ANNALES MATHÉMATIQUES BLAISE PASCAL

MUSTAPHA RACHDI
VINCENT MONSAN

Spectral density estimation for p-adic stationary processes

Annales mathématiques Blaise Pascal, tome 5, nº 1 (1998), p. 25-41
<http://www.numdam.org/item?id=AMBP_1998__5_1_25_0>

Ann. Math. Blaise Pascal, Vol. 5, N° 1, 1998, pp.25-41

Spectral density estimation for p-adic stationary processes

Mustapha RACHDI and Vincent MONSAN

Abstract

In this paper, we propose two asymptotically unbiased and consistent estimators for the spectral density of a stationary p-adic process $X = (X(t))_{t\in Q_p}$. The first estimator is constructed from observations $X = (X(t))_{t\in U_n}$, U_n being the p-adic ball with center 0 and radius $p^n, n \in \mathbb{Z}$, and the second, from observations $(X(\tau_k))_{k\in Z}$, where $(\tau_k)_{k\in Z}$ is a sequence of random variables taking their values in Q_p, associated to a Poisson counting process $\mathcal{N}$.

Key Words: p-Adic analysis; Periodogram; Quadratic-mean consistency; Spectral density.

AMS Subject Classification: 62M15-62G20.

1 Introduction

In mathematical physics, we use real and complex numbers, since space time coordinates are well described by means of real numbers.
Recently, to answer many questions in physics, an increasing interest has been given to p-adic numbers: they are used in superstrings theory (using very small distances, of the order of Planck length) where there are no grounds for believing the usual ideas to be valid.
P-adic numbers are going to be used, not only in mathematical physics, but also in other scientific grounds, where are met fractals and hierarchical structures (turbulence theory, dynamical physics, biology ...)(cf. [13] and [1]).
Brillinger, in [2], was the first to introduce spectral estimation for stationary p-adic processes, and he constructed the peridogram analogously to the real case.
This paper has two focuses developing a consistent estimates of the spectrum of a p-adic stationary process: observed on a p-adic ball and observed at the points process like in [8] and [9].
First we give some preliminaries about p-adic numbers.

2 Preliminaries

2.1 p-Adic numbers

Let p be a prime number. The norm $|.|_p$ on the field $\mathbb{Q}$ of rational numbers is defined by:

$$\forall x \in \mathbb{Q},\quad |x|_p = \begin{cases} p^{-\nu(x)} & \text{if } x = p^{\nu(x)}a/b, \text{ where } p \text{ is divisor neither of } a \text{ nor of } b. \\ 0 & \text{if } x = 0 \end{cases}$$

where $\nu(x) \in \mathbb{Z}$.

$|.|_p$ is a norm on $\mathbb{Q}$ and is called p-adic norm. The completion of $\mathbb{Q}$ for that norm is denoted Q_p, which called the field of p-adic numbers.

Theorem 2.1 (Ostrowski's theorem). *The Euclidian norms and the p-adic norms (p being a prime number), are the only non trivial (non equivalent) possible norms on the field* $\mathbb{Q}$.

Let $x \in Q_p$, $(x \neq 0)$; then x can be represented in a unique manner under the canonical form (Hansel representation):

$$x = p^{\nu(x)} \sum_{j=0}^{\infty} a_j p^j \text{ where } 0 \leq a_j < p, \quad a_0 > 0, \quad j = 0, 1, 2, ... \tag{1}$$

where the series (1) converges for the $|.|_p$ norm.

Definition 2.1 *The fractional part of a p-adic number x, denoted $\langle x \rangle_p$, or $\langle x \rangle$, is the number:*

$$\langle x \rangle = \begin{cases} 0 & \text{if } \nu(x) \geq 0, \\ p^{\nu(x)} \sum_{i=0}^{-\nu(x)-1} a_i p^i & \text{if } \nu(x) < 0. \end{cases}$$

Remark 2.1 *For all $x \in Q_p$; $0 < \langle x \rangle < 1$, if $\nu(x) < 0$.*

The ball with center x_0 and radius p^n is denoted by $U_n(x_0)$, i.e.

$$U_n(x_0) = \{x \in Q_p \ / \ |x - x_0|_p \leq p^n\}.$$

We denote $U_n = U_n(0)$, and we have the following properties:

1. $U_n(x_0)$ is compact, open in Q_p.
2. If $x_1 \in U_n(x_0)$, $U_n(x_1) = U_n(x_0)$: Every point of the ball $U_n(x_0)$ is its center.
3. If $U_n(x_0) \cap U_{n_1}(x_1) \neq \emptyset$ and $n_1 \leq n$, then $U_{n_1}(x_1) \subset U_n(x_0)$: Two balls in Q_p are either disjoint, or included one in the other.

$(Q_p, +, \times)$ is a complete separable metric space, locally compact and disconnected.

2.2 Characters of the group $(Q_p, +)$ and Fourier analysis on Q_p

$(Q_p, +)$ is an abelien locally compact group; from Haar's theorem there exists a positive measure on Q_p, uniquely determined except for a constant, denoted μ, which verifies the following properties: for $a \in Q_p$, $d(t+a) = dt$, $d(at) = |a|_p dt$ and $\mu(\mathbb{Z}_p) = 1$, where $\mathbb{Z}_p = U_0$. If $A \in \mathcal{B}_{Q_p}$; $\mu(A)$ is the Haar measure of A, where $\mathcal{B}_{Q_p}$ is the borelian σ-field on Q_p. This measure is explicited in [6](pages 202-203).

The characters γ of Q_p are defined by $\gamma : (Q_p, +) \longrightarrow (\mathbb{C}, \times)$, continuous and verifying:

1. $\gamma(t)| = \sqrt{\gamma(t)\gamma(-t)} = 1$.
2. $\forall t, s \in Q_p, \quad \gamma(t+s) = \gamma(t)\gamma(s)$

From [4], [6] (pages 400-402), we get the following expression for characters of Q_p: $\forall \gamma \in \hat{Q}_p$, $\exists \gamma \in Q_p$; $\forall t \in Q_p \quad \gamma(t) = e^{2i\pi\langle \gamma t \rangle}$, where $\langle \gamma t \rangle$ is the fractional part of the p-adic number γt and $\hat{Q}_p$ denotes the dual group of Q_p.

The Fourier transform $\mathcal{F}f$ of f is given by: $\forall u \in Q_p, \quad \mathcal{F}f(u) = \int_{Q_p} f(x)\, e^{2i\pi\langle ux \rangle} dx.$

It is defined for all absolutely integrable functions ($f \in L^1(Q_p)$).

If $f \in L^2(Q_p)$, we have the inverse relation: $f(x) = c \int_{Q_p} e^{-2i\pi\langle ux \rangle} \mathcal{F}f(u) du$, where c is a positive constant.

Moreover, Plancherel's formula is: $\displaystyle\int_{Q_p} |f(x)|^2 dx = c \int_{Q_p} |\mathcal{F}f(u)|^2 du.$

Example 2.1 *1.* $D_n(\lambda) = \int_{U_n} e^{-2i\pi\langle\lambda t\rangle}dt$, *Dirichlet kernel. If we calculate* $D_n(\lambda)$, *we obtain:* $D_n(\lambda) = \begin{cases} p^n & \text{if } |\lambda|_p \leq p^{-n} \\ 0 & \text{elsewhere} \end{cases}$

2. $\mathcal{F}\delta = 1$ *and* $\mathcal{F}1 = \delta$

Usual Fourier transforms are calculated in [4], [12].

3 Spectral density estimation for a p-adic stationary process from non-random sampling

3.1 The periodogram

Let $X = \{X(t)\}_{t\in Q_p}$ be a real valued p-adic stationary second order process, with mean zero, continuous covariance function $c_2 = \text{cum}\{X(t+u), X(t)\}$ for all t, $u \in Q_p$, element of $L^1(Q_p)$, such that

$\mathcal{H}_1$) X is stationary up to order 4, and the fourth cumulant function

$$c_4(u_1, u_2, u_3) = \text{cum}\{X(t+u_1), X(t+u_2), X(t+u_3), X(t)\},$$

is absolutely integrable.

$\mathcal{H}_2$) there exists $n_0 \in \mathrm{N}$ such that, for $n \geq n_0$, and $F_n = Q_p \setminus U_n$

$$\int_{F_n} |c_2(t)|dt \leq \text{const}/p^n,$$

where const denotes a positive constant.

