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NEW CLASSES OF DISTORTION THEOREMS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING CERTAIN FRACTIONAL DERIVATIVES

R.K. Raina and Mamta Bolia

Abstract

This paper gives new classes of distortion inequalities for various subclasses of the class K_p of analytic p-valent functions in the unit disk U. The results presented involve certain generalized fractional derivatives of functions belonging to the classes $K_p^*(a,b)$ and $S_p(a,b)$. Some special cases of the main results are also pointed out.

1. Introduction and Definitions.

Let K_p denote the class of functions defined by

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \geq 0 \; ; \; p \in N = \{1,2,...\}), \tag{1.1}$$

which are analytic and p-valent in the unit disk $U = \{ z : |z| < 1\}$.

For a and b fixed, $-1 \leq a < b \leq 1$, f(z) is said to be *in the subclass* K_p^* (a,b) *of* K_p if

$$\left| \left(\frac{zf'(z)}{f(z)} - p \right) \left(\frac{bzf'(z)}{f(z)} - ap \right)^{-1} \right| < 1, \quad z \in U. \tag{1.2}$$

Furthermore, we will denote by $S_p(a,b)$ the set of meromorphic functions f such

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that $zf' \in K_p^*$ (a,b), see [5].

We denote by N, the set of non negative integers, R stands for the set of real numbers and $R_+ = (0,\infty)$ and C is the set of complex numbers.

The Gaussian hypergeometric function ${}_2F_1[z]$ is defined by

$$(1.3) \qquad {}_2F_1[z] = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\,n!} z^n,$$

$$(z \in U,\ c \neq 0, -1, -2, \ldots).$$

Definition. The fractional derivative operator $J_{0,z}^{\alpha,\beta,\eta}$ is defined by [3] :

$$(1.4) \qquad J_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\alpha-\beta}}{\Gamma(1-\alpha)} \int_0^z (z-t)^{-\alpha} f(t)\, {}_2F_1\left(\beta-\alpha, 1-\eta; 1-\alpha; 1-\frac{t}{z}\right)dt\right),$$

$$(0 \leq \alpha < 1\ ;\ \beta\ ,\ \eta \in R),$$

where the function f(z) is analytic in a simply - connected region of the z-plane containing the origin, with the order

$$(1.5) \qquad f(z) = O\left(|z|^r\right)\ (z \to 0),$$

for

$$(1.6) \qquad r > \max\{0, \beta-\eta\}-1,$$

and the multiplicity of $(z-t)^{-\alpha}$ is removed by requiring that log (z-t) to be real when $(z-t) \in R_+$.

The operator $J_{0,z}^{\alpha,\beta,\eta}$ includes (as its special cases) the Riemann-Liouville and Erdélyi-Kober operators of fractional calculus ([2], [4]). Indeed, we have

$$J_{0,z}^{\alpha,\alpha,\eta} f(z) = {}_0D_z^{\alpha} f(z) \qquad (0 \leq \alpha < 1), \tag{1.7}$$

where ${}_0D_z^{\alpha}$ is defined by

$$ {}_0D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dz} \int_0^z (z-t)^{-\alpha} f(t)\, dt \right) \qquad (0 \leq \alpha < 1) . \tag{1.8}$$

In the limiting case, when $\alpha \to 1^-$, we have

$$\lim_{\alpha \to 1^-} {}_0D_z^{\alpha} f(z) = f'(z) . \tag{1.9}$$

Also

$$J_{0,z}^{\alpha,1,\eta}\, z f(z) = \frac{d}{dz}\left[z\, E_{0,z}^{1-\alpha,\eta} f \right], \qquad (0 \leq \alpha < 1) , \tag{1.10}$$

where $E_{0,z}^{\alpha,\eta} f(z)$ denotes the Erdèlyi-Kober operator of fractional calculus and is defined by

$$E_{0,z}^{\alpha,\eta} f(z) = \frac{z^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} t^{\eta} f(t)\, dt \tag{1.11}$$

$$(\alpha > 0,\ \eta \in R) .$$

The fractional calculus operators have found interesting applications in the theory of analytic functions. This paper is devoted to obtaining new classes of distortion theorems for fractional derivatives of functions belonging to the subclasses $K_p^*(a,b)$ and $S_p(a,b)$, thereby giving its upper and lower bounds.

