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ON DIFFERENCES OF SELF-ADJOINT SEMIGROUPS

JAN A. VAN CASTEREN

This work is dedicated to the memory of A. Badrikian.

ABSTRACT. Let K_0 be the generator of a strong Markov process $(X(t), P_x)$, with state space E, that is locally compact and second countable. Assume that the corresponding semigroup is self-adjoint and positivity preserving in the space $L^2(E,m)$. Here m is a Radon measure on E. Let V and W be Kato-Feller potentials and let $H_0 := K_0 \dot{+} V$, respectively $H_1 := K_0 \dot{+} W$ be the generators of the corresponding Feynman-Kac semigroups $\{V_j(t) : t \geq 0\}$, $j = 0, 1$. Fix $t_0 > 0$ and let $\{E_j(\xi) : \xi \in R\}$, $j = 0, 1$, be the spectral decompositions of $K_0 \dot{+} V$ and $K_0 \dot{+} W$ respectively. Suppose that the function $\exp\left(-\frac{1}{2}t_0K_0\right)|W - V|$ belongs to $L^\infty(E,m)$ or suppose that $\lim_{t\downarrow 0}\sup_{x\in E} E_x\left[\left(\int_0^t (W(X(s)) - V(X(s)))\right)^2\right] = 0$. Then the following assertions are equivalent:

(i) For all bounded intervals A_0 and A_1 the operator $E_0(A_0)(W-V)E_1(A_1)$ is compact;
(ii) The operator $V_0(t_0/2)(W - V)V_1(t_0/2)$ is compact;
(iii) The operator $V_0(t_0) - V_1(t_0)$ is compact.

Among others the identity in (2.6) below is used substantially. A number of consequences of that formula is established.

SOMMAIRE. Soit K_0 le générateur d'un processus Markovien $\{X(t), P_x\}$, avec un espace d'état E, qui est localement compact et vérifie l'axiome de deuxième dénombrabilité. On suppose que le semi-groupe associé est auto-adjoint et qu'il préserve la positivité dans l'espace $L^2(E,m)$. Ici m dénote une mesure de Radon sur E. Soient V et W des fonctions potentielles et soient $H_0 := K_0 \dot{+} V$, respectivement $H_1 := K_0 \dot{+} W$ les générateurs des semigroupes de Feynman-Kac $\{V_j(t) : t \geq 0\}$, $j = 0, 1$, associés. On fixe $t_0 > 0$ et soient $\{E_j(\xi) : \xi \in R\}$, $j = 0, 1$, les décompositions spectrales de $K_0 \dot{+} V$ et $K_0 \dot{+} W$ respectivement. On suppose que la fonction

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$\exp\left(-\frac{1}{2}t_0K_0\right)|W-V|$ appartient à $L^\infty(E,m)$ ou que la condition suivante est vérifiée: $\lim_{t\downarrow 0}\sup_{x\in e} E_x\left[\left(\int_0^t (W(X(s))-V(X(s)))\right)^2\right]=0$. Alors les assertions suivantes sont equivalentes:

(i) L'opérateur $E_0(A_0)(W-V)E_1(A_1)$ est compact pour tous les intervalles bornés A_0 et A_1;
(ii) l'opérateur $V_0(t_0/2)(W-V)V_1(t_0/2)$ est compact;
(iii) l'opérateur $V_0(t_0)-V_1(t_0)$ est compact.

Entre autres l'identité dans (2.6) au-dessous est utilisée essentiellement. Un certain nombre de conséquences de cette formule est établi.

1. Introduction.

Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be complex Hilbert spaces, let H_0 and H_1 be self-adjoint operators with domains in $\mathcal{H}_0$ respectively $\mathcal{H}_1$, and let $\mathcal{J}:\mathcal{H}_1\to\mathcal{H}_0$ be a bounded linear operator. Suppose that H_0 and H_1 are bounded below with lower bounds $-\omega_0$ and $-\omega_1$ respectively. We write $V_0(t)=\exp(-tH_0)$ and $V_1(t)=\exp(-tH_1)$. Notice that the families $\{V_j(t):\mathrm{Re}\, t\geq 0\}$, $j=0,\,1$, are strongly continuous semigroup and that the family $\{V_0(i\tau):\tau\in R\}$ is a strongly continuous unitary group. Furthermore let $\{E_0(\xi):\xi\in R\}$ and $\{E_1(\xi):\xi\in R\}$ be the spectral decomposition corresponding to H_0 and H_1 respectively. So that e.g. for f in the domain of H_0, $\langle H_0f,f\rangle=\int\langle E_0(d\xi)f,f\rangle$. Put

$$\mathcal{D}(t)T=\int_0^t V_0(u)TV_1(t-u)du,\quad \mathrm{Re}\, t\geq 0. \tag{1.1}$$

Notice the identity

$$\mathcal{D}(t)(\mathcal{J}H_1-H_0\mathcal{J})=V_0(t)\mathcal{J}-\mathcal{J}V_1(t),\quad \mathrm{Re}\, t\geq 0, \tag{1.2}$$

if $T=\mathcal{J}H_1-H_0\mathcal{J}$. We want to investigate some compactness properties of the operator $\mathcal{D}(t_0)T$. In the present our results will be based on Proposition 2.1. below. Instead of Cauchy semigroups that were crucial in [22], here we use an identically distributed, independent sequence of random variables which follow a logistic law: see Proposition 2.2. In §3 some Hilbert-Schmidt and trace properties are given for operators closely related to differences of Feynman-Kac semigroups. In §4 part of this is applied to Feynman-Kac semigroups in order to establish the result in the abstract.

2. Some operator identities

Let $\mathcal{B}_1(\mathcal{H}_1,\mathcal{H}_0)$ denote the ideal of trace class operators, let $\mathcal{B}_2(\mathcal{H}_1,\mathcal{H}_0))$ denote the ideal of Hilbert-Schmidt operators, let $\mathcal{B}_\infty(\mathcal{H}_1,\mathcal{H}_0)$ denote the ideal of compact operators, and let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$ denote the space of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_0$. If there is no ambiguity we write $\mathcal{B}_1$ for $\mathcal{B}_1(\mathcal{H}_1, \mathcal{H}_0)$, and similarly for the other spaces. As in the introduction let H_0 and H_1 be self-adjoint operators in $\mathcal{H}_0$ respectively $\mathcal{H}_1$. In the sequel we employ the following notation:

$$V_0(t) = \exp(-tH_0) \quad \text{and} \quad V_1(t) = \exp(-tH_1); \tag{2.1}$$

$$\mathcal{D}(t)T = \int_0^t V_0(u)TV_1(t-u)du; \tag{2.2}$$

$$\mathcal{F}(\sigma, t)\, T = V_0(i\sigma)\mathcal{D}(t)TV_1(-i\sigma); \tag{2.3}$$

$$\mathcal{G}(\sigma, t)\, T = tV_0(i\sigma)V_0(t/2)TV_1(t/2)V_1(-i\sigma). \tag{2.4}$$

We also use a sequence of independent identically distributed real valued random variables $(U_j : j \in N)$, where each variable U_j is *logistically* distributed:

$$\begin{aligned} P\left[U_j \in B\right] &= \frac{\pi}{2}\int_B \frac{1}{(\cosh \pi\tau)^2} d\tau \\ &= \pi^2 \int_0^\infty \frac{\sinh \pi\tau}{(\cosh \pi\tau)^3} \int_{-\tau}^{\tau} 1_B(\sigma) d\sigma d\tau. \end{aligned} \tag{2.5}$$

Let $(\Omega', \mathcal{F}', P')$ be the probability space corresponding to the sequence $(U_1, U_2, \dots)$. For more information on logistic random variables see e.g. Evans et al [13, §24]. Many of the equalities in the Proposition 2.2. will be a consequence of the identity in the following proposition.

