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WEAKLY COMPACT OPERATORS AND U-ADDITIVE MEASURES

JOSÉ AGUAYO-GARRIDO AND MIGUEL NOVA-YAÑEZ

ABSTRACT. Let X be a completely regular space and E, F locally convex Hausdorff space with a directed family of semi-norms which generates their topologies. Denote by $C_b(X,E)$ the space of all E-valued, continuous and bounded function f from X into E. $C_b(X,E)$ is endowing by locally convex topologies, so called strict topologies. We study the F-valued weakly compact operators T defined on $C_b(X,E)$. We characterize those operators which are continuous in the strict topologies, and wel find certain kind of vector measures associated with them. Necessary and sufficient conditions for a continuous and weakly compact operators to be represented by integrals with respect to $\mathcal{L}(E,F)-$valued measures on the Baire algebra generated by zero sets are obtained.

1. DEFINITIONS AND PRELIMINARIES.

Let X be a completely regular Hausdorff space. By $\mathcal{B} = \mathcal{B}(X)$ we will denote the algebra of subsets of X generated by zero sets $Z = f^{-1}(\{0\})$, where f is a real-valued continuous function defined on X and by $\mathcal{B}a = \mathcal{B}a(X)$ we will denote the σ-algebra of Baire sets, that is, $\mathcal{B}a$ is the smallest $\sigma-$algebra of subsets of X which contains the zero sets.

Let $M(X)$ denote the space of all bounded finitely-additive regular (with respect to the zero sets) measures on $\mathcal{B}$. We recall that a measure $\mu \in M(X)$ is said to be

- $\tau-$additive if, for any net $(U_\alpha)_{\alpha\in\Gamma}$ with $U_\alpha \downarrow \emptyset$, $\lim_\alpha \mu(U_\alpha) = 0$. The space of all $\tau-$additives measures is denoted by $M_\tau(X)$.
- $\sigma-$additive if, for any net $(U_n)_{n\in\Gamma}$ with $U_n \downarrow \emptyset$, $\lim_{n\to\infty} \mu(U_n) = 0$. The space of all $\sigma-$additives measures is denoted by $M_\sigma(X)$.
- u-additive if, for any partition of unity $(f_\alpha)_{\alpha\in\Gamma}$of positive functions in $C(X)$ such that $\sum_{\alpha\in\Gamma} f_\alpha = 1$ and $\{f_\alpha^{-1}(0,\infty)\}_{n\in\Gamma}$ is locally

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finite, $\sum_{\alpha}\mu(f_\alpha) = \mu(1)$. The space of all $u-$additives measures is denoted by $M_u(X)$ (see [4]).

Let E be a real locally convex Hausdorff space. For a continuous semi-norm p on E, we define $M_p(X, E')$ as the set of all E'-valued finitely additive measures m on $\mathcal{B}$ with the following two properties:

1. For every $s \in E$, the function $ms : \mathcal{B} \to \mathbb{R}$, defined by $\langle m(A), s\rangle$, is in $M(X)$.
2. $\|m\|_p = m_p(X) < \infty$, where for $A \in \mathcal{B}$ the $m_p(A)$ is defined to be the supremum of all $|\sum m(A_i)s_i|$ being taken over all finite $\mathcal{B}$-partitions $\{A_i\}$ of A and all finite collections $s_i \in B_p = \{s \in E : p(s) \leq 1\}$.

The set $M_{\sigma,p}(X, E')$ consists of those m in $M_p(X, E')$ for which $ms \in M_\sigma(X)$ for all $s \in E$. The spaces $M_{\tau,p}(X, E')$ and $M_{\infty,p}(X, E')$ are defined similarly. It is known that if m is in any of the spaces $M_p(X, E')$, $M_{\sigma,p}(X, E')$, $M_{\tau,p}(X, E')$ and $M_{\infty,p}(X, E')$, then m_p belongs to $M(X)$, $M_\sigma(X)$, $M_\tau(X)$ and $M_\infty(X)$ respectively (see [1] and [8]).

