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Quasi-invariant measures on non-Archimedean groups and semigroups of loops and paths, their representations. II.

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Abstract

Loop groups G as families of mappings of one non-Archimedean Banach manifold M into another N with marked points over the same locally compact field $\mathbf{K}$ of characteristic $char(\mathbf{K}) = 0$ are considered. Quasi-invariant measures on them are constructed. Then measures are used to investigate irreducible representations of such groups.

1 Introduction.

In the first part results on loop semigroups were exposed. This part is devoted to loop and path groups, quasi-invariant measures on them and their unitary representations. Results from Part I are used below (see also Introduction of Part I).

Irreducible components of strongly continuous unitary representations of Abelian locally compact groups are one-dimensional by Theorem 22.17 [10]. In general commutative non-locally compact groups may have infinite-dimensional irreducible strongly continuous unitary representations, for example, infinite-dimensional Banach spaces over $\mathbf{R}$ considered as additive groups (see §2.4 in [1] and §4.5 [9]).

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In §3 for the investigation of a representation's irreducibility the pseudodifferentiability and some other specific properties of the constructed quasi-invariant measures are used. Besides continuous characters separating points of the loop group (see Theorem 3.3), strongly continuous infinite-dimensional irreducible unitary representations are constructed in §3.2.

The path groups and semigroups are investigated in §4.

In the real case there are known H-groups defined with the help of homotopies [18]. A compositon on the H-group is defined relative to classes of homotopic mappings. In the non-Archimedean case homotopies are meaningless. A space of mappings $C(\xi,(M,s_0)\to(N,y_0))$ from one manifold M into another N preserving marked points (see I. §2.6) is supplied with the composition operation of families of mappings using loop semigroups. It is called a loop O-semigroup, since compositions are defined relative to certain equivalence classes, which are closures of families of certain orbits relative to the action of the diffeomorphism group of M preserving s_0. From it a loop O-group is defined with the help of the Grothendieck construction. O-groups are considered in §5.

In §6 the notation is summarized.

2 Loop groups.

2.1. Note and Definition. For a commutative monoid $\Omega_\xi(M,N)$ with the unity and the cancellation property (see Theorem I.2.7 and Condition I.2.7.(5)) there exists a commutative group $L_\xi(M,N)$ equal to the Grothendieck group. This group is the quotient group F/B, where F is a free Abelian group generated by $\Omega_\xi(M,N)$ and B is a closed subgroup of F generated by elements $[f+g]-[f]-[g]$, f and $g\in\Omega_\xi(M,N)$, $[f]$ denotes an element of F corresponding to f. In view of §9 [12] and [17] the natural mapping

$$(1)\ \gamma:\Omega_\xi(M,N)\to L_\xi(M,N)$$

is injective. We supply F with a topology inherited from the Tychonoff product topology of $\Omega_\xi(M,N)^{\mathbf{Z}}$, where each element z of F is

$$(2)\ z=\sum_f n_{f,z}[f],$$

$n_{f,z} \in \mathbf{Z}$ for each $f \in \Omega_\xi(M,N)$,

$$(3)\ \sum_f |n_{f,z}| < \infty.$$

In particular $[nf] - n[f] \in \mathsf{B}$, where $1f = f$, $nf = f \circ (n-1)f$ for each $1 < n \in \mathbf{N}$, $f + g := f \circ g$. We call $L_\xi(M,N)$ the loop group.

2.2. Proposition. *The space $L_\xi(M,N)$ from §2.1 is the complete separable Abelian Hausdorff topological group; it is non-discrete, perfect and has the cardinality* c.

Proof follows from §I.2.7 and §2.1, since in view of Formulas 2.1.(1-3) for each $f \in L_\xi(M,N)$ there are $g_j \in \Omega_\xi(M,N)$ such that $f = f_1 - f_2$, where $\gamma(g_j) = f_j$ for each $j \in \{1,2\}$. Therefeore, γ is the topological embedding such that $\gamma(f+g) = \gamma(f) + \gamma(g)$, $\gamma(e) = e$.

2.3. Theorem. *Let $G = L_\xi(M,N)$ be the same group as in §2.1, $\xi = (t,s)$ or $\xi = t$ with $0 \le t \in \mathbf{R}$, $s_0 \in \mathbf{N}_o$.*

(1) *If $At'(\tilde{M})$ has card$(\Lambda'_{\tilde{M}}) \ge 2$, then G is isomorphic with $G_1 = L_\xi(\tilde{M},N)$, where $\tilde{M} = U'_1 \cup U'_2$ (see §I.2.5). Moreover, $T_\eta G$ is the Banach space for each $\eta \in G$ and G is ultrametrizable.*

(2) *If $1 \le t + s$, then G is an analytic manifold and for it the mapping $\tilde{E} : \tilde{T}G \to G$ is defined, where $\tilde{T}G$ is the neighbourhood of G in TG such that $\tilde{E}_\eta(V) = \bar{exp}_{\eta(x)} \circ V_\eta$ from some neighbourhood $\bar{V}_\eta$ of the zero section in $T_\eta G \subset TG$ onto some neighbourhood $W_\eta \ni \eta \in G$, $\bar{V}_\eta = \bar{V}_e \circ \eta$, $W_\eta = W_e \circ \eta$, $\eta \in G$ and $\tilde{E}$ belongs to the class $C(\infty)$ by V, $\tilde{E}$ is the uniform isomorphism of uniform spaces $\bar{V}$ and W.*

(3) *There are atlases $\tilde{A}t(TG)$ and $\tilde{A}t(G)$ for which $\tilde{E}$ is locally analytic. Moreover, G is not locally compact for each $0 \le t$.*

Proof. The first statement follows immediately from Theorem I.2.17 and §2.1. Therefeore, to prove the second statement it is sufficient to consider the manifold M with a finite atlas $At(M)$.

Let $V_\eta \in T_\eta G$ for each $\eta \in G$, $V \in C_0(\xi, G \to TG)$, suppose also that $\tilde{\pi} \circ V_\eta = \eta$ be the natural projection such that $\tilde{\pi} : TG \to G$, then V is a vector field on G of class $C_0(\xi)$. The disjoint and analytic atlases $At(C_0(\xi, M \to N))$ and $At(C_0(\xi, M \to TN))$ induce disjoint clopen atlases in G and TG with the help of the corresponding equivalence relations and ultrametrics in these quotient spaces. These atlases are countable, since G and TG are separable. In view of Theorem I.2.10 the space $T_\eta G$ is Banach and not locally compact, hence it is infinite-dimensional over $\mathbf{K}$.

In view of Formulas I.2.6.2.(1-7) the multiplications

$$(1)\ R_f : G \to G,\ g \mapsto g \circ f = R_f(g) \text{ and}$$

$$(2)\ \alpha_h :\ C^0_0(\xi, (M, s_0) \to (N, y_0)) \to C^0_0(\xi, (M, s_0) \to (N, y_0)),\ \alpha_h(v) = v \circ h$$

for $f, g \in G$ and $h, v \in C^0_0(\xi, (M, s_0) \to (N, y_0))$ belong to the class $C(\infty)$.

Using Formulas (1,2) as in §I.2.10 we get, that the vector field V on G of class $C_0(\xi)$ has the form

$$(3)\ V_{\eta(x)} = v(\eta(x)),$$

where v is a vector field on N of the class $C_0(\xi)$, $\eta \in G$,

$$v(< f >_{K,\xi} (x)) := \{v(g(x)) : g \in < f >_{K,\xi}\}.$$

Since $\bar{exp} : \tilde{T}N \to N$ is analytic on the corresponding charts (see §I.2.8.). In view of Formulas I.2.8.(1-4) $\tilde{E}(V) = \bar{exp} \circ V$ has the necessary properties, where $\bar{exp}$ is considered on $At''(N)$ with $\psi''_i(V''_i)$ being K-convex in the Banach space Y. Therefore, due to Formula (3) we have

$$(4)\ \tilde{E}_\eta : T_\eta G \supset \bar{V}_\eta \to W_\eta \subset G$$

are continuous and

$$(5)\ \tilde{E}_\eta(V) = \bar{exp}_{\eta(x)} v(\eta(x)),$$

where $x \in M$, consequently, $\tilde{E}$ is of class $C(\infty)$.

2.4. Note. Let $\Omega^{\{k\}}_\xi(M, N)$ be the same submonoid as in §I.3.5 such that $c > 0$ and $c' > 0$. Then it generates the loop group $G' := L^{\{k\}}_\xi(M, N)$ as in §2.1 such that G' is the dense subgroup in $G = L_\xi(M, N)$.

2.5. Theorem. *On the group $G = L_\xi(M, N)$ from §2.1 and for each $b \in$ **C** there exist probability quasi-invariant and pseudo-differentiable of order b measures μ with values in* **R** *and* $\mathbf{K_q}$ *for each prime number q such that $q \ne p$ relative to a dense subgroup G'.*

Proof. In view of Theorem 2.3 it is sufficient to consider the case of M with the finite atlas $At'(M)$. Let the operator $\tilde{A}$ be defined on $TC^0_0(\xi, (M, s_0) \to (N, y_0))$ by Formulas I.3.6.(3,4). The factorization by the equivalence relation $\tilde{K}_\xi$ from §I.3.6 and the Grothendieck construction of §2.1 produces the following mapping $\tilde{\Upsilon}$ from the corresponding neighbourhood of the zero section in $TL_\xi(M,N)$ into a neighbourhood of the zero section either in $TL_{\xi'}(M,Y)$ for $dim_{\mathbf{K}}M < \infty$ or into $c_0(\{TL_{\xi'}(M_a,Y) : a \in \mathbf{N}\})$ for $dim_{\mathbf{K}}M = \aleph_0$.

