbers.

Proposition 2. *The function S has an asymptotic expansion in s^δ and $s^\mu(\log s)^m$, where δ and μ are strictly positive rational numbers and m is a strictly positive integer. More precisely, there exists a C^∞ function S_1 on a neighborhood of $(0,0)$ such that $S_1(0,0) = 0$ and*

$$S(s) = c \cdot s(1 + S_1(s, s^n \log s)),$$

where $c > 0$ with n is given by Proposition 1.

Hence, from (*) and the Propositions 1 and 2, it follows that

$$T(s) = k_1 s^{-n} + \tau_1(0) + k_2 c^{-n} s^{-n} + \tau_2(0) + o(s^{-n}).$$

Since $n > 0$ we have

$$T(s) = k s^{-n} + o(s^{-n}),$$

with $k > 0$. The last equality constitutes the main result of the present work. ■

Remark. If the transversal section Σ intersects the center side, the calculations are more complicated; in this case, we do not know how to calculate the function S defined on Σ.

APPENDIX

I. Proof of Propositions 1 and 2

On a neighborhood of a_k $(k = 1, 2)$, there exist C^∞ coordinates (x_k, y_k) such that the origin is a_k, and the semi-axes $x_k = 0, y_k > 0$ and $x_k > 0, y_k = 0$ are the sides of P. In the coordinates (x_1, y_1), the equation E is given by

$$E \ : \ \frac{dx_1}{dt} = x_1^{n_1+1} f_1(x_1,y_1) \ \ ; \ \ \frac{dy_1}{dt} = -y_1(n_1 + \lambda_1 x_1^{n_1}) f_1(x_1,y_1) \qquad (**)$$

where f_1 is a strictly positive C^∞ function, with $n_1 \in \mathbb{N}$ and $\lambda_1 \in \mathbb{R}$ (cf. [M]). We can consider that Σ_1 and Π_1 are the segments $0 \leq x_1 < 1,\ y_1 = 1$ and $x_1 = 1,\ 0 \leq y_1 < 1$, respectively.

In the coordinates (x_2, y_2) the equation E^* (recall that $E^* = -E$) is given by

$$E^* \ : \ \frac{dx_2}{dt} = x_2^{n_2+1} f_2(x_2,y_2) \ \ ; \ \ \frac{dy_2}{dt} = -y_2(n_2 + \lambda_2 x_2^{n_2}) f_2(x_2,y_2)$$

where f_2 is a strictly positive C^∞ function, with $n_2 \in \mathbb{N}$ and $\lambda_2 \in \mathbb{R}$. We suppose again that Σ_2 and Π_2 are the segments $0 \leq x_2 < 1,\ y_2 = 1$ and $x_2 = 1,\ 0 \leq y_2 < 1$, respectively.

1 *Proof of Proposition 1.* The definitions of σ_1 and σ_2 are similar, therefore we shall only consider the equation E and omit the subindex 1 of the coordinates $(x_1, y_1), n_1$ and λ_1.

From the equation (**), we deduce that the corner passage time function, defined on the transversal $0 < x < 1,\ y = 1$, is the line integral

$$\sigma(x_1) = \int_{\gamma_{x_1}} \frac{1}{x^{n+1} f(x,y)} dx$$

where γ_{x_1} is the orbit arc of E that joins $(x_1, 1)$ to the point $(1, y_1(x_1)) \in \Pi_1$.

A first integral of equation (**) is

$$I(x,y) = y \cdot x^\lambda \exp(-\frac{1}{x^n}),$$

hence γ_{x_1} is done by the equation $y = (\frac{x_1}{x})^\lambda \exp(\frac{1}{x^n} - \frac{1}{x_1^n}),\ x_1 \leq x \leq 1$. We obtain the equality

$$\sigma(x_1) = \int_{x_1}^{1} x^{-n-1} F(x, (\frac{x_1}{x})^\lambda \exp(\frac{1}{x^n} - \frac{1}{x_1^n}))dx \ , \ 0 < x_1 < 1,$$

where $F = \frac{1}{f}$.