The covariance c_2 is semi definite positive and continuous; then from Bochner's theorem, there exists a measure F_X with bounded variation on Q_p, such that

$$c_2(u) = \int_{Q_p} e^{2i\pi\langle ux\rangle} dF_X(x),$$

where F_X is the spectral measure of X, and is uniquely determined from c_2.
As $c_2 \in L^1(Q_p)$, the spectral density f_X is defined by

$$f_X(x) = \int_{Q_p} c_2(t)\ e^{-2i\pi\langle tx\rangle}dt, \quad \forall x \in Q_p.$$

First, we study the periodogram. The studied process is observed for all t belonging to U_n. The finite Fourier transform is then: $d_{X,n}(\lambda) = \int_{U_n} X(t)\ e^{-2i\pi\langle t\lambda\rangle}dt$, $\forall\lambda \in Q_p$,
and the periodogram is:

$$\begin{aligned} I_{X,n}(\lambda) &= \frac{1}{p^n}|d_{X,n}(\lambda)|^2 = \frac{1}{p^n}d_{X,n}(\lambda)d_{X,n}(-\lambda) \\ &= \frac{1}{p^n}\int_{U_n^2} X(t)X(s)\ e^{-2i\pi\langle(t-s)\lambda\rangle}dtds. \end{aligned} \tag{2}$$

Lemma 3.1 *Under* $\mathcal{H}_2$), *we have:* $cov\{d_{X,n}(u_1), d_{X,n}(u_2)\} = f_X(u_1)D_n(u_2 - u_1) + O(1)$.

Proof. Since, $d_{X,n}(u) = \int_{U_n} \mathrm{e}^{-2i\pi\langle tu\rangle} X(t)dt$.

$$\begin{aligned}
\operatorname{cov}\{d_{X,n}(u_1), d_{X,n}(u_2)\} &= \int_{U_n}\int_{U_n} \mathrm{e}^{-2i\pi\langle t_1u_1 - t_2u_2\rangle} c_2(t_1 - t_2)dt_1dt_2 \\
&= \int_{U_n}\left[\int_{Q_p} \mathrm{e}^{-2i\pi\langle tu_1\rangle} c_2(t)dt\right] \mathrm{e}^{2i\pi\langle t_2(u_2-u_1)\rangle} dt_2 \qquad (3)\\
&\quad -\int_{U_n}\int_{F_n} \mathrm{e}^{-2i\pi\langle tu_1\rangle} c_2(t)\, \mathrm{e}^{2i\pi\langle (u_2-u_1)t_2\rangle} dtdt_2 \qquad (4)
\end{aligned}$$

Moreover, from $\mathcal{H}_2$), (4) is less than $\int_{U_n}\int_{F_n} |c_2(t)|dtdt_2$, which is bounded. Thus (4) is $O(1)$, and

$$\begin{aligned}
\operatorname{cov}\{d_{X,n}(u_1), d_{X,n}(u_2)\} &= f_X(u_1)\int_{U_n} \mathrm{e}^{2i\pi\langle t_2(u_2-u_1)\rangle} dt_2 + O(1) \\
&= f_X(u_1)D_n(u_2 - u_1) + O(1).
\end{aligned}$$

Proposition 3.1 *Let $X = \{X(t)\}_{t\in Q_p}$ be a real variate p-adic stationary second order process, with mean zero, continuous covariance function c_2 element of $L^1(Q_p)$, such that $\mathcal{H}_2$) is satisfied, then*

$$\mathbb{E}[I_{X,n}(\lambda)] = f_X(\lambda) + O(\frac{1}{p^n}).$$

Therefore $I_{X,n}(\lambda)$ is an asymptotically unbiased estimator for $f_X(\lambda)$.

Proof. From lemma 3.1, we get

$$\mathbb{E}[I_{X,n}(\lambda)] = \frac{1}{p^n}\left[f_X(\lambda)D_n(0) + O(1)\right] = f_X(\lambda) + O(\frac{1}{p^n}).$$

Proposition 3.2 *Let $X = (X(t))_{t\in Q_p}$ be a real variate p-adic stationary second order process, with mean zero, continuous covariance function c_2 element of $L^1(Q_p)$, such that $\mathcal{H}_1$) is satisfied, then*

$$\lim_{n\to+\infty} \operatorname{var}[I_{X,n}(\lambda)] = f_X^2(\lambda) + f_X^2(0)$$

Therefore $I_{X,n}(\lambda)$ is a non consistent estimator of $f_X(\lambda)$.

Proof. We have

$$\begin{aligned}
\operatorname{var}[I_{X,n}(\lambda)] &= \frac{1}{p^{2n}}\operatorname{cum}\{d_{X,n}(\lambda)d_{X,n}(-\lambda), d_{X,n}(\lambda)d_{X,n}(-\lambda)\} \\
&= \frac{1}{p^{2n}}\operatorname{cum}\{d_{X,n}(\lambda), d_{X,n}(-\lambda), d_{X,n}(\lambda), d_{X,n}(-\lambda)\} \qquad (5)\\
&\quad +\frac{1}{p^{2n}}\operatorname{cum}\{d_{X,n}(\lambda), d_{X,n}(-\lambda)\}\operatorname{cum}\{d_{X,n}(-\lambda), d_{X,n}(\lambda)\} \qquad (6)\\
&\quad +\frac{1}{p^{2n}}\operatorname{cum}\{d_{X,n}(\lambda), d_{X,n}(\lambda)\}\operatorname{cum}\{d_{X,n}(-\lambda), d_{X,n}(-\lambda)\}. \qquad (7)
\end{aligned}$$

From [2], page 162, we can write :

$$\begin{aligned}
\operatorname{cum}\{d_{X,n}(\lambda_1), \ldots, d_{X,n}(\lambda_k)\} &= D_n(\lambda_1 + \ldots + \lambda_k)\int_{(U_n)^{k-1}} \mathrm{e}^{-2i\pi\langle \lambda_1u_1 + \cdots + \lambda_{k-1}u_{k-1}\rangle} \\
&\quad \times c_{k-1}(u_1, \ldots, u_{k-1})du_1 \ldots du_{k-1},\ k = 2, \ldots \qquad (8)
\end{aligned}$$

Then, the term (5), can be writen

$$\frac{1}{p^{2n}} D_n(0) \int_{U_n} \int_{U_n} \int_{U_n} e^{-2i\pi\langle \lambda u_1 - \lambda u_2 - \lambda u_3\rangle} c_4(u_1, u_2, u_3) du_1 du_2 du_3,$$

thus,

$$|(5)| \leq \frac{1}{p^n} \int\int\int_{Q_p^3} |c_4(u_1, u_2, u_3)| du_1 du_2 du_3.$$

Since $c_4(.,.,.) \in L^1({Q_p}^3)$ (from $\mathcal{H}_1$), we obtain that (5) is $O(1/p^n)$.
From proposition 3.1, we have $\lim_{n\to+\infty} \mathbb{E}[I_{X,n}(\lambda] = \int_{Q_p} c_2(u)\ e^{-2i\pi\langle u\lambda\rangle} du = f_X(\lambda)$,
then, the limit of (6) is $f_X^2(\lambda)$. As for (6); (7) is

$$\begin{aligned} &\frac{1}{p^{2n}} |D_n(2\lambda)|^2 \int_{U_n} e^{-2i\pi\langle u\lambda\rangle} c_2(u) du \int_{U_n} e^{2i\pi\langle v\lambda\rangle} c_2(v) dv \\ &= 1_{\{|2\lambda|\leq p^{-n}\}}(\lambda) \left[\int_{U_n} e^{-2i\pi\langle u\lambda\rangle} c_2(u) du\right]^2, \end{aligned} \tag{9}$$

and (9) converges to $f_X^2(0)$. Thus $\lim_{n\to\infty} \mathrm{var}[I_{X,n}(\lambda)] = f_X^2(\lambda) + f_X^2(0)$.

4 Smoothing the periodogram

The method is analogous to the smoothing in real time processes. Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of rational numbers, the terms of which are powers of p, and such that:

$$\lim_{n\to+\infty} M_n = 0 \quad \text{and} \quad \lim_{n\to+\infty} p^n M_n = +\infty. \tag{10}$$

For example, we choose $M_n = p^{-[\frac{n}{2}]}$, where $[x]$ is the integer part of x. That sequence verifies (10).
Let us consider a function $W : Q_p \longrightarrow \mathbb{R}$, which is continuous, positive and even, and verifies:

$$W \in L^\infty(Q_p) \cap L^1(Q_p) \text{ and } \int_{Q_p} W(\lambda) d\lambda = 1. \tag{11}$$

For each $t \in Q_p$, we denote by $\mathcal{F}W(t)$ the Fourier transform of W, that is

$$\mathcal{F}W(t) = \int_{Q_p} W(\lambda)\ e^{-2i\pi\langle t\lambda\rangle} d\lambda.$$

Let $W_n(\lambda) = 1/M_n W(M_n\lambda)$ be the spectral window.
The smoothed estimator of f_X is

$$\hat{f}_{X,n}(\lambda) = \int_{Q_p} W_n(\lambda - u) I_{X,n}(u) du.$$

Proposition 4.1 *Let $X = \{X(t)\}_{t\in Q_p}$ be a real valued p-adic stationary second order process, with mean zero, continuous covariance function $c_2 \in L^1(Q_p)$ satisfying $\mathcal{H}_2$), then*

$$\mathbb{E}\left[\hat{f}_{X,n}(\lambda)\right] = \int_{Q_p} W_n(\lambda - u) f_X(u) du + O(\frac{1}{p^n}).$$

Therefore $\hat{f}_{X,n}(\lambda)$ is an asymptotically unbiased estimator for $f_X(\lambda)$.

