

THEODORE K. BONI

On the extinction of the solution for an elliptic equation in a cylindrical domain

Annales mathématiques Blaise Pascal, tome 6, n° 1 (1999), p. 13-20

http://www.numdam.org/item?id=AMBP_1999__6_1_13_0

© Annales mathématiques Blaise Pascal, 1999, tous droits réservés.

L'accès aux archives de la revue « Annales mathématiques Blaise Pascal » (<http://math.univ-bpclermont.fr/ambp/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

ON THE EXTINCTION OF THE SOLUTION FOR AN ELLIPTIC EQUATION IN A CYLINDRICAL DOMAIN

THEODORE K. BONI

ABSTRACT. - We obtain some conditions under which there is extinction in finite time of the solution of an elliptic equation in a cylindrical domain. We also show how the solution of our elliptic equation approaches its stationary solution.

Keywords. Asymptotic behavior, extinction, dead core.

AMS classifications. 35 B 40, 35 J 60.

1. Introduction

Let Ω be a bounded domain in \mathbb{R}^n with smooth boundary $\partial\Omega$. We consider the following boundary value problem for $u(x, t)$:

$$\frac{\partial^2 u}{\partial t^2} + \Delta u - \lambda f(u) = 0 \quad \text{in } \Omega \times (0, \infty), \quad (1.1)$$

$$u = h(x) \quad \text{on } \partial\Omega \times (0, \infty), \quad (1.2)$$

$$u(x, 0) = u_o(x) \quad \text{in } \Omega, \quad \text{with } 0 \leq u_o(x) \leq 1, \quad (1.3)$$

where $h(x)$ is a nonnegative continuous function on $\partial\Omega$, $u_o(x)$ is a continuous function in Ω which satisfies the compatibility condition

$$u_o(x) = h(x) \quad \text{on } \partial\Omega. \quad (1.4)$$

For positive values of s , $f(s)$ is a positive and increasing function with $f(0) = 0$. **Proposition 1.1.** *Under the above assumptions, there exists a solution u of (1.1) – (1.3).*

The proof of the proposition is based on the following lemma

Comparison lemma 1.2 (maximum principle). *Let $u, v \in C(\bar{\Omega} \times [0, \infty))$ satisfying the following inequalities:*

$$\frac{\partial^2 u}{\partial t^2} + Lu - f(u) \leq \frac{\partial^2 v}{\partial t^2} + Lv - f(v) \quad \text{in } \Omega \times (0, \infty),$$

$$u \geq v \quad \text{on } \partial\Omega \times (0, \infty),$$

$$u(x, 0) \geq v(x, 0) \quad \text{in } \bar{\Omega}.$$

Then we have

$$u(x, t) \geq v(x, t) \quad \text{in } \Omega \times (0, \infty).$$

Proof. Assume that $w = u - v$ takes negative values and let $m = \inf w(x, t) < 0$. The function

$$m(t) = \min\{w(x, t) : x \in \bar{\Omega}\}$$

is obviously continuous and $m(0) \geq 0$, $\inf m(t) = m$. Introduce the function

$$L(t, C) = \frac{m}{2} - Ct.$$

We have

$$L(t, C) < m(t) \quad \text{in } [0, \infty) \quad \text{for } C \text{ large}$$

and the infimum of all constants C with this property, call it C_o , has the property that

$$C_o > 0, \quad L(t, C_o) \leq m(t), \quad \text{there is } t_o > 0 \quad \text{with } L(t_o, C_o) = m(t_o).$$

Let x_o be such that $m(t_o) = w(x_o, t_o)$. Since $w(x_o, t_o) < 0$, i.e. $u(x_o, t_o) < v(x_o, t_o)$, it follows from the second inequality of the lemma that x_o is in Ω . Now we have

$$f(u(x_o, t_o)) < f(v(x_o, t_o)),$$

$$w(x, t_o) \geq w(x_o, t_o) \quad \text{in } \Omega \implies (Lw)(x_o, t_o) \geq 0,$$

$$w(x_o, t) \geq L(t, C_o) \quad \text{with equality for } t = t_o \implies w_{tt}(x_o, t_o) \geq 0$$

in contradiction to the assumed first inequality of the lemma. \square

Proof of Proposition 1.1. Since Problem (1.1)–(1.3) has a comparison lemma, to prove our proposition, it suffices to show that (1.1) – (1.3) has both a supersolution and a subsolution (see [10]). Let us notice that 0 is a subsolution and 1 is a supersolution, which leads us to the result. \square

By a stationary solution of (1.1) – (1.3), we mean a function $s(x)$ satisfying

$$\Delta s(x) - \lambda f(s(x)) = 0 \quad \text{in } \Omega, \tag{1.5}$$

$$s(x) = h(x) \quad \text{on } \partial\Omega. \tag{1.6}$$

Definition 1.3. We say that there is extinction in finite time for the solution $u(x, t)$ if there exists a finite time T_o such that

$$u(x, t) = 0 \quad \text{for } x \in \Omega, \quad t \geq T_o.$$

In this paper, we are interested in the extinction of the solution u .