2.1. Proposition. *The following identity is true:*

$$\begin{aligned} & t_0 V_0(t_0/2)TV_1(t_0/2) \\ &= \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{1}{(\cosh \pi\tau)^2} V_0(i\tau t_0) \int_0^{t_0} V_0(u)TV_1(t_0-u)du V_1(-i\tau t_0) d\tau \\ &= E'\left[\mathcal{F}(\mathcal{U}_1 t_0, t_0)T\right]. \end{aligned} \tag{2.6}$$

Remark. Define the operator valued *Cauchy semigroup* $\{\exp(-t\mathcal{A}) : t \geq 0\}$ on $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_0)$ via the formula

$$\exp(-t\mathcal{A})(T) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\tau \frac{t}{\tau^2 + t^2} V_0(i\tau)TV_1(-i\tau), \quad t > 0. \tag{2.7}$$

In [22] the central identity was

$$\exp\left(-\frac{t}{2}\mathcal{A}\right)\mathcal{D}(t)(T) = \int_0^t du \exp(-u\mathcal{A})\, V_0(t/2)TV_1(t/2). \tag{2.8}$$

The basic role of equality (2.8) in [22] is taken over by equality (2.6) in the present paper. It is quite well possible, that with the semigroup in (2.7) there can be associated a quantum diffusion.

Proof. The proof will depend on the Poisson formula in the upper half-plane. From the definitions we get

$$
\begin{aligned}
&\frac{\pi}{2}\int_{-\infty}^{\infty}\frac{1}{(\cosh\pi\tau)^2}\mathcal{F}(\tau t_0,t_0)T d\tau\\
&= 2\pi\int_{-\infty}^{\infty}\frac{1}{\exp(2\pi\tau)+2+\exp(-2\pi\tau)}\\
&\qquad V_0(i\tau t_0)\int_0^{t_0}V_0(u)TV_1(t_0-u)duV_1(-i\tau t_0)d\tau\\
&= 2\pi t_0\int_0^1\int_{-\infty}^{\infty}\frac{1}{\exp(2\pi\tau)+2+\exp(-2\pi\tau)}\\
&\qquad V_0(i(\tau-iu)t_0)TV_1(t_0-i(\tau-iu)t_0)\,d\tau\,du
\end{aligned}
$$

(Γ_u is the straight line $\Gamma_u(\tau)=\tau-iu$)

$$
\begin{aligned}
&= 2\pi t_0\int_0^1\int_{\Gamma_u}\frac{1}{\exp(2\pi(\zeta+iu))+2+\exp(-2\pi(\zeta+iu))}\\
&\qquad V_0(i\zeta t_0)TV_1(t_0-i\zeta t_0)\,d\zeta\,du
\end{aligned}
$$

(Cauchy's theorem: $\zeta=-\frac{1}{2}i+\tau$)

$$
\begin{aligned}
&= 2\pi t_0\int_0^1\int_{\infty}^{\infty}\frac{1}{-\exp(2\pi(\tau+iu))+2-\exp(-2\pi(\tau+iu))}\\
&\qquad V_0(i\tau t_0)V_0(t_0/2)TV_1(t_0/2)V_1(-i\tau t_0)\,d\tau\,du\\
&= \frac{t_0}{2i}\int_0^1\frac{\partial}{\partial u}\int_{-\infty}^{\infty}\coth(\pi(\tau+iu))\\
&\qquad V_0(i\tau t_0)V_0(t_0/2)TV_1(t_0/2)V_1(-i\tau t_0)\,d\tau\,du\\
&= \lim_{\varepsilon\downarrow 0}\frac{t_0}{2i}\int_{-\infty}^{\infty}(\coth(\pi(\tau+i(1-\varepsilon)))-\coth(\pi(\tau+i\varepsilon)))\\
&\qquad V_0(i\tau t_0)V_0(t_0/2)TV_1(t_0/2)V_1(-i\tau t_0)\,d\tau\\
&= \lim_{\varepsilon\downarrow 0}t_0\int_{-\infty}^{\infty}\operatorname{Im}\frac{\exp(\pi(\tau-i\varepsilon))+\exp(-\pi(\tau-i\varepsilon))}{\exp(\pi(\tau-i\varepsilon))-\exp(-\pi(\tau-i\varepsilon))}\\
&\qquad V_0(i\tau t_0)V_0(t_0/2)TV_1(t_0/2)V_1(-i\tau t_0)\,d\tau\\
&= \lim_{\varepsilon\downarrow 0}2t_0\sin(2\pi\varepsilon)\int_{-\infty}^{\infty}\frac{1}{(\exp(\pi\tau)-\exp(-\pi\tau))^2+(2\sin(\pi\tau\varepsilon))^2}\\
&\qquad V_0(i\tau t_0)V_0(t_0/2)TV_1(t_0/2)V_1(-i\tau t_0)\,d\tau. \qquad (2.9)
\end{aligned}
$$

Next we make the substitution

$$\tau=\frac{1}{\pi}\log\left(\xi+\sqrt{\xi^2+1}\right),\qquad \text{or equivalently}\quad 2\xi=\exp(\pi\tau)-\exp(-\pi\tau).$$

We also write

$$U(\xi) = V_0\left(\frac{it_0}{\pi}\log\left(\xi+\sqrt{\xi^2+1}\right)\right)V_0(t_0/2)\,TV_1(t_0/2)\,V_1\left(-\frac{it_0}{\pi}\log\left(\xi+\sqrt{\xi^2+1}\right)\right).$$

So from (2.9) we obtain

$$\begin{aligned}&\frac{\pi}{2}\int_{-\infty}^{\infty}\frac{1}{(\cosh\pi\tau)^2}\mathcal{F}(\tau t_0,t_0)\,Td\tau\\&=\lim_{\varepsilon\downarrow 0}t_0\cos(\pi\varepsilon)\frac{\sin(\pi\varepsilon)}{\pi}\int_{-\infty}^{\infty}\frac{1}{\xi^2+(\sin(\pi\varepsilon))^2}U(\xi)\frac{d\xi}{\sqrt{1+\xi^2}}\\&=t_0U(0)=t_0V_0(t_0/2)\,TV_1(t_0/2).\end{aligned}$$

This proves the equality in (2.6).

In the following proposition we collect a number of relevant identities.

