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FORTIFIED JOIN HYPERGROUPS

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ABSTRACT : In this paper appears a study of certain fundamental properties of a new hypergroup which came into being during to approach of the theory of Language and Automata with the help of the hypercompositional structures. It is being proved that there exist two kinds of elements in it with regard to the result of their hypercomposition with zero, while the elements of this hypergroup are also being separated into two classes, according to whether they are satisfying the equality x-x= -(x-x), or not. Moreover introductory elements are presented for each one of the three kinds of subhypergroups that appear in this hypergroup.

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1 - INTRODUCTION.

It is known that the Join Spaces were introduced and studied by W. Prenowitz, though in connection with Geometry (Join Geometries) [15, 16, 17,18 19, 20]. Moreover Ch. G. Massouros in his thesis [7], has sudied these spaces in detail as an abstract algebraic hypercompositional structure, which he named Join Hypergroup, since the join spaces are hypergroups. More precisely they are commutative hypergroups $(H,\cdot)$ satisfying the so called join axiom [15 , 20] :

(J) $$(a/b)\cap(c/d) \neq \emptyset \implies (ad)\cap(bc) \neq \emptyset$$

for every $a, b, c, d \in H$, where x/y is the induced hypercomposition (division) in H, that is $x/y = \{z \in H \mid x \in zy\}$.

It can be proved that a join hypergroup having a scalar neutral element, i.e. an element e such that $xe = ex = x$ for every $x \in H$, is a Canonical Hypergroup [7 , 1, 2]. The canonical hypergroup was introduced by M. Krasner as the additive part of a multiplicative-hyperadditive structure, the Hyperfield [3, 5] which was used as the proper algebraic tool in order to define a certain approximation of a complete valued field by sequences of such fileds. The canonical hypergroup has been studied in depth by J. Mittas (who has also named it canonical [12, 13]).

Apart from its strong connection to Geometry, there appeared an equally direct conection with the join hypergroup to the Theory of Languages [10, 9 , 11]. Indeed the consideration in this theory of the "sums" $x + y$ resulting in the biset $\{x, y\}$ led to the introduction of the hypercomposition $x + y = \{x, y\}$ in the set of the words A^* over an alphabet A as for which A^* has been proved to be a join hypergroup (B-hypergroup) [10,9]. Moreover the necessity of the introduction of the "null word" in the theory of langages, an element bilaterally absorbing as for the multiplication (concatenation of the words), (i.e. $0x = x0 = 0$) has led to the introduction of a non scalar neutral element in the join hypergroup (since $0 + x = \{0, x\}$ with regard to which every element has a unique opposite. Thus the Fortified Join Hypergroup came into being. This hypergroup also appears in the attached to an automaton hypergroups and during the procedure of the minimization of a given automaton [10,9].

DEFINITION 1.1. A Fortified Join Hypergroup (FJH) is a join hypergroup $(H, +)$, satisfying the axioms :

Fj_1. There exists one and only one element in H, denoted by 0 (the zero element of H), such that :

$x \in 0 + x$ and $0 + 0 = 0$ for every $x \in H$.

and

Fj_2. For every $x \in H \setminus \{0\}$ there exists one and only one $x' \in H \setminus \{0\}$ such that

$$0 \in x + x'$$

The element x', denoted by $-x$ is the opposite or symmetrical of x.

Also $-0 = 0$.

For the sake of simplicity and when there is no danger of confusion we write the element x instead of the singleton $\{x\}$. So we write $0 + 0 = 0$, instead of $0 + 0 = \{0\}$. We also denote by ":" the induced hypercomposition in H.

EXAMPLE 1.1. Every canonical hypergroup, and hence every abelian group, is a FJH. A FJH which is not canonical, is called proper. In the folowing we consider proper FJHs.

EXAMPLE 1.2. Every set E with cardE ≥ 2, becomes a FJH if one of its elements is arbitrarily chosen to be the neutral element e and the hypercomposition is defined in the following way :

$$x+y = \{x,y\} \quad \text{if} \quad x \neq y \quad \text{and} \quad x+x = \{x,e\}.$$

EXAMPLE 1.3. If in an abelain group $(G,+)$ the hypercomposition $x \dagger y = \{x,y,x+y\}$ is defined, the resulting hypercompositional structure $(G,\dagger)$ is a FJH having the same zero with $(G,+)$ and maintaining for every element the same opposite.