2. Distortion Theorems

Before we state and prove our main results for functions belonging to subclasses $K_p^*(a,b)$ and $S_p(a,b)$ we shall need the following results :

Lemma 1 [1]. *The function* f(z) *defined by* (1.1) *belongs to the class* $K_p^*(a,b)$ *if and only if*

(2.1) $$\sum_{n=1}^{\infty} \{(1+b)n + (b-a)p\} a_{p+n} \leq (b-a)p .$$

Lemma 2 [1]. *The function* f(z) *defined by* (1.1) *belongs to the class* $S_p(a,b)$ *if and only if*

(2.2) $$\sum_{n=1}^{\infty} \left(\frac{n+p}{p}\right) \{(1+b)n + (b-a)p\} a_{p+n} \leq (b-a)p .$$

Lemma 3 [6]. *Let* $0 \leq \lambda < 1$; $\mu, \eta \in R$; $k > (\max\{0, \mu-\eta\} - 1)$, *then*

(2.3) $$J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(1+k)\,\Gamma(1-\mu+\eta+k)}{\Gamma(1-\mu+k)\,\Gamma(1-\lambda+\eta+k)} z^{k-\mu} .$$

Our first main result is contained in the following :

Theorem 1 . Let a function f(z) *defined by* (1.1) *belong to the class* $K_p^*(a,b)$. *Then*

for $p \in N$, $0 \leq \lambda < 1$; $\mu \leq p$; $\eta \in R_+$ *such that*

$$max\ (\lambda,\ \mu) - p - 1 < \eta \leq \left(1 - \frac{2+p}{\mu}\right)\lambda\ ,\ we\ have:$$

$$(2.4)\quad \left| J_{0,z}^{\lambda,\mu,\eta} f(z) \right| \geq \Delta(\lambda,\mu,\eta,p)|z|^{p-\mu} \cdot \left[1 - \frac{p(p+1)\ (1+p-\mu+\eta)\ (b-a)}{\{(1+b)+(b-a)p\}\ (1-\mu+p)\ (1-\lambda+\eta+p)}\ |z|\right],$$

and

$$(2.5)\quad \left| J_{0,z}^{\lambda,\mu,\eta} f(z) \right| \leq \Delta\ (\lambda,\mu,\eta,p)\ |z|^{p-\mu} \cdot \left[1 + \frac{p(p+1)\ (1+p-\mu+\eta)\ (b-a)}{\{(1+b)+(b-a)p\}\ (1-\mu+p)\ (1-\lambda+\eta+p)}\ |z|\right],$$

for $z \in U$, *where, for convenience,*

$$(2.6)\qquad \Delta\ (\lambda,\mu,\eta,p) = \frac{\Gamma\ (1+p)\ \Gamma\ (1-\mu+\eta+p)}{\Gamma\ (1-\mu+p)\ \Gamma\ (1-\lambda+\eta+p)}\ .$$

The equalities are attained by the function

$$(2.7)\qquad f(z) = z^p - \frac{(b-a)p}{\{(1+b)+(b-a)p\}}\ z^{p+1}\ .$$

Proof. We begin by considering the function

(2.8) $$F(z)=\frac{1}{\Delta(\lambda,\mu,\eta,p)}\, z^{\mu}\, J_{0,z}^{\lambda,\mu,\eta} f(z)\ ,$$

where $\Delta(\lambda,\mu,\eta,p)$ is given by (2.6).

Then

$$F(z) = z^p - \sum_{n=1}^{\infty} \frac{(1+p)_n\,(1+\eta-\mu+p)_n}{(1-\mu+p)_n\,(1-\lambda+\eta+p)_n}\, a_{p+n}\, z^{p+n}\ ,$$

(2.9) $$= z^p - \sum_{n=1}^{\infty} \phi(n)\, a_{p+n} z^{p+n}\ ,$$

where

(2.10) $$\phi(n) = \frac{(1+p)_n\,(1-\mu+\eta+p)_n}{(1-\mu+p)_n\,(1-\lambda+\eta+p)_n}\ ,$$

and $(a)_n = a(a+1)...(a+n-1) = \Gamma(a+n) / \Gamma(a)$ denotes the factorial function.

We observe the following:

The function $\phi(n)$ defined by (2.10) satisfies the inequality $\phi(n+1) \leq \phi(n)$, $\forall\, n \in N$, provided that

$$\eta \leq \left\{1-\frac{(n+p+1)}{\mu}\right\} \lambda\ ,$$

thereby showing that $\phi(n)$ is non-increasing.

Thus, under the conditions stated in the hypothesis, we have

$$0 \le \phi(n) \le \phi(1) = \frac{(1+p)(1-\mu+\eta+p)}{(1-\mu+p)(1-\lambda+\eta+p)}, \quad \forall\ n \ge 1. \tag{2.11}$$