Let $\{p : p \in I\}$ be a generating family of continuous semi-norms on E which is directed, i.e., given p_1, p_2 in I there exists $p_3 \in I$ such that $p_3 \geq \max\{p_1, p_2\}$. We will denote by $M(X, E')$ the space $\cup_{p\in I} M_p(X, E')$. Analogous notation we will have for $M_\sigma(X, E')$, $M_\tau(X, E')$ and $M_\infty(X, E')$.

We shall denote by $C_b(X, E)$ (if $E = \mathbb{R}$, simply $C_b(X)$) the space of all E-valued continuous and bounded functions f defined on X, $C_{rc} = C_{rc}(X, E)$ the space of all $f \in C_b(X, E)$ for which $f(X)$ is relatively compact. We will denote by u the uniform topology generated by the family of semi-norms $\left\{\|.\|_p\right\}_{p\in I}$, where $\|f\|_p = \sup\{p(f(x)) : x \in X\}$.

For A in $\mathcal{B}$ and for $m \in M_p(X, E')$, we define

$$\int_A f dm = \lim \sum m(A_i)(f(x_i)),$$

where the limit is taken over the directed set of all finite $\mathcal{B}-$partitions $\{A_i\}$ of A. This limit always exists when $f \in C_{rc}$ and the map $f \to T_m(f) = \int_X f dm$ is a uniformly continuous linear functional on C_{rc}; even more,

$$\|m\|_p = \sup\left\{|T_m(f)| : \|f\|_p \leq 1\right\}$$

and the mapping $m \to T_m$ is one-one linear mapping from $M(X, E')$ onto C_{rc} (see [5]).

Every f in C_{rc} has a unique extension $\widehat{f}$ to all the Stone-*Cech* compactification βX. Let Ω and Ω_1 be the class of all compact and all zero sets in $\beta X \setminus X$, respectively. Ω_∞ will be the class of all compact subsets of $\beta X \setminus X$ satisfying any of the equivalent assertions of the following lemma given in [4]..

Lemma 1. *For a compact $K \subset \beta X \setminus X$ the following are equivalent:*

1. *There is a co-zero cover $(U_\alpha)_{\alpha \in A}$ of X which is (a) locally finite, (b) σ-locally finite or (c) $\sigma-$discrete such that*
$$cl_{\beta X} U_\alpha \cap K = \Phi \text{ for all } \alpha \in A$$
2. *There is a continuous function f from X onto a metric space Y such that $\widehat{f}(K) \subset \beta Y \setminus Y$, where $\widehat{f} : \beta X \to \beta Y$ is the continuous extension of f.*
3. *There exists a partition of unity $(f_\alpha)_{\alpha \in A}$ for X such that $\widehat{f_{\alpha}}_{|K} \equiv 0$ for all $\alpha \in A$.*
4. *There exists a partition of unity $(f_\alpha)_{\alpha \in A}$ for X and $0 < \epsilon < 1$ such that*
$$\sum_{\alpha \in A} \widehat{f}_\alpha(x) \leq 1 - \epsilon \text{ for all } x \in K$$

It can be proved that $\Omega_1 \subset \Omega_\infty \subset \Omega$. If K belongs to some of these collections, then we define β_K to be the locally convex topology on $C_b(X, E)$ generated by the family of the semi-norms
$$f \to \|gf\|_p = \sup\{p(g(x)f(x)) : x \in X\},$$
where $p \in I$ and $g \in C_K = \left\{h \in C_b(X) : \widehat{h}_{|K} \equiv 0\right\}$.

The locally convex topologies β, β_u and β_1 defined on $C_b(X, E)$ will be the inductive limits topologies of the topologies β_K as K ranges over Ω, Ω_∞ and Ω_1 respectively.

For a fixed $p \in I$, $\beta_{p,K}$ denotes the locally convex topology on $C_b(X, E)$ generated by the semi-norms $f \to \|gf\|_p$, $g \in C_K$. The locally convex topologies β_p, $\beta_{u,p}$ and $\beta_{1,p}$ will be the inductive limits topology on $C_b(X, E)$ of the topologies $\beta_{p,K}$ as K ranges over Ω, Ω_∞ and Ω_1 respectively.

The locally convex topologies β', β'_u and β'_1 are the projective limits topologies on $C_b(X, E)$ of the topologies β_p, $\beta_{u,p}$ and $\beta_{1,p}$ respectively, as p ranges over I.