Therefore they are continuously strongly differentiable with $(D\tilde{\Upsilon}(f))(v) = \tilde{\Upsilon}(f)(v)$, where f and $v \in V_N \subset T_e L_\xi(M,N)$, V_N is the corresponding neighbourhoods of zero sections for the element $e =< \omega_0 >_{K,\xi}$. In view of the existence of the mapping $\tilde{E}$ (see Formulas 2.3.(4,5)) for $\tilde{T}G$ there exists the local diffeomorphism

$$(1) \Upsilon : W_e \to V'_0$$

induced by $\tilde{E}$ and $\tilde{\Upsilon}$, where W_e is a neighbourhood of e in G, V'_0 is a neighbourhood of zero either in the Banach subspace $\tilde{H}$ of $T_e L_{\xi'}(M,Y)$ for $dim_{\mathbf{K}}M < \infty$ or in the Banach subspace $\tilde{H}$ of $c_0(\{T_e L_{\xi'}(M_a,Y) : a \in \mathbf{N}\})$ for $dim_{\mathbf{K}}M = \aleph_0$.

Let now W'_e be a neighbourhood of e in G' such that $W'_e W_e = W_e$. It is possible, since the topology in G and G' is given by the corresponding ultrametrics and there exists W_e with $W_e W_e = W_e$, hence it is sufficient to take $W'_e \subset W_e$. For $g \in W_e$, $v = \tilde{E}^{-1}(g)$, $\phi \in W'_\xi$ the following operator

$$(2)\ S_\phi(v) := \Upsilon \circ L_\phi \circ \Upsilon^{-1}(v) - v$$

is defined for each $(\phi, v) \in W'_e \times V'_0$, where $L_\phi(g) := \phi \circ g$. Then $S_\phi(v) \in V''_0 \subset V'_0$, where V''_0 is an open neighbourhood of the zero section either in the Banach subspace $\tilde{H}'$ of $T_e G'$ for $dim_{\mathbf{K}}M < \infty$ or in the Banach subspace $\tilde{H}'$ of $c_0(\{T_e G'_a : a \in \mathbf{N}\})$ for $dim_{\mathbf{K}}M = \aleph_0$, where $G'_a = L^{\{k\}}_\xi(M_a,N)$. Moreover, $S_\phi(v)$ is the $C(\infty)$-mapping by ϕ and v. As in §I.3.6 a quasi-invariant and pseudo-differentiable of order b measure ν on $V'_0 \subset \tilde{H}$ exists relative to $\phi \in W'_e$, where

$$(3)\ \nu(dx) = \bigotimes_{j=1}^{\infty} \nu_{l(j)}(dx^j)$$

and Conditions I.3.6.(13,14,17-20) are satisfied.

More general classes of quasi-invariant and pseudo-differentiable of order b measures ν with values in $[0,\infty)$ or in $\mathbf{K_q}$ exist on V'_0 relative to the action of $\phi \in W'_e$, $(\phi, v) \mapsto v + S_\phi(v)$, where $v \in V'_0$.

In view of Formulas $(1-3)$ the measure ν induces a measure $\tilde{\mu}$ on W_e with the help of Υ such that

$$(4)\ \tilde{\mu}(A) = \nu(\Upsilon(A)) \text{ for each } A \in Bf(W_e),$$

since $\|\nu\|(V_0') > 0$. The groups G and G' are separable and ultrametrizable, hence there are locally finite coverings $\{\phi_i \circ W_i : i \in \mathbf{N}\}$ of G and $\{\phi_i \circ W'_i : i \in \mathbf{N}\}$ of G' with $\phi_i \in G'$ such that W_i are open subsets in W_e and W'_i are open subsets in W'_e, that is,

$$\bigcup_{i=1}^{\infty} \phi_i \circ W_i = G \text{ and } \bigcup_{i=1}^{\infty} \phi_i \circ W'_i = G',$$

where $\phi_1 = e$, $W_1 = W_e$ and $W'_1 = W'_e$ [6]. Then $\tilde{\mu}$ can be extended onto G by the following formula

$$(5)\ \mu(A) := (\sum_{i=1}^{\infty} \tilde{\mu}((\phi_i^{-1} \circ A) \cap W_i) r^i) / (\sum_{i=1}^{\infty} \tilde{\mu}(W_i) r^i)$$

for each $A \in Bf(G)$, where $0 < r < 1$ for real $\tilde{\mu}$ or $r = q$ for $\tilde{\mu}$ with values in $\mathbf{K_q}$. In view of Formulas $(4, 5)$ this μ is the desired measure, which is quasi-invariant and pseudo-differentiable of order b relative to the subgroup $G'' = G'$ (see also §§I.3.2-4).

3 Representations of loop groups.

3.1. Let μ be a real non-negative quasi-invariant relative to G' measure on $(G, Bf(G))$ as in Theorem 2.5. Assume also that $H := L^2(G, \mu, \mathbf{C})$ is the standard Hilbert space of equivalence classes of functions $f : G \to \mathbf{C}$ for which absolute values $|f|$ are square-integrable by μ. Suppose that $U(H)$ is the unitary group on H in a topology induced from a Banach space $L(H \to H)$ of continuous linear operators supplied with the operator norm.

Theorem. *There exists a strongly continuous injective homomorphism* $T : G' \to U(H)$.

Proof. Let f and h be in H, their scalar product is given by the standard formula

$$(1)\ (f, h) := \int_G \bar{h}(g) f(g) \mu(dg),$$

where f and $h : G \to \mathbf{C}$, $\bar{h}$ denotes the complex conjugated function h. There exists the regular representation

$$(2)\ \mathsf{T} :\ G' \to U(H)$$

defined by the following formula:

$$(3)\ \mathsf{T}_z f(g) := [\rho(z,g)]^{1/2} f(z^{-1}g),$$

where

$$(4)\ \rho(z,g) = \mu_z(dg)/\mu(dg),\ \mu_z(S) := \mu(z^{-1}S)$$

for each $S \in Bf(G)$, $z \in G'$. For each fixed z the quasi-invariance factor $\rho(z,g)$ is continuous by g, hence $\mathsf{T}_z f(g)$ is measurable, if $f(g)$ is measurable (relative to $Af(G,\mu)$ and $Bf(\mathbf{C})$). Therefore,

$$(5)\ (\mathsf{T}_z f(g), \mathsf{T}_z h(g)) = \int_G \bar{h}(z^{-1}g) f(z^{-1}g)\rho(z,g)\mu(dg) = (f,h),$$

consequently, T_z is the unitary operator for each $z \in G'$. From

$$(6)\ \rho(z'z,g) = \rho(z,(z')^{-1}g)\rho(z',g) = [\mu_{z'z}(dg)/\mu_{z'}(dg)][\mu_{z'}(dg)/\mu(dg)]$$

it follows that

$$(7)\ \mathsf{T}_{z'}\mathsf{T}_z = \mathsf{T}_{z'z},\ \mathsf{T}_{id} = I \text{ and } \mathsf{T}_{z^{-1}} = \mathsf{T}_z^{-1},$$

where I is the unit operator on H.

The embedding of T_eG' into T_eG is the compact operator. The measure μ on G is induced by the measure on $c_0(\omega_0,\mathbf{K})$, where ω_0 is the first countable ordinal. In view of Theorems 3.12 and 3.28 [13] for each $\delta > 0$ and $\{f_1,...,f_n\} \subset H$ there exists a compact subset B in G such that

$$(8)\ \sum_{i=1}^n \int_{G\setminus B} |f_i(g)|^2 \mu(dg) < \delta^2.$$

Therefore, there exists an open neighbourhood W' of e in G' and an open neighbourhood S of e in G such that $\rho(z,g)$ is continuous and bounded on $W' \times W' \circ S$, where $S \subset W' \circ S \subset G$. In view of Formulas (5-8), Theorems 2.3 and 2.5 and the Hölder inequality we have

$$\lim_{j\to\infty} \sum_{i=1}^n \|(\mathsf{T}_{z_j} - I)f_i\|_H = 0$$

for each sequence $\{z_j : j \in \mathbf{N}\}$ converging to e in G'. Indeed, for each $v > 0$ and a continuous function $f : G \to \mathbf{C}$ with $\|f\|_H = 1$ there is an open neighbourhood V of id in G' (in the topology of G'), such that $|\rho(z,g)-1| < v$ for each $z \in V$ and each $g \in F$ for some open F in G, $id \in F$ with

$$\mu_z^f(G \setminus F) < v \text{ for each } z \in V, \text{ where } \mu^f(dg) := |f(g)|\mu(dg)$$

and $f \in \{f_1, ..., f_n\}$, $n \in \mathbf{N}$. At first this can be done analogously for the corresponding Banach space from which μ was induced.