Proof. From proposition 3.1, and with the change of variates $v = M_n\lambda - M_n u$, we get:

$$\mathbb{E}\left[\hat{f}_{X,n}(\lambda)\right] = \int_{Q_p} W(v) f_X(\lambda - \frac{v}{M_n}) dv + O(1/p^n).$$

Since $c_2 \in L^1(Q_p)$, f_X is continuous; and as $\int_{Q_p} W(u)du = 1$, then by dominated convergence theorem, we get the result.

In the sequel, for $x \in Q_p$, we denote by δ_x the p-adic Dirac delta function, given for all $\lambda \in Q_p$ by:

$$\delta_x(\lambda) = \begin{cases} 1 & \text{if } \lambda = x \\ 0 & \text{otherwise.} \end{cases} \quad \text{and shortly, } \delta(\lambda) = \delta_0(\lambda).$$

Proposition 4.2 *Let $X = \{X(t)\}_{t \in Q_p}$ be a real variate p-adic stationary second order process, with mean zero, continuous covariance function c_2 element of $L^1(Q_p)$, such that hypothesis $\mathcal{H}_1$) and $\mathcal{H}_2$) are satisfied, then for all λ_1, λ_2 in Q_p*

$$\begin{aligned} & p^n M_n cov[\hat{f}_{X,n}(\lambda_1), \hat{f}_{X,n}(\lambda_2)] \\ & = [\delta(\lambda_1 + \lambda_2) + \delta(\lambda_1 - \lambda_2)] f_X^2(\lambda_1) \int_{Q_p} W^2(u)du + o(1) + O(M_n). \end{aligned}$$

Therefore $\hat{f}_{X,n}(\lambda)$ is a consistent estimator for $f_X(\lambda)$.

To prove proposition 4.2, we need the following lemma:

Lemma 4.1 *We denote $\Delta_n(u) = 1/p^n |D_n(u)|^2$ the p-adic Fejer kernel, and $J_n(\lambda) = \int_{Q_p} W_n(\lambda - u)\Delta_n(u)du - W_n(\lambda)$. Then $J_n(\lambda) = o(1/M_n)$, uniformly in λ.*

Proof of lemma 4.1. With $t = (1/p^n)\lambda$, we obtain

$D_n(vp^n) = \int_{|\lambda|_p \le 1} e^{-2i\pi\langle v\lambda\rangle} |1/p^n|_p \, d\lambda = p^n D_0(v)$, where $D_0(v) = \int_{|\lambda|_p \le 1} e^{-2i\pi\langle v\lambda\rangle} d\lambda$.

As, $\int_{Q_p} \Delta_n(u)du = p^n \mu\{|u|_p \le p^{-n}\}$, moreover $\mu\{|u|_p \le p^{-n}\} = p^{-n}$, then $\int_{Q_p} \Delta_n(u)du = 1$. With the change of variates $u = vp^n$,

$$\begin{aligned} J_n(\lambda) &= \frac{1}{M_n}\int_{Q_p} [W(M_n\lambda - p^n M_n v) - W(M_n\lambda)]\, \Delta_n(vp^n)\frac{dv}{p^n} \\ &= \frac{1}{M_n}\int_{Q_p} [W(M_n\lambda - p^n M_n v) - W(M_n\lambda)]\, |D_0(v)|^2 dv. \end{aligned}$$

As, $W(\lambda) = \int_{Q_p} e^{2i\pi\langle\lambda t\rangle} \mathcal{F}W(t)dt$, we have

$$|W(M_n\lambda - p^n M_n v) - W(M_n\lambda)| \le \int_{Q_p} \left| e^{-2i\pi\langle tp^n M_n v\rangle} - 1\right| |\mathcal{F}W(t)|\, dt,$$

thus, $|M_n J_n(\lambda)| \le \int_{Q_p} \left[\int_{Q_p} \left| e^{-2i\pi\langle tM_n p^n v\rangle} - 1\right| |\mathcal{F}W(t)|dt\right] |D_0(v)|^2 dv.$

Then, $|M_n J_n(\lambda)| \le \int_{Q_p} \theta(v)dv$, where $\theta(v) = 2|D_0(v)|^2 \int_{Q_p} |\mathcal{F}W(t)|dt \in L^1(Q_p)$.

Indeed, $\int_{Q_p} |\theta(v)|dv = 2\,|\mathcal{F}W|_{L^1(Q_p)} < \infty$. Then, by dominated convergence theorem, we get: $\lim_{n\to+\infty} |M_n J_n(\lambda)| \le \int_{Q_p}\int_{Q_p} \lim_{n\to+\infty} \left| e^{-2i\pi\langle tM_n p^n v\rangle} - 1\right| |\mathcal{F}W(t)|\, |D_0(v)|\, dv.$

Since, $(tvM_np^n)_{n\in N}$ is a p-adic sequence, and, $\lim_{n\to+\infty} p^nM_n = +\infty$,

$$\lim_{n\to+\infty} |tvM_np^n|_p = \lim_{n\to+\infty} |t|_p|v|_p|M_n|_p|p^n|_p = \lim_{n\to+\infty} |t|_p|v|_p/M_np^n = 0,$$

thus, $\lim_{n\to+\infty} M_nJ_n(\lambda) = 0$ uniformly in λ, i.e. $J_n(\lambda) = o(1/M_n)$.

Proof of proposition 4.2. Let $\lambda_1, \lambda_2 \in Q_p$. We have

$\mathrm{cov}\{\hat{f}_{X,n}(\lambda_1), \hat{f}_{X,n}(\lambda_2)\} = \int_{Q_p}\int_{Q_p} W_n(\lambda_1-u_1)W_n(\lambda_2-u_2)\mathrm{cov}\{I_{X,n}(u_1), I_{X,n}(u_2)\}du_1du_2$.
Let us comput $\mathrm{cov}\{I_{X,n}(u_1), I_{X,n}(u_2)\}$.
For this, we have $\mathrm{cov}\{I_{X,n}(u_1), I_{X,n}(u_2)\} = A_1 + A_2 + A_3$, where

$$A_1 = \frac{1}{p^{2n}}\mathrm{cum}\{d_{X,n}(u_1), d_{X,n}(-u_1), d_{X,n}(u_2), d_{X,n}(-u_2)\} \quad (12)$$

$$A_2 = \frac{1}{p^{2n}}\mathrm{cum}\{d_{X,n}(u_1), d_{X,n}(u_2)\}\mathrm{cum}\{d_{X,n}(-u_1), d_{X,n}(-u_2)\} \quad (13)$$

$$A_3 = \frac{1}{p^{2n}}\mathrm{cum}\{d_{X,n}(u_1), d_{X,n}(-u_2)\mathrm{cum}\{d_{X,n}(u_2), d_{X,n}(-u_1)\} \quad (14)$$

Thus, $\mathrm{cov}\{\hat{f}_{X,n}(\lambda_1), \hat{f}_{X,n}(\lambda_2)\} = I_1 + I_2 + I_3$, where I_i is the contribution of A_i for $i = 1,2,3$.
For the first term I_1 : From the proof of proposition 2.2, we have $A_1 = O(1/p^n)$, thus

$$|I_1| \le \frac{1}{p^n}\int_{Q_p} |W_n(\lambda_1-u_1)|\,du_1 \int_{Q_p} |W_n(\lambda_2-u_2)|\,du_2 \int\int\int_{Q_p^3} |c_4(v_1,v_2,v_3)|dv_1dv_2dv_3.$$

W is integrable, and from $x = M_n\lambda - M_nu_i$, for $i = 1,2$, we get

$$\int_{Q_p} |W_n(\lambda_i-u_i)|du_i = \int_{Q_p} |W(x)|\,dx < \infty.$$

From $\mathcal{H}_1$), we get $p^nI_1 = O(1)$ i.e. $p^nM_nI_1 = O(M_n)$.
The second term I_2 : From lemma 3.1, we get

$$\begin{aligned} A_2 &= \frac{1}{p^{2n}}\{[f_X(u_1)D_n(u_2-u_1) + O(1)][\overline{f_X(u_1)D_n(u_1-u_2)} + O(1)]\} \\ &= \frac{1}{p^{2n}}|f_X(u_1)D_n(u_2-u_1)|^2 + \frac{1}{p^{2n}}O(1)f_X(u_1)D_n(u_2-u_1) \\ &\quad + \frac{1}{p^{2n}}O(1)\overline{f_X(u_1)D_n(u_2-u_1)} + O(\frac{1}{p^{2n}}). \end{aligned}$$