2. CANONICAL AND ATTRACTIVE ELEMENTS

For the following let $(H,+)$ be a FJH. In the beginning we have the fundamental theorem for the FJHs :

THEOREM 2.1. *For every $x \in H$ the following inclusion holds : $0+x \subseteq \{0,x\}$.*

Proof. It is obvious for $x=0$. Let $x \neq 0$ and $y \in 0+x$. First we remark that in a proper FJH there can possibly exist an element $x \in H \setminus \{0\}$ such that $0 \in 0+x$, as it can be seen in the examples. So let $y \in 0+x$ and $y \neq 0$. Then, on the one hand $x \in y:0$ and on the orther $x \in 0:(-x)$, since $0 \in x-x$. Hence $(y:0) \cap [0:(-x)] \neq \emptyset$, from which, according to the join axiom, $(y-x) \cap (0+0) \neq \emptyset$, that is $0 \in y-x$, thus , $y=x$. Therefore if $0 \notin 0+x$, then $0+x=x$ and if this last equality holds for every $x \in H$, then 0 is a scalar element of H and the FJH is canonical [1,2,7]. So there exist $x \in H$, such that $0 \in 0+x$ and therefore $x+0=\{x,0\}$.

Consequently there exist two kinds of elements in every FJH, according to the definition :

DEFINITION 2.1. An element $x \in H$ is called canonical or c-element if $0+x$ is the singleton $\{x\}$, while it is called attractive or a-element if $0+x$ is the biset $\{0,x\}$.

Obviously the 0 is a c-element and in the above Examples 1.2 and 1.3, 0 is the only c-element. Nevertheless there exist FJHs with c-elements different from 0 as in the example:

EXAMPLE 2.1. Let $(E,+)$ be a FJH and $w \notin E$. Then the set $H = E \cup \{w\}$ with hypercomposition :

$$\begin{array}{llll} x \dagger y &=& x+y, & \text{for every } x,y \in E \\ x \dagger w &=& w \dagger x = w & \text{for every } x \in E \\ w \dagger w &=& H & \end{array}$$

is a FJH with w being a c-element.

Next let us denote the sets of the canonical and the attractive elements of the FJH $(H,+)$ with C and A respectively. These two sets are obviously forming a partition in H. For the canonical and the attractive elements we have the properties

PROPOSITION 2.1.

i) *If $x \in A$ then $-x \in A$.*

ii) *If $x,y \in A$, then $x,y \in x+y$.*

iii) *If $x,y \in A$, then $x+y \subseteq A \cup \{0\}$.*

Proof.

i) If $x \in A$ then $0 \in 0+x$ whence $x \in 0:0$. Yet $x \in 0:(-x)$, thus $(0:0) \cap [0:(-x)] \neq \emptyset$, from where $0 \in 0+(-x)$, and so $-x \in A$.

ii) Obviously $(x+y)+0 = (x+y) \cup \{0\}$. But
$$(x+y)+0 \;=\; x+(y+0) \;=\; (x+y) \cup (x+0) \;=\; (x+y) \cup \{x,0\}.$$
So $(x+y) \cup \{0\} = (x+y) \cup \{x,0\}$. Thus $x \in x+y$. Similary $y \in x+y$.

iii) Let us assume that $z \in x+y$ and $z \notin A \cup \{0\}$. Then $x \in z:y$. Yet $x \in 0:(-x)$. Thus $(z:y) \cap [0:(-x)] \neq \emptyset$ therefore $(0+y) \cap (z-x) \neq \emptyset$, so $\{0,y\} \cap (z-x) \neq \emptyset$. Also $z \neq x$, which implies $0 \notin z-x$ and so $y \in z-x$, in which case $0+y \subseteq (0+z)-x$ or equivalently $\{0,y\} \subseteq z-x$, that is $0 \in z-x$, from where $z = x$, which is absurd. Hence $x+y \subseteq A \cup \{0\}$.