The following properties are satisfied on $C_b(X, E)$ and will be used in sequel:

- $\beta \leq \beta_u \leq \beta_1 \leq u$ and $\beta' \leq \beta$; $\beta'_u \leq \beta_u$ and $\beta'_1 \leq \beta_1$; all of these topologies have the same bounded subset (see [6] and [1]).
- the dual of C_{rc}, provided by β_1 and β'_1, is $M_\sigma(X, E')$ (see [6]);
- the dual of C_{rc}, provided by β and β', is $M_\tau(X, E')$ (see [6]).
- $C_b(X) \otimes E$ is β_u- dense in $C_b(X, E)$ (see [1]).

- the dual of $C_b(X,E)$, provided by β_u and β'_u, is $M_\infty(X,E')$ (see [1]).

Definition 1. *Let F be another real locally convex Hausdorff space and let $\{q : q \in J\}$ be a generating directed family of continuous semi-norms on F. We denote by $M(X,\mathcal{L}(E,F))$ the space of all finitely-additive $\mathcal{L}(E,F)$-valued measures m on $\mathcal{B}$ with the following two properties:*

1. *For each $x' \in F'$, the set function $x'm : \mathcal{B} \to E'$, defined by $(x'm)(A)s = x'(m(A)s)$, $s \in E$, is in $M(X,E')$.*
2. *Given $q \in J$, there exists $p \in I$ such that for all x' on the polar B_q^o of the unit ball B_q, $x'm \in M_p(X,E')$ and $\|m\|_{p,q} = m_{p,q}(X) < \infty$, where $m_{p,q}(X)$ $=sup\{(x'm)_p(X) : x' \in B_q^o\}$.*
   The definition of $M_\sigma(X,\mathcal{L}(E,F))$, $M_\tau(X,\mathcal{L}(E,F))$ and $M_\infty(X,\mathcal{L}(E,F))$ are given analogously to 1. and 2.

Definition 2. *Let $m \in M(X,\mathcal{L}(E,F))$ and let $f : X \to E$ be a function. We will say that f is $m-$ integrable over A in $\mathcal{B}$ if:*

1. *For each $x' \in F'$, the integral $\int_A fd(x'm)$ exists*
2. *There exists a vector in F, denoted by $\int_A fdm$, such that for all $x' \in F'$ we have $x'(\int_A fdm) = \int_A fd(x'm)$.*

Remark 1. *Since F is a locally convex Hausdorff space, the $\int_A fdm$ is unique whenever it exists. If f is $m-$integrable over all $A \in \mathcal{B}$, we will say that f is $m-$integrable. If f is $m-$integrable on X, $\int_X fdm$ will be denoted by simply $\int fdm$. The following Representation Theorem is found in* [5].

Theorem 1. *If T is a continuous weakly compact operator from (C_{rc},u) into F, then there exists a unique $m \in M(X,\mathcal{L}(E,F))$ such that:*

1. *Every f in C_{rc} is $m-$integrable and $T(f) = \int fdm$*
2. *If $p \in I$ and $q \in J$ are such that $\|T\|_{p,q} = \sup\left\{q(T(f)) : \|f\|_p \leq 1\right\} < \infty$, then $\|m\|_{p,q} = \|T\|_{p,q}$.*
3. *For every $x' \in F'$, we have $T'x' = x'm$.*
4. *The set $V_{m,S} = \{\sum m(A_i)s_i : \{A_i\}$ is a finite $\mathcal{B}$-partition of X, $s_i \in S\}$ is weakly relatively compact in F, for every bounded set S in E.*
   Conversely, if $m \in M(X,\mathcal{L}(E,F))$ is such that (4) holds, then

every $f \in C_{rc}$ is $m-$integrable and the operator $T(f) = \int f dm$ is $u-$continuous and weakly compact.