Since $1/p^{2n} \to 0$ and $1/p^{2n}f_X(u_1)D_n(u_2-u_1) \to 0$ as $n \to +\infty$, we deduce

$$A_2 = \frac{1}{p^{2n}}|f_X(u_1)D_n(u_2-u_1)|^2 + O(\frac{1}{p^n}).$$

As, $\Delta_n(u) = 1/p^n|D_n(u)|^2$, we have

$$I_2 = \frac{1}{p^n}\int_{Q_p} W_n(\lambda_1-u_1)|f_X(u_1)|^2\,[J_n(\lambda_2-u_1) + W_n(\lambda_2-u_1)]\,du_1 + O(\frac{1}{p^n}).$$

From lemma 4.1, and by dominated convergence theorem, we get

$$\begin{aligned} I_2 &= \frac{1}{p^n}\int_{Q_p} W_n(\lambda_1-u_1)|f_X(u_1)|^2W_n(\lambda_2-u_1)du_1 \\ &\quad + \frac{1}{p^n}\int_{Q_p} W_n(\lambda_1-u_1)|f_X(u_1)|^2o(\frac{1}{M_n})du_1 + O(\frac{1}{p^n}) \\ &\hat{=}\ I_{2,1} + I_{2,2} + O(\frac{1}{p^n}), \end{aligned}$$

where, $p^n M_n I_{2,2} = o(1) \int_{Q_p} W_n(\lambda_1 - u_1)|f_X(u_1)|^2 du_1$.
As, $W \in L^1(Q_p)$ and $f_X \in L^\infty(Q_p)$, we have, $p^n M_n I_{2,2} = o(1)$.
With $v = M_n\lambda_1 - M_n u_1$, we get:

$$p^n M_n I_{2,1} = \int_{Q_p} W(v) W\left(M_n(\lambda_1 - \lambda_2) - v\right) f_X^2(\lambda_1 - \frac{v}{M_n}) dv.$$

If $\lambda_1 = \lambda_2$: Note that we have, W is a even function, f_X is a continuous and bounded function and $\forall v \in Q_p$; $\lim_{n\to+\infty} v/M_n = 0$ in Q_p, then

$$p^n M_n I_{2,1} = \int_{Q_p} W(v)^2 \left[f_X^2(\lambda_1 - \frac{v}{M_n}) - f_X^2(\lambda_1)\right] dv + \int_{Q_p} W(v)^2 f_X^2(\lambda_1) dv.$$

Thus, by dominated convergence theorem, $p^n M_n I_{2,1} = o(1) + f_X^2(\lambda_1) \int_{Q_p} W(v)^2 dv$.

If $\lambda_1 \neq \lambda_2$: Since $W(x) = \int_{Q_p} e^{-2i\pi(xt)} \mathcal{F}W(t) dt$, the Fourier transform of $\mathcal{F}W$, it is uniformly continuous; and since $\lim_{n\to+\infty} |M_n(\lambda_2 - \lambda_1)|_p = \lim_{n\to\infty} |(\lambda_2 - \lambda_1)|_p / M_n = +\infty$. Then from the dominated convergence theorem, $\lim_{n\to+\infty} p^n M_n I_{2,1} = 0$.
Thus we get, $\forall \lambda_1, \lambda_2 \in Q_p$, $p^n M_n I_{2,1} = \delta(\lambda_2 - \lambda_1) f_X^2(\lambda_1) \int_{Q_p} W^2(v) dv + o(1)$.
So, $p^n M_n I_2 = \delta(\lambda_2 - \lambda_1) f_X^2(\lambda_1) \int_{Q_p} W^2(v) dv + o(1) + O(M_n)$.
With analogous calculations, we get

$$p^n M_n I_3 = \delta(\lambda_2 + \lambda_1) \int_{Q_p} W(v)^2 dv f_X^2(\lambda_1) + o(1) + O(M_n).$$

Thus,

$$p^n M_n \mathrm{cov}\left[\hat{f}_{X,n}(\lambda_1), \hat{f}_{X,n}(\lambda_2)\right] = \left[\delta(\lambda_2 + \lambda_1) + \delta(\lambda_2 - \lambda_1)\right] f_X^2(\lambda_1) \int_{Q_p} W^2(v) dv + o(1) + O(M_n),$$

and $\hat{f}_{X,n}$ is consistent, thus proposition 4.2 is proved.

Corollary 4.1 *Under $\mathcal{H}_1) - \mathcal{H}_2)$, we have $\hat{f}_{X,n}(\lambda)$ converges to $f_X(\lambda)$ in quadratic mean as $n \to \infty$.*

Proof of corollary 4.1. We know that, the mean square error is:

$$MSE(n) = \ \mathrm{bias}^2(\hat{f}_{X,n}) + \ \mathrm{var}\{\hat{f}_{X,n}(\lambda)\}.$$

From propositions 4.1 and 4.2, we get $MSE(n) \to 0$ as $n \to +\infty$.
This implies mean quadratic convergence.

5 Spectral density estimation of a p-adic stationary process from random sampling

5.1 Preliminaries

Let $X = \{X(t), t \in Q_p\}$ be a $p-$adic stationary second order process, with mean zero, continuous covariance function element of $L^1(Q_p)$, and with spectral density function f_X. From [3] and [7], there exists a counting process, denoted by $\mathcal{N}$, which is associated to a

sequence $(\tau_k)_{k\in\mathbb{Z}}$ of random variables taking their values in Q_p.
The process $\mathcal{N}$ is defined by

$$\begin{array}{rccl} \mathcal{N}: & \mathcal{B}_{Q_p}\times\Omega & \longrightarrow & \mathbb{N} \\ & (A,\omega) & \longmapsto & \mathcal{N}(A,\omega)=\sum_{k\in\mathbb{Z}} 1_A(\tau_k(\omega)) \end{array}$$

and $\mathcal{N}(A,\omega)$ is the number of τ_k's belonging to A.
We suppose that, for every A element of $\mathcal{B}_{Q_p}$, the random variable $\mathcal{N}(A)$ has a Poisson distribution $\mathcal{P}(\Lambda(A))$ (such a process exists by [3, 7]), where $\Lambda(A)=\beta\mu(A)$ and μ is the Haar measure on Q_p. In the sequel, we suppose also that also the mean intensity $\beta=\mathbb{E}\{U_0\}$ is known.
For every A, B disjoint in Q_p; $\mathcal{N}(A)$ and $\mathcal{N}(B)$ are independent.
Thus $\mathbb{E}[\mathcal{N}(A)]=\beta\mu(A)=\beta\int_{Q_p}1_A(x)dx$ which implies $\mathbb{E}[\mathcal{N}(dt)]=\beta dt$.
Let $A,B\in\mathcal{B}_{Q_p}$, we have

$$\begin{aligned}\mathbb{E}[\mathcal{N}(A)\mathcal{N}(B)] &= \mathbb{E}[\mathcal{N}(A\cap B)^2]+\mathbb{E}[\mathcal{N}(A\cap B)\mathcal{N}(A\cap\bar{B})]\\ &\quad+\mathbb{E}[\mathcal{N}(A\cap B)\mathcal{N}(\bar{A}\cap B)]+\mathbb{E}[\mathcal{N}(A\cap\bar{B})\mathcal{N}(\bar{A}\cap B)],\end{aligned}$$

where $\bar{A}$ denotes the complement of A in Q_p.
Since $\mathcal{N}(A)$ is Poisson for every A in $\mathcal{B}_{Q_p}$, we get:

$$\mathbb{E}\left[\mathcal{N}(A\cap B)^2\right]=\beta^2\mu(A\cap B)^2+\beta\mu(A\cap B).$$

Thus, since $\int_{Q_p}f(x,x)dx=\int_{Q_p}f(x,y)\left[\int_{Q_p}d\delta_x(y)\right]dx$, we have

$$\begin{aligned}\mathbb{E}\left[\mathcal{N}(A)\mathcal{N}(B)\right] &= \beta^2\mu(A)\mu(B)+\beta\mu(A\cap B)\\ &= \int_{A\times B}\beta^2dxdy+\int_{A\times B}\beta d\delta_x(y)dx,\end{aligned}$$