PROPOSITION 2.2.

i) *If $x \in C$ then $-x \in C$.*

ii) *If $x \in C \setminus \{0\}$, then $A \subseteq x-x$.*

iii) *If $x,y \in C$, with $y \neq -x$, then $x+y \subseteq C$.*

iv) *If $x,y \in C$, and if $x \dagger y = (x+y) \cap C$, then $(C,\dagger)$ is a canonical hypergroup.*

Proof.

i) It derives directly from (i) of the previous Proposition.

ii) Let z be an arbitrary element of A. Then from the reproductive axiom of the hypergroup [14], i.e. $x+H = H$, it derives that there exits $y \in H$ such that $z \in x+y$, from where $0+z \subseteq (0+x)+y$ and so $\{0,z\} \subseteq x+y$. Thus $0 \in x+y$, and since $y \neq 0$ (otherwise $x=0$, which is absurd) it derives that $y=-x$ and therefore $z \in x-x$.

iii) Let $z \in x+y$ and let us assume that $z \notin C$. Then $0+z \subseteq (0+z)+y$ implies that $\{0,z\} \subseteq x+y$. Thus $0 \in x+y$ and therefore $y=-x$, which contradicts the supposition. So $x+y \subseteq C$.

PROPOSITION 2.3.

i) *If* $y \in C \setminus \{0\}$ *and* $x \in A$, *then* $y = x+y$.

ii) *If* $x \in A \cup \{0\}$, *then* $x : x = A \cup \{0\}$.

Proof.

i) $y = y+0$ thus $0 \in y : y$. Moreover $0 \in x+0$, so $0 \in 0 : x$. Consequently $(y : y) \cap (0 : x) \neq \emptyset$ implies that $(y+x) \cap (0+y) \neq \emptyset$ whence $(x+y) \cap \{y\} \neq \emptyset$, and therefore $y \in x+y$. Next it will be proved that apart from y no other c-element belongs to $x+y$. Indeed, let $y' \in C$ and $y' \in x+y$. Then $x \in y' : y$. Moreover $x \in 0 : 0$ so $(y' : y) \cap (0 : 0) \neq \emptyset$ from where $(y'+0) \cap (y+0) \neq \emptyset$, and therefore $y' = y$. Finally it will be proved that none of the a-elements belongs to $x+y$. Indeed, let $x' \in A$ and $x' \in x+y$. Then from $x'+0 \subseteq (x+y)+0$ we get $\{0,x'\} \subseteq x+(y+0)$ whence $\{0,x'\} \subseteq x+y$. Therefore $0 \in x+y$, that is $y=-x$, absurd. Consequently, $y = x+y$.

ii) According to Proposition 2.1 (ii) $x \in x+t$, for every $t \in A \cup \{0\}$. So $t \in x : x$ and therefore $A \cup \{0\} \subseteq x : x$. Also, because of (i) holds $s \notin x : x$ for every $s \in C \setminus \{0\}$, since $s = x+s$. Therefore $x : x = A \cup \{0\}$.

COROLLARY 2.1.

i) *If* $x, y \in A$, *then* $x : y \subseteq A \cup \{0\}$.

ii) *If* $z \in C$, *then* $A \subseteq z : z$.

Having proved the above we can give the following construction which extends a canonical hypergroup by a FJH in which non zero element is attractive :

CONSTRUCTION *Let* $(K,+)$ *be a canonical hypergroup and let* $(E,+)$ *be a FJH in which all the non zero elements are attractive and such that* $K \cap E = \{0\}$. *Define, on* $H = K \cup E$ *a hypercomposition* $\dagger$ *by :*

$$x \dagger y = \begin{cases} x\dot{+}y & \textit{if} \quad (x,y) \in E^2 \\ x\dot{+}y & \textit{if} \quad (x,y) \in K^2 \quad \textit{and} \quad y \neq -x \\ (x\dot{+}y) \cup E & \textit{if} \quad (x,y) \in (K \setminus \{0\})^2 \quad \textit{and} \quad y = -x \\ y & \textit{if} \quad x \in E \quad \textit{and} \quad y \in K \setminus \{0\}. \end{cases}$$

Then $(H, \dagger)$ *is a FJH.*

Using this construction we get the following result :

THEOREM 2.2. *Every FJH can be obtained by the previous construction.*

As it is know [12,13] the reversibility, that is the implication :

$$z \in x + y \implies y \in z - x, \quad \text{for every} \quad x, y, z \in H$$

is one of the axioms of the canonical hypergroup. In the FJH though, it can not be generally valid, since, if it were, for every $x \in A$, it would be : $0 \in 0 + x \implies x \in 0 - 0$, so $x = 0$, that is A would be an empty set and the FJH would be a canonical hypergroup. For the FJHs, we conclude with :