In H continuous functions $f(g)$ are dense, hence for each $0 < v < 1$ there exists V" such that

$$\int_G |f(g) - f(zg)(\rho(z,g))^{1/2}|^2 \mu(dg) < 4v$$

for each finite family $\{f_j\}$ with $\|f_j\|_H = 1$ and $z \in V' = V \cap V$", where V" is an open neighbourhood of id in G' such that $\|f(g) - f(zg)\|_H < v$ for each $z \in V$", consequently T is strongly continuous (that is, T is continuous relative to the strong topology on $U(H)$ induced from $L(H \to H)$, see its definition in [8]).

Moreover, T is injective, since for each $g \neq id$ there is $f \in C^0(G, \mathbf{C}) \cap H$, such that $f(id) = 0$, $f(g) = 1$, and $\|f\|_H > 0$, so $\mathrm{T}_f \neq I$.

Note. In general T is not continuous relative to the norm topology on $U(H)$, since for each $z \neq id \in G'$ and each $1/2 > v > 0$ there is $f \in H$ with $\|f\|_H = 1$, such that $\|f - \mathrm{T}_z f\|_H > v$, when $supp(f)$ is sufficiently small with $(z \circ supp(f)) \cap supp(f) = \emptyset$.

3.2. Theorem. *Let G be a loop group with a real probability quasi-invariant measure μ relative to a dense subgroup G' as in Theorem 2.5. Then μ may be chosen such that the associated regular unitary representation (see §3.1) of G' is irreducible.*

Proof. Let ν on $c_0(\omega_0, \mathbf{K})$ be of the same type as in §3.23 or §3.30 [13] or it is given by Formulas I.3.6.(13-20). For example, ν is generated by a weak distribution such that

$$(1)\ \nu_j(dx^j) := c_j exp(-|x^j/\xi^j|^\gamma) v(dx^j),$$

where $c_j > 0$, $\nu_j(\mathbf{K}) = 1$, v is the Haar non-negative measure on $\mathbf{K}$,

$$(2)\ \lim_{j\to\infty} \xi^j = 0,$$

$0 \neq \xi^j \in \mathbf{K}$, $\gamma > 0$ is fixed with

$$(3)\ \sum_{j=1}^{\infty} |\xi^j|^{-\gamma} p^{-k(i_j, m_j)} < \infty$$

(see about $k(i,m)$ in §I.3.5). Let a ν-measurable function $f : c_0(\omega_0, \mathbf{K}) \to \mathbf{C}$ be such that $\nu(\{x \in c_0(\omega_0,\mathbf{K}) : f(x+y) \ne f(x)\} = 0$ for each $y \in sp_{\mathbf{K}}(e_j : j \in \mathbf{N}) =: X_o$ with $f \in L^1(c_0(\omega_0,\mathbf{K}),\nu,\mathbf{C})$. Let also $P_k : c_0(\omega_0,\mathbf{K}) \to L(k)$ be projectors such that $P_k(x) = x_k$ for each $x = (\sum_{j\in\mathbf{N}} x^j e_j)$, where $x_k := \sum_{j=1}^k x^j e_j$ and $L(k) := sp_{\mathbf{K}}(e_1,...,e_k)$. Then analogously to the proof of Proposition II.3.1 [4] in view of Fubini theorem there exists a sequence of cylindrical functions

$$(4)\ f_k(x) = f_k(x_k) = \int_{c_0(\omega_0,\mathbf{K})\ominus L(k)} f(P_k x + (I-P_k)y)\nu_{I-P_k}(dy)$$

which converges to f in $L^1(c_0(\omega_0,\mathbf{K}),\nu,\mathbf{C})$, where $\nu = \nu_{L(k)} \otimes \nu_{I-P_k}$, ν_{I-P_k} is the measure on $c_0(\omega_0,\mathbf{K}) \ominus L(k)$. Each cylindrical function f_k is ν-almost everywhere constant on $c_0(\omega_0,\mathbf{K})$, since $L(k) \subset X_o$ for each $k \in \mathbf{N}$, consequently, f is ν-almost everywhere constant on $c_0(\omega_0,\mathbf{K})$. Let Υ be the local diffeomorphism from Formula 2.5.(1). In view of Theorems 5.13 and 5.16 [16] these Banach spaces are topologically K-linearly isomorphic with $c_0(\omega_0,\mathbf{K})$. From the construction of G' and μ with the help of Υ and ν as in §2.5 it follows that if a function $f \in L^1(G,\mu,\mathbf{C})$ satisfies the following condition $f^h(g) = f(g)\ (mod\ \mu)$ by $g \in G$ for each $h \in G'$, then $f(x) = const\ (mod\ \mu)$, where $f^h(g) := f(hg)$, $g \in G$.

Let $f(g) = ch_U(g)$ be the characteristic function of a subset U, $U \subset G$, $U \in Af(G,\mu)$, then $f(hg) = 1 \Leftrightarrow g \in h^{-1}U$. If $f^h(g) = f(g)$ is true by $g \in G$ μ-almost everywhere, then

$$(5)\ \mu(\{g \in G : f^h(g) \ne f(g)\}) = 0,$$

that is $\mu((h^{-1}U)\Delta U) = 0$, consequently, the measure μ satisfies the condition (P) from §VIII.19.5 [8], where $A\Delta B := (A\setminus B)\cup(B\setminus A)$ for each $A, B \subset G$. For each subset $E \subset G$ the outer measure $\mu^*(E) \le 1$, since $\mu(G) = 1$ and μ is non-negative [2], consequently, there exists $F \in Bf(G)$ such that $F \supset E$ and $\mu(F) = \mu^*(E)$. This F may be interpreted as the least upper bound in $Bf(G)$ relative to the latter equality. In view of Proposition VIII.19.5 [8] the measure μ is ergodic, that is for each $U \in Af(G,\mu)$ and $F \in Af(G,\mu)$ with $\mu(U) \times \mu(F) \ne 0$ there exists $h \in G'$ such that $\mu((h \circ E) \cap F) \ne 0$.

From Theorem I.1.2 [4] it follows that $(G, Bf(G)$ is a Radon space, since G is separable and complete. Therefore, a class of compact subsets approximates from below each measure $|f(g)|\mu(dg)$, where $f \in L^2(G,\mu,\mathbf{C})$. Due to

Egorov Theorem 2.3.7 [7] for each $\epsilon > 0$ and for each sequence $f_n(g)$ converging to $f(g)$ for μ-almost every $g \in G$, when $n \to \infty$, there exists a compact subset K in G such that $\mu(G \setminus \mathsf{K}) < \epsilon$ and $f_n(g)$ converges on K uniformly by $g \in \mathsf{K}$, when $n \to \infty$. Hence in view of the Stone-Weierstrass Theorem A.8 [8] an algebra $\mathsf{V}(Q)$ of finite pointwise products of functions from the following space

$$(6)\ sp_{\mathbf{C}}\{\psi(g) := \rho^{1/2}(h,g) : h \in G'\} =: Q$$

is dense in H, since $\rho(e,g) = 1$ for each $g \in G$ and $L_h : G \to G$ are diffeomorphisms of the manifold G, where $L_h(g) := hg$.

For each $m \in \mathbf{N}$ there are locally analytic curves $S(\zeta, \phi_j)$ in G' with analytic restrictions $S(\zeta,\phi_j)|_{B(\mathbf{K},0,1)}$, where $j = 1,...,m$ and $\zeta \in \mathbf{K}$ is a parameter, such that

$$S(0,\phi_j) = e \text{ and } (\partial S(\zeta,\phi_j)/\partial\zeta)|_{\zeta=0} \text{ are linearly independent in } T_eG'$$

for $j = 1,...,m$, since G' is the infinite-dimensional group, which is complete relative to its own uniformity. In accordance with §2.5 there exists infinitely pseudo-differentiable μ on G (that is, of order l for each $l \in \mathbf{N}$) relative to $S(\zeta,\phi_j)$ for each j. If two real non-negative quasi-invariant relative to G' measures μ and λ on G are equivalent, then the corresponding regular representations T^μ and T^λ are equivalent, since the mapping

$$f(g) \mapsto (\mu(dg)/\lambda(dg))^{1/2} f(g)$$

establishes an isomorphism of $L^2(G,\mu,\mathbf{C})$ with $L^2(G,\lambda,\mathbf{C})$, where $f \in L^2(G,\mu,\mathbf{C})$. Then the following condition $det(\Psi(g)) = 0$ defines an analytic submanifold G_Ψ in G of codimension over $\mathbf{K}$ no less than one:

$$(7)\ codim_{\mathbf{K}} G_\Psi \geq 1,$$

where $\Psi(g)$ is a matrix function of the variable $g \in G$ with matrix elements

$$(8)\ \Psi_{l,j}(g) := PD_c(l, \rho^{1/2}(S(\zeta,\phi_j), g))$$

for $l \geq 1$. If $f \in H$ is such that

$$(9)\ (f(g), \rho^{1/2}(\phi, g))_H = 0$$

for each $\phi \in G' \cap W$, then

$$(10)\ PD_c(l,(f(g),\rho^{1/2}(S(\zeta,\phi_j),g))_H)=0.$$

But $V(Q)$ is dense in H and in view of Formulas $(6-10)$ this means that $f=0$, since for each m there are $S(\zeta,\phi_j) \in G' \cap W$ such that $det\Psi(g) \neq 0$ μ-almost everywhere on G. If $\|f\|_H > 0$, then $\mu(supp(f)) > 0$, consequently, $\mu((G' supp(f)) \cap W) = 1$, since $G'U = G$ for each open U in G and for each $\epsilon > 0$ there exists an open U such that $U \supset supp(f)$ and $\mu(U \setminus supp(f)) < \epsilon$.

