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# From binomial expectations to the Black-Scholes formula : the main ideas

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# From binomial expectations to the Black-Scholes formula: the main ideas.

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**Résumé**

Nous montrons comment une variable aléatoire discrète sur un espace de probabilité fini muni d'une distribution binomiale peut être proche d'une variable aléatoire continue, dans un sens qui respecte les espérances. Comme application, nous approximerons la variable aléatoire d'un processus multiplicatif discret par des exponentielles continues, et par conséquent, nous dériverons une formule donnant les valeurs des options qui généralisera la formule de Black et Scholes.

**Abstract**

We show how a discrete random variable on a finite probability space endowed with a binomial distribution may be close to a random variable on the continuum, in a way which respects the expectation. As an application, we approximate the random variables of a discrete multiplicative binomial process by continuous exponentials, and thus derive an option pricing formula, which contains the formula of Black and Scholes as a special case.

*Keywords:* Binomial distribution, standard normal distribution, expectation, Rieman-sums , shadow, Wiener walk, option

## 1 Introduction

We study continuous approximations of discrete expressions in the context of elementary probability theory and option pricing. The main result is a sort of extension of the De Moivre-Laplace central limit theorem, and concerns the approximation of the expectation of a random variable with respect to a binomial distribution by an expectation with respect to the standard normal distribution.

Our study is motivated by the derivation of the Black-Scholes formula (see [?]) for the pricing of European call options. In [?], J. C. Cox, S. A. Ross and M.

presented an option pricing formula in the form of a discrete binomial expectation, and then they showed that in the limit it converged to the Black-Scholes formula.

As a consequence of our main theorem we obtain a pricing formula for continuous options, of which the Black-Scholes formula is a special case. Our derivation is both more direct and more general than the derivation of Cox, Ross and Rubinstein: we reduce their sum formula to a Riemann-sum of the Black-Scholes integral formula. However, our setting is still their simple discrete pricing model, and thus avoids entirely the complications of limits of stochastic processes, continuous stochastic processes and measure theory. Instead, we apply nonstandard analysis, and following N. G. Cutland, E. Kopp and W. Willinger [6], we assume that the time steps of the discrete model are infinitesimal. With respect to their approach to option pricing, we obtained a further simplification, by avoiding the transitions between a standard and a nonstandard model, and Loeb-measure theory.

# 2 Notations and an informal presentation of the main result

Let

$$B_{N,p}(j) = \begin{pmatrix} N \\ j \end{pmatrix} p^j(1-p)^{N-j}$$

be the $j^{\text{th}}$ binomial coefficient and put

$$\begin{aligned} \mu_p &= N \cdot p \\ \sigma_p &= \sqrt{N \cdot p(1-p)} \\ x_j &= \frac{j-\mu_p}{\sigma_p} \\ \Omega_p &= \{x_j \mid j = 0, 1, \ldots, N\}. \\ dx_j &= x_{j+1} - x_j \end{aligned} \tag{2.1}$$

Notice that the $x_j$ are "normalized" with respect to the probability distribution $B_{N,p}(j)$: their mean is 0 and their standard deviation equals 1. For large $N$ we have the well-known approximation

$$B_{N,p}(j) \sim \frac{1}{\sqrt{2\pi}\sigma_p} \cdot \mathrm{e}^{-\left(\frac{j-\mu_p}{\sigma_p}\right)^2/2} = \frac{1}{\sqrt{2\pi}} \cdot \mathrm{e}^{-x_j^2/2} dx_j$$

It may be expected that the approximation carries over to sums:

$$\sum_{i\leq j} B_{N,p}(i) \sim \frac{1}{\sqrt{2\pi}} \cdot \sum_{i\leq j} e^{-x_i^2/2} dx_i$$

Thus we sketched a derivation of the De Moivre-Laplace central limit theorem

$$\sum_{i\leq j} B_{N,p}(i) \sim \mathcal{N}\left(\frac{j-\mu_p}{\sigma_p}\right)$$

where $\mathcal{N}$ is the normal distribution function given by

$$\mathcal{N}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \mathrm{e}^{-x^2/2} dx. \tag{2.2}$$

Our main result concerns expectations of the form

$$\mathrm{E}(h) = \sum_{i=0}^{N} h(x_i) B_{N,p}(i) \tag{2.3}$$

where $h$ is a discrete random variable defined on the $x_i$'s.