PROPOSITION 2.4. *For every* $x, y, z \in H$ *with* $z \in x + y$*, the reversibility generally holds, except for the cases :*

i) $z = y = 0$, $x \in A$ *and*

ii) $z = x \neq y$ *with* $x, y \in A$,

for which respectively holds :

$0 \in x + 0$ *implies that* $0 \in 0 - x$, *while* $x \notin 0 - 0$

$x \in x + y$ *implies that* $x \in x - y$, *while, generally,* $y \notin x - x$.

Before proceeding with the proof, we prove the Lemma :

LEMMA 2.1. *If* $x, y, z \in H$ *and* $z \in x+y$ *then* $(z-x)\cap(0+y) \neq \emptyset$ *and* $(z-y)\cap(0+x) \neq \emptyset$.

Proof. If $z \in x + y$ then $x \in z : y$ and $y \in z : x$. Also $x \in 0 : (-x)$ and $y \in 0 : (-y)$. So $(z : y) \cap [0 : (-x)] \neq \emptyset$ and $(z : x) \cap [0 : (-y)] \neq \emptyset$, and, because of the join axiom, the Lemma is proved.

Proof of the PROPOSITION 2.4. We distinguish the three different cases. Let

α) $x, y \in C$. From the Lemma, and since $0 + x = x$, $0 + y = y$, derives $y \in z - x$ and $x \in z - y$.

β) $x \in A$, $y \in C$. If $y \neq 0$, then, because of Proposition 2.3, $z \in x + y$ implies that $z = y$ and therefore, from the same Proposition, it derives that $y = y - x$, while, because of

Proposition 2.2. (ii), $x \in y - y$. If $y = 0$, then $z \in x + 0$ implies that either $z = 0$ or $z = x$.

From $0 \in x+0$ follows that $0 \in 0-x$, while $x \notin 0-0$ [since $x \in A$ implies that $-x \in A$ (Prop. 2.1.) and $x \neq 0$] and from $x \in x + 0$ results that $0 \in x - x$ and $x \in x - 0$.

γ) $x, y \in A$. From the Lemma we have :

$$(z - x) \cap \{0, y\} \neq \emptyset \quad \text{and} \quad (z - y) \cap \{0, x\} \neq \emptyset$$

and for $0 \neq z \neq x \neq 0$ holds $y \in z - x$, while for $0 \neq z \neq y \neq 0$ is $x \in z - y$. Now, if $z = 0$, then from $0 \in x + y$ derives $y = -x$, and since $x \in 0 + x$ and $-x \in 0 - x$, the reversibility holds again. Next if $z = x$, then $(x - x) \cap \{0, y\} \neq \emptyset$ and $(x - y) \cap \{0, x\} \neq \emptyset$ and therefore, if $y \neq x$, then from the second relation it derives that $x \in z - y$, while the first one does not imply that $y \in x - x$.Thus in the Example 1.2, where all the non zero elements of the hypergroup are attractive, we have that the implication $x \in x + y \implies x \in x - y$ holds, since $x - y = x + y = \{x, y\}$, while $y \notin x - x = x + x = \{0, x\}$, for every $y \neq x$. Also, in Example 1.3, we generally have the implication $x \in x \dagger y \implies x \in x - y$, since $x - y = \{x, -y, x - y\}$, while, on the contrary, $y \in x - x = \{x, -x, 0\}$ only for $y = -x$, and $y = x$. Finally, if $z = x = y$, then, according to Proposition 2.1. (ii), we have the implication : $x \in x - x \implies x \in x + x$.

Consequently we can talk about "partial" reversibility and so put forward the Proposition :

PROPOSITION 2.5. *Every FJH $(H, +)$ is partially reversible, that is, for every x, y, $z \in H$, if $z \in x + y$, then either $y \in z - x$ or $x \in z - y$ (without the one always rulling out the other).*

COROLLARY 2.2. *If $z \in x + y$ then $y \in z - x$, for every $x, y \in C$.*

As it is know [12,13], in a canonical hypergroup $(K, +)$ the reversibility is equivalent to each one of the properties :

i) $-(x + y) = -x - y$ for every $x, y \in K$

ii) $(x + y) \cap (z + w) \neq \emptyset \implies (x - z) \cap (w - y) \neq \emptyset$ for every $x, y, z, w \in K$.