Elliptic equations in cylindrical domains have been the subject of research of many authors (see [1], [2], [5], [6], [7], [8], [11], [12] and the references cited therein). Concerning the extinction, Kondratiev and Veron in [8] have shown that under some assumptions, there is extinction in finite time for the solution u in the case where $f(s) = s^p$ with $p < 1$ and the Dirichlet condition replaced by that of Neumann. Our aim in this paper is to generalize this result considering the problem described in (1.1) – (1.3). We also obtain the asymptotic behavior as $t \rightarrow \infty$ of the solution u showing how it approaches its stationary solution. We shall see in this paper that the stationary solution of u plays an important

role in its asymptotic behavior as $t \rightarrow \infty$. This phenomenon is well known in the parabolic case (see, for instance [3], [4]). Our paper is organized as follows. In the next section, we give some conditions of extinction and in the last section, we study the asymptotic behavior of the solution u as $t \rightarrow \infty$.

2. Conditions of extinction

In this section, we show that under some conditions, there is extinction in finite time for the solution u of (1.1) – (1.3).

Introduce the function $F(s) = \int_0^s f(\sigma) d\sigma$. Let

$$J(s) = \int_s^1 \frac{d\sigma}{\sqrt{F(\sigma)}} \quad (2.1)$$

and let $K(s)$ be the inverse function of $J(s)$. In this notation, the IVP

$$\alpha'(t) = -\sqrt{2\lambda F(\alpha(t))}, \quad \alpha(0) = 1, \quad (2.2)$$

has the solution

$$\alpha(t) = K(t\sqrt{2\lambda}) \quad \text{for } t > 0 \quad \text{if } \int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty, \quad (2.3)$$

$$\alpha(t) = K(t\sqrt{2\lambda}) \quad \text{for } t < \frac{1}{\sqrt{2\lambda}} \int_0^1 \frac{ds}{\sqrt{F(s)}} \quad \text{if } \int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty, \quad (2.4)$$

$$\alpha(t) = 0 \quad \text{for } t \geq \frac{1}{\sqrt{2\lambda}} \int_0^1 \frac{ds}{\sqrt{F(s)}} \quad \text{if } \int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty. \quad (2.5)$$

Remark 2.1. If $f(s) = s^p$, then $\alpha(t)$ is defined as follows:

$$\alpha(t) = \left[1 + t\sqrt{\lambda} \frac{\sqrt{p+1}(p-1)}{\sqrt{2}} \right]^{-\frac{2}{p-1}} \quad \text{if } p > 1,$$

$$\alpha(t) = e^{-2\lambda t} \quad \text{if } p = 1,$$

$$\alpha(t) = \left[1 - t\sqrt{\lambda} \frac{\sqrt{p+1}(1-p)}{\sqrt{2}} \right]_+^{\frac{2}{1-p}} \quad \text{if } p < 1,$$

where $[x]_+ = \max\{x, 0\}$.

Theorem 2.2. (a) Suppose that $h(x) = 0$.

If

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty,$$

then there is extinction in finite time for $u(x, t)$.

(b) Assume that $\inf_{x \in \bar{\Omega}} u_o(x) = \rho > 0$. If

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty,$$

then $u(x, t) > 0$ in $\Omega \times (0, \infty)$.

Proof. (a) Let $\alpha(t)$ be a solution of (2.2). Then, we easily show that $\alpha(t)$ is a supersolution of (1.1) – (1.3) and the maximum principle implies that

$$0 \leq u(x, t) \leq \alpha(t) \quad \text{in } \Omega \times (0, T).$$

Therefore, the result follows from (2.5).

(b) Let $v(t)$ be a solution of the following differential equation

$$v'(t) = -\sqrt{2\lambda F(v(t))}, \quad v(0) = \rho. \quad (2.6)$$

Then $v(t)$ is a subsolution of (1.1) – (1.3). The maximum principle implies that $u(x, t) \geq v(t)$. It follows from $\int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty$ that $v(t) > 0$ for $t > 0$, which leads us to the result. \square

In the above theorem, we have obtained the extinction in finite time of the solution u of (1.1) – (1.3) in the case where the boundary values are zero. The question which appears is what happens if the boundary values are positive? To answer this question, we shall need the following definition:

Definition 2.3. Let $\varphi(x)$ be the stationary solution of the solution u of (1.1) – (1.3). We say that there exists a dead core for $\varphi(x)$ if there exists a set $\Omega_o \subset \Omega$ such that $\varphi(x) = 0$ for $x \in \Omega_o$.