Therefore, Q is dense in H. This means that the unit vector f_0 is cyclic, where $f_0 \in H$ and $f_0(g) = 1$ for each $g \in G$. The group G is Abelian, hence there exists a unitary operator $U : H \to H$ such that

$$(11)\ U^{-1}T_hU = F_h$$

are operators of multiplication on functions $F_h \in L^\infty(G,\mu,\mathbf{C})$ for each $h \in G'$, where

$$(12)\ F_h(g) = exp(2\pi i f_h(g)),$$

$g \in G$, $f_h \in L^0(G,\mu,\mathbf{R})$, $L^0(G,\mu,\mathbf{R})$ is a Frechét space of classes of equivalent μ-measurable functions $f : G \to \mathbf{R}$, which is supplied with a metric

$$(13)\ d(f,v) := \int_G \min(1,|f(g)-v(g)|)\mu(dg),$$

$i = (-1)^{1/2}$ (see §IV.8 and Theorem X.2.1 and Theorem X.4.2 and Segal Theorem in §X.9 [5]). The following set $(cl\ sp_{\mathbf{C}}\{F_h : h \in G'\})$ is not contained in any ideal of the form $\{F :\ supp(F) \subset G \setminus A\}$ with $A \in Af(G,\mu)$ and $\mu(A) > 0$, since $|F_h(g)| = 1$ for each $(h,g) \in G' \times G$, where $cl(E)$ is taken in $L^\infty(G,\mu,\mathbf{C})$ for its subset E. Then $\{F_h : h \in G'\}$ is not contained in any set

$$\{F = exp(2\pi i f) :\ f \in L^0(G,\mu,\mathbf{C}),\ supp(f) \subset G \setminus A\}$$

with $A \in Af(G,\mu)$ and $\mu(A) > 0$, since μ is ergodic relative to G'. From the construction of μ (see Formulas (1-3) and I.3.6.(13-17,21-24)) it follows that for each $f_{1,j}$ and $f_{2,j} \in H$, $j = 1,...,n$, $n \in \mathbf{N}$ and each $\epsilon > 0$ there exists $h \in G'$ such that

$$|(T_h f_{1,j}, f_{2,j})_H| \leq \epsilon |(f_{1,j}, f_{2,j})_H|,$$

when $|(f_{1,j}, f_{2,j})_H| > 0$, hence

$$|(F_h U^{-1} f_{1,j}, U^{-1} f_{2,j})_H| \le \epsilon |(U^{-1} f_{1,j}, U^{-1} f_{2,j})_H| = \epsilon |(f_{1,j}, f_{2,j})_H|,$$

since G is the Radon space by Theorerm I.1.2 [4] and G is not locally compact. Therefore, for each $\tilde{f}_{1,j}$ and $\tilde{f}_{2,j} \in H$, $j = 1, ..., n$, $n \in \mathbf{N}$ and $\epsilon > 0$ there exists $h \in G'$ for which $|(F_h \tilde{f}_{1,j}, \tilde{f}_{2,j})_H| \le \epsilon |(\tilde{f}_{1,j}, \tilde{f}_{2,j})_H|$ for each $j = 1, ..., n$, when $|(\tilde{f}_{1,j}, \tilde{f}_{2,j})_H| > 0$, since $UH = H$. This means that there is not any finite-dimensional G'-invariant subspace H' in H, that is, $F_h H' \subset H'$ for each $h \in G'$.

We suppose that λ is a probability Radon measure on G' such that λ has not any atoms and $supp(\lambda) = G'$. In view of the strong continuity of the regular representation there exists the S. Bochner integral $\int_G T_h f(g) \mu(dg)$ for each $f \in H$, which implies its existence in the weak (B. Pettis) sence. The measures μ and λ are non-negative and bounded, hence $H \subset L^1(G, \mu, \mathbf{C})$ and $L^2(G', \lambda, \mathbf{C}) \subset L^1(G', \lambda, \mathbf{C})$ due to the Cauchy inequality. Therefore, we can apply below Fubini theorem (see §II.16.3 [8]). Let $f \in H$, then there exists a countable orthonormal base $\{f^j : j \in \mathbf{N}\}$ in $H \ominus \mathbf{C}f$. Then for each $n \in \mathbf{N}$ the following set

$$B_n := \{q \in L^2(G', \lambda, \mathbf{C}) : \ (f^j, f)_H = \int_{G'} q(h)(f^j, \mathrm{T}_h f_0)_H \lambda(dh) \text{ for } j = 0, ..., n\}$$

is non-empty, since the unit vector f_0 is cyclic, where $f^0 := f$. There exists $\infty > R > \|f\|_H$ such that $B_n \cap B^R =: B_n^R$ is non-empty and weakly compact for each $n \in \mathbf{N}$, since B^R is weakly compact, where

$$B^R := \{q \in L^2(G', \lambda, \mathbf{C}) : \|q\| \le R\}$$

(see the Alaoglu-Bourbaki theorem in §(9.3.3) [15]). Therefore, B_n^R is a centered system of closed subsets of B^R, that is,

$$\cap_{n=1}^m B_n^R \ne \emptyset \text{ for each } m \in \mathbf{N},$$

hence it has a non-empty intersection, consequently, there exists $q \in L^2(G', \lambda, \mathbf{C})$ such that

$$(14)\ f(g) = \int_{G'} q(h) \mathrm{T}_h f_0(g) \lambda(dh)$$

for μ-almost each $g \in G$. If $F \in L^\infty(G, \mu, \mathbf{C})$, f_1 and $f_2 \in H$, then there exist q_1 and $q_2 \in L^2(G', \lambda, \mathbf{C})$ satisfying equation (14). Therefore,

$$(15)\ (f_1, Ff_2)_H = \int_G \int_{G'} \int_{G'} \bar{q}_1(h_1) q_2(h_2) F(g) \rho_\mu^{1/2}(h_1, g) \rho_\mu^{1/2}(h_2, g) \lambda(dh_1) \lambda(dh_2) \mu(dg).$$

Let

(16) $\xi(h) := \int_G \int_{G'} \int_{G'} \bar{q}_1(h_1) q_2(h_2) \rho_\mu^{1/2}(h_1, g) \rho_\mu^{1/2}(hh_2, g) \lambda(dh_1) \lambda(dh_2) \mu(dg)$.

Then there exists $\beta(h) \in L^2(G', \lambda, \mathbf{C})$ such that

$$(17) \int_{G'} \beta(h) \xi(h) \lambda(dh) = (f_1, F f_2)_H =: c.$$

To prove this we consider two cases. If $c = 0$ it is sufficient to take β orthogonal to ξ in $L^2(G', \lambda, \mathbf{C})$. Each function $q \in L^2(G', \lambda, \mathbf{C})$ can be written as $q = q^1 - q^2 + iq^3 - iq^4$, where $q^j(h) \geq 0$ for each $h \in G'$ and $j = 1, ..., 4$, hence we obtain the corresponding decomposition for ξ:

$$(18)\ \xi = \sum_{j,k} b^{j,k} \xi^{j,k},$$

where $\xi^{j,k}$ corresponds to a pair (q_1^j, q_2^k), where $b^{j,k} \in \{1, -1, i, -i\}$. If $c \neq 0$ we can choose (j_0, k_0) for which $\xi^{j_0,k_0} \neq 0$ and

(19) β is orthogonal to others $\xi^{j,k}$ with $(j, k) \neq (j_0, k_0)$.

Otherwise, if $\xi^{j,k} = 0$ for each (j, k), then $q_l^j(h) = 0$ for each (l, j) and λ-almost every $h \in G'$, since due to Formula (16):

$$\xi(0) = \int_G \mu(dg) (\int_{G'} \bar{q}_1(h_1) \rho_\mu^{1/2}(h_1, g) \lambda(dh_1)) (\int_{G'} q_2(h_2) \rho_\mu^{1/2}(h_2, g) \lambda(dh_2)) = 0$$

and this implies $c = 0$, which is the contradiction with the assumption $c \neq 0$. Hence due to Formula (18) there exists β satisfying Formula (17) and Condition (19).