Similar properties appear in the FJH as well. More precisely :

PROPOSITION 2.6. *The partial reversibility in the FJH is equivalent to the property :*

$$-(x + y) = -x - y, \quad \text{for every} \quad x, y \in H \quad \text{with} \quad y \neq -x.$$

Proof. It is obvious for $x = 0$ (or $y = 0$). Let $x, y \neq 0$ and $z \in x + y$. Then, because of the partial reversibility, it will be for instance, $y \in z - x$ from where $y - y \subseteq z + (-x - y)$. So it is $0 \in z + (-x - y)$, and since $y \neq -x$, then $0 \notin -x - y$ and therefore $-z \in -x - y$. Thus $-(x+y) \subseteq -x-y$. Now if $z \in -x-y$, then, because again of the partial reversibility, it will be for instance $-y \in z + x$, and following the same steps as above it derives that $z \in -(x + y)$ and so $-(x + y) = -x - y$.

Conversely now, this property, for $y \neq -x$, gives the partial reversibility.

Indeed, for $z \in x + y$ we get $-z \in -(x + y) = -x - y$ and so $z - z \subseteq (z - x) - y = (z - y) - x$. Thus $0 \in (z - x) - y$ and $0 \in (z - y) - x$. So for $z \neq x$ or $z \neq y$ it will be $y \in z - x$ or $x \in z - y$ respectively. Yet, if $z = x = y$, then again $x \in x + x$ implies that $x \in x - x$

REMARK 2.1. The property $-(x + y) = -x - y$ is not generally valid for $y = -x$. For instance, it holds in the FJH of the Examples 1.1 - 1.3, as it is also generally valid in FJHs with selfopposite elements. Yet, there exist FJHs in which this property is not valid.

EXAMPLE 2.2. Let H be a set totally ordered, dense as for the order and symmetrical around a center denoted by $0 \in H$, as for which a partition $H = H_1 \cup \{0\} \cup H_2 = H - \cup\{0\} \cup H^+$ can be defined, such that :

$x < 0 < y$ for every $x \in H^-$ and $y \in H^+$,

$x \leq y$ implies that $-y \leq -x$ for every $x, y \in H$ (where $-x$ is the symmetrical of x with regard to 0.

We introduce in H the hypercomposition :

$$\begin{aligned} x + y &= \{x, y\}, \quad \text{if} \quad y \neq -x \\ \text{and} \qquad x + (-x) &= [0, |x|] \cup \{-|x|\}. \end{aligned}$$

The deriving structure is a FJH in which $x - x \neq -(x - x)$, for every $x \neq 0$, since $-(x - x) = [-|x|, 0] \cup \{|x|\}$. The proof, which is achieved by the verification of the axioms, is quite complicated and it can be found in [10].

Relatively we also have :

PROPOSITION 2.7. *If x is a c-element of a FJH, then $-(x - x) = x - x$.*

Proof. Let $x \in C$. Since all the a-elements belong to $x - x$ [Prop. 2.2 (ii)] and since $-A = A$ [Prop. 2.1 (i)] we have $A \subseteq x - x$, which implies that $-A \subseteq x - x$ and so $A \subseteq -(x - x)$. Next let z be a c-element such that $z \in x - x$. Then (Corol. 2.2) $x \in z + x$, so $x - x \subseteq z + (x - x)$, thus $0 \in z + (x - x)$, hence $-z \in x - x$ and therefore $z \in -(x - x)$. So

$x - x \subseteq -(x-x)$. Furthermore $x-x \subseteq -(x-x)$ implies that $-(x-x) \subseteq -[-(x-x)] = x-x$ and so the Proposition.

COROLLARY 2.3. *For every, $x, y \in C$, the equality $-(x+y) = -x-y$ holds.*

PROPOSITION 2.8. *In a FJH the non empty result of the intersection $(x+y) \cap (z+w)$ implies that $(z-x) \cap (y-w) \neq \emptyset$ for every $x, y, z, w \in H$ with $x \neq z$ and $y \neq -x, w$.*

Proof. If any of the x, y, z, w is 0, then the Proposition derives from the partial reversibility. If none of them is 0, then consider an element $t \in (x+y) \cap (z+w)$. Thus (Prop.2.6) :

$t - t \subseteq (z+w)-(x+y) \; = \; z+w-x-y \; = \; (z-x)+(w-y) \; = \; (z-x)-(y-w)$.

