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# Approximation Results in the Strict Topology

João B. Prolla and Samuel Navarro*

**Abstract:** In this paper we prove results of the Weierstrass-Stone type for subsets $W$ of the vector space $V$ of all continuous and bounded functions from a topological space $X$ into a real normed space $E$, when $V$ is equipped with the strict topology $\beta$. Our main results characterize the $\beta$-closure of $W$ when (1) $W$ is $\beta$-truncation stable; (2) $E = \mathbb{R}$ and $W$ is a subalgebra; (3) $E = \mathbb{R}$ and $W$ is the convex cone of all positive elements of some algebra; (4) $W$ is uniformly bounded; (5) $X$ is a completely regular Hausdorff space and $W$ is convex.

## §1. Introduction and definitions

Let $X$ be a topological space and let $E$ be a real normed space. We denote by $B(X; E)$ the normed space of all bounded $E$-valued functions on $X$, equipped with the supremum norm

$$||f||_X = \sup\{||f(x)||; x \in X\}$$

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for each $f \in B(X;E)$. We denote by $B_0(X;E)$ the subset of all $f \in B(X;E)$ that vanish at infinity, i.e., those $f$ such that for every $\varepsilon > 0$, the set $K = \{t \in X; ||f(t)|| \geq \varepsilon\}$ is compact (or empty). And we denote by $B_{00}(X;E)$ the subset of all $f \in B(X;E)$ which have compact support. We denote by $C(X;E)$ the vector space of all continuous $E$-valued functions on $X$, and set

$$\begin{aligned} C_b(X;E) &= C(X;E) \cap B(X;E), \\ C_0(X;E) &= C(X;E) \cap B_0(X;E), \\ C_{00}(X;E) &= C(X;E) \cap B_{00}(X;E) \end{aligned}$$

We denote by $I(X)$ the set of all $\varphi \in B(X;\mathbb{R})$ such that $0 \leq \varphi(x) \leq 1$, for all $x \in X$. We then define

$$\begin{aligned} D(X) &= C_b(X;\mathbb{R}) \cap I(X), \\ D_0(X) &= B_0(X;\mathbb{R}) \cap I(X). \end{aligned}$$

The **strict topology** $\beta$ on $C_b(X;E)$ is the locally convex topology determined by the family of seminorms

$$p_\varphi(f) = \sup\{\varphi(x)||f(x)||; x \in X\}$$

for $f \in C_b(X;E)$, when $\varphi$ ranges over $D_0(X)$. Clearly, given $\varphi \in D_0(X)$ there is a compact subset $K$ such that $\varphi(x) < \varepsilon$ for all $x \notin K$. Therefore, our strict topology is coarser than the strict topology introduced by R. Giles [3]. To see that they actually coincide, let $\psi \in B(X;\mathbb{R})$ be such that, for each $\varepsilon > 0$ there is a compact subset $K$ such that $\psi(x) < \varepsilon$ for all $x \notin K$. We may assume $||\psi||_X < 1$. Choose compact sets $K_n$ with $\phi = K_0 \subset K_1 \subset K_2 \subset ...$ such that $|\psi(x)| < 2^{-n}$, for all $x \notin K_n$.

Let $\psi_n \in B_0(X;\mathbb{R})$ be the characteristic function of $K_n$ multiplied by $2^{-n}$, i.e., $\psi_n(x) = 2^{-n}$, if $x \in K_n$; and $\psi_n(x) = 0$ if $x \notin K_n$. Let $\varphi = \sum_{n=1}^{\infty}\psi_n$. For each $\varepsilon > 0$, we claim that the set $K = \{x \in X; \varphi(x) \geq \varepsilon\}$ is compact (or empty). If $\varepsilon > 1$, then $K = \phi$. If $\varepsilon = 1$, then $K = K_1$, because $\varphi(t) = 1$ precisely for $t \in K_1$. If $\varepsilon < 1$, let $n \geq 0$ be such that $2^{-(n+1)} \leq \varepsilon < 2^{-n}$. Then $K = K_{n+1}$. Hence $\varphi \in D_0(X)$. We claim now that $\psi(x) \leq \varphi(x)$ for all $x \in X$. We first notice that $\varphi(x) = 0$ if, and only if $x \notin \bigcup_{n=1}^{\infty} K_n$. Indeed, if the point $x \notin \bigcup_{n=1}^{\infty} K_n$, then $\psi_k(x) = 0$ for all $n = 1, 2, 3, \ldots$, and so $\varphi(x) = 0$. Conversely, if $\varphi(x) = 0$, then $\psi_n(x) = 0$ for all $n = 1, 2, 3, \ldots$ and therefore $x \notin K_n$ for all $n = 1, 2, 3, \ldots$ Hence $x \notin \bigcup_{n=1}^{\infty} K_n$. Let now $x \in X$. If $\varphi(x) = 0$, then $x \notin K_n$ for all $n = 1, 2, 3, \ldots$ and so $|\psi(x)| < 2^{-n}$ for all $n = 1, 2, 3, \ldots$. Hence $\psi(x) = 0$ and so $\psi(x) = \varphi(x)$. Suppose now $\varphi(x) > 0$. Then $x \in \bigcup_{n=1}^{\infty} K_n$. Let $N$ be the smallest positive integer $n \geq 1$ such that $x \in K_n$. If $N = 1$, then $x \in K_1$ and so $\varphi(x) = 1 > \psi(x)$. If $N > 1$, then $x \in K_N$ and $x \notin K_{N-1}$. Hence

$$\varphi(x) = \sum_{n=N}^{\infty} 2^{-n} = 2^{-(N-1)}$$

and $\psi(x) < 2^{-(N-1)}$, since $x \notin K_{N-1}$. Therefore $\psi(x) < \varphi(x)$, whenever $\varphi(x) > 0$.