Since $L^2(G', \lambda, \mathbf{C})$ is infinite-dimensional, then for each finite families

$$\{a_1, ..., a_m\} \subset L^\infty(G, \mu, \mathbf{C}) \text{ and } \{f_1, ..., f_m\} \subset H$$

there exists $\beta(h) \in L^2(G', \lambda, \mathbf{C})$, such that

$$\beta \text{ is orthogonal to } \int_G \bar{f}_s(g) [f_j(h^{-1}g)(\rho_\mu(h, g))^{1/2} - f_j(g)] \mu(dg)$$

for each $s, j = 1, ..., m$. Hence each operator of multiplication on $a_j(g)$ belongs to A_G", since due to Formula (17) and Condition (19) there exists $\beta(h)$ such that

$$(f_s, a_j f_l) = \int_G \int_{G'} \bar{f}_s(g)\beta(h)(\rho_\mu(h,g))^{1/2} f_l(h^{-1}g)\lambda(dh)\mu(dg)$$

$$= \int_G \int_{G'} \bar{f}_s(g)\beta(h)(\mathrm{T}_h f_l(g))\lambda(dh)\mu(dg) \text{ and}$$

$$\int_G \bar{f}_s(g) a_j(g) f_l(g)\mu(dg) = \int_G \int_{G'} \bar{f}_s(g)\beta(h) f_l(g)\lambda(dh)\mu(dg) = (f_s, a_j f_l).$$

Hence A_G" contains subalgebra of all operators of multiplication on functions from $L^\infty(G, \mu, \mathbf{C})$.

Let us remind the following. A Banach bundle B over a Hausdorff space G' is a bundle $< B, \pi >$ over G', together with operations and norms making each fiber B_h $(h \in G')$ into a Banach space such that

$BB(i)$ $x \mapsto \|x\|$ is continuous from B into $\mathbf{R}$;

$BB(ii)$ the operation $+$ is continuous as a function from $\{(x,y) \in B \times B : \pi(x) = \pi(y)\}$ into B;

$BB(iii)$ for each $\lambda \in \mathbf{C}$ the map $x \mapsto \lambda x$ is continuous from$B$ into B;

$BB(iv)$ if $h \in G'$ and $\{x_i\}$ is any net of elements of B such that $\|x_i\| \to 0$ and $\pi(x_i) \to h$ in G', then $x_i \to 0_h$ in B,

where $\pi : B \to G'$ is a bundle projection, $B_h := \pi^{-1}(h)$ is the fiber over h (see §II.13.4 [8]). If G' is a Hausdorff topological group, then a Banach algebraic bundle over G' is a Banach bundle $\mathsf{B} =< B, \pi >$ over G' together with a binary operation $\bullet$ on B satisfying the following Conditions $AB(i - v)$:

$AB(i)$ $\pi(b \bullet c) = \pi(b)\pi(c)$ for b and $c \in B$;

$AB(ii)$ for each x and $y \in G'$ the product $\bullet$ is bilinear from $B_x \times B_y$ into B_{xy};

$AB(iii)$ the product $\bullet$ on B is associative;

$AB(iv)$ $\|b \bullet c\| \le \|b\| \times \|c\|$ for each $b, c \in B$;

$AB(v)$ the map $\bullet$ is continuous from $B \times B$ into B

(see §VIII.2.2 [8]). With G' and a Banach algebra A the trivial Banach bundle $\mathsf{B} = \mathsf{A} \times G'$ is associative, in particular let $\mathsf{A} = \mathbf{C}$ (see §VIII.2.7 [8]).

The regular representation T of G' gives rise to a canonical regular $L^2(G, \mu, \mathbf{C})$-projection-valued measure $\bar{P}$:

$$(20)\quad \bar{P}(W)f := Ch_W f,$$

where $f \in L^2(G, \mu, \mathbf{C})$, $W \in Bf(G)$, Ch_W is the characteristic function of W. Therefore,

$$(21)\ \mathsf{T}_h\bar{P}(W) = \bar{P}(h \circ W)\mathsf{T}_h$$

for each $h \in G'$ and $W \in Bf(G)$, since $\rho(h, h^{-1} \circ g)\rho(h, g) = 1 = \rho(e, g)$ for each $(h, g) \in G' \times G$,

$$Ch_W(h^{-1} \circ g) = Ch_{h\circ W}(g) \text{ and}$$

$$(22)\ \mathsf{T}_h(\bar{P}(W)f(g)) = \rho(h^{-1}, g)^{1/2}\bar{P}(h \circ W)f(h^{-1} \circ g).$$

Thus $< T, \bar{P} >$ is a system of imprimitivity for G' over G, which is denoted T^μ, that is,

$SI(i)$ T is a unitary representation of G';

$SI(ii)$ $\bar{P}$ is a reguialr $L^2(G, \mu, \mathbf{C})$–projection-valued Borel measure on G and

$SI(iii)$ $\mathsf{T}_h\bar{P}(W) = \bar{P}(h \circ W)\mathsf{T}_h$ for all $h \in G'$ and $W \in Bf(G)$.

For each $F \in L^\infty(G, \mu, \mathbf{C})$ let α_F be the operator in $L(L^2(G, \mu, \mathbf{C}))$ consisting of multiplication by F:

$$\alpha_F(f) = Ff \text{ for each } f \in L^2(G, \mu, \mathbf{C}),$$

where $L(Z) := L(Z \to Z)$ (see §3.1). The map $F \mapsto \alpha_F$ is an isometric $*$-isomorphism of $L^\infty(G, \mu, \mathbf{C})$ into $L(L^2(G, \mu, \mathbf{C}))$ (see §VIII.19.2[8]). Therefore, using the approach of this particular case given above we get, that Propositions VIII.19.2,5[8] are applicable in our situation.

If $\bar{p}$ is a projection onto a closed H^μ-stable subspace of $L^2(G, \mu, \mathbf{C})$, then due to Formulas (20-22) $\bar{p}$ commutes with all $\bar{P}(W)$. Hence $\bar{p}$ commutes with α_F for each $F \in L^\infty(G, \mu, \mathbf{C})$, so by §VIII.19.2 [8] $\bar{p} = \bar{P}(V)$, where $V \in Bf(G)$. Also $\bar{p}$ commutes with T_h for each $h \in G'$, consequently, $(h \circ V) \setminus V$ and $(h^{-1} \circ V) \setminus V$ are μ-null for each $h \in G'$, hence $\mu((h \circ V) \Delta V) = 0$

for all $h \in G'$. In view of the ergodicity of μ and Proposition VIII.19.5 [8] either $\mu(V) = 0$ or $\mu(G \setminus V) = 0$, hence either $\bar{p} = 0$ or $\bar{p} = I$.

3.3. Theorem. *On the loop group $G = L_\xi(M, N)$ from §2.1 there exists a family of continuous characters $\{\Xi\}$, which separate points of G.*

Proof. In view of Lemma I.2.17 it is sufficient to consider the case of the submanifold $\tilde{M}$ having no more than two charts. Then $\tilde{\tilde{M}}$ is clopen in $c_0(\alpha, \mathbf{K})$, where $\tilde{\tilde{M}} = \tilde{M} \setminus \{s_0\}$.

Let at first $dim_{\mathbf{K}} M < \aleph_0$. The Haar measure $\lambda_\alpha : Bf(\mathbf{K}^\alpha) \to \mathbf{Q_q}$ with a prime number $q \ne p$ (see the Monna-Springer theorem in §8.4 [16]) induces the measure $\lambda_\alpha : Bf(\tilde{M}) \to \mathbf{Q_q}$, analogously for

$$(1)\ N_J := N \cap sp_{\mathbf{K}}\{e_j :\ j \in J\}$$

for each $\mathbf{N} \ni n \ge \alpha$ and $h \in L(N_J, \lambda_n, \mathbf{Q_q})$ there corresponds a measure $\nu_{J,h}$ on $Bf(N_J)$ for which

$$(2)\ \nu_{J,h}(dy) = h(y)\lambda_n(dy)$$

and to $\nu_{J,h}$ there corresponds a differential form

$$(3)\ w_{J,h}(y) = h(y)dy^{j_1} \wedge \ldots \wedge dy^{j_n},$$

where $y \in N_J$ and $J := \{j_1, \ldots, j_n\}$. Hence there exists its pull back $(\pi_J \tilde{f})^* w$, where $\pi_J : c_0(\beta, \mathbf{K}) \to sp_{\mathbf{K}}\{e_j : j \in J\}$ is the projection for each $J \subset \beta$, $f \in C^0_0(\xi, \tilde{M} \to N)$, $\tilde{f} = P(l, s+1)f$, $l = [t] + 1$ (see §I.2.11 and Corollary I.2.16).

As usually, for a mapping $h : \tilde{M} \to N_J$ of class $C(1)$ and a tensor T of the type $(0, k)$ with components $T_{i_1,\ldots,i_k}$ defined for N_J we have:

$$(4)\ (h^*T)_{l_1,\ldots,l_k}(x^1, \ldots, x^\alpha) = [\sum_{i_1,\ldots,i_k} T_{i_1,\ldots,i_k}(\partial y^{i_1}/\partial x^{l_1})\ldots(\partial y^{i_k}/\partial x^{l_k})](y(x^1, \ldots, x^\alpha))$$

such that h^*T is defined for $\tilde{M}$, where $(x^1, .., x^\alpha)$ are coordinates in $\tilde{M}$ induced from $\mathbf{K}^\alpha$ and $(y^1, \ldots, y^n) = y$ are coordinates in N_J induced from $\mathbf{K}^n$, $y^j = y^j(x^1, \ldots, x^\alpha) = h^j(x^1, \ldots, x^\alpha)$, x^j and $y^j \in \mathbf{K}$.