Then

$$\mathbb{E}\left[\mathcal{N}(dt)\mathcal{N}(ds)\right]=\beta^2dtds+\beta d\delta_t(s)dt$$

$$\mathbb{E}\left[\mathcal{N}(t+dt)\mathcal{N}(t+s+ds)\right]=\beta^2dtds+\beta d\delta(s)dt.$$

Since $\mathcal{N}(A)$ is Poisson, $\mathbb{E}\left[\mathcal{N}(t+dt)\right]=\mathbb{E}\left[\mathcal{N}(dt)\right]=\beta dt$ and

$$\begin{aligned}\operatorname{cov}\left[\mathcal{N}(t+dt),\mathcal{N}(t+s+ds)\right] &= \beta^2dtds+\beta d\delta(s)dt-\beta^2dtds\\ &= \beta d\delta(s)dt.\end{aligned}$$

5.2 Construction of the spectral density estimator

Definition 5.1 *The sample process Z is defined by*

$$Z(A)=\int_A X(t)\mathcal{N}(dt)=\sum_{k\in\mathbb{Z}}X(\tau_k)1_A(\tau_k)=\sum_{\tau_k\in A}X(\tau_k),\quad \forall A\in\mathcal{B}_{Q_p}.$$

This definition may be written, too: $Z(t+dt) = X(t)\mathcal{N}(t+dt)$.
Since X and $\mathcal{N}$ are independent, we get

$$\mathbb{E}[Z(dt)] = \mathbb{E}[X(t)\mathcal{N}(dt)] = \mathbb{E}[X(t)]\,\mathbb{E}[\mathcal{N}(dt)] = 0$$

and

$$\begin{aligned}\operatorname{cov}[Z(t+dt), Z(t+s+ds)] &= \mathbb{E}[X(t)X(t+s)]\,\mathbb{E}[\mathcal{N}(t+dt)\mathcal{N}(t+s+ds)] \\ &= c_2(s)[\beta^2 ds + \beta d\delta(s)]dt.\end{aligned}$$

Thus the increment process Z is second order stationary.
Since $c_2 \in L^1(Q_p)$, we define the measure θ_Z, for all B element of $\mathcal{B}_{Q_p}$, by:

$$\theta_Z(B) = \int_B \theta_Z(dt) = \int_B c_2(u)\left[\beta^2 dt + \beta d\delta(t)\right],$$

then

$$\theta_Z(dt) = c_2(t)\left[\beta^2 dt + \beta d\delta(t)\right].$$

Defining the measure $\theta_{\mathcal{N}}$, for every B in $\mathcal{B}_{Q_p}$ by: $\theta_{\mathcal{N}}(B) = \int_B [\beta^2 du + \beta d\delta(u)]$ (under differential form $\theta_{\mathcal{N}}(dt) = [\beta^2 dt + \beta d\delta(t)]$), then we have

$$\begin{aligned}\theta_Z(B) &= \int_B c_2(u)\beta d\delta(u) + \int_B c_2(u)\beta^2 du \\ &= \beta c_2(0)\delta(B) + \int_B c_2(u)\beta^2 du.\end{aligned}$$

The Haar measure being σ-finite on Q_p, then the measures $\theta_{\mathcal{N}}$ and θ_Z are also σ-finite.
The spectral density f_Z associated to the process Z is defined, for $\lambda \in Q_p$ by:

$$f_Z(\lambda) = \int_{Q_p} \mathrm{e}^{-2i\pi\langle t\lambda\rangle}\theta_Z(dt).$$

By definition of θ_Z, we get

$$\begin{aligned}f_Z(\lambda) &= \beta\int_{Q_p} \mathrm{e}^{-2i\pi\langle\lambda t\rangle}c_2(0)d\delta(t) + \beta^2\int_{Q_p} \mathrm{e}^{-2i\pi\langle\lambda t\rangle}c_2(t)dt \\ &= \beta c_2(0) + \beta^2 f_X(\lambda).\end{aligned} \tag{15}$$

In order to estimate f_X, we write from the formula (15): $f_X(\lambda) = 1/\beta^2\,[f_Z(\lambda) - \beta c_2(0)]$.
So we have to estimate $c_2(0)$ and f_Z.

1. Estimation of $c_2(0)$.
   We propose the estimator: $\hat{c}_{2,n}(0) = 1/(\beta p^n)\int_{U_n} X^2(t)\mathcal{N}(dt)$.

2. Estimation of $f_Z(\lambda)$.
   First we introduce a sequence $(M_n)_{n\in\mathbb{N}}$ of rational numbers, a p-adic kernel W, like in section 2.1, formula (11) ; and the same spectral window, i.e. $W_n(\lambda) = 1/M_n W(M_n\lambda)$.
   Let $d_{Z,n}(\lambda)$ be the finite Fourier transform associated to the observations $Z(t)$, $t \in U_n$, i.e.

$$d_{Z,n}(\lambda) = \int_{U_n} \mathrm{e}^{-2i\pi\langle\lambda t\rangle}Z(dt) = \int_{U_n} \mathrm{e}^{-2i\pi\langle\lambda t\rangle}X(t)\mathcal{N}(dt).$$

   The associated periodogram is:

$$I_{Z,n}(\lambda) = \frac{1}{p^n}|d_{Z,n}(\lambda)|^2 = \frac{1}{p^n}\left|\int_{U_n} \mathrm{e}^{-2i\pi\langle\lambda t\rangle}X(t)\mathcal{N}(dt)\right|^2.$$

   We estimate $f_Z(\lambda)$ by: $\hat{f}_{Z,n}(\lambda) = \int_{Q_p} W_n(\lambda - u)I_{Z,n}(u)du$.

Then we propose the following estimator for $f_X(\lambda)$:

$$\hat{f}_{X,n}(\lambda) = \frac{1}{\beta^2}\left[\hat{f}_{Z,n}(\lambda) - \beta\hat{c}_{2,n}(0)\right].$$

5.3 Asymptotic behaviour of the estimators

The asymptotic behaviour of $\hat{f}_{X,n}(\lambda)$ will be established from the properties of the estimators $\hat{f}_{Z,n}(\lambda)$ and $\hat{c}_{2,n}(0)$.
Our hypothesis are the following:

$\mathcal{H}_3)$ $c_2 \in L^1(Q_p)$	$\mathcal{H}_4)$ $c_2 \in L^2(Q_p)$
$\mathcal{H}_5)$ $\mathcal{F}W \in L^1(Q_p)$	$\mathcal{H}_6)$ $c_4(u,u,u) \in L^1(Q_p)$
$\mathcal{H}_7)$ $c_4(u,0,0) \in L^1(Q_p)$	$\mathcal{H}_8)$ $c_4(u,u,0) \in L^1(Q_p)$
$\mathcal{H}_9)$ $c_4(u,v,0) \in L^1(Q_p^2)$	$\mathcal{H}_{10})$ $c_4(u,v,v) \in L^1(Q_p^2)$
$\mathcal{H}_{11})$ $c_4(u,u+v,0) \in L^1(Q_p^2)$	$\mathcal{H}_{12})$ $c_4(u,u+v,w) \in L^1(Q_p^3)$.

Proposition 5.1 *Under $\mathcal{H}_4)$ and $\mathcal{H}_8)$, $\hat{c}_{2,n}(0)$ is unbiased and consistent.*

Proof of proposition 5.1.

a. As, X and $\mathcal{N}$ are independent,

$$\mathbb{E}[\hat{c}_{2,n}(0)] = \frac{1}{\beta p^n}\int_{U_n}\mathbb{E}[X^2(t)\mathcal{N}(dt)] = \frac{1}{\beta p^n}c_2(0)\int_{U_n}\beta dt = c_2(0).$$

Then, $\hat{c}_{2,n}(0)$ is unbiased.

b. $\hat{c}_{2,n}(0)$ is consistent. Indeed:

$$\begin{aligned} p^{2n}\beta^2\mathrm{var}[\hat{c}_{2,n}(0)] &= \int_{U_n}\int_{U_n}\mathbb{E}\left[X^2(t)X^2(s)\mathcal{N}(dt)\mathcal{N}(ds)\right] - p^{2n}\beta^2c_2^2(0)\\ &= \int_U\int_{U_n}\mathbb{E}\left[X^2(t)X^2(s)\right]\mathbb{E}\left[\mathcal{N}(dt)\mathcal{N}(ds)\right] - p^{2n}\beta^2c_2^2(0).\end{aligned}$$

As X and $\mathcal{N}$ are independent, with $s = t+u$, we get

$$\begin{aligned} p^{2n}\beta^2\mathrm{var}[\hat{c}_{2,n}(0)] &= \int_{U_n}\int_{U_n}\left[c_4(0,u,u) + 2c_2^2(u) + c_2^2(0)\right][\beta^2du + \beta d\delta(u)]dt - p^{2n}\beta^2c_2^2(0)\\ &= \beta^2p^n\int_{U_n}c_4(0,u,u)du + \beta p^nc_4(0,0,0) + 2\beta^2p^n\int_{U_n}c_2^2(u)du + 3\beta p^nc_2^2(0).\end{aligned}$$

From $\mathcal{H}_4)$ and $\mathcal{H}_8)$, we deduce: $\mathrm{var}[\hat{c}_{2,n}(0)] = O(1/p^n)$.