The following lemma which may be found in [4] will be used later. It gives us information on the following steady-state problem:

$$\Delta \varphi(x) = \lambda f(\varphi(x)) \quad \text{in } \Omega, \quad (2.7)$$

$$\varphi(x) = h(x) \quad \text{on } \partial\Omega. \quad (2.8)$$

Lemma 2.4. Let $h(x) > 0$. Suppose that

$$f \in C^2(0, 1), \quad f(0) = 0, \quad f(1) = 1, \quad f'(s) \geq 0 \quad \text{on } (0, 1),$$

$$\text{either } f''(s) \geq 0 \quad \text{on } (0, 1) \text{ or } f''(s) \leq 0 \quad \text{on } (0, 1).$$

If

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty,$$

then there is no dead core for any λ .

If

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty,$$

then a dead core exists for sufficiently large λ , and for any $x_o \in \Omega$, x_o belongs to the dead core.

If $\int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty$, for $x_o \in \Omega$, define λ_o as follows:

$$\lambda_o = \inf_{\lambda} \{\varphi(x_o, \lambda) = 0\}, \quad (2.9)$$

where $\varphi(x, \lambda)$ is the solution of (2.7) – (2.8).

Now, introduce our result in the case where the boundary values are positive.

Theorem 2.5. *Assume that assumptions of Lemma 2.4 hold. For fixed $x_o \in \Omega$, choose $\lambda > \lambda_o$, where λ_o is defined in (2.9).*

(a) If

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty,$$

then $u(x_o, t) = 0$ for

$$t \geq \frac{1}{\sqrt{2(\lambda - \lambda_o)}} \int_0^1 \frac{ds}{\sqrt{F(s)}}.$$

(b) If

$$\int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty,$$

and $\min_{x \in \bar{\Omega}} u_o(x) > 0$, then $u(x_o, t) > 0$ for all t .

Proof. (a) Put

$$w(x, t) = \beta(t) + \varphi(x),$$

where $\beta(t)$ is a solution of

$$\beta'(t) = -\sqrt{2(\lambda - \lambda_o)F(\beta(t))}, \quad \beta(0) = 1$$

and $\varphi(x)$ the solution of (2.7) – (2.8) with $\lambda = \lambda_o$. We compute that

$$\begin{aligned} w_{tt} + \Delta w - \lambda f(w) &= \beta''(t) + \Delta \varphi(x) - \lambda f(w) \\ &= (\lambda - \lambda_o)f(\beta) + \lambda_o f(\varphi(x)) - \lambda f(w) \\ &\leq (\lambda - \lambda_o + \lambda_o - \lambda)f(w) = 0. \end{aligned}$$

It is easy to see that

$$w(x, t) \geq u(x, t) \quad \text{on} \quad \partial\Omega \times (0, \infty),$$

$$w(x, 0) \geq u_o(x) \quad \text{in } \Omega.$$

The maximum principle implies that

$$w(x, t) \geq u(x, t) \quad \text{in } \Omega \times (0, \infty),$$

and the result follows from (2.5) and (2.9).

The proof of Theorem 2.5 (b) is as the proof of Theorem 2.2 (b). \square

3. Asymptotic behavior.

In this section, we show how the solution u of (1.1) – (1.3) approaches its stationary solution φ . We assume that $u_o(x) \geq \varphi(x)$ in Ω . The maximum principle implies that $u(x, t) \geq \varphi(x)$ in $\Omega \times (0, \infty)$.

The following lemma will be used in the proof of the theorem in below.

Lemma 3.1. *Let $f(s)$ be a convex function with $f(0) = 0$ and let a and b be two nonnegative numbers. Then we have*

$$f(a + b) \geq f(a) + f(b).$$

Proof. Put $\alpha = \inf\{a, b\}$ and $\beta = \sup\{a, b\}$. Since $f(s)$ is convex with $f(0) = 0$, then using Taylor's formula, we get

$$f(a+b) = f(\beta+\alpha) \geq f(\beta) + \alpha f'(\beta) \geq f(\beta) + \alpha f'(\alpha) \geq f(\beta) + f(\alpha) = f(a) + f(b).$$

Hence the result. \square

Theorem 3.2. (a) *If f is convex, then*

$$0 \leq u(x, t) - \varphi(x) \leq \alpha(t),$$

where $\alpha(t)$ is a solution of (2.2).