So $0 \in (z-x)-(y-w)$ and since $z \neq x$ and $w \neq y$, the 0 does not belong to $z-x$ and to $y-w$. Consequently there exists an element $v \in (z-x) \cap (y-w)$, and so $(z-x) \cap (y-w) \neq \emptyset$.

COROLLARY 2.4. *$(x+y) \cap (z+w) \neq \emptyset$ always implies that $(z-x) \cap (y-w) \neq \emptyset$, for every $x, y, z, w \in C$.*

REMARK 2.2. In the canonical hypergoups the reversibility-or the equivalent property of Prop.2.8- as an axiom is equivalent to the join axiom, that is axiom (J). It is not the same though in FJHs, where the join axiom is always valid for the entire hypergroup by default, while the reversibility is generally valid in its partial form.

Now regarding the induced hypercomposition (division), which is of a great significance in the theory of the join hypergroups, we initially remark that in the canonical hypergroups, as it is known [7, 10] , $x : y = x - y$ for every $x, y \in H$. This is not valid though in the (proper) FJHs, because, for example, in those hypergroups $0 : 0 = A \cup \{0\}$ while $0 - 0 = \{0\}$. For the FJHs we have :

PROPOSITION 2.9. *If one of the x, y is a c-element, different than 0, then*

$$x : y \; = \; (-y):(-x) \; = \; x - y.$$

Proof. $x : y = \{z \mid x \in z+y\}$ which, due to the partial reversibility and the supposition, is equal to $\{z \mid z \in x-y\}$ that is to $x-y$. Similarly $(-y):(-x) \; = x-y$.

PROPOSITION 2.10. *For every $x, y \in H$:*

i) $x - y = (x : y) \cup [(-y) : (-x)]$ if $x \neq y$

ii) $x - x \subseteq (x : x) = (-x) : (-x)$

Proof.

i) Let $z \in x - y$. Then, according to Proposition 2.5, we have that either $x \in z + y$, from where derives that $z \in x : y$ or $-y \in z - x$, from where derives that $z \in (-y) : (-x)$ and so $x - y \subseteq (x : y) \cup [(-y) : (-x)]$. Conversely now, $z \in x : y$ implies $x \in z + y$, whence $z \in x - y$ and $z \in (-y) : (-x)$ implies $-y \in z - x$, whence $z \in x - y$. Thus $(x : y) \cup [(-y) : (-x)] \subseteq x - y$ and so (i).

ii) If $x \in C \setminus \{0\}$, then ii) is valid because of Proposition 2.9. Moreover, if $x \in A \cup \{0\}$, then, because of Proposition 2.3. $x : x = (-x) : (-x) = A \cup \{0\}$ and $x - x \subseteq A \cup \{0\}$.

3. SUBHYPERGROUPS OF A FJH

From Proposition 2.6., Remark 2.1. and Example 2.2., there derives a new distinction of the elements of a FJH (H,+), according to the definition:

DEFINITION 3.1. An element $x \in H$ such that $-(x - x) = x - x$ is called normal. Otherwise it is called abnormal. A FJH having only normal elements is called normal FJH while if it contains at least one abnormal element it is called obnormal FJH.

It derives that every c-element of H is normal. Moreover the hypergroup of the Example 1.2 and the FJH of the Example 1.3 are normal FJHs. As an example of an abnormal FJH we mention Example 2.2 in which every element $x \neq 0$ is abnormal. From the Definition and the Proposition 2.6 we come to the conclusion that :

PROPOSITION 3.1. *In a normal FJH $(H,+)$:*

$$-(x + y) = -x - y$$

for every $x, y \in H$ and, by induction :

$$-(x_1 + \ldots + x_n) = -x_1 - \ldots - x_n$$

for every $x_1, \ldots, x_n \in H$.