Given any non-empty subset $S \subset C(X; E)$ we denote by $x \equiv y$ (mod. S) the equivalence relation defined by $f(x) = f(y)$ for all $f \in S$. For each $x \in X$, the equivalence class of $x$ (mod. S) is denoted by $[x]_S$, i.e.,

$$[x]_S = \{t \in X \ ; \ x \equiv t (\text{mod. S})\}$$

For any non-empty subset $K \subset X$ and any $f : X \to E$, we denote by $f_K$ its restriction to $K$. If $S \subset C(X; E)$ and $K \subset X$, then for each $x \in K$ one has

$$[x]_{S_K} = K \cap [x]_S.$$

If $S \subset C_b(X; \mathbb{R})$, we define $S^+$ by

$$S^+ = \{f \in S \ ; \ f \geq 0\}.$$

If $S = C_b(X; \mathbb{R})$, we write $S^+ = C_b^+(X; \mathbb{R})$.

**Definition 1.** Let $S \subset C_b(X;\mathbb{R})$ and let $W \subset C_b(X;E)$ be given. We say that $W$ is **$\beta$-localizable under $S$** if, for every $f \in C_b(X;E)$, the following are equivalent:

(1) $f$ belongs to the $\beta$-closure of $W$;
(2) for every $\varphi \in D_0(X)$, every $\varepsilon > 0$ and every $x \in X$,there is some $g_x \in W$ such that $\varphi(t)||f(t) - g_x(t)|| < \varepsilon$ for all $t \in [x]_S$.

**Remark.** Clearly, (1) $\Rightarrow$ (2) in any case. Hence a set $W$ is $\beta$-localizable under $S$ if, and only if, (2) $\Rightarrow$ (1). Notice also that if $W$ is $\beta$-localizable under $S$ and $T \subset S$, then $W$ is $\beta$-localizable under $T$. Indeed, $T \subset S$ implies $[x]_S \subset [x]_T$.

**Definition 2.** We say that a set $W \subset C_b(X,E)$ is **$\beta$-truncation stable** if, for every $f \in W$ and every $M > 0$, the function $T_M \circ f$ belongs to the $\beta$-closure of $W$, where $T_M : E \to E$ is the mapping defined by

$$\begin{aligned} T_M(v) &= v, \text{ if } ||v|| < 2M; \\ T_M(v) &= \frac{v}{||v||}.2M, \text{ if } ||v|| \geq 2M \end{aligned}$$

Notice that, when $E = \mathbb{R}$, the mapping $T_M : \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned} T_M(r) &= r, \text{ if } ||r|| < 2M; \\ T_M(r) &= 2M, \text{ if } r > 2M; \\ T_M(r) &= -2M, \text{ if } r > -2M. \end{aligned}$$

Remark that, for every $f \in C_b(X;E)$, one has $||T_M \circ f||_X \leq 2M$.

Notice that when $W \subset C_b^+(X;\mathbb{R})$, then $W$ is $\beta$-truncation stable if, for every $f \in W$ and every constant $M > 0$, the function $P_M \circ f$ belongs to the $\beta$-closure of $W$, where $P_M : \mathbb{R} \to \mathbb{R}_+$ is the mapping defined by $P_M = \max(0, T_M)$, i.e.,

$$\begin{aligned} P_M(r) &= 0, \text{ if } r < 0; \\ P_M(r) &= r, \text{ if } 0 \leq r \leq 2M; \\ P_M(r) &= 2M, \text{ if } r > 2M. \end{aligned}$$

**Definition 3.** Let $W \subset C_b(X;E)$ be a non-empty subset. A function $\psi \in D(X)$ is called a **multiplier of** $W$ if $\psi f + (1-\psi)g$ belongs to $W$, for each pair, $f$ and $g$, of elements of $W$.

**Definition 4.** A subset $S \subset D(X)$ is said to have **property** $V$ if

(a) $\psi \in S$ implies $(1-\psi) \in S$;
(b) the product $\varphi\psi$ belongs to $S$, for any pair, $\varphi$ and $\psi$, of elements of $S$.

Notice that the set of all multipliers of a subset $W \subset C_b(X;E)$ has property $V$. Indeed, condition (a) is clear and the equation

$$(\varphi\psi)f + (1-\varphi\psi)g = \varphi[\psi f + (1-\psi)g] + (1-\varphi)g$$

show that (b) holds as well.

When $X$ is locally compact, R.C. Buck [1] proved a Weierstrass-Stone Theorem for subalgebras of $C_b(X;\mathbb{R})$ equipped with the strict topology. This result was extended and generalized by Glicksberg [4], Todd [7], Wells [8] and Giles [3]. See also Buck [2], where modules are dealt with, and Prolla [5], where the strict topology is considered as an example of weighted spaces.

Our versions of the Weierstrass-Stone Theorem are analogues of Chapter 4 of Prolla [6] for arbitrary subsets of $C(X;E)$ equipped with the uniform convergence topology, $X$ compact. Whereas the previous results dealt only with algebras or vector spaces which are modules over an algebra, our results now go much further: we are able to cover the case of convex sets (when $X$ is completely regular) or $\beta$-truncation stable sets (when $X$ is just a topological space). The latter case cover both algebras and the convex cones obtained by taking the set of positive elements of an algebra.

## §2. $\beta$-truncation stable subsets

**Theorem 1.** *Let $W \subset C_b(X;E)$ be a $\beta$-truncation stable non-empty subset, and let $A$ be the set of all multipliers of $W$. Then $W$ is $\beta$-localizable under $A$.*

**Proof.** Let $f \in C_b(X;E)$ be given and assume condition (2) of Definition 1, with $S = A$. Let $\varphi \in D_0(X)$ and $\varepsilon > 0$ be given. Without loss of generality we may assume that $\varphi$ is not identically zero. Choose $M > 0$ so big that $M > ||f||_X, M > \varepsilon$ and the compact set $K = \{t \in X; \varphi(t) \geq \varepsilon/(6M)\}$ is non-empty. Consider the non-empty subset $W_K \subset C(K;E)$. Clearly, the set $A_K \subset D(K)$ is a set of multipliers of $W_K$. Take a point $x \in K$. By condition (2) applied to $\varepsilon^2/(12M)$, there exists $g_x \in W$ such that $\varphi(t)||f(t) - g_x(t)|| < \varepsilon^2/(12M)$ for all $t \in [x]_A$. Let $M \subset D(K)$ be the set of all multipliers of $W_K \subset C(K;E)$. Then $M$ has property $V$. Now $A_K \subset M$ implies

$$[x]_M \subset [x]_{A_K} = [x]_A \cap K.$$

Hence $\varphi(t)||f(t) - g_x(t)|| < \varepsilon^2/(12M)$ holds for all $t \in K$ such that $t \in [x]_M$. Now $\varphi(t) \geq \varepsilon/(6M)$ for all $t \in K$ and therefore

$$||f(t) - g_x(t)|| < \varepsilon/2$$

for all $t \in [x]_M$. By Theorem 1, Chapter 4, of Prolla [6] applied to $W_K \subset C(K;E)$ and to the set $M \subset D(K)$, there is $g_1 \in W$ such that

$$||f(t) - g_1(t)|| < \varepsilon/2$$

for all $t \in K$. Let $h = T_M \circ g_1$. By hypothesis, $h$ belongs to the $\beta$-closure of $W$, and there is $g \in W$ such that $p_\varphi(h - g) < \varepsilon/2$. We claim that $p_\varphi(f - h) < \varepsilon/2$. Let