Proposition 5.2 *Under $\mathcal{H}_3) - \mathcal{H}_{12})$, $\hat{f}_{Z,n}(\lambda)$ is asymptotically unbiased and consistent.*

Proof of proposition 5.2.

a. $\hat{f}_{Z,n}(\lambda)$ is asymptotically unbiased.
With change of variate $v = M_n\lambda - M_nu$, such that: $dv = |M_n|_p\,du = 1/M_ndu$, we get

$$\begin{aligned}\mathbb{E}[\hat{f}_{Z,n}(\lambda)] &= \frac{1}{M_n}\mathbb{E}\left[\int_{Q_p}W(M_n\lambda - M_nu)I_{Z,n}(u)du\right]\\ &= \int_{Q_p}W(v)\mathbb{E}\left[I_{Z,n}(\lambda - \frac{v}{M_n})\right]dv.\end{aligned}$$

As, $\int_{Q_p} W(v)dv = 1$, we obtain

$$\mathbb{E}[\hat{f}_{Z,n}(\lambda)] - f_Z(\lambda) = \int_{Q_p} W(v)\mathbb{E}\left[I_{Z,n}(\lambda - \frac{v}{M_n}) - f_Z(\lambda)\right] dv.$$

Let $g_{n,\lambda}(v) = W(v)\mathbb{E}\left[I_{Z,n}(\lambda - \frac{v}{M_n}) - f_Z(\lambda)\right]$. With $w = u - t$, we can write

$$\begin{aligned}
\mathbb{E}\left[I_{Z,n}(\lambda - \frac{v}{M_n})\right] &= \frac{1}{p^n}\int_{U_n}\int_{U_n} e^{-2i\pi\langle(\lambda - v/M_n)(u-t)\rangle}\mathbb{E}\left[Z(dt)Z(du)\right] \\
&= \frac{\beta^2}{p^n}\int_{U_n}\int_{U_n} e^{-2i\pi\langle(\lambda - v/M_n)(u-t)\rangle}c_2(t-u)dudt \\
&\quad + \frac{\beta}{p^n}\int_{U_n}\int_{U_n} e^{-2i\pi\langle(\lambda - v/M_n)(u-t)\rangle}c_2(t-u)d\delta_t(u)dt \\
&= \frac{\beta^2}{p^n}\int_{U_n}\left[\int_{U_n} e^{-2i\pi\langle(\lambda - v/M_n)w\rangle}c_2(w)dw\right]dt + c_2(0)\frac{\beta}{p^n}\mu\{U_n\} \\
&= \beta^2\int_{U_n} e^{-2i\pi\langle(\lambda - v/M_n)w\rangle}c_2(w)dw + c_2(0)\beta. \qquad (16)
\end{aligned}$$

From dominated convergence theorem, and $\lim_{n\to+\infty} v/M_n = 0$ in Q_p, we obtain:

$$\lim_{n\to+\infty}\mathbb{E}\left[I_{Z,n}(\lambda - \frac{v}{M_n})\right] = \beta c_2(0) + \beta^2 f_X(\lambda) = f_Z(\lambda).$$

Thus $\lim_{n\to\infty} g_{n,\lambda}(v) = 0$. Moreover, from (16), and since $|f_Z(\lambda)| \le \beta^2|c_2|_{L^1(Q_p)} + |c_2(0)/\beta|$, we have

$$|g_{n,\lambda}(v)| \le W(v)\left[|c_2(0)|\beta^2 + \beta|c_2|_{L^1(Q_p)} + \beta^2|c_2|_{L^1(Q_p)} + |\frac{c_2(0)}{\beta}|\right]. \qquad (17)$$

The right hand side of (17) is integrable, since c_2 and W are; from dominated convergence theorem, we get the asymptotic unbiasedness of $\hat{f}_{Z,n}$.

b. With the change of variates $v = \lambda - u$, we get

$$\begin{aligned}
&\hat{f}_{Z,n}(\lambda) \\
&= \frac{1}{p^n}\int_{Q_p}\int_{U_n}\int_{U_n} W_n(\lambda - u)\, e^{-2i\pi\langle u(t-s)\rangle}X(t)X(s)\mathcal{N}(t+dt)\mathcal{N}(s+ds)du \\
&= \frac{1}{p^n}\int_{U_n}\int_{U_n}\left[\int_{Q_p} W_n(v)\, e^{2i\pi\langle v(t-s)\rangle}dv\right] e^{-2i\pi\langle\lambda(t-s)\rangle}X(t)X(s)\mathcal{N}(t+dt)\mathcal{N}(s+ds),
\end{aligned}$$

and, with $u = M_n v$, which implies $dv = M_n du$, we can write

$$\int_{Q_p}\frac{1}{M_n}W(M_n v)\, e^{2i\pi\langle v(t-s)\rangle}dv = \int_{Q_p} W(u)\, e^{2i\pi\langle u((t-s)/M_n)\rangle} = \mathcal{F}W\left((t-s)/M_n\right).$$

Let $V_n(u) = \mathcal{F}W\left(u/M_n\right)$. We obtain

$$\hat{f}_{Z,n}(\lambda) = 1/p^n\int_{U_n}\int_{U_n} V_n(t-s)\, e^{-2i\pi\langle\lambda(t-s)\rangle}X(t)X(s)\mathcal{N}(t+dt)\mathcal{N}(s+ds).$$

Let λ_1 and λ_2 be elements of Q_p. We have from the independence of X and $\mathcal{N}$:

$$\begin{aligned}
&p^{2n}\mathrm{cov}\left[\hat{f}_{Z,n}(\lambda_1), \hat{f}_{Z,n}(\lambda_2)\right] \\
&= \int\int\int\int_{U_n^4} V_n(t-s)V_n(u-v)\, e^{-2i\pi\langle\lambda_1(t-s)-\lambda_2(u-v)\rangle} \\
&\quad\times\left[\mathbb{E}\left[X(s)X(t)X(u)X(v)\right]\mathbb{E}\left[\mathcal{N}(ds)\mathcal{N}(dt)\mathcal{N}(du)\mathcal{N}(dv)\right]\right. \\
&\quad\left.-\mathbb{E}\left[X(s)X(t)\right]\mathbb{E}\left[\mathcal{N}(ds)\mathcal{N}(dt)\right]\mathbb{E}\left[X(u)X(v)\right]\mathbb{E}\left[\mathcal{N}(du)\mathcal{N}(dv)\right]\right].
\end{aligned}$$

As,

$$\mathbb{E}\left[X(s)X(t)X(u)X(v)\right] = c_4(t-s,u-s,v-s)+c_2(t-s)c_2(u-v) + c_2(t-u)c_2(s-v)+c_2(t-v)c_2(s-u),$$

we have, $p^{2n}\mathrm{cov}\left[\hat{f}_{Z,n}(\lambda_1),\hat{f}_{Z,n}(\lambda_2)\right] = \sum_{i=1}^{4} J_i$, where

$$\begin{aligned}
J_1 &= \int\int\int\int_{U_n^4} c_2(t-s)c_2(u-v)V_n(t-s)V_n(u-v)\ \mathrm{e}^{-2i\pi\langle\lambda_1(t-s)-\lambda_2(u-v)\rangle} \\
&\quad\times\left(\mathbb{E}\left[\mathcal{N}(ds)\mathcal{N}(dt)\mathcal{N}(du)\mathcal{N}(dv)\right]-\mathbb{E}\left[\mathcal{N}(ds)\mathcal{N}(dt)\right]\mathbb{E}\left[\mathcal{N}(du)\mathcal{N}(dv)\right]\right), \\
J_2 &= \int\int\int\int_{U_n^4} c_2(t-v)c_2(s-u)V_n(t-s)V_n(u-v) \\
&\quad\times \mathrm{e}^{-2i\pi\langle\lambda_1(t-s)-\lambda_2(u-v)\rangle}\,\mathbb{E}\left[\mathcal{N}(ds)\mathcal{N}(dt)\mathcal{N}(du)\mathcal{N}(dv)\right], \\
J_3 &= \int\int\int\int_{U_n^4} c_2(t-u)c_2(s-v)V_n(t-s)V_n(u-v) \\
&\quad\times \mathrm{e}^{-2i\pi\langle\lambda_1(t-s)-\lambda_2(u-v)\rangle}\,\mathbb{E}\left[\mathcal{N}(ds)\mathcal{N}(dt)\mathcal{N}(du)\mathcal{N}(dv)\right], \\
J_4 &= \int\int\int\int_{U_n^4} c_4(t-s,u-s,v-s)V_n(t-s)V_n(u-v) \\
&\quad\times \mathrm{e}^{-2i\pi\langle\lambda_1(t-s)-\lambda_2(u-v)\rangle}\,\mathbb{E}\left[\mathcal{N}(ds)\mathcal{N}(dt)\mathcal{N}(du)\mathcal{N}(dv)\right].
\end{aligned}$$