We show that under a suitable condition the above reasoning can be extended to this sum, leading to the approximations

$$\begin{aligned}\sum_{i=0}^{N} h(x_i) B_{N,p}(i) &\sim \frac{1}{\sqrt{2\pi}} \cdot \sum_{i=0}^{N} h(x_i) \mathrm{e}^{-x_i^2/2} dx_i \\ &\sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{h}(x) \cdot \mathrm{e}^{-x^2/2} dx\end{aligned}$$

where $\overline{h}$ is a continuous real function, closely related to $h$. Indeed we have the following main result

**Theorem 2.1 (Main Theorem).** *Let $N \simeq +\infty$, $0 \lneq p \lneq 1$ and $\Omega_p$ be the probability space given by ( 2.1 ) and endowed with the binomial distribution $B_{N,p}$. Let $h : \Omega_p \to \mathbb{R}$ be a random variable of class $S^o$, and of S-exponential order in $+\infty$. Then*

$$E(h) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} {}^o h(x) \cdot \mathrm{e}^{-x^2/2} dx \tag{2.4}$$

■

The theorem transforms an expectation with respect to the binomial distribution into an expectation with respect to the standard normal distribution. We remark that the formal nonstandard proof is very similar to the observations above. See [3].

# 3 Discrete arithmetic and geometric Brownian motions

Our application concerns the approximation of the expectation of a random variable with respect to a discrete geometric binomial process $S(t,x)$. This process will be defined on an *arithmetic binomial network*. Let $T>0$, $N\in\mathbb{N}$, and $dt>0$ be such that $Ndt=T$. Then $W_{T,dt}$ is the network given by

$$W_{T,dt}=\left\{(t,x)\in[0,T]\times\mathbb{R}\;\middle|\;\begin{array}{c}\exists m,n\in\mathbb{N},\quad 0\le m\le n\le N\\ \text{and}\\ t=ndt,\quad x=(-n+2m)\sqrt{dt}\end{array}\right\} \tag{3.5}$$

We call $dt$ the *period* of the network and

$$\mathbf{T}\equiv\{0,dt,2dt,\ldots,Ndt=T\} \tag{3.6}$$

the *time line* of the network. Notice that $W_{T,dt}$ is the union of all trajectories of the discrete arithmetic Brownian motion ("Wiener walk") on the time line $\mathbf{T}$. Sometimes we simply write $W_T$ instead of $W_{T,dt}$. We write $W_T(t)$ for the vertical sections of the network; they correspond to the values reached at time $t$ by the sample paths of the discrete arithmetic Brownian motion. Usually $dt$ is infinitesimal, and then we speak also of an *infinitesimal arithmetic binomial network*. Notice that in this case the vertical step $\sqrt{dt}$, though still infinitesimal is infinitely large with respect to the horizontal step $dt$. See also Figure 1.

The process $S(t,x)$, called the *discrete geometric Brownian motion*, is defined by induction on $W_T$. Let $S_0>0$, $\mu\in\mathbb{R}$, $\sigma>0$ and $0<p<1$. We put

$$S(0,0)=S_0$$

and for $t\in\mathbf{T}$, $t<T$

$$\left\{\begin{array}{rcl}S(t+dt,\;x+\sqrt{dt})&=&S(t,x)\cdot(1+\mu dt+\sigma\sqrt{dt})\\ S(t+dt,\;x-\sqrt{dt})&=&S(t,x)\cdot(1+\mu dt-\sigma\sqrt{dt})\end{array}\right. \tag{3.7}$$