$t \in K$. Then

$$\|g_1(t)\| \leq \|f(t) - g_1(t)\| + \|f(t)\| < \varepsilon/2 + M < 2M$$

and so $h(t) = T_M(g_1(t)) = g_1(t)$. Hence

$$\begin{aligned} \varphi(t)\|f(t) - h(t)\| &= \varphi(t)\|f(t) - g_1(t)\| \\ &\leq \|f(t) - g_1(t)\| < \varepsilon/2. \end{aligned}$$

Suppose now $t \notin K$. Then

$$\begin{aligned} \varphi(t)\|f(t) - h(t)\| &< \frac{\varepsilon}{6M}\|f(t) - h(t)\| \\ &\leq \frac{\varepsilon}{6M}\Big(\|f\|_X + \|h\|_X\Big) < \frac{\varepsilon}{6M}.3M = \frac{\varepsilon}{2}, \end{aligned}$$

because $\|h\|_X \leq 2M$, and $\|f\|_X < M$.

This establishes our claim that $p_\varphi(f - h) < \frac{\varepsilon}{2}$. Hence $p_\varphi(f - g) < \varepsilon$, and $f$ belongs to the $\beta$-closure of $W$. □

**Theorem 2.** *Let $W \subset C_b(X; E)$ be a $\beta$-truncation stable non-empty subset, and let $B$ be any non-empty set of multipliers of $W$. Then $W$ is $\beta$-localizable under $B$.*

**Proof.** Let $A$ be the set of all multipliers of $W$. By Theorem 1 the set $W$ is $\beta$-localizable under $A$. Now $B \subset A$, so $W$ is also $\beta$-localizable under $B$. □

## §3. The case of subalgebras

**Lemma 1.** *If $B \subset C_b(X; \mathbb{R})$ is a uniformly closed subalgebra, and $T : \mathbb{R} \to \mathbb{R}$ is a continuous mapping, with $T(0) = 0$, then $T \circ f$ belongs to $B$, for every $f \in B$.*

**Proof.** Let $f \in B$ and $\varepsilon > 0$ be given. Choose $k \geq ||f||_X$. By Weierstrass' Theorem, there exists an algebraic polynomial $p$ such that $|T(t) - p(t)| < \varepsilon$ for all $t \in \mathbb{R}$ with $|t| \leq k$, and we may assume $p(0) = T(0) = 0$. Hence, for every $x \in X$, we have $|T(f(x)) - p(f(x))| < \varepsilon$, because $|f(x)| \leq k$. Now $p \circ f$ belongs to $B$, and therefore $T \circ f$ belongs to the uniform closure of $B$, that is $B$ itself. □

**Corollary 1.** *Every subalgebra $W \subset C_b(X;\mathbb{R})$ is $\beta$-truncation stable.*

**Proof.** Let $f \in W$ and $M > 0$ be given. Let $B$ be the $\beta$-closure of $W$ in $C_b(X;\mathbb{R})$. We know that $B$ is then a uniformly closed subalgebra. By Lemma 1 applied to $T = T_M$, we see that $T_M \circ f$ belongs to the $\beta$-closure of $W$ as claimed. □

**Corollary 2.** *Every uniformly closed subalgebra of $C_b(X;\mathbb{R})$ is a lattice.*

**Proof.** Since

$$\begin{aligned} \max(f,g) &= \frac{1}{2}\Big[f + g + |f - g|\Big] \\ \min(f,g) &= \frac{1}{2}\Big[f + g - |f - g|\Big] \end{aligned}$$

it suffices to show that $|f| \in B$, for every $f \in B$. This follows from Lemma 1, by taking $T : \mathbb{R} \to \mathbb{R}$ to be the mapping $T(t) = |t|$, for $t \in \mathbb{R}$. □

**Theorem 3.** *Every subalgebra $W \subset C_b(X;\mathbb{R})$ is $\beta$-localizable under itself.*

**Proof.** Let $f \in C_b(X;\mathbb{R})$ and assume that condition (2) of Definition 1 holds with $S = W$. Notice that for every $x \in X$ one has

$$[x]_W = [x]_B$$

where $B$ is the $\beta$-closure of $W$. Let now

$$V = \{\psi \in B;\ ||\psi||_X \leq 1\} \quad \text{and} \quad A = \{\psi \in B;\ 0 \leq \psi \leq 1\}.$$

It is easy to see that

$$[x]_B = [x]_V \subset [x]_A ,$$

for each $x \in X$. Notice that, by Corollary 2, every $\psi \in V$ can be written in the form $\psi = \psi^+ - \psi^-$, with $\psi^+$ and $\psi^-$ in $A$. Hence $[x]_A \subset [x]_V$ is also true. Hence $f$ satisfies condition (2) of Definition 1 with respect to $S = A$. Now $A$ is a set of multipliers of $B$, and the algebra $B$, by Corollary 1, is $\beta$-truncation stable. Hence, by Theorem 3, the function $f$ belongs to the $\beta$-closure of $B$, that is $B$ itself. We have proved that $f$ belongs to the $\beta$-closure of $W$. Hence $W$ is $\beta$-localizable under $S = W$. □

**Corollary 3.** *Let $W \subset C_b(X; \mathbb{R})$ be a subalgebra, and let $f \in C_b(X; \mathbb{R})$ be given. Then $f$ belongs to the $\beta$-closure of $W$ if, and only if, the following conditions are satisfied:*

(1) *for each pair, $x$ and $y$, of elements of $X$ such that $f(x) \neq f(y)$, there is some $g \in W$ such that $g(x) \neq g(y)$;*
(2) *for each $x \in X$ such that $f(x) \neq 0$ there is some $g \in W$ such that $g(x) \neq 0$.*

**Proof.** Clearly, if $f \in \overline{W}^{\beta}$, then (1) and (2) are satisfied. Conversely, assume that conditions (1) and (2) are verified.