From [10], we get the following formulas :

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{N}(ds)\mathcal{N}(dt)\mathcal{N}(du)\mathcal{N}(dv)\right] \\
&= \beta d\delta_t(s)d\delta_s(u)d\delta_s(v)dt+\beta^2 d\delta_t(v)d\delta_v(u)dtds+\beta^2 d\delta_u(s)d\delta_s(v)dtdu \\
&+\beta^2 d\delta_t(s)d\delta_s(u)dtdv+\beta^2 d\delta_t(s)d\delta_s(v)dtdu+\beta^2 d\delta_t(u)d\delta_s(v)dsdt \\
&+\beta^2 d\delta_t(v)d\delta_s(u)dsdt+\beta^2 d\delta_t(s)d\delta_v(u)dtdv+\beta^3 d\delta_s(v)dsdtdu \\
&+\beta^3 d\delta_s(u)dsdtdv+\beta^3 d\delta_u(v)dsdtdu+\beta^3 d\delta_t(s)dtdudv \\
&+\beta^3 d\delta_t(v)dsdtdu+\beta^3 d\delta_t(u)dsdtdv+\beta^4 dsdtdudv
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{N}(ds)\mathcal{N}(dt)\mathcal{N}(du)\mathcal{N}(dv)\right]-\mathbb{E}\left[\mathcal{N}(ds)\mathcal{N}(dt)\right]\mathbb{E}\left[\mathcal{N}(du)\mathcal{N}(dv)\right] \\
&= \beta d\delta_t(s)d\delta_s(u)d\delta_s(v)dt+\beta^2 d\delta_t(s)d\delta_v(u)dtds+\beta^2 d\delta_u(s)d\delta_s(v)dtdu \\
&+\beta^2 d\delta_t(s)d\delta_s(u)dtdv+\beta^2 d\delta_t(s)d\delta_s(v)dtdu+\beta^2 d\delta_t(u)d\delta_s(v)dsdt \\
&+\beta^2 d\delta_t(v)d\delta_s(u)dsdt+\beta^3 d\delta_s(v)dsdtdu+\beta^3 d\delta_s(u)dsdtdv \\
&+\beta^3 d\delta_t(v)dsdtdu+\beta^3 d\delta_t(u)dsdtdv.
\end{aligned}$$

Remark 5.1 Without any supplementary condition on X, we get

$$J_{1,1} = J_{2,1} = J_{3,1} = J_{4,1} = O(p^n).$$

Calculating J_i for $i = 1,\ldots,4$ we obtain the following lemmas.

Lemma 5.1

i) *Under $\mathcal{H}_3$)*

$$J_{1,j} = O(p^n)\ \text{for}\ j = 2,\cdots,11,$$

$J_{2,j} = O(p^n)$ *for* $j = 2,3,4,5,6,7,9,10,11,12,13,14,$
$J_{3,j} = O(p^n)$ *for* $j = 2,3,4,5,9,10,11,12,13.$

ii) *Under* $\mathcal{H}_3$) *and* $\mathcal{H}_4$),
$J_{2,8} = J_{2,15} = J_{3,15} = J_{3,8} = O(p^n/M_n).$

Proof of lemma 5.1.

i) We give a detailed proof for the result: $J_{1,3} = O(p^n)$, the other results follow analogously. First, by definition, we get

$$\begin{aligned} J_{1,3} &= \beta^2 \int_{U_n}\int_{U_n}\int_{U_n}\int_{U_n} c_2(t-s)c_2(u-v)V_n(t-s)V_n(u-v) \\ &\quad \times \mathrm{e}^{-2i\pi\langle\lambda_1(t-s)-\lambda_2(u-v)\rangle} d\delta_u(s)d\delta_s(v)dtdu \\ &= \beta^2 c_2(0)V_n(0)\int_{U_n}\int_{U_n} c_2(t-u)V_n(t-u)\ \mathrm{e}^{-2i\pi\langle\lambda_1(t-u)\rangle}dtdu. \end{aligned}$$

with $x = (t-u)/M_n$, from Fubini's theorem and definition of V_n, we get

$$|J_{1,3}| \le \beta^2\,|c_2(0)|\,\mathcal{F}W(0)\frac{p^n}{M_n}\int_{Q_p}|c_2(xM_n)|\,\mathcal{F}W(x)dx.$$

Let $y = xM_n$. Then as $\mathcal{F}W$ is bounded, $|J_{1,3}| \le$ const $\beta^2\,|c_2(0)|\,\mathcal{F}W(0)p^n\int_{Q_p}|c_2(y)|\,dy$. From $\mathcal{H}_3$), we get the result.

ii) We are going to prove: $J_{2,15} = O(p^n/M_n)$.
First, we can write

$$\begin{aligned} J_{2,15} &= \beta^4 \int_{U_n}\int_{U_n}\int_{U_n}\int_{U_n} c_2(t-v)c_2(s-u)V_n(t-s)V_n(u-v) \\ &\quad \times \mathrm{e}^{-2i\pi\langle\lambda_1(t-s)-\lambda_2(u-v)\rangle} dsdtdudv. \end{aligned}$$

Successively, let $x = \dfrac{t-s}{M_n}$; $y = s - v + xM_n$; $z = \dfrac{u-v}{M_n}$; and $v = s-u$.
By Fubini's theorem and V_n being bounded, we get

$$|J_{2,15}| \le \text{const}\ \frac{p^n}{M_n}|c_2|_{L^1(Q_p)}|\mathcal{F}W|_{L^1(Q_p)}\int_{Q_p}|c_2(v)|\,dv.$$

Thus, $\mathcal{H}_3$) and $\mathcal{H}_4$) imply: $J_{2,15} = O(\frac{p^n}{M_n})$.

Lemma 5.2

i) *Under* $\mathcal{H}_4$), $J_{3,7} = J_{3,8} = O(p^n)$.
ii) *Under* $\mathcal{H}_5$), $J_{3,14} = O(p^n/M_n)$.
iii) *Under* $\mathcal{H}_6$), $J_{4,2} = J_{4,6} = O(p^n)$.
iv) *Under* $\mathcal{H}_7$), $J_{4,3} = J_{4,4} = J_{4,5} = O(p^n)$.
v) *Under* $\mathcal{H}_8$), $J_{4,7} = J_{4,8} = O(p^n)$.
vi) *Under* $\mathcal{H}_9$), $J_{4,9} = J_{4,10} = O(p^n)$.
vii) *Under* $\mathcal{H}_{10}$), $J_{4,11} = J_{4,13} = O(p^n)$.
viii) *Under* $\mathcal{H}_{11}$), $J_{4,12} = O(p^n)$.
ix) *Under* $\mathcal{H}_{12}$), $J_{4,15} = O(p^n)$.

Proof. We only prove i) and iii); the other results are proved analogously.

i) We can write

$$\begin{aligned} J_{3,7} &= \beta^2 \int_{U_n}\int_{U_n}\int_{U_n}\int_{U_n} c_2(t-u)c_2(s-v)V_n(t-s)V_n(u-v) \\ &\qquad \times \mathrm{e}^{-2i\pi\langle \lambda_1(t-s)-\lambda_2(u-v)\rangle} d\delta_t(v)d\delta_s(u)dsdt \\ &= \beta^2 \int_{U_n}\int_{U_n} c_2(t-s)^2 V_n(t-s)^2\, \mathrm{e}^{-2i\pi\langle(\lambda_1+\lambda_2)(t-s)\rangle} dtds. \end{aligned}$$

From $x = (t-s)/M_n$, $y = xM_n$, we get

$$\begin{aligned} |J_{3,7}| &\leq \beta^2 p^n \int_{Q_p} \left|c_2(xM_n)^2\right| \mathcal{F}W(x)^2 \frac{dx}{M_n} \\ &\leq \beta^2 p^n |\mathcal{F}W|^2_{L^\infty(Q_p)} |c_2|^{\frac{1}{2}}_{L^2(Q_p)}, \end{aligned}$$

thus, $J_{3,7} = O(p^n)$ by $\mathcal{H}_2$).

ii) We can write:

$$\begin{aligned} J_{4,2} &= \beta^2 \int_{U_n}\int_{U_n}\int_{U_n}\int_{U_n} c_4(t-s, u-s, v-s)V_n(t-s)V_n(u-v) \\ &\qquad \times \mathrm{e}^{-2i\pi\langle \lambda_1(t-s)-\lambda_2(u-v)\rangle} d\delta_t(v)d\delta_v(u)dtds \\ &= \beta^2 \int_{U_n}\int_{U_n} c_4(t-s, t-s, t-s)V_n(t-s)\mathcal{F}W(0)\, \mathrm{e}^{-2i\pi\langle \lambda_1(t-s)\rangle} dtds, \end{aligned}$$

with $t - s/M_n = x$ and $u = xM_n$, we obtain

$$\begin{aligned} |J_{4,2}| &\leq \text{const } \beta^2 \int_{U_n} \left[\int_{Q_p} |c_4(xM_n, xM_n, xM_n)|\, \mathcal{F}W(x)dx\right] ds \\ &\leq \text{const } \beta^2 p^n \int_{Q_p} |c_4(u,u,u)|\, du. \end{aligned}$$

Then, $J_{4,2} = O(p^n)$ comes from $\mathcal{H}_6$).
We come back to the proof of proposition 5.2.