Remark 2. *Since T can be extended to $C(\beta X, E)$ and if $\widehat{T}$ denotes its extension, we have $\widehat{T}(\widehat{f}) = T(f)$ which is $\widehat{u}-$continuous, where $\widehat{u}$ denotes the uniform topology on $C(\beta X, E)$. The dual of $(C(\beta X, E), \widehat{u})$ is $M_\tau(\beta X, E')$. Using the above theorem, we can get a vector measure representation of $\widehat{T}$, say, $\widehat{T}(\widehat{f}) = \int \widehat{f} d\widehat{m}$.*

The proof of the next assertions can be found in [1] :

Theorem 2. *Let $m \in M_p(X, E')$. Then,*

1. *if $m \in M_\infty(X, E')$, then $m_p \in M_\infty(X)$.*
2. *$m \in M_{\infty,p}(X, E')$ if, and only if, $\widehat{m}_p(K) = 0$ for all $K \in \Omega_\infty$*

2. Weakly compact operators and integral representation

In this section we will study the F-valued weakly compact operators T defined on $C_b(X, E)$, where F is another locally convex Hausdorff space with a directed family of semi-norms J which generates its topology. We shall characterize those operators which are β'_u- continuous and we shall find certain kind of vector measures associated with them.

In section 1, we announced the Katsaras' Theorem which gives us an integral representation of weakly compact operators. The following theorem characterizes the β'_u-continuous operators defined on $C_{rc}(X, E)$. First we need the following very well known Grothendieck's Lemma [3].

Lemma 2. *Let T be an operator from a topological vector space V into another topological vector space W and let T' and T'' denote, respectively, the transpose and the second transpose of T. The following statements are equivalent:*

1. *T is weakly compact*
2. *T'' maps V'' into W.*

Theorem 3. *Let T be a weakly compact and u-continuous operator defined on C_{rc}. Then, if m and $\widehat{m}$ are the associated measures of T and $\widehat{T}$ respectively given by Th. 5 and Remark 2, then the following statements are equivalent:*

1. *T is β'_u-continuous*
2. *Given $q \in J$, there exists $p \in I$ such that*

$$\|T\|_{p,q} < \infty \text{ and}$$

$$\inf\{\widehat{m}_{p,q}(V) : V \text{ is a co-zero set such that } K \subset V\} = 0$$

for all $K \in \Omega_\infty$

3. *Given* $q \in J$, *there exists* $p \in I$ *such that* $\|T\|_{p,q} < \infty$ *and* $\widehat{m}_{p,q}(K) = 0$, *for all* $K \in \Omega_\infty$.
4. *Given* $q \in J$, *there exists* $p \in I$ *such that* $\|T\|_{p,q} < \infty$ *and, for any partition of unity* $\{f_\alpha\}_{\alpha\in A}$ *of* X *and for any* $\epsilon > 0$, *there exists a finite subset* L_o *of* A *such that for all finite subset* L *of* A *containing* L_o,

$$(x'm)_p(1 - \sum_{\alpha\in L} f_\alpha) < \epsilon \text{ uniformly for } x' \in B_q^o,$$

where $B_q = \{x \in F : q(x) \leq 1\}$.

Proof. (1) $\Rightarrow$ (2) The arguments are very similar to those given in ([5] , Th. 4, p. 553) and we omit them.

(2) $\Rightarrow$ (3) Obvious

(3) $\Rightarrow$ (4) Let $q \in J$; since $\|T\|_{p,q} < \infty$, we only need to prove the second part of the statement.

Let $\{f_\alpha\}_{\alpha\in A}$ be a partition of unity of X, and let $\epsilon > 0, 0 < \epsilon < 1$. For any finite subset L of A, we put $Z_L = \left\{x \in \beta X : \sum_{\alpha\in L} \widehat{f}_\alpha(x) \leq 1 - \epsilon\right\}$ and then we denote by $K = \cap_L Z_L$. We already know that $K \in \Omega_\infty$ (see [1]).