Let $x \in X$ be given. By condition (1) the function $f$ is constant on $[x]_W$. Let $f(x)$ be its value. If $f(x) = 0$, then $g_x = 0$ belongs to $W$ and $f(t) = f(x) = 0 = g_x(t)$ for all $t \in [x]_W$. If $f(x) \neq 0$, by condition (2) there is $g \in W$ such that $g(x) \neq 0$. Define $g_x = [f(x)/g(x)]g$. Then $g_x \in W$ and $g_x(t) = f(x) = f(t)$ for all $t \in [x]_W$. Hence $f$ satisfies condition (2) of Definition 1 with respect to $S = W$. By Theorem 3, we conclude that $f$ belongs to the $\beta$-closure of $W$. □

Corollary 3 implies the following results.

**Corollary 4.** *Let $A$ be a subalgebra of $C_b(X; \mathbb{R})$ which for each $x \in X$ contains a function $g$ with $g(x) \neq 0$, and let $f \in C_b(X; \mathbb{R})$ be given. Then $f$ belongs to the $\beta$-closure of $A$ if, and only if, for each pair, $x$ and $y$, of elements of $X$ such that $f(x) \neq f(y)$, there is some $g \in A$ such that $g(x) \neq g(y)$.*

**Corollary 5.** *Let $A$ be a subalgebra of $C_b(X; \mathbb{R})$ which separates the points of $X$ and for each $x \in X$ contains a function $g$ with $g(x) \neq 0$. Then $A$ is $\beta$-dense in $C_b(X; \mathbb{R})$.*

**Corollary 6.** *If $X$ is a locally compact Hausdorff space, then $C_{00}(X; \mathbb{R})$ is $\beta$-dense in $C_b(X; \mathbb{R})$.*

**Lemma 2.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(t) \geq 0$ for all $t \in \mathbb{R}$ and $f(0) = 0$. If $k > 0$ and $\varepsilon > 0$ are given, there is a real algebraic polynomial $p$ such that $p(t) \geq 0$ for all $0 \leq t \leq k, p(0) = 0$ and $|p(t) - f(t)| \leq \varepsilon$ for all $0 \leq t \leq k$.*

**Proof.** Define $g : [0,1] \to \mathbb{R}$ by setting $g(u) = f(ku)$, for each $u \in [0,1]$. Clearly, $g(u) \geq 0$, for all $0 \leq u \leq 1$ and $g(0) = 0$. Now, given $\varepsilon > 0$, choose $n$ so that the $n$-th Bernstein polynomial of $g$, written $B_n g$, is such that

$$|(B_n g)(u) - g(u)| < \varepsilon$$

for all $0 \leq u \leq 1$. For $t \in \mathbb{R}$, define $p(t) = (B_n g)(t/k)$. Since $B_n g \geq 0$ in $[0,1]$, it follows that $p(t) \geq 0$, for $t \in [0, k]$. Since $(B_n g)(0) = g(0) = f(0) = 0$, we see that $p(0) = 0$. It remains to notice that, for any $0 \leq t \leq k$ we have $0 \leq t/k \leq 1$ and

$$|p(t) - f(t)| = |(B_n g)(t/k) - g(t/k)| < \varepsilon \qquad \square$$

**Lemma 3.** *If $A \subset C_b(X; \mathbb{R})$ is a subalgebra, then $A^+$ is $\beta$-truncation stable.*

**Proof.** Let $f \in A^+$ and $M > 0$ be given. We claim that $P_M \circ f$ belongs to the $\beta$-closure of $A^+$. Let $k > 0$ be such that $0 \leq f(x) \leq k$ for all $x \in X$. Let $\varphi \in D_0(X)$ and $\varepsilon > 0$ be given. By Lemma 2 above there exists a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $p(t) \geq 0$ for all $0 \leq t \leq k, p(0) = 0$ and $|p(t) - P_M(t)| < \varepsilon$ for all $0 \leq t \leq k$. Let $x \in X$. Then $\varphi(x) \leq 1$ and so $\varphi(x)|p(f(x)) - P_M(f(x))| < \varepsilon$. Now $p \circ f$ belongs to $A$ (since $p(0) = 0$) and $p(f(x)) \geq 0$ for all $x \in X$, since $0 \leq f(x) \leq k$. Hence $p \circ f \in A^+$. This ends the proof that $P_M \circ f$ belongs to the $\beta$-closure of $A^+$ as claimed. □

**Theorem 4.** *If $A \subset C_b(X; \mathbb{R})$ is a subalgebra, then $A^+$ is localizable under itself.*

**Proof.** Let $f \in C_b(X; \mathbb{R})$ be given satisfying condition (2) of Definition 1 with respect to $S = A^+$. Define $B = \{f \in A; 0 \leq f \leq 1\}$. It is easy to see that $[x]_S = [x]_B$, for every $x \in X$. Hence $f$ satisfies condition (2) of Definition 1 with respect to $B$, which is a set of multipliers of $A^+$. By Lemma 3, the set $A^+$ is $\beta$-truncation stable. Therefore $A^+$ is $\beta$-localizable under $B$, by Theorem 2. Hence $f$ belongs to the $\beta$-closure of $A^+$.