From lemmas 5.1 and 5.2, we get

$$J = \sum_{i=1}^{4} J_i = O(p^n) + O(\frac{p^n}{M_n}) + O(\frac{1}{M_n}) = O(\frac{p^n}{M_n}),$$

and thus $p^{2n}\text{cov}\left[\hat{f}_{Z,n}(\lambda_1), \hat{f}_{Z,n}(\lambda_2)\right] = O(\frac{p^n}{M_n})$,
which implies, $\text{cov}\left[\hat{f}_{Z,n}(\lambda_1), \hat{f}_{Z,n}(\lambda_2)\right] = O(\frac{1}{p^n M_n})$.
So, $\lim_{n\to+\infty} \text{cov}\left[\hat{f}_{Z,n}(\lambda_1), \hat{f}_{Z,n}(\lambda_2)\right] = 0$, for every $\lambda_1,\ \lambda_2 \in Q_p$..
This proves that $\hat{f}_{Z,n}$ is consistent.

Our main result in this section is the following:

Theorem 5.1 *Under the hypothesis of propositions 5.1 and 5.2, $\hat{f}_{X,n}(\lambda)$ is asymptotically unbiased and consistent.*

Proof of theorem 5.1 .

a) $\hat{f}_{X,n}(\lambda)$ is asymptotically unbiased.
Indeed, $\mathbb{E}\left[\hat{f}_{X,n}(\lambda)\right] = \frac{1}{\beta^2}\left[\mathbb{E}\left[\hat{f}_{Z,n}(\lambda)\right] - \beta c_2(0)\right]$.
Under conditions of lemma 5.1, as $\hat{f}_{Z,n}$ is asymptotically unbiased, then

$$\lim_{n\to+\infty} \mathbb{E}\left[\hat{f}_{X,n}(\lambda)\right] = \frac{1}{\beta^2}\mathbb{E}\left[\hat{f}_Z(\lambda) - \beta c_2(0)\right] = f_X(\lambda).$$

b) $\hat{f}_{X,n}(\lambda)$ is consistent.

Let λ_1 and λ_2 be elements of Q_p. We have

$$\begin{aligned}\operatorname{cov}\left[\hat{f}_{X,n}(\lambda_1), \hat{f}_{X,n}(\lambda_2)\right] &= \frac{1}{\beta^4}\operatorname{cov}\left[\hat{f}_{Z,n}(\lambda_1), \hat{f}_{Z,n}(\lambda_2)\right] - \frac{1}{\beta^3}\operatorname{cov}\left[\hat{f}_{Z,n}(\lambda_1), \hat{c}_{2,n}(0)\right] \\ &\quad - \frac{1}{\beta^3}\operatorname{cov}\left[\hat{f}_{Z,n}(\lambda_2), \hat{c}_{2,n}(0)\right] + \frac{1}{\beta^2}\operatorname{var}\left[\hat{c}_{2,n}(0)\right].\end{aligned}$$

For $i = 1, 2$, we can write $\left|\operatorname{cov}\left[\hat{f}_{Z,n}(\lambda_i), \hat{c}_{2,n}(0)\right]\right| \le \sqrt{\operatorname{var}\left[\hat{f}_{Z,n}(\lambda_i)\right]}\sqrt{\operatorname{var}\left[\hat{c}_{2,n}(0)\right]}$.
The two propositions 5.1, 5.2 imply:

$$\operatorname{cov}\left[\hat{f}_{X,n}(\lambda_1), \hat{f}_{X,n}(\lambda_2)\right] = O(\frac{1}{p^n}) + O(\frac{1}{p^n M_n}) + O(\frac{1}{p^n\sqrt{M_n}})$$

The theorem is then proved.

Corollary 5.1 *Under $\mathcal{H}_3$)–$\mathcal{H}_{12}$), $\hat{f}_{X,n}(\lambda)$ converges to $f_X(\lambda)$ in quadratic mean as $n \to +\infty$.*

Proof of corollary 5.1 . We know that $MSE(n,\lambda) = \operatorname{bias}^2(\hat{f}_{X,n}(\lambda)) + \operatorname{var}\{\hat{f}_{X,n}(\lambda)\}$.
Then, from theorem 5.1, we obtain $MSE(n) \to 0$ as $n \to +\infty$.
This implies mean quadratic convergence.

6 Discussion and extensions

This paper has been concerned with the case of real-valued process. Extensions to the complex-valued and r-vector valued cases are immediate. We think that, it will be very important to treat the case of p-adic valued process, and afterward, observe the almost sure convergence and give the asymptotic distributions of the estimates.
As the convergence rate of the estimators depends on the sequence $(M_n)_{n\in\mathbb{N}}$, we think that the choice of this sequence is crucial, and methods like Cross-Validation procedure's (cf. [11] and [5]), will solve this problem.

Aknowledgements :

We would like to thank professor M. Bertrand, the editor and the anonymous referee for their careful reading of an earlier version. Their comments and criticisms substantially improved the paper.

References

[1] A.Yu.KHRENNIKOV. *p-adic valued distributions in mathematical physics.* Kluwer Academic, Dordrecht, 1994.

[2] D. R. BRILLINGER. Some asymptotics of finite fourier transforms of a stationary p-adic process. *Journal of Combinatorics, Information & System*, 16(2-3):155–169, 1991.

[3] D. DACUNHA-CASTELLE and M. DUFLO. *Probabilités et statistiques (2. Problèmes à temps mobile).* MASSON, 1983.

[4] I. M. GELFAND, M. I. GRAEV, and PYATETSKII-SHAPIRO. *Representation theory and automorphic functions.* Saunders, Philadelphia, 1969.

[5] L. GYÖRFI, W. HÄRDLE, P. SARDA, and P. VIEU. *Nonparametric curve estimation from time series.* Lecture notes in statistics, Springer-Verlag, 1990.

[6] E. HEWITT and K. A. ROSS. *Abstract harmonic analysis*, volume 1. Academic Press, New York, 1963.

[7] Alan. F. KARR. *Point processes and their statistical inference (Second Edition Revised and Expanded).* Marcel Dekker, inc, 1991.

[8] Keh-Shin. LII and Elias. MASRY. Spectral estimation of continuous-time stationary processes from random sampling. *Stochastic processes and their applications*, 52:39–64, 1993.

[9] E. MASRY. Poisson sampling and spectral estimation of continuous-time processes. *IEEE trans. Inform. Theory*, IT-24(2):173–183, 1978.

[10] V. MONSAN. *Estimation spectrale dans les processus périodiquement corrélés.* Univ de Rouen, (Thèse de doctorat), 1994.

[11] M. RACHDI. *Choix de la largeur de fenêtre spectrale par validation croisée. Analyse spectrale p-adique.* Univ de Rouen, (Thèse de doctorat), 1998.

[12] V. S. VLADIMIROV. Generalized functions over the field of p-adic numbers. *Russian Math. Surveys*, 43:19–64, 1988.

[13] I.V.VOLOVICH V.S.VLADIMIROV and E.I.ZELENOV. *p-adic analysis and mathematical physics.* World. Sc. Publ, Singapure, 1994.

Mustapha RACHDI
Université de Rouen
UFR des Sciences-Mathématiques
URA CNRS 1378 Site Colbert
76 821 Mont Saint Aignan Cedex
FRANCE
e-mail: rachdi@univ-rouen.fr

Monsan VINCENT
Université de Cocody
UFR de Maths et Informatique
22 BP 582 Abidjan 22
COTE d'IVOIRE
e-mail: monsan@syfed.ci.refer.org

Manuscrit reçu en Janvier 1997