We know that two C^∞ functions F_1 and F_2 exist on $\mathbb{R}^2$ such that

$$F(x,y) = F(0,0) + xF_1(x,y) + yF_2(x,y).$$

Therefore

$$\sigma(x_1) = \frac{F(0,0)}{n}x_1^{-n} - \frac{F(0,0)}{n} + H_1(x_1) + H_2(x_1),$$

where

$$H_1(x_1) = \int_{x_1}^{1} x^{-n} F_1(x, (\frac{x_1}{x})^{\lambda} \exp(\frac{1}{x^n} - \frac{1}{x_1^n}))dx$$

and

$$H_2(x_1) = x_1^{\lambda} \exp(-\frac{1}{x_1^n}) \int_{x_1}^{1} x^{-n-\lambda-1} \exp(\frac{1}{x^n}) F_2(x, (\frac{x_1}{x})^{\lambda} \exp(\frac{1}{x^n} - \frac{1}{x_1^n}))dx.$$

Since the set

$$\{(x, (\frac{x_1}{x})^{\lambda} \exp(\frac{1}{x^n} - \frac{1}{x_1^n})) \mid x_1 \leq x \leq 1, \; x_1 \in]0,1[\,\}$$

is contained in $[0,1] \times [0,1]$, a constant $K > 0$ exists such that

$$\mid F_1(x, (\frac{x_1}{x})^{\lambda} \exp(\frac{1}{x^n} - \frac{1}{x_1^n})) \mid \leq K$$

for all $x_1 \in]0,1[$ and $x \in [x_1, 1]$. Consequently

$$\mid x_1^n H_1(x_1) \mid = \mid x_1^n \int_{x_1}^{1} x^{-n} F_1(x, (\frac{x_1}{x})^{\lambda} \exp(\frac{1}{x^n} - \frac{1}{x_1^n}))dx \mid \leq K x_1^n \int_{x_1}^{1} x^{-n} dx.$$

We conclude that $H_1(x_1) = o(x_1^{-n})$. By a similar calculation, we have $H_2(x_1) = o(x_1^{-n})$.

Thus,

$$\sigma(x_1) = \frac{F(0,0)}{n}x_1^{-n} + o(x_1^{-n}).$$

Since the C^{∞} coordinate change $s \mapsto x_1$ fixes zero, we obtain that $\sigma_1(s)$ satisfies the equality

$$\sigma_1(s) = k_1 s^{-n} + o(s^{-n}),$$

with $k_1 > 0$.

In the same way,

$$\sigma_2(\xi) = k_2\xi^{-n_2} + o(\xi^{-n_2}),$$

with $k_2 > 0$. This proves Proposition 1. ■

2 *Proof of Proposition 2.* Let $h(\kappa)$ be the number such that the integral curve through $\pi_1(\kappa) \in \Pi_1$ intersects the transversal Π_2 for the first time at $\pi_2(h(\kappa))$. The function h is a strictly increasing analytic function with $h(0) = 0$. Recall that the integral curve of E^* through $\gamma_2(\xi) \in \Sigma_2$ intersects the transversal Π_2 at $\pi_2(g_2(\xi))$. This defines the function $\xi \mapsto g_2(\xi)$. Thus, the function S satisfies the equality

$$S = g_2^{-1} \circ h \circ g_1.$$

Next, consider the expression of the functions g_1, g_2 and h in the coordinates x_1, x_2, y_1 and y_2. That is,

$$g_1 : x_1 \mapsto y_1, \quad g_2 : x_2 \mapsto y_2 \quad \text{and} \quad h : y_1 \mapsto y_2.$$

Hence, the function S is given by

$$S = g_2^{-1} \circ h \circ g_1 \ : \ x_1 \mapsto x_2.$$

To find the expression $x_2 = S(x_1)$, consider the equality

$$g_2(x_2) = (h \circ g_1)(x_1).$$

Recall that, on a neighborhood of a_1, a first integral of equation E is $I(x_1, y_1) = y_1 \cdot x_1^{\lambda_1} \exp(-\frac{1}{x_1^{n_1}})$. So that $y_1 = g_1(x_1)$ is the solution of the equation $I(1, y_1) = I(x_1, 1)$.

Thus, $g_1(x_1)$ is given by

$$g_1(x_1) = x_1^{\lambda_1} \exp(1 - \frac{1}{x_1^{n_1}}).$$

Similarly, for $g_2(x_2)$:

$$g_2(x_2) = x_2^{\lambda_2} \exp(1 - \frac{1}{x_2^{n_2}}).$$

Moreover, since $h : y_1 \mapsto y_2$ is a strictly increasing C^∞ function with $h(0) = 0$ (y_1 and y_2 are C^∞ coordinates of Π_1 and Π_2 respectively, and the coordinates change are strictly increasing), there exists a C^∞ function h_1 such that $h_1(0) = 0$ and

$$h(y_1) = \beta y_1(1 + h_1(y_1)),$$

with $\beta > 0$.