2.2. Proposition. (a) *Let $V_0(t)$ and $V_1(t)$ be as above and also let the other notation be as above. Let $T:\mathcal{H}_0\to\mathcal{H}_1$ be a continuous linear operator. The function f is a function in $L^1(R)$ and μ is a Borel measure of bounded variation on the real line. The following identities are true:*

$$\mathcal{G}(\sigma t_0,t_0)\,T=E'\left[\mathcal{F}(\sigma+U_1)\,T\right];\tag{2.10}$$

$$\begin{aligned}&\int_{-\infty}^{\infty}E'f\left(\sigma-(U_1+\cdots+U_j)\right)\mathcal{G}(\sigma t_0,t_0)\,Td\sigma\\&=\int_{-\infty}^{\infty}E'f\left(\sigma-(U_1+\cdots+U_{j+1})\right)\mathcal{F}(\sigma t_0,t_0)\,Td\sigma.\end{aligned}\tag{2.11}$$

In addition the following equality is true:

$$\begin{aligned}&\int f(\sigma)\mathcal{F}(\sigma t_0,t_0)\,Td\sigma\\&=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\int_{-\infty}^{\infty}E'f\left(\sigma-(U_1+\cdots+U_j)\right)\mathcal{G}(\sigma t_0,t_0)\,Td\sigma &(2.12)\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\widehat{f}(\xi)\frac{\sinh(\xi/2)}{\xi/2}e^{i\sigma\xi}d\xi\right)\mathcal{G}(\sigma t_0,t_0)\,Td\sigma &(2.13)\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\widehat{f}(\xi)\exp\left(\frac{1}{2}|\xi|\right)e^{i\tau\xi}d\xi\right)\\&\quad\times\frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left(1+\frac{1}{(\sigma-\tau)^2}\right)\mathcal{G}(\sigma t_0,t_0)\,Td\sigma\,d\tau. &(2.14)\end{aligned}$$

Some other equalities are

$$\int f(\sigma)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma - \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^j \int_{-\infty}^{\infty} E' f(\sigma-(U_1+\cdots+U_j))\,\mathcal{G}(\sigma t_0, t_0)\,T d\sigma$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\frac{\xi/2}{\sinh(\xi/2)}\right)^{n+1}\widehat{f}(\xi)e^{i\sigma\xi}d\xi\,\mathcal{F}(\sigma t_0, t_0)\,T d\sigma; \tag{2.15}$$

$$\int_{-\infty}^{\infty} f(\sigma)\mathcal{G}(\sigma t_0, t_0)\,T d\sigma$$

$$= \pi^2\int_0^{\infty}\frac{\sinh \pi\tau}{(\cosh \pi\tau)^3}\int_{-\tau}^{\tau}\int_{-\infty}^{\infty} f(\sigma)\mathcal{F}((u+\sigma)t_0, t_0)\,T d\sigma\, du\, d\tau; \tag{2.16}$$

$$\int_{-\infty}^{\infty}\mathcal{G}(\sigma t_0, t_0)\,T d\mu(\sigma)$$

$$= \pi^2\int_0^{\infty}\frac{\sinh \pi\tau}{(\cosh \pi\tau)^3}\int_{-\tau}^{\tau}\int_{-\infty}^{\infty}\mathcal{F}((u+\sigma)t_0, t_0)\,T d\mu(\sigma)\, du\, d\tau; \tag{2.17}$$

(b) *If the function f is n times continuously differentiable, then the following identity is valid as well:*

$$\int f(\sigma)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma$$

$$-\sum_{k=0}^{n}(-1)^k E' U_1\ldots U_k\int_0^1 ds_1\ldots\int_0^1 ds_k\int_{-\infty}^{\infty} f^{(k)}(\sigma-(s_1U_1+\cdots+s_kU_k))\,\mathcal{G}(\sigma t_0, t_0)\,T d\sigma$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\frac{\xi/2}{\sinh(\xi/2)}\right)^{n+1}\widehat{f}(\xi)e^{i\sigma\xi}d\xi\,\mathcal{F}(\sigma t_0, t_0)\,T d\sigma. \tag{2.18}$$

(c) *In addition the following identity is true ($N \in N$):*

$$\sum_{k=0}^{N}\sum_{j=0}^{k}\binom{k}{j}(-1)^j E'\left[\mathcal{G}((U_1+\cdots+U_j)t_0, t_0)\,T\right]$$

$$= \mathcal{F}(0, t_0)\,T - \sum_{j=0}^{N+1}\binom{N+1}{j}(-1)^j E'\left[\mathcal{F}((U_1+\cdots+U_j)t_0, t_0)\,T\right]. \tag{2.19}$$

(d) *Let $u(\tau, s)$ be a harmonic function on $R\times(-1/2, 1/2)$ with the property that the following expression*

$$\limsup_{\epsilon\downarrow 0}\int_{-\infty}^{\infty}\left|\int_{-1/2+\epsilon}^{1/2-\epsilon} u(\tau, s)ds\right| d\tau$$

is finite. Then the following identity is true as well:

$$\int_{-\infty}^{\infty} u(\tau, 0)\mathcal{F}(\tau t_0, t_0))\,T d\tau = \int_{-1/2}^{1/2}\int_{-\infty}^{\infty} u(\tau, s)\mathcal{G}(\tau t_0, t_0)\,T d\tau\, ds. \tag{2.20}$$

Proof. (a) The equality in (2.10) coincides with the one in (2.6) for $\sigma = 0$. The one in (2.10) then follows from the definitions of $\mathcal{G}(\sigma, t)$ and $\mathcal{F}(\sigma, t)$. Since the random variable U_{j+1} does not depend on the variables $U_1, \ldots, U_j$, the equality in (2.11) is a consequence of Fubini's theorem, of equality (2.11), and of the fact that the P'-distribution of U_{j+1} coincides with that of U_1. The equality of the expressions in (2.12) and in (2.13) follows from the following argument:

$$\begin{aligned}
&E' f\left(\sigma - (U_1 + \cdots + U_j)\right) \\
&= \frac{1}{2\pi} E' \int_{-\infty}^{\infty} \widehat{f}(\xi) \exp\left(i\left(\sigma - (U_1 + \cdots + U_j)\right)\xi\right) d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\sigma\xi} E' \exp\left(-i\left(U_1 + \cdots + U_j\right)\xi\right) d\xi
\end{aligned}$$

(the variables $U_1, \ldots, U_j$ are mutually independent and identically distributed)

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\sigma\xi} \left(E' \left[\exp\left(-iU_1\xi\right)\right]\right)^j d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\sigma\xi} \left(\frac{\xi/2}{\sinh(\xi/2)}\right)^j d\xi. \qquad (2.21)
\end{aligned}$$