Then indeed the process is defined on $W_{T,dt}$. We assume that the upper increment of (3.7) has conditional probability $p$, and the lower increment has conditional probability $1-p$, and that the increments are independent in time. Then $S(t,x)$ is properly defined as a stochastic process, and up to elementary transformations its random variables $S(t)\equiv S(t,\cdot)$ have binomial distributions. In particular

$$Pr\left\{S(T)=S_0\left(1+\mu dt+\sigma\sqrt{dt}\right)^j\left(1+\mu dt-\sigma\sqrt{dt}\right)^{N-j}\right\}=B_{N,p}(j).$$

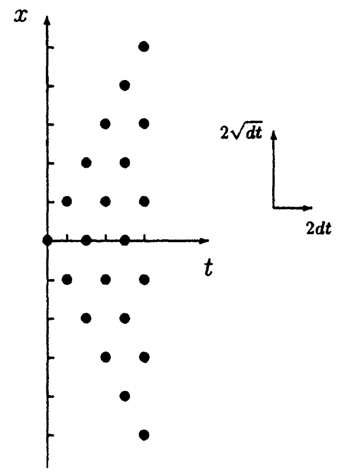


Figure 1: An arithmetical binomial network for $N = 5$.

Note that if $p = 1/2$, then $\mu$ is the relative conditional expectation, or *drift rate* of the process and $\sigma^2$ its relative conditional variance, or *volatility*.

# 4 Expectations and option pricing

In the economic context of option pricing, the process $S(t, x)$ endowed with the conditional probability $p = \frac{1}{2}$ is considered as a model describing the possible movements in time of the price of a share of some stock; trading is allowed at the times $\{0, dt, 2dt, \ldots, T\}$, the drift rate of the stock price being equal to $\mu$, and its volatility $\sigma$. Given a real-valued function $f$, the random variable $f(S(T))$ models a claim on that share at the future time $T$. For instance, let $K > 0$. Then the claim

$$f(S(T)) = (S(T) - K)^+$$

is called *the European call option with exercise date $T$ and with striking price $K$.* It models the payoff of a contract giving its owner the right to buy the share $S$ at time $T$ for the price $K$.

In fact, we described a stochastic process which is suitable for the discrete option pricing model of Cox, Ross and Rubinstein. They argue (see also [4]) that if $r$ is the risk-free rate of interest, the correct price $C_{dt}$ of the claim $f$ must be the Present Value (henceforth $PV_r$) of the expectation of the random variable $f(S(T))$ in a risk-neutral world (that is, the drift rate $\mu$ of the process $S$ must be $r$). Let then

$$\mathrm{E}_r f(S(T)) \tag{4.8}$$

denote the expectation of the random variable $f(S(T))$ in a risk-neutral world. Then

$$C_{dt} = PV_r(\mathrm{E}_r f(S(T))) \tag{4.9}$$

Recall that the present value in a risk-neutral world of an asset $A$ equals its future value $A(T)$ at time $T$ discounted at the risk-free rate of interest. That is to say

$$PV_r(A) = A(T)/(1+rdt)^{T/dt}$$

If the process $S(t,x)$ is in a risky world, (that is, its drift rate $\mu$ is different from $r$) then it is always possible to adjust its conditional probability $p$ to some value $p(r)$ which will change its drift rate to the prescribed risk-free rate of interest $r \in \mathbb{R}$. Note that $p(r)$ must satisfy

$$p(r)(1+\mu dt+\sigma\sqrt{dt}) + (1-p(r))(1+\mu dt-\sigma\sqrt{dt}) = 1+rdt$$

so

$$p(r) = \frac{1}{2} + \frac{r-\mu}{2\sigma}\sqrt{dt} \tag{4.10}$$

The expectation ( 4.8 ) can be written in the form ( 2.3 ); Indeed, let $p := p(r)$ and define the affine transformation $\nu_p : \Omega_p \to W_T(T)$ by

$$\nu_p(x) = 2\sqrt{T}\left[\sqrt{p(1-p)}x + \left(p-\frac{1}{2}\right)\frac{\sqrt{T}}{\sqrt{dt}}\right] \tag{4.11}$$