(b) *If f is concave, then*

$$0 \leq u(x, t) - \varphi(x) \leq \zeta(t),$$

where $\zeta(t)$ is a solution of

$$\zeta'(t) = -\sqrt{\lambda f'(2)}\zeta(t), \quad \zeta(0) = 1.$$

Proof. (a) Put

$$w(x, t) = \alpha(t) + \varphi(x).$$

We compute that

$$\begin{aligned} w_{tt} + \Delta w - \lambda f(w) &= \alpha''(t) + \Delta \varphi(x) - \lambda f(w) \\ &= \lambda[f(\alpha) + f(\varphi(x)) - f(w)]. \end{aligned}$$

Since $f(s)$ is convex, it follows from Lemma 3.1 that

$$w_{tt} + \Delta w - \lambda f(w) \leq 0 \quad \text{in } \Omega \times (0, \infty).$$

Obviously, we have

$$\begin{aligned} w(x, t) &\geq u(x, t) \quad \text{on } \partial\Omega \times (0, \infty), \\ w(x, 0) &\geq u_o(x) \quad \text{in } \Omega. \end{aligned}$$

It follows from the maximum principle that

$$w(x, t) \geq u(x, t) \quad \text{in } \Omega \times (0, \infty),$$

which yields the first result.

(b) Let

$$z(x, t) = \zeta(t) + \varphi(x).$$

We compute that

$$\begin{aligned} z_{tt} + \Delta z - \lambda f(z) &= \zeta''(t) + \Delta\varphi(x) - \lambda f(z) \\ &= -\lambda f'(2)\zeta(t) + \lambda[f(\varphi(x)) - f(z)]. \end{aligned}$$

Since λf is concave, then using Taylor's formula, we get

$$\lambda[f(z) - f(\varphi(x))] \geq \lambda f'(z)(z - \varphi(x)) = \lambda f'(z)\zeta(t).$$

It follows from $z \leq 2$ that $\lambda[f(z) - f(\varphi(x))] \geq \lambda f'(2)\zeta(t)$, which implies that

$$z_{tt} + \Delta z - \lambda f(z) \leq 0 \quad \text{in } \Omega \times (0, \infty).$$

We easily show that

$$\begin{aligned} z(x, t) &\geq u(x, t) \quad \text{on } \partial\Omega \times (0, \infty), \\ z(x, 0) &\geq u_o(x) \quad \text{in } \Omega. \end{aligned}$$

The maximum principle implies that

$$z(x, t) \geq u(x, t) \quad \text{in } \Omega \times (0, \infty),$$

which gives the second result. \square

References

- [1] T. K. Boni, *On the asymptotic behavior of solutions for some semilinear parabolic and elliptic equations of second order with nonlinear boundary conditions*, Nonl. Anal. TMA, to appear.

- [2] **T. K. Boni**, *Sur le comportement asymptotique des solutions pour certaines équations paraboliques et elliptiques semi-linéaires du second ordre*, C. R. Acad. Sci. Paris, t. 327, Série I, 5 (1998), p. 455-460.
- [3] **C. Bandle - I. Stakgold**, *The formation of the dead core in parabolic reaction diffusin problems* Trans. Amer. Math. Soc. Vol. 286, 1 (1984) 275-293.
- [4] **C. Bandle - T. Nanbu - I. Stakgold**, *Porous medium equation with absorption* SIAM J. Math. Anal. Vol. 20, 5 (1998) 1268-1278.
- [5] **G. V. Grishina**, *Asymptotic behavior of solutions to weakly nonlinear equations in tube domains*, Diff. Equat., Vol. 31, 3 (1995), 429-434.
- [6] **V. A. Kondratiev - O. A. Oleinik**, *Some results for nonlinear elliptic equations in cylindrical domains*. in: Operational calculus spectral theory, Birkhauser, 1992, 185-195.
- [7] **V. A. Kondratiev - O. A. Oleinik**, *On asymptotic of solutions of nonlinear second order elliptic equations in cylindrical domains*. in: "Nonlinear. Diff. Equat. Appl.", 22, Birkhauser, 1996, 160-173.
- [8] **V. A. Kondratiev - L. Véron**, *Asymptotic behavior of solutions of some nonlinear parabolic or elliptic equations*, Asymptotic Analysis, 14 (1997) 117-156.
- [9] **M. H. Protter - H. F. Weinberger**, "Maximum principles in differential equations", Prentice Hall, Englewood Cliffs, NJ, 1967.
- [10] **D. H. Sattinger**, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J. Vol. 21, 11 (1972), 979-1000.
- [11] **L. Veron**, *Comportement asymptotique des solutions d'équations elliptiques semi-linéaires dans R^n* . Annali Math. Pura Appl., 127 (1981), 25-50.
- [12] **L. Veron**, *Equations d'évolution semi-linéaires du second ordre dans L^1* Rev. Roumaine Math. Pures Appl., XXVII (1982), 95-123.
- [13] **W. Walter**, "Differential- und Integral-Ungleichungen", Springer, Berlin, 1964.

Théodore K. BONI
 Université Paul Sabatier
 UFR-MIG, MIP
 118 route de Narbonne
 31062 Toulouse, France.
 E. Mail: boni@mip.ups-tlse.fr