Here we used the fact that the Fourier transform of the P'-distribution of the variable U_1 is given by the function $\xi \mapsto \dfrac{\xi/2}{\sinh(\xi/2)}$. Equality (2.21) together with a limiting argument will show the equality of the expressions in (2.12) and in (2.13). Next we proceed with proving the equality of (2.13) and (2.14). From the identity

$$\frac{u}{\pi} \int_{-\infty}^{\infty} \frac{1}{u^2 + \tau^2} e^{-i\tau\xi} d\tau = \exp\left(-u|\xi|\right), \quad u > 0,$$

we infer

$$\begin{aligned}
&\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \widehat{f}(\xi) \exp\left(\frac{1}{2}|\xi|\right) e^{i\tau\xi} d\xi\right) \frac{1}{2\pi} \int_{-\infty}^{\infty} \log\left(1 + \frac{1}{(\sigma - \tau)^2}\right) \mathcal{G}(\sigma t_0, t_0) T d\sigma\, d\tau \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \widehat{f}(\xi) \exp\left(\frac{1}{2}|\xi|\right) e^{i\tau\xi} d\xi\right) \frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^1 \frac{u}{u^2 + (\sigma - \tau)^2} du \mathcal{G}(\sigma t_0, t_0) T d\sigma\, d\tau \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f}(\xi) \exp\left(\frac{1}{2}|\xi|\right) e^{i\sigma\xi} d\xi \int_{-\infty}^{\infty} \int_0^1 \exp\left(-u|\xi|\right) du \mathcal{G}(\sigma t_0, t_0) T d\sigma \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \widehat{f}(\xi) \frac{\sinh(\xi/2)}{\xi/2} e^{i\sigma\xi} d\xi\right) \mathcal{G}(\sigma t_0, t_0) T d\sigma \\
&= \int_{-\infty}^{\infty} f(\sigma) \mathcal{F}(\sigma t_0, t_0) T d\sigma.
\end{aligned}$$

This shows that the expressions in (2.13) and in (2.14) are equal. The first equality in (2.12) is a consequence of the identity in (2.15). So we only need to prove the identity in (2.15). This may be achieved as follows:

$$\int f(\sigma)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma - \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^j \int_{-\infty}^{\infty} E' f\left(\sigma - (U_1 + \cdots + U_j)\right)\mathcal{G}(\sigma t_0, t_0)\,T d\sigma$$

(see equality (2.11))

$$= \int f(\sigma)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma$$
$$- \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}(-1)^j \int_{-\infty}^{\infty} E' f\left(\sigma - (U_1 + \cdots + U_{j+1})\right)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma$$
$$= \int f(\sigma)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma$$
$$- \sum_{k=0}^{n}\sum_{j=0}^{k}\left(\binom{k+1}{j+1} - \binom{k}{j+1}\right)(-1)^j \int_{-\infty}^{\infty} E' f\left(\sigma - (U_1 + \cdots + U_{j+1})\right)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma$$
$$= \int f(\sigma)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma$$
$$+ \sum_{k=0}^{n}\sum_{j=1}^{k+1}\binom{k+1}{j}(-1)^j \int_{-\infty}^{\infty} E' f\left(\sigma - (U_1 + \cdots + U_j)\right)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma$$
$$- \sum_{k=1}^{n}\sum_{j=1}^{k}\binom{k}{j}(-1)^j \int_{-\infty}^{\infty} E' f\left(\sigma - (U_1 + \cdots + U_{j+1})\right)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma$$
$$= \int f(\sigma)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma$$
$$+ \sum_{k=1}^{n+1}\sum_{j=1}^{k}\binom{k}{j}(-1)^j \int_{-\infty}^{\infty} E' f\left(\sigma - (U_1 + \cdots + U_j)\right)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma$$
$$- \sum_{k=1}^{n}\sum_{j=1}^{k}\binom{k}{j}(-1)^j \int_{-\infty}^{\infty} E' f\left(\sigma - (U_1 + \cdots + U_{j+1})\right)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma$$
$$= \sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j \int_{-\infty}^{\infty} E' f\left(\sigma - (U_1 + \cdots + U_j)\right)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1 - \frac{\xi/2}{\sinh(\xi/2)}\right)^{n+1} \widehat{f}(\xi) e^{i\sigma\xi}\, d\xi\, \mathcal{F}(\sigma t_0, t_0)\,T d\sigma. \tag{2.22}$$

Notice that (2.22) is the same as (2.15). The equalities in (2.16) and in (2.17) follow from the equality in (2.11) together with translation invariance of one-dimensional Lebesgue measure and Fubini's theorem.

(b) The identity

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j f(\sigma-(U_1+\cdots+U_j))$$
$$=(-1)^k U_1\ldots U_k\int_0^1 ds_1\ldots\int_0^1 ds_k f^{(k)}(\sigma-(s_1U_1+\cdots+s_kU_k))$$

implies the equality in (2.18).

(c) The equality in (2.19) is proved in the same manner as the equality in (2.15).

(d) First we prove the equality in (2.20) for harmonic function $u(\tau,s)$ of the form:

$$u(\tau,s)=\frac{\pi}{2}\int_{-\infty}^{\infty}\mathrm{Re}\,\frac{1}{(\cosh\pi(\tau-\sigma+is))^2}f(\sigma)d\sigma),$$

where f is a function in $L^1(R)$. For such a function $u(\tau,s)$ we get, for $0<\varepsilon<1/2$,

$$\int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon}u(\tau,s)ds$$
$$=\frac{\pi}{2}\int_{-\infty}^{\infty}\mathrm{Re}\int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon}\frac{1}{(\cosh\pi(\tau-\sigma+is))^2}ds f(\sigma)d\sigma$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{Re}\left(\frac{1}{i}\tanh\pi(\tau-\sigma+is)\,\Big|_{s=-\frac{1}{2}+\varepsilon}^{s=\frac{1}{2}-\varepsilon}\right)f(\sigma)d\sigma$$
$$=\int_{-\infty}^{\infty}\mathrm{Re}\left(\frac{1}{i}\frac{-1}{\exp(2\pi(\tau-\sigma+is))+1}\right)\Big|_{s=-\frac{1}{2}+\varepsilon}^{s=\frac{1}{2}-\varepsilon}f(\sigma)d\sigma$$
$$=\int_{-\infty}^{\infty}\frac{4\cos\pi\varepsilon\sin\pi\varepsilon}{(\exp(\pi\sigma)-\exp(-\pi\sigma))^2+4\sin^2\pi\varepsilon}f(\tau-\sigma)d\sigma$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\pi\varepsilon}{1+\xi^2}f\left(\tau-\frac{1}{\pi}\log\left(\xi\sin\pi\varepsilon+\sqrt{1+\xi^2\sin^2\pi\varepsilon}\right)\right)\frac{d\xi}{\sqrt{1+\xi^2\sin^2\pi\varepsilon}}.$$

So that

$$\lim_{\varepsilon\downarrow 0}\int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon}u(\tau,s)ds=f(\tau).$$

On the other hand we also see

$$\int_{-\infty}^{\infty}u(\tau,0)\mathcal{F}(\tau t_0,t_0)Td\tau=\frac{\pi}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(\cosh\pi(\tau-\sigma))^2}f(\sigma)d\sigma\mathcal{F}(\tau t_0,t_0)Td\tau$$
$$=\int_{-\infty}^{\infty}\frac{\pi}{2}\int_{-\infty}^{\infty}\frac{1}{(\cosh\pi(\tau-\sigma))^2}\mathcal{F}(\tau t_0,t_0)Td\tau f(\sigma)d\sigma$$

(equality (2.10))

$$=\int_{-\infty}^{\infty}f(\sigma)\mathcal{G}(\sigma t_0,t_0)Td\sigma.$$

This shows the equality in (2.20) for harmonic functions with the indicated properties. In order to complete the proof we shall also prove assertion (d) for a harmonic function $u(\tau, s)$ of the form

$$u(\tau, s) = \int_{-\infty}^{\infty} \frac{\cos \pi s}{\cosh \pi(\tau - \sigma) - \sin \pi s} f(\sigma) d\sigma.$$