Then we have

$$\mathrm{E}_r f(S(T)) = \sum_{j=0}^{N} f(S(T,\ \nu_p(x_j))B_{N,p}(j) \tag{4.12}$$

Due to the rapid trading at stock markets, economists prefer a market model with a continuous time line: the Black-Scholes market, for which the option price $C_0$ becomes the Black-Scholes formula [4]. Now both the formulation of a Black-Scholes market model, and a derivation of the Black-Scholes formula within such a model are very intricate (see [8] for a survey). Instead, as argued by Cutland, Kopp and Willinger in [6], the Cox-Ross-Rubinstein model is a good alternative, provided the period $dt$ is infinitesimal: it expresses rapid trading, it has the simplicity of a discrete model, and the option price $C_{dt}$ almost does not depend on the length of $dt$. In fact, the difference between $C_{dt}$ and the Black-Scholes price $C_0$ is infinitesimal under some natural conditions on the order of magnitude of the parameters involved. Indeed, using straightforward approximations of $\nu_p$

$$\nu_p(x) \simeq \left(\frac{r-\mu}{\sigma}\right) \cdot T + x\sqrt{T}$$

(see Lemma 2.1 of [3]), of $S(T,x)$

$$S(T,x) \simeq S_0 \cdot \mathrm{e}^{(\mu-\sigma^2/2)T+\sigma x}$$

(see Proposition 3.1 of [3]), and the main theorem, we prove that

$$PV_r\left(\mathrm{E}_r f(S(T))\right) \simeq \frac{\mathrm{e}^{-rT}}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f\left(S_0 \mathrm{e}^{(r-\sigma^2/2)T+\sigma\sqrt{T}x}\right) \cdot \mathrm{e}^{-x^2/2} dx \tag{4.13}$$

The integral of the right-hand side of ( 4.13 ) is the Feynman-Kac formula (see [8]). From this we obtain the Black-Scholes formula by a straightforward standard transformation.

**Theorem 4.1 (Black-Scholes formula).** *Let $T > 0$ be appreciable and $W_T$ be an infinitesimal arithmetic binomial network. Let $S(t,x)$ be the discrete geometric Brownian motion on $W_T$ with appreciable initial value $S_0 > 0$, limited drift rate $\mu$ and appreciable volatility $\sigma^2$. Let $r$ be a limited risk-free rate of interest. Let $C_0 = {}^\circ(PV_r(E_r f(S(T))))$ be the shadow of the price of a the European call option $(S(T)-K)^+$ with striking price $K$ and exercise date $T$. Put*

$$x_0 = \frac{\log(S_0/K) - (r-\sigma^2/2)T}{\sigma\sqrt{T}}$$

*Then*

$$C_0 \simeq S_0 \cdot \mathcal{N}\left(x_0 + \sigma\sqrt{T}\right) - K\mathrm{e}^{-rT} \cdot \mathcal{N}(x_0). \tag{4.14}$$

Notice that ( 4.14 ) becomes an identity if $S_0$, $K$, $r$, $\sigma$ and $T$ are standard.

There are three main differences between the work of Cutland, Kopp and Willinger [6] and our approach in [3]. First to estimate $S(T)$ they use a nonstandard

Itô-calculus, while we use a "method of lines." Second to relate the discrete and the continuous they use the Loeb-measure and Loeb-spaces [12], while we use Riemann-sums, such as sketched above, and the external numbers of [10] and [11]. Third, their setting is Robinsonian nonstandard analysis [16], while our setting is axiomatic nonstandard analysis IST [14]. The main difference is that in the latter approach the infinitesimals are included within the set of real numbers $\mathbb{R}$, while in the former approach they are included in a nonstandard extension of $\mathbb{R}$.

The reader is referred to [7] or [13] for an introduction to the axiomatic nonstandard analysis IST, to [1], [2] or [15] for an account of discrete stochastic processes from a nonstandard point of view, to [8] or [9] for the classical option pricing theory, and to [3] for formal proofs of our results.

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