**Theorem 4.** *Let $A \subset C_b(X; \mathbb{R})$ be a subalgebra and let $f \in C_b^+(X; \mathbb{R})$ be given. Then $f$ belongs to the $\beta$-closure of $A^+$ if, and only if, the following two conditions hold:*

(1) *for each pair, $x$ and $y$, of elements of $X$ such that $f(x) \neq f(y)$, there is some $g \in A^+$ such that $g(x) \neq g(y)$;*
(2) *for each $x \in X$ such that $f(x) > 0$ there is some $g \in A^+$ such that $g(x) > 0$.*

**Proof.** If $f$ belongs to the $\beta$-closure of $A^+$ the two conditions (1) and (2) above are easily seen to hold. Conversely, assume that conditions (1) and (2) above hold. Let $x \in X$ be given. By condition (1), the function $f$ is constant on $[x]_S$ where $S = A^+$. Let $f(x) \geq 0$ be its constant value. If $f(x) = 0$, then $g_x = 0$ belongs to

$A^+$ and $f(t) = f(x) = 0 = g_x(t)$ for all $t \in [x]_S$. If $f(x) > 0$, then by condition (2) there is $g_x \in A^+$ such that $g(x) > 0$. Let $g_x = [f(x)/g(x)]g$. Then $g_x \in A^+$ and $g_x(t) = f(x) = f(t)$ for all $t \in [x]_S$. Hence $f$ satisfies condition (2) of Definition 1 with respect to $W = A^+$ and $S = A^+$. By Theorem 4, we conclude that $f$ belongs to the $\beta$-closure of $A^+$. □

## §4. The case of uniformly bounded subsets

**Theorem 5.** *Let $W$ be a uniformly bounded subset of $C_b(X;E)$ and let $A$ be the set of all multipliers of $W$. Then $W$ is $\beta$-localizable under $A$.*

**Proof.** Let $f \in C_b(X;E)$ be given and assume that condition (2) of Definition 1 holds with $S = A$. Let $\varepsilon > 0$ and $\varphi \in D_0(X)$ be given. Choose $M > 0$ so big that $M > ||f||_X$ and $M > k = \sup\{||g||_X; g \in W\}$, and the compact set $K = \{t \in X; \varphi(t) \geq \varepsilon/(2M)\}$ is non-empty. (Without loss of generality we may assume that $\varphi$ is not identically zero). Consider the non-empty set $W_K \subset C(K;E)$. Clearly, the set $A_K$ is a set of multipliers of $W_K$. Take a point $x \in K$. By condition (2) applied to $\varepsilon^2/(2M)$, there exists some $g_x \in W$ such that

$$\varphi(t)||f(t) - g_x(t)|| < \varepsilon^2/(2M)$$

for all $t \in [x]_A$. Hence $||f(t) - g_x(t)|| < \varepsilon$ for all $t \in [x]_{A_K}$, since $\varphi(t) \geq \varepsilon/(2M)$ for all $t \in K$. Let now $M$ be the set of all multipliers of $W_K \subset C(K;E)$. Since $A_K \subset M$, it follows that $[x]_M \subset [x]_{A_K}$ and so $||f(t) - g_x(t)|| < \varepsilon$ for all $t \in [x]_M$. By Theorem 1, Chapter 4 of Prolla [6] there is $g \in W$ such that $||f(t) - g(t)|| < \varepsilon$ for all $t \in K$. We claim that $p_\varphi(t - g) < \varepsilon$. Let $x \in X$. If $x \in K$, then $\varphi(x) \leq 1$ and

$$\varphi(x)||f(x) - g(x)|| \leq ||f(x) - g(x)|| < \varepsilon.$$

If $x \notin K$, then

$$\varphi(x)||f(x) - g(x)|| \leq \frac{\varepsilon}{2M}[||f||_X + ||g||_X] < \varepsilon.$$

Hence $f$ belongs to the $\beta$-closure of $W$ and so $W$ is $\beta$-localizable under $A$. □

**Theorem 6.** *Let $W$ be a uniformly bounded subset of $C_b(X;E)$ and let $B$ be any non-empty set of multipliers of $W$. Then $W$ is $\beta$-localizable under $B$.*

**Proof.** Let $A$ be the set of all multipliers of $W$. Since $B \subset A$ and by Theorem 5 the set $W$ is $\beta$-localizable under $A$, it follows that $W$ is also $\beta$-localizable under $B$.□

**Theorem 7.** *Let $A$ be a non-empty subset of $D(X)$ with property $V$ and let $f \in D(X)$. Then $f$ belongs to the $\beta$-closure of $A$ if, and only if, the following two conditions hold:*

(1) *for every pair of points, $x$ and $y$, of $X$ such that $f(x) \neq f(y)$, there exists $g \in A$ such that $g(x) \neq g(y)$;*

(2) *for every $x \in X$ such that $0 < f(x) < 1$, there exists $g \in A$ such that $0 < g(x) < 1$.*

**Proof.** It is easy to see that conditions (1) and (2) are necessary for $f$ to belong to the $\beta$-closure of $A$. Conversely, assume that $f$ satisifes conditions (1) and (2).

Let $\varphi \in D_0(X)$ and $\varepsilon > 0$ be given. Without loss of generality we may assume that $\varphi$ is not identically zero. Choose $\delta > 0$ so small that $2\delta < \varepsilon$ and the compact set $K = \{t \in X; \varphi(t) \geq \delta\}$ is non-empty. Clearly, $A_K$ has property $V$. Since conditions (1) and (2) hold, we may apply Theorem 1, Chapter 8, Prolla [6] to conclude that $f_K$ belongs to the uniform closure of $A_K$. Hence there is some $g \in A$ such that $|f(t) - g(t)| < \varepsilon$ for all $t \in K$. We claim that $p_\varphi(f - g) < \varepsilon$. Let $x \in X$. If $x \in K$, then $\varphi(x) \leq 1$ and $\varphi(x)|f(x) - g(x)| \leq |f(x) - g(x)| < \varepsilon$.

If $x \notin K$, then $\varphi(x) < \delta$ and

$$\varphi(x)|f(x) - g(x)| \leq \delta[||f||_X + ||g||_X] \leq 2\delta < \varepsilon.$$

Hence $f$ belongs to the $\beta$-closure of $A$. □

**Remark.** We say that a subset $A \subset D(X)$ has property $VN$ if $fg + (1 - f)h \in A$

for all $f, g, h \in A$. Clearly, if $A$ has property $VN$ and contains 0 and 1, then $A$ has property $V$.