Let now $dim_{\mathbf{K}} M = dim_{\mathbf{K}} N = \aleph_0$. Let λ be equivalent with a probability $\mathbf{Q_q}$-valued measure either on the entire T_yN or on its Banach infinite-dimensional over $\mathbf{K}$ subspace P (see Formulas I.3.6.(13-20)). Each such λ

induces a family of probability measures ν on $Bf(N)$ or its cylinder subalgebra induced by the projection of T_yN onto P, which may differ by their supports.

Let $T_yN =: \mathsf{L}$ be an infinite-dimensional separable Banach space over $\mathbf{K}$, so there exists a topological vector space $\mathsf{L}^{\mathbf{N}} := \prod_{j=1}^{\infty} \mathsf{L}_j$, where $\mathsf{L}_j = \mathsf{L}$ for each $j \in \mathbf{N}$ [15]. Consider a subspace Λ^∞ of a space of continuous ∞-multilinear functionals $\eta : \mathsf{L}^{\mathbf{N}} \to \mathbf{K}$ such that

$$\eta(x+y) = \eta(x) + \eta(y),\ \eta(\sigma x) = (-1)^{|\sigma|}\eta(x) \text{ and } \eta(x) = \lambda\eta(z)$$

for each $x, y \in \mathsf{L}^{\mathbf{N}}$, $\sigma \in S_\infty$ and $\lambda \in \mathbf{K}$, where

$$x = \{x^j :\ x^j \in \mathsf{L}, j \in \mathbf{N}\} \in \mathsf{L}^{\mathbf{N}},\ z^j = x^j \text{ for each } j \ne k_0 \text{ and } \lambda z^{k_0} = x^{k_0},$$

S_∞ is a group of all bijections $\sigma : \mathbf{N} \to \mathbf{N}$ such that $card\{j : \sigma(j) \ne j\} < \aleph_0$, $|\sigma| = 1$ for $\sigma = \sigma_1 ... \sigma_n$ with odd $n \in \mathbf{N}$ and pairwise transpositions $\sigma_l \ne I$, that is,

$$\sigma_l(j_1) = j_2,\ \sigma_l(j_2) = j_1 \text{ and } \sigma_l|_{\mathbf{N}\setminus\{j_1, j_2\}} = I$$

for the corresponding $j_1 \ne j_2$, $|\sigma| = 2$ for even n or $\sigma = I$. Then Λ^∞ (or Λ^j) induces a vector bundle $\Lambda^\infty N$ (or $\Lambda^j N$) on a manifold N of ∞-multilinear skew-symmetric mappings over $\mathsf{F}(N)$ of $\Psi(N)^\infty$ (or $\Psi(N)^j$ respectively) into $\mathsf{F}(N)$, where $\Psi(N)$ is a set of differentiable vector fields on N and $\mathsf{F}(N)$ is an algebra of $\mathbf{K}$-valued C^1-functions on N. This $\Lambda^\infty N$ is the vector bundle of differential ∞-forms on N. Then there exist a subfamily $\Lambda^\infty_G N$ of differential forms w on N induced by the family $\{\nu\}$.

Let $\Lambda^j N$ be the space of differential j-forms w on N such that $w = \sum_{|J|=j} w_J dx^J$, where $dx^J = dx^{j_1} \wedge ... \wedge dx^{j_n}$ for a multi-index $J = (j_1, ..., j_n)$, $n \in \mathbf{N}$, $|J| = j_1 + .. + j_n$, $0 \le j_i \in \mathbf{Z}$, $w_J : N \to \mathbf{K}$ are C^∞-mappings, $B^k N := \bigoplus_{j=0}^k \Lambda^j N$. Here the manifold $B^k N$ is considered to be of classes of smoothness C^∞.

Let $\bar{B}^\infty N := (\bigoplus_{0 \le j \in \mathbf{Z}} \Lambda^j N) \oplus \Lambda^\infty_G N$ for $dim_{\mathbf{K}} N = \infty$ and $\bar{B}^k N = \bigoplus_{j=0}^k \Lambda^j N$ for each $k \in \bar{\mathbf{N}}$. We choose $w \in \bar{B}^k N$, where $k = \min(dim_{\mathbf{K}} N, dim_{\mathbf{K}} M)$. There exists its pull back $\tilde{f}_\kappa^* w$ for each $f \in C_0(\xi, M \to N)$ (see for comparison the classical case in §§1.3.10, 1.4.8 and 1.4.15 in [11] and the non-Archimedean case in [3]), where

$$\tilde{f}_\kappa := \sum_{a=1}^{\infty} \kappa_a \{A_a(f|_{M_a}) - A_{a-1}(f|_{M_{a-1}})\},$$

$|\kappa_a| \times \|A_a\| \le 1$ and $\kappa_a \in \mathbf{K}$ for each $a \in \mathbf{N}$, $A_0 := 0$ (see Formula I.3.6.(1)). This series is correctly defined and converges due to Lemma I.2.4.2 and Formulas I.2.4.3.b.(1-4). When $f \ne 0$ there exists $\kappa := \{\kappa_a : a \in \mathbf{N}\}$ such that $\tilde{f}_\kappa \ne 0$. Let $E_j : S_j \to P$ be a family of continuous linear operators from Banach spaces S_j into a Banach space P, then there exists a continuous linear operator

$$E : c_0(\{S_j : j \in \mathbf{N}\}) \to P \text{ such that}$$

$$Ex = \sum_{j=1}^{\infty} E_j x^j,$$

where $x = \{x^j : x^j \in S_j, j \in \mathbf{N}\} \in c_0(\{S_j : j \in \mathbf{N}\})$. We take $w \in C_0(\infty, \tilde{M} \to B^k N)$, when $dim_{\mathbf{K}} M \le dim_{\mathbf{K}} N$. When $\aleph_0 > dim_{\mathbf{K}} M > dim_{\mathbf{K}} N$ we take $w \in C_0(\infty, \tilde{M} \to B^k(N^m))$, where $N^m = N_1 \times ... \times N_m$ with $N_j = N$ for each $j = 1, ..., m$ such that $\mathbf{N} \ni m \ge dim_{\mathbf{K}} M / dim_{\mathbf{K}} N$. A mapping $F \in C_0(t, \tilde{M} \to N)$ generates a mapping $F^{\otimes m} := (F, ..., F) : \tilde{M} \to N^m$ and the pull back $(F^{\otimes m})^*$ which is also denoted simply by F^*, where $F^* w$ is a $C_0(t-1)$-mapping, when $1 \le t \in \mathbf{R}$, $(F, ..., F)$ is an m-tuplet. When $\aleph_0 = dim_{\mathbf{K}} M > dim_{\mathbf{K}} N$ we take instead of N or N^m a submanifold $\tilde{N}$ of $N^\infty := \bigotimes_{j=1}^{\infty} N_j$ modelled on $c_0(\{S_j : j \in \mathbf{N}\})$, where $S_j = T_y N$ for each j, that is, in accordance with our notation $\tilde{N} := c_0(N_j : j \in \mathbf{N})$. Therefore, there exists a pull back $\tilde{f}^* w$ for ν and w either on N^s or on $\tilde{N}$ instead of N in the corresponding cases of $dim_{\mathbf{K}} M$ and $dim_{\mathbf{K}} N$

Moreover, to $(\pi_J \tilde{f}_\kappa)^* w$ a $\mathbf{Q_q}$-valued measure μ_w on $\tilde{M}$ corresponds, since ν is the $\mathbf{Q_q}$-valued measure. When $dim_{\mathbf{K}} M < \aleph_0$ we take $\tilde{f}$ instead of $\tilde{f}_\kappa$. Then there exists a $\mathbf{Q_q}$-valued functional:

$$(5)\ F_{J,w,\kappa}(f) := \int_{\tilde{M}} (\pi_J \tilde{f}_\kappa)^* w = \int_{\tilde{M}} (\pi_J \tilde{f}_\kappa \circ \psi)^* w$$

for each $f \in C_0^0(\xi, (\tilde{M}, s_0) \to (\bar{N}, y_0))$ and $\psi \in G_0(\xi, \tilde{\bar{M}})$, consequently, $F_{J,w,\kappa}$ is continuous and constant on each class $< f >_{K,\xi}$, where either $\bar{N} = N$ or $\bar{N} = N^m$ or $\bar{N} = \tilde{N}$ in the corresponding cases. If h is not locally constant then h^* is not zero operator, hence the family $\{F_{J,w,\kappa} : J, w, \kappa\}$ separates points in the loop semigroup, where κ is omitted in the case $dim_{\mathbf{K}} M < \aleph_0$.

Let $\tilde{\Xi}_y : \mathbf{Q_q} \to S^1$ be a continuous character of $\mathbf{Q_q}$ as the additive group (see §25.1 [10]), where $S^1 := \{z \in \mathbf{C} : |z| = 1\}$ is the unit circle, x and

$y \in \mathbf{Q_q}$,

$$(6)\ \tilde{\Xi}_y(x) = exp[2\pi i(\sum_{n=-\infty}^{\infty}(\sum_{s=n}^{\infty} y_{-s}q^{(n-s-1)}))],$$

$x = \sum_{n=-\infty}^{\infty} x_n q^n$, $x_n \in \{0,1,...,q-1\}$. For a given x and y this sum in [*] is finite, where y is fixed. In view of Formulas (1-6)

$$\Xi(g) := \tilde{\Xi}(\binom{+}{-} F_{J,w,\kappa}(f))$$

is a continuous character on $L_\xi(M,N) = L_\xi(\tilde{M},N)$, where $F_{J,w,\kappa}(f)$ [or $-F_{J,w,\kappa}(f)$] corresponds to g [or $-g$ respectively], for g being the image of $< f >_{K,\xi}$ relative to the embedding

$$\gamma : \Omega_\xi(\tilde{M},N) \hookrightarrow L_\xi(\tilde{M},N)$$

(see also §2.2).