Also, from Lemma 1, we have

$$\sum_{n=1}^{\infty} a_{p+n} \le \frac{(b-a)p}{\{(1+b)+(b-a)p\}}. \tag{2.12}$$

Hence (2.9), (2.11) and (2.12) yield

$$|F(z)| \ge |z^p| - \phi(1)\ |z|^{p+1} \sum_{n=1}^{\infty} a_{p+n}$$

$$\ge |z|^p - \frac{p(b-a)}{\{(1+b)+(b-a)p\}}\ \phi(1)\ |z|^{p+1},$$

where $\phi(1)$ is given by (2.11). This readily gives the assertion (2.4) of Theorem 1. The assertion (2.5) can be proved similarly. It can be easily verified that equalities in (2.4) and (2.5) are attained by the function f(z) defined by (2.7). By applying Lemma 2 (instead of Lemma 1) to the function f(z) belonging to the class $S_p(a,b)$, we can prove the following :

Theorem 2. *Let a function* f(z) *defined by* (1.1) *belong to the class* $S_p(a,b)$. *Then for* $p \in N$, $0 \le \lambda < 1$; $\mu \le p$; $\eta \in R_+$ *such that*

$$max\ (\lambda,\mu)-p-1 < \eta \le \left(1-\frac{2+p}{\mu}\right)\lambda, \text{ we have :}$$

$$(2.13)\qquad \left|J_{0,z}^{\lambda,\mu,\eta}f(z)\right| \geq \Delta(\lambda,\mu,\eta,p)\,|z|^{p-\mu}$$

$$\cdot\left[1-\frac{p^2\,(b-a)\,(1-\mu+\eta+p)}{(1+p-\mu)\,(1-\lambda+\eta+p)\{(1+b)\,+\,(b-a)p\}}\,|z|\right],$$

and

$$(2.14)\qquad \left|J_{0,z}^{\lambda,\mu,\eta}f(z)\right| \geq \Delta(\lambda,\mu,\eta,p)\,|z|^{p-\mu}$$

$$\cdot\left[1+\frac{p^2\,(b-a)\,(1-\mu+\eta+p)}{(1+p-\mu)\,(1-\lambda+\eta+p)\{(1+b)\,+\,(b-a)p\}}\,|z|\right],$$

where Δ $(\lambda, \mu, \eta, \mathrm{p})$ *is given by* (2.6), *and* $z \in U$. *The equalities are attained by the function*

$$(2.15)\qquad f(z)=z^p - \frac{p^2(b-a)}{(p+1)\{(1+b)+(b-a)p\}}z^{p+1}.$$

3. Deductions of Theorems 1 and 2

By noting the relations (1.7) and (1.10), we point out below the deductions of Theorems 1 and 2, which give the corresponding distortion inequalities involving the operators ${}_0D_z^{\alpha}$ (the familiar Riemann - Liouville operator), and the operator $E_{0,z}^{\alpha,\eta}$ (the Erdélyi - Kober operator).

Thus, in the special case when $\lambda = \mu = \alpha$, then in view of (1.7), Theorems 1 and 2 would yield, respectively, the Corollaries 1 and 2 given below:

Corollary 1. Let a function f(z) *defined by* (1.1) *belong to the class* $K_p^*(a,b)$, *then*

(*under conditions easily obtainable from Theorem* 1):

$$(3.1)\qquad |{}_0D_z^\alpha f(z)| \geq \frac{\Gamma(1+p)}{\Gamma(1-\alpha+p)}\,|z|^{p-\alpha}\left[1-\frac{p(b-a)\;(p+1)}{(1-\alpha+p)\{(1+b)+(b-a)p\}}\,|z|\right],$$

and

$$(3.2)\qquad |{}_0D_z^\alpha f(z)| \leq \frac{\Gamma(1+p)}{\Gamma(1-\alpha+p)}\,|z|^{p-\alpha}\left[1+\frac{p(b-a)\;(p+1)}{(1-\alpha+p)\{(1+b)+(b-a)p\}}\,|z|\right].$$

Corollary 2. Let a function f(z) *defined by* (1.1) *belong to the class* $S_p(a,b)$, *then*

(under conditions easily obtainable from Theorem 1):

$$(3.3)\qquad |{}_0D_z^\alpha f(z)| \geq \frac{\Gamma(1+p)}{\Gamma(1-\alpha+p)}\,|z|^{p-\alpha}\left[1-\frac{p^2(b-a)}{(1-\alpha+p)\{(1+b)+(b-a)p\}}\,|z|\right],$$

and

$$(3.4)\qquad |{}_0D_z^\alpha f(z)| \leq \frac{\Gamma(1+p)}{\Gamma(1-\alpha+p)}\,|z|^{p-\alpha}\left[1+\frac{p^2(b-a)}{(1-\alpha+p)\{(1+b)+(b-a)p\}}\,|z|\right].$$

Similarly, distortion inequalities involving the operator (1.10) can be deduced from Theorems 1 and 2, and we do not record it here.

It may be observed that in the limiting case when $\alpha \to 1^-$, the results (3.1) and (3.2) of Corollary 1 give the bounds for the function $f'(z)$, which correspond to the results obtained in [1] (see also [5]).

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