**Corollary 6.** *Let $A$ be a non-empty subset of $D(X)$ with property $V$, and let $W$ be its $\beta$-closure. Then $W$ has property $VN$ and $W$ is a lattice.*

**Proof.** (a) *$W$ has property $VN$*: Let $f, g, \varphi$ belong to $W$, and let $h = \varphi f + (1-\varphi)g$. Assume $h(x) \neq h(y)$. Then at least one of the following three equalities is necessarily false: $\varphi(x) = \varphi(y), f(x) = f(y)$ and $g(x) = g(y)$. Since $\varphi, f$ and $g$ belong all three to $W$, there exists $a \in A$ such that $a(x) \neq a(y)$. Hence $h$ satisfies condition (1) of Theorem 7. Suppose now that $0 < h(x) < 1$. If $0 < \varphi(x) < 1$, then $0 < a(x) < 1$ for some $a \in A$, because $\varphi$ belongs to the $\beta$-closure of $A$. Assume that $\varphi(x) = 0$. Then $h(x) = g(x)$ and so $0 < g(x) < 1$. Since $g \in W$, it follows that $0 < a(x) < 1$ for some $a \in A$. Similarly, if $\varphi(x) = 1$ then $h(x) = f(x)$ and so $0 < f(x) < 1$. Since $f \in W$, there is $a \in A$ such that $0 < a(x) < 1$. Hence $h$ satisfies condition (2) of Theorem 7. By Theorem 7 above, the function $h$ belongs to $W$.

(b) *$W$ is lattice*: Let $f$ and $g$ belong to $W$. Let $h = \max(f, g)$. Let $x$ and $y$ be a pair of points of $X$ such that $h(x) \neq h(y)$. Then at least one of the two equalities $f(x) = f(y), g(x) = g(y)$ must be false. Since $f$ and $g$ both belong to the $\beta$-closure of $A$, there exists $a \in A$ such that $a(x) \neq a(y)$. On the other hand, let $x \in X$ be such that $0 < h(x) < 1$. If $f(x) \geq g(x)$, then $h(x) = f(x)$ and so $0 < f(x) < 1$. Since $f \in W$, there exists $a \in A$ such that $0 < a(x) < 1$. Assume now $f(x) < g(x)$. Then $h(x) = g(x)$ and so $0 < g(x) < 1$. Since $g \in W$, there exists $a \in A$ such that $0 < a(x) < 1$. By Theorem 7 above, the function $h$ belongs to $W$. Similarly, one shows that the function $\min(f, g)$ belongs to $W$. □

**Corollary 7.** *Let $A$ be a $\beta$-closed non-empty subset of $D(X)$ with property $V$. Then $A$ has property $VN$ and $A$ is a lattice.*

**Proof.** Immediate from Corollary 6. □

## §5. The case of convex subsets

In this section we suppose that $X$ is a completely regular Hausdorff space. We denote its Stone-Čech compactification by $\beta X$, and by $\beta : C_b(X; \mathbb{R}) \to C(\beta X; \mathbb{R})$ the linear isometry which to each $f \in C_b(X; \mathbb{R})$ assigns its (unique) continuous extension to $\beta X$. Since $\beta$ is an algebra (and lattice) isomorphism, the image $\beta(A)$ of any subset $A \subset C_b(X, \mathbb{R})$ with property $V$ is contained in $D(\beta X)$ and has property $V$. If $B = \beta(A)$, then for each $x \in X$ one has

$$[x]_A = [x]_B \cap X.$$

If $Y$ denotes the quotient space of $\beta X$ by the equivalence relation $x \equiv y$ if and only if $\varphi(x) = \varphi(y)$, for all $\varphi \in B$, then $Y$ is a compact Hausdorff space.

If $x \in X$ and $K_x \subset X$ is a compact subset disjoint from $[x]_A$, then $\pi(K_x)$ is a compact subset in $Y$ which does not contain the point $\pi(x)$. (Here we have denoted by $\pi$ the canonical projection $\pi : \beta X \to Y$. Indeed, if $\pi(x) \in \pi(K_x)$, then $\pi(x) = \pi(y)$ for some $y \in K_x$. Now $y \in [x]_B$ because that $y \in [x]_A$. But $K_x \cap [x]_A = \phi$, and we have reached a contradiction. Hence $\pi(x) \notin \pi(K_x)$. We will apply these remarks in the proof of the following lemma.

**Lemma 4.** *Let $A \subset D(X)$ be a subset with property $V$ and containing some constant $0 < c < 1$. Let $x \in X$ and let $K_x \subset X$ be a compact subset, disjoint from $[x]_A$. Then, there exists an open neighborhood $W(x)$ of $[x]_A$ in $X$, disjoint from $K_x$ and such that given $0 < \delta < 1$ there is $\varphi \in A$ such that*

(1) $\varphi(t) < \delta$, *for all* $t \in K_x$;
(2) $\varphi(t) > 1 - \delta$, *for all* $t \in W(x)$.

**Proof.** Let $N(x)$ be the complement of $K_x$ in $\beta X$. Then $N(x)$ is an open neighborhood of $[x]_A$ in $\beta X$. We know that $\pi(K_x)$ is a compact subset of $Y$ which does not contain the point $y = \pi(x)$. Let $f \in C(Y;\mathbb{R})$ be a mapping such that $0 \leq f \leq 1, f(y) = 0$ and $f(t) = 1$ for all $t \in \pi(K_x)$. Let $g = f \circ \pi$. By Theorem 1, Chapter 8, Prolla [6], the function $g$ belongs to the uniform closure of $B$ in $D(\beta X)$. Notice that $a(x) = 0$ and $g(u) = 1$, for all $u \in K_x$. Define $N(x) = \{t \in \beta X; g(t) < 1/4\}$. Clearly, $[x]_B \subset N(x)$, since $g(t) = 0$ for all $t \in [x]_B$. It is also clear that $N(x)$ is disjoint from $K_x$. Let us define $W(x) = N(x) \cap X$. Then $W(x)$ is an open neighborhood of $[x]_A$ in $X$, which is disjoint from $K_x$.