3.4. Note. The loop groups and semigroups were considered above for analytic manifolds with disjoint clopen charts. Each metrizable manifold M on a Banach space X over a local field $\mathbf{K}$ is a disjoint union of clopen subsets diffeomorphic with balls in X, since the value group $\Gamma_\mathbf{K} := \{|x|_\mathbf{K} : 0 \ne x \in \mathbf{K}\}$ is discrete in $(0,\infty)$ (see [14] and Lemma 7.3.6 [6]).

Suppose now that a new atlas $At'(M)$ is with open charts (U'_j, ϕ'_j) such that there are $U_j \cap U'_i \ne \emptyset$ for some $i \ne j$. Using spaces $C_0(\xi, \phi'_j(U'_j) \to Y)$ we can define $C_0(\xi, M \to N)$ correctly only if connecting mappings $\phi_i \circ \phi'^{-1}_j$ on $\phi'_j(U'_j \cap U'_i)$ are of class of smoothness not less than $C_0(\xi)$ for each $i \ne j$ with $U'_j \cap U'_i \ne \emptyset$. Here the atlases $At'(M)$ and $At'(N)$ need not be disjoint. The same condition need to be imposed on $\psi'_i \circ \psi'^{-1}_j$ for each $V'_j \cap V'_i \ne \emptyset$ for a new atlas $At'(N)$ of N with open charts (V'_j, ψ'_j). This is also necessary for the definition of $G(\xi, M)$. Let $\phi : M \to M'$ be a diffeomorphism for $1 \le \xi = t$ or $\xi = (t,s)$ with $0 \le t$ and $1 \le s$ (a homeomorphism for $0 \le \xi = t < 1$) of class not less than $C_0(\xi)$ of two manifolds (may be one set with two different atlases), then $G(\xi, M)$ and $G(\xi, M')$ are diffeomorphic (or homeomorphic) topological groups with the diffeomorphism (the homeomorphism respectively)

$$g \mapsto \phi \circ g \circ \phi^{-1},$$

since $G(\xi, M)$ have a Banach manifold structure for $1 \le t$ or $1 \le s$, where $g \in G(\xi, M)$. If $\psi : N \to N'$ is a diffeomorphism (homeomorphism) of class at least $C_0(\xi)$, then $C_0(\xi, M \to N)$ and $C_0(\xi, M' \to N')$ are diffeomorphic (homeomorphic) due to the following map

$$g \mapsto \psi \circ g \circ \phi^{-1},$$

where $g \in C_0(\xi, M \to N)$. If $\{f_n\}$ and $\{g_n\}$ are sequences in $C_0(\xi, (M, s_0) \to (N, y_0))$ converging to f and g respectively, $\{\eta_n\}$ is a sequence in $G_0(\xi, M)$ such that $g_n = f_n \circ \eta_n$ for each $n \in \mathbf{N}$, then

$$\psi \circ f_n \circ \phi^{-1} \circ \phi \circ \eta_n \circ \phi^{-1} = \psi \circ g_n \circ \phi^{-1}.$$

This gives a bijective correspondence between classes $< g >_{K,t}$ and $< \tilde{g} >_{K,t}$ in $C_0(\xi, (M, s_0) \to (N, y_0))$ and $C_0(\xi, (M', s'_0) \to (N', y'_0))$ respectively, where

$$\tilde{g} = \psi \circ g \circ \phi^{-1} \in C_0(\xi, (M', s'_0) \to (N', y'_0)),$$

$s'_0 = \phi(s_0)$, $y'_0 = \psi(y_0)$. Therefore, $\Omega_\xi(M, N)$ and $\Omega_\xi(M', N')$ are diffeomorphic (homeomorphic respectively) topological semigroups, consequently, $L_\xi(M, N)$ and $L_\xi(M', N')$ are diffeomorphic (homeomorphic) topological groups due to Theorems I.2.7, I.2.10, 2.3 and Proposition 2.2. This means independence of these semigroups and groups relative to a choice of equivalent atlases of manifolds.

4 Path groups.

4.1. Definition and Note. In view of Equations I.2.9.(1-3) each space N^ξ has the additive group structure, when $N = B(Y, 0, R)$, $0 < R \leq \infty$.

Therefore, the factorization by the equivalence relation $K_\xi \times id$ produce the monoid of paths $C_0^\theta(\xi, \bar{M} \to N)/(K_\xi \times id) =: \mathrm{S}_\xi(M, N)$ in which compositions are defined not for all elements, where $y_1 id y_2$ if and only if $y_1 = y_2 \in N$. There exists a composition $f_1 f_2 = (g_1 g_2, y)$ if and only if $y_1 = y_2 = y$, where $f_i = (g_i, y_i)$, $g_i \in \Omega_\xi(M, N)$ and $y_i \in N^\xi$, $i \in \{1, 2\}$. The latter semigroup has elements e_y such that $f = e_y \circ f = f \circ e_y$ for each f, when their composition is defined, where $y \in N^\xi$, $f = (g, y)$, $g \in \Omega_\xi(M, N)$, $e_y = (e, y)$. If N^ξ is a monoid, then $\mathrm{S}_\xi(M, N)$ can be supplied with the structure of a direct product of two monoids. Therefore, $P_\xi(M, N) := L_\xi(M, N) \times N^\xi$ is called the path group.

4.2. Theorem. *On the monoid $G = S_\xi(M,N)$ from §4.1, when $N = B(Y,0,R)$ and N^ξ is supplied with the additive group structure, and each $b \in \mathbf{C}$ there are probability quasi-invariant and pseudo-differentiable of order b measures μ with values in $\mathbf{R}$ and $\mathbf{Q_q}$ for each prime number $q \ne p$ relative to a dense submonoid G'.*

Proof. In view of Formulas 2.9.(1-3) there is the following isomorphism $S_\xi(M,N) = \Omega_\xi(M,N) \times N^\xi$. Hence it is sufficient to construct $\mu = \mu_1 \times \mu_2$, where μ_2 is a quasi-invariant and pseudo-differentiable measure on N^ξ and μ_1 on $\Omega_\xi(M,N)$, since μ_1 was constructed in Theorem I.3.6. The desired measure μ_2 on N^ξ exists due to Theorems 3.23, 3.27 and 4.3 [13].

4.3. Theorem. *On the path group $G = P_\xi(M,N)$ from §4.1, when $N = B(Y,0,R)$ and N^ξ is supplied with the additive group structure, and each $b \in \mathbf{C}$ there are probability quasi-invariant and pseudo-differentiable of order b measures μ with values in $\mathbf{R}$ and $\mathbf{Q_q}$ for each prime number $q \ne p$ relative to a dense subgroup G'.*

Proof. Since $P_\xi(M,N) = L_\xi(M,N) \times N^\xi$, it is sufficient to construct $\mu = \mu_1 \times \mu_2$, where μ_2 is a quasi-invariant and pseudo-differentiable measure on N^ξ and μ_1 on $L_\xi(M,N)$, since μ_1 was constructed in Theorem 2.5 and μ_2 in §4.2.

4.4. Remark. Loop and path groups can be defined also for manifolds modelled on locally K-convex spaces.

In general for locally K-convex spaces X and Y a mapping $F : U \to Y$ is called of class $C(t)$ if the partial difference quotient $\Phi^v F$ has a bounded continuious extension $\bar{\Phi}^v F : U \times V^s \times S^s \to Y_{\Lambda_p}$ for each $0 \le v \le t$ and each derivative $F^{(k)}(x) : X^k \to Y$ is a continuous k-linear operator for each $x \in U$ and $0 < k \le [t]$, where U and V are open neighbourhoods of 0 in X, $U + V \subset U$, $k \in \mathbf{N_o}$, Y_{Λ_p} is a locally Λ_p-convex space obtained from Y by extension of a scalar field from K to Λ_p, $s = [v] + sign\{v\}$. If F is of class $C(n)$ for each $n \in \mathbf{N}$ then it is called of class $C(\infty)$.

For $C(m)$-manifolds M and N modelled on locally K-convex spaces X and Y with atlases $At(M) = \{(U_i, \phi_i) : i \in \Lambda_M\}$ and $At(N) = \{(V_i, \psi_i) : i \in \Lambda_N\}$ a mapping $F : M \to N$ is called of class $C(n)$ if $F_{i,j}$ are of class $C(n)$ for each i and j, where $F_{i,j} = \psi_i \circ F \circ \phi_j^{-1}$, $\phi_i \circ \phi_j^{-1}$ and $\psi_i \circ \psi_j^{-1}$ are of class $C(m)$, $\infty \ge m \ge n \ge 0$.