Therefore we need the following identities:

$$\int_{-1/2}^{1/2} \frac{\cos \pi s}{\cosh \pi(\tau - \sigma) - \sin \pi s} ds = \int_{-1/2}^{1/2} \frac{1}{\cosh \pi(\tau + is - \sigma)} ds$$
$$= \frac{1}{\pi} \log \frac{\cosh \pi(\tau - \sigma) + 1}{\cosh \pi(\tau - \sigma) - 1}.$$

Then we get

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(\tau - \sigma, 0) f(\sigma) d\sigma \mathcal{F}(\tau t_0, t_0) T d\tau$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\cosh \pi(\tau - \sigma)} f(\sigma) d\sigma \mathcal{F}(\tau t_0, t_0) T d\tau$$

(Cauchy's theorem, see the argument for the proof of equality (3.6))

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-1/2}^{1/2} \frac{1}{\cosh \pi(\tau + is - \sigma)} ds f(\sigma) d\sigma \mathcal{G}(\tau t_0, t_0) T d\tau$$
$$= \int_{-\infty}^{\infty} \int_{-1/2}^{1/2} u(\tau, s) ds \mathcal{F}(\tau t_0, t_0) T d\tau.$$

Since the same argument can be used for the function $u(\tau, -s)$, this suffices for a proof via a density argument and the fact that solutions for the Dirichlet problem on the strip $R \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ are of the form

$$u(\tau, s) = \int_{-\infty}^{\infty} \frac{\cos \pi s}{\cosh \pi(\tau - \sigma) - \sin \pi s} f_1(\sigma) d\sigma$$
$$+ \int_{-\infty}^{\infty} \frac{\cos \pi s}{\cosh \pi(\tau - \sigma) + \sin \pi s} f_2(\sigma) d\sigma$$

3. Some inequalities.

Let the notation be as in §2. In this section we consider a sub-additive homogeneous function (a semi-norm) $p : \mathcal{B}_{\text{fin}}(\mathcal{H}_1, \mathcal{H}_0) \to [0, \infty)$, where $\mathcal{B}_{\text{fin}}(\mathcal{H}_1, \mathcal{H}_0)$ denotes the space of all finite rank operators from between the Hilbert spaces $\mathcal{H}_1$

and $\mathcal{H}_0$. We suppose that $p(V_0(i\tau)TV_1(-i\tau)) = p(T)$ for all $\tau \in R$. We also introduce the following convex sets of harmonic functions on the strip $R \times \left(-\frac{1}{2}, \frac{1}{2}\right)$:

$$\begin{aligned} &\mathrm{harm}_1 \\ &= \left\{u : R \times \left(-\frac{1}{2}, \frac{1}{2}\right) \to R : u \text{ harmonic and } \limsup_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} \left| \int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon} u(\tau, s) ds \right| d\tau \leq 1 \right\}; \end{aligned} \tag{3.1}$$

$$\mathrm{harm}_1^+ = \left\{u \in \mathrm{harm}_1 : \int_{-a}^{a} u(\tau, s) ds \geq 0 \text{ for all } 0 \leq a < \frac{1}{2}\right\}. \tag{3.2}$$

3.1. Proposition. *(a) The following equalities are valid:*

$$\begin{aligned} &t_0 p(V_0(t_0/2)TV_1(t_0/2)) \\ &= \sup\left\{p\left(\int_{-\infty}^{\infty} u(\tau, 0) V_0(i\tau t_0)\mathcal{D}(t_0)TV_1(-i\tau t_0)\right) : u \in \mathrm{harm}_1\right\} \qquad (3.3) \\ &= \sup\left\{p\left(\int_{-\infty}^{\infty} u(\tau, 0) V_0(i\tau t_0)\mathcal{D}(t_0)TV_1(-i\tau t_0)\right) : u \in \mathrm{harm}_1^+\right\}. \qquad (3.4) \end{aligned}$$

(b) Let $f : R \to R$ be any C^∞-function with the property that the integrals $\int_{-\infty}^{\infty} \left|f^{(k)}(\sigma)\right| d\sigma$ are finite for all $k \in N$. The following inequality is valid:

$$\begin{aligned} &p\left(\int_{-\infty}^{\infty} f(\tau) V_0(i\tau t_0)\mathcal{D}(t_0)TV_1(-i\tau t_0) d\tau\right) \\ &\leq \sum_{k=0}^{\infty} \left(\frac{\log 2}{\pi}\right)^k \int_{-\infty}^{\infty} \left|f^{(k)}(\tau)\right| d\tau\, t_0 p(V_0(t_0/2)TV_1(t_0/2)). \end{aligned} \tag{3.5}$$

(c) Next let μ be a real (or complex) Borel measure on R. The following inequality is valid:

$$p\left(\int_{-\infty}^{\infty} \mathcal{G}(\sigma t_0, t_0) T d\mu(\sigma)\right) \leq p\left(\int_{-\infty}^{\infty} \mathcal{F}(\sigma t_0, t_0) T d\mu(\sigma)\right). \tag{3.6}$$

(d) In addition the inequalities

$$p(\mathcal{G}(0, t_0)T) \leq \sup_{\tau > 0} p\left(\frac{1}{2\tau} \int_{-\tau}^{\tau} \mathcal{F}(u t_0, t_0) T du\right) \leq p(\mathcal{F}(0, t_0)T) \tag{3.7}$$

hold true.

(e) Let $f \in L^1(R)$ be a function with the property that the function

$$\sigma \mapsto \int_{-\infty}^{\infty} \widehat{f}(\xi) \frac{\sinh(\xi/2)}{\xi/2} e^{i\xi\sigma} d\xi$$

belongs to $L^1(R)$. Then the following inequalities are true:

$$\begin{aligned} &p\left(\int_{-\infty}^{\infty} f(\sigma)\mathcal{F}(\sigma t_0, t_0)\,T d\sigma\right) \\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\widehat{f}(\xi)\frac{\sinh(\xi/2)}{\xi/2}e^{i\sigma\xi}d\xi\right| d\sigma p(\mathcal{G}(0,t_0)\,T) &(3.8)\\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\widehat{f}(\xi)\exp\left(\frac{1}{2}|\xi|\right)e^{i\sigma\xi}d\xi\right| d\sigma p(\mathcal{G}(0,t_0)T). &(3.9)\end{aligned}$$

Proof. (a) From equality (2.20) in assertion (d) of Proposition 2.2. we obtain the inequality

$$\begin{aligned} &p\left(\int_{-\infty}^{\infty} u(\tau,0)\mathcal{F}(\tau t_0,t_0))\,T d\tau\right) = \lim_{\varepsilon\downarrow 0} p\left(\int_{-1/2+\varepsilon}^{1/2-\varepsilon}\int_{-\infty}^{\infty} u(\tau,s)\mathcal{G}(\tau t_0,t_0)\,T d\tau\, ds\right)\\ &= \lim_{\varepsilon\downarrow 0} p\left(\int_{-\infty}^{\infty}\int_{-1/2+\varepsilon}^{1/2-\varepsilon} u(\tau,s)\mathcal{G}(\tau t_0,t_0)\,T ds\, d\tau\right)\\ &\le \lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\left|\int_{-1/2+\varepsilon}^{1/2-\varepsilon} u(\tau,s)ds\right| p(\mathcal{G}(\tau t_0,t_0)\,T)\,d\tau\\ &= \lim_{\varepsilon\downarrow 0}\int_{-\infty}^{\infty}\left|\int_{-1/2+\varepsilon}^{1/2-\varepsilon} u(\tau,s)ds\right| p(\mathcal{G}(0,t_0)\,T)\,d\tau. &(3.10)\end{aligned}$$