Given $0 < \delta < 1$, let $p$ be a polynomial determined by Lemma 1, Chapter 1, Prolla [6], applied to $a = 1/4$ and $b = 3/4$, and $\varepsilon = \delta/2$. Let $h(t) = p(g(t))$, for all $t \in \beta X$. Since $\overline{B}$ has property $V$, it follows that $h \in \overline{B}$. If $t \in K_x$, then $g(t) = 1$ and so $h(t) < \delta/2$. If $t \in W(x)$, then $g(t) < 1/4$ and so $h(t) > 1 - \delta/2$. Choose now $\psi \in B$ with $||\psi - h||_X < \delta/2$, and let $\varphi \in A$ be such that $\beta(\varphi) = \psi$. Then $\varphi \in A$ satisfies conditions (1) and (2). $\square$

**Theorem 8.** *Let $W \subset C_b(X;E)$ be a non-empty subset and let $A$ be a set of multipliers of $W$ which has property $V$ and contains some constant $0 < c < 1$. Then $W$ is $\beta$-localizable under $A$.*

**Proof.** Assume that condition (2) of Definition 1 is true with $S = A$. For each $x \in X$, there is some $g_x \in W$ such that, for all $t \in [x]_A$, one has $\varphi(t)||f(t) - g_x(t)|| < \varepsilon/2$. Consider the compact subset $K_x$ of $X$ defined by

$$K_x = \{t \in X; \varphi(t)||f(t) - g_x(t)|| \geq \frac{\varepsilon}{2}\}.$$

Clearly, $K_x$ is disjoint from $[x]_A$. Now for each $x \in X$, select an open neighborhood $W(x)$ of $[x]_A$, disjoint from $K_x$, according to Lemma 4.

Select and fix a point $x_1 \in X$. Let $K = K_{x_1}$. By compactness of $K$, there exists a finite set $\{x_2, \dots, x_m\} \subset K$ such that

$$K \subset W(x_2) \cup W(x_3) \cup \dots \cup W(x_m)$$

Let $k = \sum_{i=1}^{m} p_\varphi(f - g_{x_i})$ and let $0 < \delta < 1$ be so small that $\delta k < \varepsilon/2$.

By Lemma 4, there are $\varphi_2, \ldots, \varphi_m \in A$ such that

(a) $\varphi_i(t) < \delta$, *for all* $t \in K_{x_i}$ ;

(b) $\varphi_i(t) > 1 - \delta$, *for all* $t \in W(x_i)$

for $i = 2, \ldots, m$. Define

$$\begin{aligned}
&\psi_2 = \varphi_2 \\
&\psi_3 = (1 - \varphi_2)\varphi_3 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\psi_m = (1 - \varphi_2)(1 - \varphi_3)\ldots(1 - \varphi_{m-1})\varphi_m.
\end{aligned}$$

Clearly, $\psi_i \in A$ for all $i = 2, \ldots, m$. Now

$$\psi_2 + \ldots + \psi_j = 1 - (1 - \varphi_2)(1 - \varphi_3)\ldots(1 - \varphi_j)$$

for all $j \in \{2, \ldots, m\}$, can be easily seen by induction. Define

$$\psi_1 = (1 - \varphi_2)(1 - \varphi_3)\ldots(1 - \varphi_m)$$

then $\psi_1 \in A$ and $\psi_1 + \psi_2 + \ldots + \psi_m = 1$.

Notice that

(c) $\psi_i(t) < \delta$ for all $t \in K_{x_i}$

for each $i = 1, 2, \ldots, m$. Indeed, if $i \geq 2$ then (c) follows from (a). If $i = 1$, then for $t \in K$, we have $t \in W(x_j)$ for some $j = 2, \ldots, m$. By (b), one has $1 - \varphi_j(t) < \delta$ and so

$$\psi_1(t) = (1 - \varphi_j(t)) \prod_{i \neq j} (1 - \varphi_i(t)) < \delta.$$

Let us write $g_i = g_{x_i}$ for $i = 1, 2, \ldots, m$.

Define $g = \psi_1 g_1 + \psi_2 g_2 + \ldots + \psi_m g_m$.

Notice that

$$\begin{aligned}
g \;=\; & \varphi_2 g_2 + (1 - \varphi_2)[\varphi_3 g_3 + (1 - \varphi_3)[\varphi_4 g_4 + \ldots + \\
& +(1 - \varphi_{m-1})[\varphi_m g_m + (1 - \varphi_m) g_1] \ldots]].
\end{aligned}$$

Hence $g \in W$. Let $x \in X$ be given. Then

$$\begin{aligned}\varphi(x)\|f(x)-g(x)\| &= \varphi(x)\|\sum_{i=1}^{m}\psi_i(x)(f(x)-g_i(x))\| \\ &\leq \varphi(x)\|\sum_{i=1}^{m}\psi_i(x)\|(f(x)-g_i(x))\|\end{aligned}$$

Define $I = \{1 \leq \tau \leq m; x \notin K_{x_i}\}$; $J = \{1 \leq i \leq m; x \in K_{x_i}\}$.
If $i \in I$, then $x \notin K_{x_i}$ and

$$\varphi(x)\|f(x)-g_i(x)\| < \frac{\varepsilon}{2}$$

and therefore

$$(*)\sum_{i\in I}\varphi(x)\psi_i(x)\|f(x)-g_i(x)\| \leq \frac{\varepsilon}{2}\sum_{i\in I}\psi_i(x) \leq \frac{\varepsilon}{2}.$$

If $i \in J$, then by (c), $\psi_i(x) < \delta$ and so

$$(**)\sum_{i\in J}\varphi(x)\psi_i(x)\|f(x)-g_i(x)\| \leq \delta k < \frac{\varepsilon}{2}.$$

From $(*)$ and $(**)$ we get $\varphi(x)\|f(x)-q(x)\| < \varepsilon$. □

**Theorem 9.** *Let $W \subset C_b(X;E)$ be a non-empty convex subset and let $A$ be the set of all multipliers of $W$. Then $W$ is $\beta$-localizable under $A$.*

**Proof.** The set $A$ has property $V$ and, since $W$ is convex, every constant $0 < c < 1$ belongs to $A$. □

**Theorem 10.** *Let $W \subset C_b(X;E)$ b a non-empty convex subset and let $B$ be any non-empty set of multipliers of $W$. Then $W$ is $\beta$-localizable under $B$.*

**Proof.** Similar to that of Theorem 6, using now Theorem 9 instead of Theorem 5.

□

**Corollary 8.** *Let $W \subset C_b(X;E)$ be a non-empty convex subset such that the set of all multipliers of $W$ separates the points of $X$. Then, for each $f \in C_b(X;\mathbb{R})$ the following are equivalent:*