We claim that

$$\widehat{m}_{p,q}(K) = \inf_L \widehat{m}_{p,q}(Z_L).$$

Clearly,

$$\widehat{m}_{p,q}(K) \leq \inf_L \widehat{m}_{p,q}(Z_L).$$

On the other hand, since for each $x' \in F'$, $(x'\widehat{m})_p \in M_\tau(\beta X)$, we have that

$$(x'\widehat{m})_p(K) = \inf_L (x'\widehat{m})_p(Z_L)$$

and then,

$$\begin{aligned}\widehat{m}_{p,q}(K) &= \sup\{(x'\widehat{m})_{p,q}(K) : x' \in B_q^o\} \\ &= \sup\left\{\inf_L (x'\widehat{m})_p(Z_L) : x' \in B_q^o\right\}\end{aligned}$$

Thus, to prove the claim it is enough to prove that

$$\inf_L \sup\{(x'\widehat{m})_p(Z_L) : x' \in B_q^o\} \leq \sup\left\{\inf_L (x'\widehat{m})_p(Z_L) : x' \in B_q^o\right\}$$

Take a finite subset L of A; hence there exists $x'_{\epsilon,L} \in B^o_q$ such that

$$\widehat{m}_{p,q}(Z_L) < (x'_{\epsilon,L}\widehat{m})_p(Z_L) + \epsilon.$$

This implies that

$$\begin{aligned}\inf_L \widehat{m}_{p,q}(Z_L) &< \inf_L (x'_{\epsilon,L}\widehat{m})_p(Z_L) + \epsilon \\ &\leq \sup\left\{\inf_L (x'\widehat{m})_p(Z_L) : x' \in B^o_q\right\} \\ &< \widehat{m}_{p,q}(K) + \epsilon\end{aligned}$$

From this and (3), we get $0 \leq \inf \widehat{m}_{p,q}(Z_L) \leq \widehat{m}_{p,q}(K) = 0$, and then, by this $\epsilon > 0$, there exists a finite subset L_o of A such that $\widehat{m}_{p,q}(Z_{L_o}) < \epsilon$. Take any $x' \in B^o_q$ and a finite subset L of A containing L_o; then,

$$(x'm)_p(1 - \sum_{\alpha \in L} f_\alpha) = (x'\widehat{m})_p(1 - \sum_{\alpha \in L} \widehat{f}_\alpha)$$

$$\begin{aligned}&= \int_{Z_L} (1 - \sum_{\alpha\in L} \widehat{f}_\alpha) d(x'\widehat{m})_p + \int_{\beta X \setminus Z_L} (1 - \sum_{\alpha \in L} \widehat{f}_\alpha) d(x'\widehat{m})_p \\ &\leq (x'\widehat{m})_p(Z_L) + \epsilon(x'\widehat{m})_p(\beta X) \\ &\leq \widehat{m}_{p,q}(Z_L) + \epsilon \widehat{m}_{p,q}(\beta X) \\ &\leq \epsilon(1 + \widehat{m}_{p,q}(\beta X))\end{aligned}$$

This proves the statement.

(4) $\Rightarrow$ (1) Let $\{g_\alpha\}_{\alpha\in D}$ be a β'_u null-convergent net. We shall prove that $T(g_\alpha) \to 0$ in F.

If $q \in J$ and $\epsilon > 0$ are given, then there exists $p \in I$ such that $\|T\|_{p,q} < \infty$. Thus, the subset $\{(x'm)_p \in M_\infty(X) : x' \in B^o_q\}$ is bounded.

Now, if $\{f_\lambda\}_{\lambda\in A}$ is a partition of unity in X, then there is a finite subset L_o of A such that, for all $L_o \subset L$ and L a finite subset of A,

$$(x'm)_p(1 - \sum_{\alpha\in L} f_\alpha) < \epsilon, \text{ uniformly for } x' \in B^o_q$$

Then, by Prop. 3.6 ([4], p.474), $\{(x'm)_p \in M_\infty(X) : x' \in B^o_q\}$ is β_u-equicontinuous on $C_b(X)$. On the other hand, since $g_\alpha \xrightarrow{\beta_u} 0$, by Th. 1.3 ([7], p. 162), we have $p \circ g \xrightarrow{\beta'_u} 0$. Thus, there exists α_o such that

$$\alpha \geq \alpha_o \Rightarrow p \circ g_\alpha \in \epsilon \{(x'm)_p \in M_\infty(X) : x' \in B^o_q\}^o$$

Therefore, if $\alpha \geq \alpha_o$ and $x' \in B^o_q$, then

$$|x'(Tg_\alpha)| \leq \int p \circ g_\alpha d(x'm)_p < \epsilon$$

and then $q(T(g_\alpha)) < \epsilon$, for all $\alpha \geq \alpha_o$. This prove that T is β'_u-continuous.