From (3.10) it follows that the quantity $p(\mathcal{G}(0,t_0)T)$ dominates the expression in (3.3). In turn, since $\text{harm}_+ \subseteq \text{harm}_1$, the expression in (3.4) is trivially dominated by the one in (3.3). Upon inserting the function

$$u(\tau,s) = \operatorname{Re}\frac{\pi}{2}\frac{1}{(\cosh\pi(\tau+is))^2}$$

together with employing the following facts:

$$\begin{aligned} &\int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon} u(\tau,s)ds = \frac{\pi}{2}\operatorname{Re}\int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon}\frac{1}{(\cosh\pi(\tau+is))^2}ds\\ &= \frac{4\cos\pi\varepsilon\sin\pi\varepsilon}{(\exp(\pi(\tau-\sigma))-\exp(-\pi(\tau-\sigma)))^2+4\sin^2\pi\varepsilon}\\ &\ge 0 \quad\text{and}\end{aligned}$$

$$\int_{-\infty}^{\infty}\int_{-\frac{1}{2}+\varepsilon}^{\frac{1}{2}-\varepsilon} u(\tau,s)ds\,d\tau = 1-2\varepsilon \le 1$$

in conjunction with Proposition 2.1. yields the inequality

$$p(\mathcal{G}(0,t_0)\,T) \le \sup\left\{p\left(\int_{-\infty}^{\infty} u(\tau,0)V_0(i\tau t_0)\mathcal{D}(t_0)TV_1(-i\tau t_0)\right) : u \in \text{harm}_1^+\right\}.$$

Altogether this shows the equalities in (3.3) and (3.4).

(b) Since $E'|U_1| = \frac{\log 2}{\pi}$ the inequality in (3.5) is a consequence of assertion (b) in Proposition 2.2.

(c) The inequality in (3.6) is a consequence of the equality in (2.17): notice the equality

$$\pi^2 \int_{-\infty}^{\infty} \frac{2\tau \sinh \pi\tau}{(\cosh \pi\tau)^3} d\tau = 1. \tag{3.11}$$

(d) Let μ be the Dirac measure at 0. An appeal to (2.17) together with (3.11) yields the first inequality in (3.7). The second inequality follows from the subadditivity, positive homogeneity, and continuity of the function p together with the standard hypothesis that $p(V_0(i\tau)TV_1(-i\tau)) = p(T)$ for all $\tau \in R$.

(e) The inequalities in (3.8) and (3.9) follow from the corresponding equalities in (2.13) and (2.14) respectively. We also notice the identities (see the proof of equality (2.13) and (2.14) in Proposition 2.1.):

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)\frac{\sinh(\xi/2)}{\xi/2}e^{i\sigma\xi}d\xi$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\widehat{f}(\xi)\exp\left(\frac{1}{2}|\xi|\right)e^{i\tau\xi}d\xi\right)\frac{1}{2\pi}\log\left(1+\frac{1}{(\sigma-\tau)^2}\right)d\tau, \quad \text{and}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left(1+\frac{1}{(\sigma-\tau)^2}\right)d\sigma = 1.$$

3.2. Corollary. *(a) Let T be linear operator with the property that, for every $\tau > 0$, the operator $\int_{-\tau}^{\tau} \mathcal{F}(ut_0, t_0)T du$ is compact, Hilbert-Schmidt, or of trace class respectively. Then the operator $\mathcal{G}(0, t_0)T$ possesses the corresponding property as well.*

(b) Let f be a C^∞-function with the property that the expression

$$\sum_{k=0}^{\infty}\left(\frac{\log 2}{\pi}\right)^k \int_{-\infty}^{\infty}\left|f^{(k)}(\tau)\right| d\tau$$

is finite or let $f(\tau) = u(\tau, 0)$, where $u(\tau, s)$ is a harmonic function belonging to harm$_1$. Let T be an operator for which the operator $V_0(t_0/2)TV_1(t_0/2)$ is a Hilbert-Schmidt operator (or a trace class operator, or a compact operator), then this property is shared with the operator $\int_{-\infty}^{\infty} f(\sigma)V_0(i\sigma t_0)\mathcal{D}(t_0)TV_1(-i\tau\sigma t_0)d\sigma$. The same conclusion is true if f is a function belonging to $L^1(R)$ which possesses the property that the integral

$$\sigma \mapsto \int_{-\infty}^{\infty}\widehat{f}(\xi)\frac{\sinh(\xi/2)}{\xi/2}e^{i\sigma\xi}d\xi$$

is a member of $L^1(R)$ as well.

(c) Let $T : \mathcal{H}_1 \to \mathcal{H}_0$ be an operator with the property that for every $f \in L^1(R)$ there exists a function $g \in L^1(R)$ such that

$$\int_{-\infty}^{\infty} f(\sigma)\mathcal{F}(\sigma t_0, t_0)\, T d\sigma = \int_{-\infty}^{\infty} g(\sigma)\mathcal{G}(\sigma t_0, t_0)\, T d\sigma.$$

Suppose that the operator $V_0(t_0/2)TV_1(t_0/2)$ is compact, or Hilbert-Schmidt, or trace class. Then so are operators $\int_{-\infty}^{\infty} f(\sigma)\mathcal{F}(\sigma t_0, t_0)\, T d\sigma$, for f belonging to $L^1(R)$.

Remark 1. Let $H_0 = \int \xi E_0(d\xi)$ and $H_1 = \int \eta E_1(d\eta)$ be the spectral decompositions of H_0 and H_1 respectively. Suppose that there exist bounded intervals A_0 and A_1 in R with the property that $E_0(A_0)TE_1(A_1) = T$. Then the hypothesis in (c) is satisfied. in fact let $u : R \to R$ be a compactly supported C^{∞}-function with the property that

$$1_{A_0 - A_1}(\xi) \le u(t_0\xi/2) \le 1.$$

Then

$$\int_{\infty}^{\infty} f(\sigma)\mathcal{F}(\sigma t_0, t_0)\, T d\sigma \int_{\infty}^{\infty} g(\sigma)\mathcal{G}(\sigma t_0, t_0)\, T d\sigma,$$

where the function g is chosen in such a way that

$$\widehat{g}(\xi) = \widehat{f}(\xi)\frac{\sinh(\xi/2)}{\xi/2}u(\xi).$$

This can be shown by noticing that, for $f \in L^1(R)$, the operator

$$\int_{-\infty}^{\infty} f(\sigma)\mathcal{F}(\sigma t_0, t_0)\, T d\sigma$$

equals the so-called *double Stieltjes operator integral*

$$\begin{aligned}
&\int\int \widehat{f}\left(\frac{1}{2}t_0(\xi - \eta)\right) \frac{\sinh \frac{1}{2}t_0(\xi - \eta)}{\frac{1}{2}t_0(\xi - \eta)} E_0(d\xi)\mathcal{G}(0, t_0)TE_1(d\eta) \\
&= \int\int \widehat{f}\left(\frac{1}{2}t_0(\xi - \eta)\right) \frac{\sinh \frac{1}{2}t_0(\xi - \eta)}{\frac{1}{2}t_0(\xi - \eta)} u\left(\frac{t_0}{2}(\xi - \eta)\right) E_0(d\xi)\mathcal{G}(0, t_0)TE_1(d\eta) \\
&= \int\int \widehat{g}\left(\frac{1}{2}t_0(\xi - \eta)\right) E_0(d\xi)\mathcal{G}(0, t_0)TE_1(d\eta) = \int_{-\infty}^{\infty} g(\sigma)\mathcal{G}(\sigma t_0, t_0)\, T d\sigma.
\end{aligned}$$