The notion of the normal elements of a FJH becomes very useful in the study of its subhypergroups. Let, for instance , X be a set of normal elements of a FJH $(H, +)$ and let us denote by $\Omega(X)$ the union of all the sums $(x_1 - x_1) + \ldots + (x_n - x_n)$, where n is an arbitrary non negative integer and $x_1, \ldots, x_n$ are chosen independently from X. Then :

LEMMA 3.1. *If $w \in (x_1 - x_1) + \ldots + (x_n - x_n)$, then $-w \in (x_1 - x_1) + \ldots + (x_n - x_n)$, for every $x_1, x_2, \ldots, x_n \in X$.*

Proof. This Lemma will be proved by induction. So, for $n = 1$, $w \in x_1 - x_1$ implies that $-w \in -(x_1 - x_1) = x_1 - x_1$, since x_1 is normal. Now suppose that it is true for $n = k$ and let $w \in (x_1 - x_1) + \ldots + (x_k - x_k) + (x_{k+1} - x_{k+1})$. Then there exists $s \in (x_1 - x_1) + \ldots + (x_k - x_k)$ such that $w \in s + (x_{k+1} - x_{k+1}) = (s + x_{k+1}) - x_{k+1}$. Therefore there exists $t \in s + x_{k+1}$ such that $w \in t - x_{k+1}$. So $-w \in -(t - x_{k+1})$. Now if $t = x_{k+1}$, then it is a normal element and so $-(t - x_{k+1}) = -t + x_{k+1}$. The last equality is always valid if $t \neq x_{k+1}$ (Prop. 2.6). Thus $-w \in -t + x_{k+1} \subseteq -(s + x_{k+1}) - x_{k+1} = -s + (x_{k+1} - x_{k+1})$, according to the same reasoning as above, which due to the induction gives :
$-w \in (x_1 - x_1) + \ldots + (x_k - x_k) + (x_{k+1} - x_{k+1})$.

PROPOSITION 3.2. *$\Omega(X)$ is a subhypergroup of H.*

Proof. Apparently $0 \in \Omega(X)$ and because of the Lemma , $-x \in \Omega(X)$ for every $x \in \Omega(X)$. next for every $x \in \Omega(X)$:

$$x + \Omega(X) \subseteq (x_1 - x_1) + \ldots + (x_n - x_n) + \Omega(X) \subseteq \Omega(X)$$

Yet for $y \in \Omega(X)$ it will be :

$$y - x \subseteq [(y_1 - y_1) + \ldots + (y_m - y_m)] - [(x_1 - x_1) + \ldots + (x_n - x_n)] \subseteq \Omega(X)$$

so $y - x \subseteq \Omega(X)$ implies $(y - x) + x \subseteq x + \Omega(X)$ whence $y + (x - x) \subseteq x + \Omega(X)$. Therefore $y + 0 \subseteq x + \Omega(X)$, thus $y \in x + \Omega(X)$ and consequently $\Omega(X) \subseteq x + \Omega(X)$. Thus for every $x \in \Omega(X)$ we have $x + \Omega(X) = \Omega(X)$ and so $\Omega(X)$ is a subhypergroup of H.

The subhypergroup $\Omega(X)$ is not always a FJH, because even though it satisfies the axioms FJ_1 and FJ_2 of the Definition 1.1., it is not generaly closed (stable) under the induced hypercomposition [that is $x : y \subseteq \Omega(X)$ is not valid for every $x, y \in \Omega(X)$], which is a necessary prerequisite for the validity of the join axiom inside the set $\Omega(X)$. This is the case in Example 1.2., where for an arbitrary non void subset X of E we have $\Omega(X) = X \cup \{0\}$. Obviously in this FJH $x : y = x$ for $x \neq y$ and $x : x = E$, thus $\Omega(X)$ is not closed under the induced hypercomposition, when X is a proper subset of E. So there exist subhypergroups h of any FJH $(H, +)$, with the property $-x \in h$ for every $x \in h$. Such subhypergroups will be called symmetrical subhypergroups. Generally the symmetrical subhypergroups are not FJH themselves. Yet a FJH is a join hypergroup and so there also exist join subhypergroups in it, i.e. subhypergroups that are join hypergroups themselves. Thus in a FJH there are :

a) Join subhypergroups,

b) symmetrical subhypergoups and

c) subhypergroups that are neither join nor symmetrical.