Theorem 4. *Let T be a $F-$valued, weakly compact and β'_u- continuous operator on C_{rc}. Then, the unique m, given by Th. 4. of section 1, is in $M_\infty(X, \mathcal{L}(E, F))$. Conversely, if $m \in M_\infty(X, \mathcal{L}(E, F))$ satisfies (4), given in Th. 4, then the operator $T(f) = \int f dm$ is weakly compact and β'_u-continuous.*

Proof. Since, for every $x' \in F'$, $x'm = x'T'$ and T is β'_u- continuous, we have $x'm = x'T' \in (C_{rc}, \beta'_u)' = M_\infty(X, E')$ and then $m \in M_\infty(X, \mathcal{L}(E, F))$.

Conversely, we already know that T is weakly compact under the duality $\langle C_{rc}, M(X, E')\rangle$; since β' and the uniform topology u have the same bounded sets ([6], Th. 5.4, p. 225) and since $\beta' \leq \beta'_u \leq u$, we have T is weakly compact under the duality $\langle C_{rc}, M_\infty(X, E')\rangle$.

The β'_u-continuity of T is coming from the fact that $\widehat{m}_{p,q}(K) = 0$, for all $K \in \Omega_\infty$, and Th. 7.

Theorem 5. *Let T be a $F-$valued, weakly compact and β'_u- continuous operator on $C_b(X, E)$, where F is, in addition, complete. Then, there exists a unique $m \in M_\infty(X, \mathcal{L}(E, F))$ such that:*

1. *Every $f \in C_b(X, E)$ is $m-$integrable and $T(f) = \int f dm$.*
2. *Given $q \in J$, there exists $p \in I$ such that $\|T\|_{p,q} = \|m\|_{p,q}$.*
3. *For every $x' \in F'$, we have $T'x' = x'm$.*
4. *For every bounded set S of E, the set*

$$V_{m,S} = \left\{\sum m(G_i)s_i : \{G_i\} \text{ is a finite } \mathcal{B} - \text{partition of } X \text{ and } s_i \in S\right\}$$

is weakly relatively compact.

Conversely, if $m \in M_\infty(X, \mathcal{L}(E, F))$ is such that (4) holds, then every $f \in C_b(X, E)$ is $m-$integrable and the operator $T(f) = \int f dm$ is β'_u-continuous and weakly compact.

Proof. Suppose that T is weakly compact and β'_u- continuous operator on $C_b(X, E)$. If we put $\widetilde{T} = T_{|C_{rc}}$, then it is clear that $\widetilde{T}$ is also weakly compact and β'_u- continuous operator on C_{rc}. So, there exists a unique $m \in M_\infty(X, \mathcal{L}(E, F))$ which satisfies (1), (2), (3) and (4).

Now, we shall prove that $f \in C_b(X, E)$ is $m-$integrable. Fix $x' \in F'$ and $G \in \mathcal{B}$, we define $L_G : C_{rc} \to \mathbb{R}$, by $L_G(g) = \int_G g d(x'm)$. We claim that L_G is β'_u- continuous. In fact, let $g_\alpha \xrightarrow{\beta'_u} 0$; since $m \in$

$M_\infty(X, \mathcal{L}(E,F))$, there exists $p \in I$ such that $x'm \in M_{\infty,p}(X, E')$. This implies that $(x'm)_p \in M_\infty(X)$. Also, by ([7], Th. 1.3), $p \circ g_\alpha \overset{\beta_u = \beta_\infty}{\longrightarrow} 0$ in $C_b(X)$. Now, since $|L_G(g)| \leq \int p \circ g d(x'm)_p$, we get $L_G(g_\alpha) \to 0$.

By the β'_u-denseness of C_{rc} in $C_b(X,E)$ (see [1]), L_G can be uniquely extended to $C_b(X,E)$ and then $\int_G fd(x'm)$ exists.

The next step is to prove that there exists a vector in F denoted by $\int_G fdm$ such that for all $x' \in F'$, $x'(\int_G fdm) = \int_G fd(x'm)$.