(1) *$f$ belongs to the $\beta$-closure of $W$;*
(2) *for each $\varepsilon > 0$ and each $x \in X$, there is some $g \in W$ such that $||f(x) - g(x)|| < \varepsilon$.*

**Proof.** Clearly, (1) $\Rightarrow$ (2). Suppose now that (2) holds. Let $\varphi \in D_0(X), \varepsilon > 0$ and $x \in X$ be given. Notice that $[x]_W = \{x\}$. If $\varphi(x) = 0$, for any $g \in W$ one has $\varphi(x)||f(x) - q(x)|| = 0 < \varepsilon$. If $\varphi(x) > 0$, by (2) there is $g \in W$ such that $||f(x) - g(x)|| < \varepsilon/\varphi(x)$. Hence $\varphi(x)||f(x) - q(x)|| < \varepsilon$, and by Theorem 9, (1) is true. □

**Corollary 9.** *Let $S \subset X$ be a non-empty closed subset and let $V \subset E$ be a non-empty convex subset. Let $W = \{g \in C_b(X;E); g(S) \subset V\}$. Then, for each $f \in C_b(X;E)$ the following are equivalent:*

(1) *$f$ belongs to the $\beta$-closure of $W$;*
(2) *for each $x \in S$, $f(x)$ belongs to the closure of $V$ in $E$*

*Hence, $\overline{W}^{\beta} = \{f \in C_b(X;E); f(S) \subset \overline{V}\}$, where $\overline{V}$ is the closure of $V$ in $E$.*

**Proof.** Clearly, (1) $\Rightarrow$ (2). Conversely, assume that (2) holds. Clearly, $W$ is a convex set such that $D(X)$ is the set of all multipliers of $W$. Since $X$ is a completely regular Hausdorff space, $D(X)$ separates the points of $X$. Let $\varepsilon > 0$ and $x \in X$ be given. If $x \in S$ there is $v \in V$ such that $||f(x) - v|| < \varepsilon$, and the constant mapping on $X$ whose value is $v$ belongs to $W$ and $g(x) = v$. If $x \notin S$, choose $\varphi \in C_b(X;\mathbb{R}), 0 \leq \varphi \leq 1, \varphi(t) = 1$ for all $t \in S$ and $\varphi(x) = 0$; and let $g \in C_b(X;E)$

be defined by $g = \varphi \otimes v_0 + (1-\varphi) \otimes f(x)$, where $v_0 \in V$ is chosen arbitrarily. Then $g(t) = v_0$ for all $t \in S$, and therefore $g \in W$, and $g(x) = f(x)$. Hence (2) of Corollary 8 is verified and so $f$ belongs to the $\beta$-closure of $W$. □

**Corollary 10.** *Let $W \subset C_b(X;E)$ be a non-empty convex subset such that the set of all multipliers of $W$ separates the points of $X$ and, for each $x \in X$, the set $W(x) = \{g(x); g \in W\}$ is dense in $E$. Then $W$ is $\beta$-dense in $C_b(X;E)$.*

**Proof.** Apply Corollary 8. □

**Corollary 11.** *The vector subspace $W = C_b(X;I\!R) \otimes E$ is $\beta$-dense in $C_b(X;E)$.*

**Proof.** The set $A$ of all multipliers of $W$ is $D(X)$, and $W(x) = E$, for each $x \in X$. It remains to apply Corollary 10. □

**Corollary 12.** *If $X$ is a locally compact Hausdorff space, then $C_{00}(X;I\!R) \otimes E$ is $\beta$-dense in $C_b(X;E)$.*

**Proof.** Let $W = C_{00}(X;I\!R) \otimes E$. As in the previous corollary, the set $A$ of all multipliers of $W$ is $D(X)$, and for each $x \in X, W(x) = E$. □

**Theorem 11.** *Let $A \subset C_b(X;I\!R)$ be a subalgebra and let $W \subset C_b(X;E)$ be a vector subspace which is an $A$-module, i.e., $AW \subset W$. Then $W$ is $\beta$-localizable under $A$.*

**Proof.** Let $f \in C_b(X;E)$ be given . Assume that condition (2) of Definition 1 holds with $S = A$. Without loss of generality we may assume that $A$ is $\beta$-closed and contains the constants. Let $M$ be the set of all multipliers of $W$. We claim that, for each $x \in X$, one has $[x]_M \subset [x]_A$. Indeed, let $t \in [x]_M$ and let $\varphi \in A$. If $\varphi = 0$, then $\varphi \in M$ and $\varphi(t) = \varphi(x)$. Assume $\varphi \neq 0$. Write $\varphi = \varphi^+ - \varphi^-$, where $\varphi^+ = \max(\varphi, 0)$ and $\varphi^- = \max(-\varphi, 0)$. By Corollary 2, §3, both $\varphi^+$ and $\varphi^-$ belong to $A$. If $\varphi^+ = 0$, then $\varphi^+$ belongs to $M$ and $\varphi^+(t) = \varphi^+(x)$. If $\varphi^+ \neq 0$, let $\psi = \varphi^+/||\varphi^+||_X$. Now $\psi$ belongs to $A$ and $0 \leq \psi \leq 1$. Hence $\psi \in M$ and therefore $\psi(t) = \psi(x)$. Consequently, one has $\varphi^+(t) = \varphi^+(x)$. Similarly, one proves that $\varphi^-(t) = \varphi^-(x)$. Hence $\varphi(t) = \varphi(x)$. This ends the proof that $[x]_M \subset [x]_A$ for all $x \in X$. Hence condition (2) of Definition 1 is verified with $S = M$. By Theorem 9, $W$ is $\beta$-localizable under $M$. Hence $f$ belongs to the $\beta$-closure of $W$. □

**Corollary 13.** *Let $W \subset C_b(X; E)$ be a vector subspace, and let*

$$A = \{\psi \in C_b(X; \mathbb{R}); \psi g \in W \text{ for all } g \in W\}.$$

*Then $W$ is $\beta$-localizable under $A$.*

**Proof.** Clearly $A$ is a subalgebra of $C_b(X; \mathbb{R})$ and $W$ is an $A$-module. □

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