Remark 2. In [1, 2, 3, 4] Birman and Solomyak make a detailed study of operators of the form

$$T \mapsto \int\int \varphi(\xi, \eta)E_0(d\xi)TE_1(d\eta).$$

In [23, p. 225-228] Yafaev gives some information as well on these so-called *double Stieltjes operator integrals* and so do the authors of [16] on page 66. Related results and estimates can be found in Farforovskaya [14, 15] and in Peller [17].

Proof. Assertion (a) follows from the equality in (2.17) together with the inequality in (3.7) for an appropriate choice of p and of μ. The measure μ is supposed to be the Dirac measure at the origin. For p we either take the usual operator norm, the Hilbert-Schmidt, or the trace norm.

(b) This assertion is a consequence of inequality (3.5) together with the equality in (2.18) in case we are dealing with an appropriate C^{∞}-function and of (3.4) together with (2.20) in case we are dealing with a harmonic function in harm_1. In case the function f possesses the property that the inverse Fourier transform of the function $\xi \mapsto \widehat{f}(\xi)\dfrac{\sinh(\xi/2)}{\xi/2}$ belongs to $L^1(R)$ we employ inequality (3.8) in assertion (e) of Proposition 3.1.

(c) Since

$$\mathcal{G}(\sigma t_0, t_0)T = t_0 V_0(i\sigma t_0)V_0(t_0/2)TV_1(t_0/2)V_1(-i\sigma t_0),$$

this assertion is readily established.

3.3. Theorem. *Let $T : \mathcal{H}_1 \to \mathcal{H}_0$ be a linear operator. The following assertions are equivalent:*

(1) *The operator $\mathcal{D}(t_0)T$ is compact;*
(2) *The operator $\mathcal{D}(t_0)T$ is bounded, the operator $V_0(t_0/2)TV_1(t_0/2)$ is compact and*

$$\lim_{a\downarrow 0} \|V_0(a)\mathcal{D}(t_0)TV_1(a) - \mathcal{D}(t_0)T\| = 0. \tag{3.12}$$

Proof. (1) $\Rightarrow$ (2). A compact operator is automatically bounded. The compactness of the operator $V_0(t_0/2)TV_1(t_0/2)$ follows from equality (2.6) in Proposition 2.1. The equality in (3.12) also follows from the compactness of the operator $\mathcal{D}(t_0)T$ in conjunction with the strong continuity of the semigroups $\{V_0(a) : a \geq 0\}$ and $\{V_1(a) : a \geq 0\}$. In fact, assume that the equality in (3.12) were not true. Then there exists a sequence of elements $(x_n : n \in N)$ in $\mathcal{H}_1$ of norm one, which converges to x_0 weakly, together with a sequence of positive real numbers $(a_n : n \in N)$, which decreases to 0, such that for some $\varepsilon > 0$, the inequality

$$\|V_0(a_n)\mathcal{D}(t_0)TV_1(a_n)x_n - \mathcal{D}(t_0)Tx_n\| \geq \varepsilon$$

is true for all $n \in N$. It follows that x_0 is also the weak limit of the sequence $(V_1(a_n)x_n : n \in N)$. Since, by compactness, the equalities

$$\begin{aligned}\mathcal{D}(t_0)Tx_0 &= \lim_{n\to\infty} \mathcal{D}(t_0)Tx_n = \lim_{n\to\infty} \mathcal{D}(t_0)TV_1(a_n)x_n \\ &= \lim_{n\to\infty} V_0(a_n)\mathcal{D}(t_0)TV_1(a_n)x_n\end{aligned}$$

are valid in norm sense, this is a contradiction.

(2) $\Rightarrow$ (1). The compactness of the operator $V_0(t_0/2)TV_1(t_0/2)$ implies that of $V_0(a/2)\mathcal{D}(t_0)TV_1(a/2)$. This can be seen as follows. Put

$$\ell_a(\tau) = \frac{\pi}{2a}\frac{1}{\left(\cosh\left(\pi\tau/a\right)\right)^2}.$$

A Tauberian argument (see e.g. Theorem 3, page 357 of Yosida [24]) applies to the effect that the subspace $\left\{\ell_1 * \varphi : \varphi \in L^1(R)\right\}$ is dense in $L^1(R)$. So there exists a sequence of functions in $L^1(R)$ such that

$$\lim_{n\to\infty}\left\|\ell_{a/t_0} - \ell_1 * \varphi_{a,n}\right\|_1 = 0.$$

So we obtain

$$aV_0(a/2)\mathcal{D}(t_0)TV_1(a/2) = \int_{-\infty}^{\infty}\ell_1(\tau)\mathcal{F}(\tau a, a)\mathcal{D}(t_0)Td\tau$$

$$= \mathcal{D}(a)\left(\int_{-\infty}^{\infty}\ell_{a/t_0}(\tau)\mathcal{F}(\tau t_0, t_0)\,Td\tau\right)$$

(Tauberian theorem)

$$= \lim_{n\to\infty}\mathcal{D}(a)\left(\int_{-\infty}^{\infty}\ell_1 * \varphi_{a,n}(\tau)\mathcal{F}(\tau t_0, t_0)\,Td\tau\right)$$

(equality (2.10) in Proposition 2.2.)

$$= \lim_{n\to\infty}\mathcal{D}(a)\left(\int_{-\infty}^{\infty}\varphi_{a,n}(\tau)\mathcal{G}(\tau t_0, t_0)\,Td\tau\right). \tag{3.13}$$

The operators in (3.13) are compact, and hence we see that the operators $V_0(a)\mathcal{D}(t_0)TV_1(a)$, $a > 0$, are compact as well. So (1) follows from the equality in (3.12).

Remark. Let $\mathcal{J} : \mathcal{H}_1 \to \mathcal{J}_0$ be a continuous linear operator. Suppose that the operators H_0 $\mathcal{J}$-subordinate with respect to H_1. By definition this means that there exist locally bounded functions f_0 and f_1, defined on R, with the following properties:

(a) $|f_0(\lambda)| \geq 1$, $|f_1(\lambda)| \geq 1$, $\lim_{\lambda\to\infty}|f_0(\lambda)| = \lim_{\lambda\to\infty}|f_1(\lambda)| = \infty$;

(b) The operator $\mathcal{J}$ is a continuous linear operator from $D\left(f_1(H_1)\right)$ (with its graph-norm) to the space $D\left(f_0(H_0)\right)$ (with its graph-norm).