The following Propositions are relevant to these subhypergroups for the proof of which it is necessary to give the Lemma :

LEMMA 3.2. *A subhypergroup of a join hypergroup (H,+) (not necessarily fortified) is join if and only if it is closed.*

Proof. As it is known, a subhypergroup h of a hypergroup (H,.) is called closed when, for every $a \in H \setminus h, ah \cap h = \emptyset$ [right closed] and $ha \cap h = \emptyset$[left closed] [4,6, 12, 13]. Moreover it is known [7] that a subhypergroup of H is closed if and only if it closed under the induced hypercomposition. So for a join subhypergroup h and for every $x, y, z, \omega \in h$, if $(x : y) \cap (z : \omega) \neq \emptyset$, then this non void intersection is contained in h, which is also valid for the intersection $(x : y) \cap (x : y)$. Thus it will be $x : y \subseteq h$ and so h is closed. Conversely now, if h is closed, then $x : y \subseteq h$, $z : \omega \subseteq h$ and apparently, the join axiom is satisfied inside h as well. Hence h is join.

PROPOSITION 3.3. *Every join subhypergroup h of a FJH (H,+) is a FJH itself with the same zero.*

Proof. As it is known [6,12,13], every closed subhypergroup of any hypergroup contains all its neutral elements. Therefore h contains 0 and so it satisfies the axiom FJ_1 of the Definition 1.1. Moreover, for every $x \in h$ it will be $0 : x \subseteq h$ and since $-x \in 0 : x$ it derives that $-x \in h$. Thus FJ_2 is satisfied as well.

COROLLARY 3.1. *Every join subhypergroup of a FJH is symmetrical as well.*

PROPOSITION 3.4. *The set $A \cup \{0\} = \widehat{A}$ is the minimun (in the sense of inclusion) join subhypergroup of H.*

Proof. $\widehat{A}$ is a subhypergroup of H. Indeed for every $x \in \widehat{A}$, and according to Proposition 2.1.,$-x \in \widehat{A}$ and $x + \widehat{A} \subseteq \widehat{A}$. Next if y is an arbitrary element of $\widehat{A}$, then $-x + y \subseteq \widehat{A}$ and $0 + y \subseteq (x - x) + y$ implies $\{0, y\} \subseteq x + (-x + y)$, thus $y \in x + (-x + y)$. Therefore there exists $z \in -x + y$ such that $y \in x + z \subseteq x + \widehat{A}$. So $\widehat{A} \subseteq x + \widehat{A}$ and consequently $x + \widehat{A} = \widehat{A}$. Hence $\widehat{A}$ is a subhypergroup of H and a symmetrical one too.

$\hat{A}$ is closed and so join (Lemma 3.2). Indeed, let z be an element of $H \setminus \hat{A}$. Then z is a non zero canonical element and so (Prop. 2.3) $(z + \hat{A}) \cap \hat{A} = \{z\} \cap \hat{A} = \emptyset$.

Since every join subhypergroup of a FJH contains the 0 element and since $0 : 0 = A \cup \{0\}$ [Prop 2.3 (ii)], $\hat{A}$ does not contain a join subhypergroup of H. Moreover a join subhypergroup h of H,different from $\hat{A}$ must contain a c-element. Let x be this c-element. Then, because of Proposition 2.2 (ii), $\hat{A} \subseteq x - x$, so $\hat{A} \subseteq h$ and therefore the Proposition.

COROLLARY 3.2. *The FJHs that contain (apart from 0) only a-element have no proper join subhypergroups.*

PROPOSITION 3.5. *If a symmetrical subhypergroup h of H contains a c-element other than zero, then it is a join one.*

Proof. Let z be a c-element belonging to h. Then $\hat{A} \subseteq z - z$ and therefore $x : y \subseteq h$, for every $x, y \in \hat{A}$. Moreover if at least one of the $x, y \in h$ is a c-element, then according to Proposition 2.9 $x : y = x - y \subseteq h$. Hence , as h is closed [8], it is also join (Lemma 3.2).

COROLLARY 3.3. *The symmetrical subhypergroups that are not join consist only of a-elemnts.*

Finaly, as mentioned above, apart from the symmetrical and the join subhypergroups there exists FJH having subhypergroups that are not symmetrical, and much less join. This case appears in Example 2.2, where every subset h of H with the property $-x \notin h$ for every $x \in h$, is such a subhypergroup.

The thorough study of the subhypergroups of a FJH and more generally of the join hypergroups, is very extensive and constitutes the subject of other, forthcoming papers of ours.

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