Let $\{f_\alpha\}_{\alpha \in \Lambda}$ be a net in C_{rc} such that $f_\alpha \overset{\beta'_u}{\to} f$. We claim that $\left\{\int_G f_\alpha dm\right\}$ is a Cauchy net in F. In fact, since $\widetilde{T}$ is β'_u-continuous, we have that, for a given $q \in J$, there exists $p \in I$ such that $\left\{(x'm)_p : x' \in B^o_q\right\} \subset M_\infty(X)$ is $\beta_u = \beta_\infty$-equicontinuous and $p \circ (f_\alpha - f) \overset{\beta_\infty}{\to} 0$ in $C_b(X)$ (see [7]). Thus, there exists $\alpha_o \in \Lambda$ such that

$$\int p \circ (f_\alpha - f) d(x'm)_p \leq \frac{\epsilon}{2} \text{ for } \alpha \geq \alpha_o \text{ and for all } x' \in B^o_q$$

Therefore, if $\alpha, \beta \geq \alpha_o$ and $x' \in B^o_q$, then

$$\begin{aligned}\left|x'(\int_G f_\alpha dm - \int_G f_\beta dm)\right| &\leq \int p \circ (f_\alpha - f_\beta) d(x'm)_p \\ &\leq \int p \circ (f_\alpha - f) d(x'm)_p + \int p \circ (f - f_\beta) d(x'm)_p \\ &< \epsilon\end{aligned}$$

Thus, there exists a vector in F denoted by $\int_G fdm$ such that $\int_G f_\alpha dm \to \int_G fdm$. It is easy to see that $x'(\int_G fdm) = \int_G fdx'm$.

Now, the continuity of T shows that

$$T(f) = \int fdm,$$

$$\|T\|_{p,q} = \left\|\widetilde{T}\right\|_{p,q} = \|m\|_{p,q}$$

and $T'x' = x'm$ for all $x' \in F'$..

Conversely, we will suppose that $m \in M_\infty(X, \mathcal{L}(E,F))$ and we will prove that there exists a β'_u-continuous and weakly compact operator T such that $T(f) = \int fdm$. By Th. 7, we have a β'_u-continuous and weakly compact operator $\widetilde{T}$ defined on C_{rc} such that $\widetilde{T}(f) = \int f dm$. Now, the β'_u-denseness of C_{rc} gives us a unique β'_u-continuous operator T defined on $C_b(X,E)$ such that $T(f) = \widetilde{T}(f)$, for all $f \in C_{rc}$. Using the same arguments given before, we prove that each $f \in C_b(X,E)$ is $m-$integrable and then $T(f) = \int f dm$. The remainder of the proof, that is, T is weakly compact, follows easily from the fact that $T' = \widetilde{T}'$ which implies that

$$T''((C_b(X,E),\beta'_u)'') = \widetilde{T}''((C_{rc},\beta'_u)'' \subset F.$$

Remark 3. *Since $\beta' \leq \beta'_u$, we can rewrite Th. 5, p. 554, given by Katsaras* [6], *as follows:*

Theorem 6. *Let T be a $F-$valued, weakly compact and $\beta'-$ continuous operator on $C_b(X,E)$, where F is, in addition, complete. Then, there exists a unique $m \in M_\tau(X,\mathcal{L}(E,F))$ such that:*

1. *For every $f \in C_b(X,E)$ is $m-$integrable and $T(f) = \int f dm$.*
2. *Given $q \in J$, there exists $p \in I$ such that $\|T\|_{p,q} = \|m\|_{p,q}$.*
3. *For every $x' \in F'$, we have $T'x' = x'm$.*
4. *For every bounded set S of E, the set*

$$V_{m,S} = \left\{\sum m(G_i)s_i : \{G_i\} \text{ is a finite } \mathcal{B} - \text{partition of } X \text{ and } s_i \in S\right\}$$

is weakly relatively compact.

Conversely, if $m \in M_\tau(X,\mathcal{L}(E,F))$ is such that (4) *holds, then every $f \in C_b(X,E)$ is $m-$integrable and the operator $T(f) = \int f dm$ is $\beta'-$continuous and weakly compact.*

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