Then quite analogously to §I.2.6 and §2.1 loop and path semigroups and groups can be defined. For the construction of quasi-invariant measures in addition there can be used closed subspaces S of separable type over

K in dual spaces to nuclear locally K-convex spaces. From such spaces S quasi-invariant measures can be induced on containing them locally K-convex spaces Z with the help of the standard procedure based on algebras of cylindrical subsets with the subsequent extension onto the Borel σ-field. Then measures on groups can be constructed analogously to the considered above cases. If a group G is non-separable, then a non-zero Borel measure μ may be quasi-invariant relative to a subgroup G' which is not dense in G. Nevertheless, with the help of μ a regular representation of G' associated with μ can be induced.

5 Quasi-invariant measures on O-groups.

5.1. Definition. The space $C^0_0(\xi,(M,s_0) \to (N,y_0))$ is not a semigroup itself, but compositions are defined for the families $< f >_{K,\xi}$, that is, relative to the equivalence relation K_ξ. Henceforth, let the topology of $\Omega_\xi(M,N)$ be defined relative to countable $At(M)$ as in §I.2.5 and §I.2.6. If F is the free Abelian group corresponding to $\Omega_\xi(M,N)$ from §2.1, then there exists a set $\bar{W}$ generated by formal finite linear combinations over $\mathbf{Z}$ of elements from $C^0_0(\xi,(M,s_0) \to (N,y_0))$ and a continuous extension $\bar{K}_\xi$ of K_ξ onto $W_\xi(M,N)$ and a subset $\bar{B}$ of $\bar{W}$ generated by elements $[f+g]-[f]-[g]$ such that $W_\xi(M,N)/\bar{K}_\xi$ is isomorphic with $L_\xi(M,N)$, where

$$W_\xi(M,N) := \bar{W}/\bar{B},$$

f and $g \in C^0_0(\xi,(M,s_0) \to (N,y_0))$, $[f]$ is an element in $\bar{W}$ corresponding to f, $\bar{W}$ is in a topology inherited from the space $C^0_0(\xi,(M,s_0) \to (N,y_0))^{\mathbf{Z}}$ in the Tychonoff product topology. We call $W_\xi(M,N)$ an O-group. Clearly the composition in $C^0_0(\xi,(M,s_0) \to (N,y_0))$ induces the composition in $W_\xi(M,N)$. Then $W_\xi(M,N)$ is not the algebraic group, but associative compositions are defined for its elements due to the homomorphism χ^* given by Formulas 2.6.2.(5,6), hence $W_\xi(M,N)$ is the monoid without the unit element.

Let $\mu_h(A) := \mu(h \circ A)$ for each $A \in Bf(W_\xi(M,N))$ and $h \in W_\xi(M,N)$, then as in §§I.3.3 and I.3.4 we get the definition of quasi-invariant and pseudo-differentiable measures.

Let now $G' := W^{\{k\}}_\xi(M,N)$ be generated by $C^0_{0,\{k\}}(\xi,(M,s_0) \to (N,0))$ as in §I.3.5, then it is the dense O-subgroup in $W_\xi(M,N)$, where $c > 0$ and

$c' > 0$.

5.2. Theorem. *Let $G := W_\xi(M,N)$ be the O-group as in §5.1 and $At(M)$ be finite. Then there exist quasi-invariant and pseudo-differentiable measures μ on G with values in $[0,\infty)$ and in $\mathbf{Q_q}$ (for each prime number q such that $q \ne p$) relative to a dense O-subgroup G'.*

Proof. In view of the definition of the space $C_0^0(\xi, M \to Y)$ the mapping $\tilde{A}$ given by Formula I.3.6.(3) for $At(M)$ instead of $At'(M)$ is the isomorphism of $T_0C_0^0(\xi,(M,s_0) \to (N,0))$ onto the Banach subspace of $\tilde{Z}$ for $\xi = (t,s)$, since $At(M)$ is finite and $\phi_j(U_j)$ are bounded in X (see §I.2.4.1). In view of the existence of the mapping $w_{\epsilon sp}(V)$ given by Formulas I.2.8.(3,4) there exists the local diffeomorphism $\Upsilon : W_\epsilon \to V'_0$ induced by $\tilde{A}$ and $\tilde{K}_\xi$, where W_ϵ is a neighbourhood of 0 in $W_\xi(M,N)$, V'_0 is a neighbourhood of zero either in the Banach subspace $\tilde{H}$ of $T_0W_{\xi'}(M,Y)$ for $dim_{\mathbf{K}}M < \infty$ or in the Banach subspace $\tilde{H}$ of $c_0(\{T_0W_{\xi'}(M_a,Y) : a \in \mathbf{N}\})$ for $dim_{\mathbf{K}}M = \aleph_0$.

Let now W'_ϵ be a neighbourhood of 0 in G' such that $W'_\epsilon W_\epsilon = W_\epsilon$. It is possible, since the topology in G and G' is given by the corresponding ultrametrics and there exists W_ϵ with $W_\epsilon W_\epsilon = W_\epsilon$, hence it is sufficient to take $W'_\epsilon \subset W_\epsilon$. For $g \in W_\epsilon$, $v = w_{\epsilon sp}^{-1}(g)$, $\phi \in W'_\epsilon$ the following operator $S_\phi(v) := \Upsilon \circ L_\phi \circ \Upsilon^{-1}(v) - v$ is defined for each $(\phi, v) \in W'_\epsilon \times V'_0$, where $L_\phi(g) := \phi \circ g$. Then $S_\phi(v) \in V''_0 \subset V'_0$, where V''_0 is an open neighbourhood of the zero section either in the Banach subspace $\tilde{H}'$ of T_eG' for $dim_{\mathbf{K}}M < \infty$ or in the Banach subspace $\tilde{H}'$ of $c_0(\{T_eG'_a : a \in \mathbf{N}\})$ for $dim_{\mathbf{K}}M = \aleph_0$, where $G'_a = W_\xi^{\{k\}}(M_a, N)$. Moreover, $S_\phi(v)$ is the $C(\infty)$-mapping by ϕ and v. The rest of the proof is quite analogous to that of Theorem I.3.6.

5.3. Note. O-groups can be defined in another topology with the help of $c_0(\{H_j : j \in \mathbf{N}\})$, where $H_j := C_0(\xi; U_j \to Y)$. Then on such O-groups quasi-invariant and pseudo-differentiable measures can be constructed quite analogously.

6 Notation.

$\mathbf{K}$ is a local field; $\mathbf{N} := \{1,2,3,...\}$; $\mathbf{N_o} := \{0,1,2,...\}$;

$B(X,x,r)$ and $B(X,x,r^-)$ are balls §I.2.2;

$\bar{Q}_m$ are polynomials §I.2.2;

$X = c_0(\alpha, \mathbf{K})$, $Y = c_0(\beta, \mathbf{K})$, $\{e_i : i \in \alpha\}$ and $\{q_i : i \in \beta\}$ are orthonormal bases in Banach spaces X and Y; M and N are manifolds on X and Y

respectively §I.2.4;

$At(M) = \{(U_j, \phi_j) : j \in \Lambda_M\}$ and $AT(N) = \{(V_k, \psi_k) : k \in \Lambda_N\}$ are atlases §I.2.4;

$C(t, M \to Y)$ and $C_0(t, M \to Y)$ are spaces, $\|f\|_{C(t,M\to Y)} = \|f\|_t$ and $\|f\|_{C_0(t,M\to Y)}$ are norms §I.2.4;

$\rho^\xi(f,g)$ and $\rho^\xi_0(f,g)$ are ultrametrics in $C^\theta(\xi, M \to N)$ and $C^\theta_0(\xi, M \to N)$ respectively, $\xi = t$ or $\xi = (t,s)$, for $s > 0$ the manifold M is locally compact, for $s = 0$ the manifold M may be non-locally compact §I.2.4.3;

$Hom(M)$ is a homeomorphism group §I.2.4.4;

$G(\xi, M)$ and $Diff(\xi, M)$ are diffeomorphism groups §I.2.4.4;

$M = \bar{M} \setminus \{0\}$, $\bar{M} \hookrightarrow c_0(\omega_0, \mathbf{K})$, $At'(\bar{M}) = \{(\bar{U}_j, \bar{\phi}'_j) : j \in \Lambda'_{\bar{M}}\}$, $s_0 = 0$ and $y_0 = 0$ are marked points of $\bar{M}$ and N respectively §I.2.5;

$\chi : M \vee M \to M$ is a mapping §I.2.6;

$G_0(\xi, M)$ is a subgroup and $C_0(\xi, (M, s_0) \to (N, y_0))$ is a subspace preserving marked points, K_ξ is an equivalence relation, $< f >_{K,\xi}$ is a class of equivalent elements §I.2.6;

$\Omega_\xi(M, N)$ is a loop semigroup §I.2.6;

$P(l, s)$ is an antiderivation §I.2.11;

$Bf(X')$, $Af(X', \mu)$ and $Bco(X')$ are algebras of subsets of X', N_μ is a function §I.3.1;

$\rho_\mu(h, g)$ is a quasi-invariance factor §I.3.3;

$S_\xi(M, N)$ is a path semigroup §II.4.1;

$L_\xi(M, N)$ is a loop group §II.2.1;

$P_\xi(M, N)$ is a path group §II.4.1;

$W_\xi(M, N)$ is an O-group §II.5.1.

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