Hence, the equation $g_2(x_2) = (h \circ g_1)(x_1)$ is equivalent to

$$x_2^{\lambda_2} \exp(1 - \frac{1}{x_2^{n_2}}) = \beta x_1^{\lambda_1} \exp(1 - \frac{1}{x_1^{n_1}})(1 + h_1(x_1^{\lambda_1} \exp(1 - \frac{1}{x_1^{n_1}})).$$

Applying the logarithmic function, we obtain

$$\lambda_2 \log x_2 - \frac{1}{x_2^{n_2}} = \log\beta + \lambda_1 \log x_1 - \frac{1}{x_1^{n_1}} + \epsilon(x_1) \tag{1}$$

where $\epsilon(x_1) = \log(1 + h_1(x_1^{\lambda_1} \exp(1 - \frac{1}{x_1^{n_1}})))$ is a C^∞ function, which is flat at zero (that is, its Taylor series at zero is equal to zero).

Consider the variable z through

$$x_2^{n_2} = \frac{x_1^{n_1}}{1+z}. \tag{2}$$

From (1), we obtain

$$\lambda_2 n_1 x_1^{n_1} \log x_1 - \lambda_2 x_1^{n_1} \log(1+z) - n_2 z = n_2 x_1^{n_1} \log\beta + \lambda_1 n_2 x_1^{n_1} \log x_1 + n_2 x_1^{n_1} \epsilon(x_1).$$

Next, put $w = x_1^{n_1} \log x_1$ and consider the function G defined as

$$G(x, w, z) = \lambda_2 n_1 w - \lambda_2 x_1^{n_1} \log(1+z) - n_2 z - n_2 x_1^{n_1} \log\beta - \lambda_1 n_2 w - n_2 x_1^{n_1} \epsilon(x_1).$$

The function G is C^∞ on a neighborhood of the origin $(x_1, w, z) = (0,0,0)$, with $G(0,0,0) = 0$ and $\frac{\partial G}{\partial z}(0,0,0) = -n_2 \neq 0$. Thus, from the Implicit Function Theorem, there exists a C^∞ function $z = z(x_1, w)$, defined on a neighborhood of $(x_1, w) = (0,0)$ such that $z(0,0) = 0$ and $G(x_1, w, z(x_1, w)) = 0$. Therefore,

$$x_2^{n_2}(x_1) = \frac{x_1^{n_1}}{1 + z(x_1, x_1^{n_1} \log x_1)}.$$

From this, it follows that the function S is given by

$$S(x_1) = x_1^{\frac{n_1}{n_2}}(1 + v_1(x_1, x_1^{n_1} \log x_1)) \tag{3}$$

where v_1 is a C^∞ function on a neighborhood of $(0,0)$ such that $v_1(0,0) = 0$.

Now, the coordinate change $s \mapsto x_1$ is a C^∞ diffeomorphism on a neighborhood of zero and fixes zero, that is, $x_1 = c_1 s(1 + b(s))$, where $c_1 > 0$ and b is a C^∞ function with $b(0) = 0$. Hence, substituting this expression in (3), we obtain that the function S satisfies the equality

$$S(s) = c \cdot s^{\frac{n_1}{n_2}}(1 + v(s, s^{n_1} \log s)),$$

where $c > 0$ and v is a C^∞ function with $v(0,0) = 0$.

Note that the return function R is the composition of the function S plus a C^∞ diffeomorphism from Σ_2 to Σ_1. Since R is the identity map it follows that $n_1 = n_2$ and therefore

$$S(s) = c \cdot s(1 + v(s, s^n \log s)),$$

where $c > 0$ and v is as above. This proves the Proposition 2 and the proof of the main result is now complete. ■

Remark. If the polycycle have an arbitrary number of vertices, it seems possible to find a principal part of the period function. The composition of the corner passage functions (g_1 and g_2 in the present work), involves the composition of exponential functions. Therefore, the asymptotic expansion of the return function (function S here) is not anymore in $\{s^\mu\}$ and $\{s^\delta\} \log s$. This case will be the object of another paper.

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