It will then follow that the operator $f_0(H_0)\mathcal{J}f_1(H_1)^{-1}$ is bounded from $\mathcal{H}_1$ to $\mathcal{H}_0$ and that condition (b) is satisfied: see e.g. Lemma 7, page 207 of Yafaev [23]. For the more details see e.g. Definition 6, page 207 of [23]. If, in addition, H_1 is $\mathcal{J}^*$-subordinate with respect to H_0, then the operators H_0 and H_1 are said to be mutually subordinate. In case $\mathcal{J} = I$, so that $\mathcal{H}_0 = \mathcal{H}_1$, then H_0 and H_1 are mutually subordinate with respect to each other if e.g. $D(H_1) = D(H_0)$ or if, for $\gamma > 0$ large enough, $D\left(\gamma I + H_0\right)^{1/2} = D\left(\gamma I + H_1\right)^{1/2}$. If $\mathcal{H}_1 = \mathcal{H}_0$ and if $\mathcal{J}$ is the identity operator we just say that H_0 is subordinate to H_1, provided they have the properties (a) and (b). Hamiltonians which are subordinate to each other play a central role in mathematical scattering theory: see e.g. Yafaev [23].

4. An application to Feynman-Kac semigroups.

In the present section we want to apply some of the results obtained in the previous section to Feynman-Kac or generalized Schrödinger semigroups. In this context we let E be a locally compact second countable Hausdorff space with a non-negative Radon measure m. The operator K_0 is supposed to generate a strongly continuous, positivity preserving, semigroup $\{\exp(-tK_0) : t \geq 0\}$ with the property that $0 \leq f \leq 1$ implies $0 \leq \exp(-tK_0) f \leq 1$. It is well-known that in the present circumstances there exists a strong Markov process

$$\{(\Omega, \mathcal{F}, P_x), (X(t) : t \geq 0), (X(t) : t \geq 0), (\vartheta_t : t \geq), (E, \mathcal{E})\}$$

with the property that $\exp(-tK_0) f(x) = E_x [f(X(t)]$. We may assume that for the paths $t \mapsto X(t)$ are P_x-almost surely right-continuous and have left limits in E on their *life time* $\zeta := \inf\{s > 0 : X(s) = \Delta\}$. Here Δ is an extra point, which we add to the state space, in order to make genuine probability measures on the space $E^{\Delta} := E \cup \Delta$ out of the sub-probabilities $B \mapsto \int_B p_0(t, x, y) dm(y)$. The proof of this result is quite technical. It combines the well-known theorem of Kolmogorov on projective systems of measures with a proof of the fact that the indicated path space has full measure, so that no information is lost. Proofs are omitted. They can be found in for example Blumenthal and Getoor [5, Theorem 9.4. p. 46]. Another good reference for Markov processes is the book by Ethier and Kurtz [12]. The Markov property means that for all $t > 0$ and for all x in the state space E the equality

$$E_x \left[Y \circ \vartheta_t \mid \mathcal{F}_t \right] = E_{X(t)} [Y], \quad P_x\text{-almost surely.}$$

Here $\mathcal{F}_t$ is the "information" from the past: $\mathcal{F}_t$ is the σ-field generated by the state variables $(X(s) : s \leq t)$. The relevant relation between the state variables $X(t)$, $t \geq 0$, and the translation or shift variables ϑ_s, $s \geq 0$, is the following identity: $X(t) \circ \vartheta_s = X(s + t)$, for all s and $t \geq 0$. We will also need the strong Markov property, where instead of "fixed" times t we consider adapted times, so-called *stopping* times S. Such a time S has the property that for every $s > 0$ the event $\{S \leq s\}$ belongs to the σ-field $\mathcal{F}_s$. Since the paths of our process may be chosen to be continuous from the right and to possess left limits in E on their life time, we may enlarge the σ-fields $\mathcal{F}_t$ in such a way that the filtration $\{\mathcal{F}_t : t \geq 0\}$ is right continuous and that for any Borel subset B of E the hitting time of B is a stopping time for this new filtration. These extended σ-fields $\overline{\mathcal{F}}_t$ in this enlarged filtration will again be denoted by $\mathcal{F}_t$. For more details see e.g. Chapter 2 in Blumenthal and Getoor [5]. The σ-field corresponding to the stopping time S is then defined by

$$\mathcal{F}_S := \bigcap_{t>0} \{A \in \mathcal{F} : A \cap \{S \leq t\} \in \mathcal{F}_t\}.$$

In the sequel we will need the notion of *Kato-Feller* potential. A function $V : E \to [-\infty,\infty]$ is called a Kato-Feller potential if its negative part V_- satisfies

$$\limsup_{t\downarrow 0}\left\|\int_0^t \exp(-\sigma K_0)\, V_- d\sigma\right\|_\infty = 0, \tag{4.1}$$

and if for every compact subset K its positive part V_+ satisfies

$$\limsup_{t\downarrow 0}\left\|\int_0^t \exp(-\sigma K_0)\,(1_K V_+)\, d\sigma\right\|_\infty = 0.$$

In fact for our purposes it will be sufficient to assume that V_+ is locally an L^1-function together with the fact that (4.1) is replaced with

$$\limsup_{t\downarrow 0}\left\|\int_0^t \exp(-\sigma K_0)\, V_- d\sigma\right\|_\infty < \frac{1}{2}, \tag{4.2}$$

It can be proved that a function which is (locally) of Kato class is locally an L^1-function. For the Laplace operator a proof can be found in [20]. If V is such a Kato-Feller potential, then we may use the Feynman-Kac formula to define a self-adjoint semigroup in $L^2(E,m)$:

$$\left[\exp\left(-t(K_0\dot{+}V)\right) f\right](x) = E_x\left[\exp\left(-\int_0^t V(X(u))du\right) f(X(t))\right],$$

where f belongs to $L^2(E,m)$. In fact the same formula may be employed to define a strongly continuous semigroup in $C_0(E)$ and in the spaces $L^p(E,m)$, $1 \le p < \infty$. For more details the reader is referred to the literature: Simon [18, 19], [7, 8, 9, 10, 11, 21]. Suppose that the function V belongs to L^1_{loc}. If V_- has form bound 0, then $K_0 + V$ extends to the generator of a strongly continuous semigroup in $L^p(E,m)$, $1 < p < \infty$. This result is due to W. Arendt (private communication). Our proofs require a condition like (4.2). It is not clear if and how that this can be relaxed. In what follows we write $\|T\|_{p,q}$ to indicate the norm of the operator T considered as an operator from $L^q(E,m)$ to $L^p(E,m)$.

4.1. Theorem. *Let V and W be (Kato-Feller) potentials and let the constants M_{2V} and b_{2V} be chosen in such a way that*

$$E_x\left[\exp\left(-2\int_0^{t_0/2} V(X(s))ds\right)\right] \le M_{2V}\exp\left(\frac{t_0}{2}b_{2V}\right),$$

for all $x \in E$. Let $\{E_0(d\xi) : \xi \in R\}$ be the spectral decomposition of $K_0\dot{+}V$. Suppose that $\|\exp(-t_0K_0)\,|W-V|\|_\infty$ is finite.

(1) *For every bounded interval A_0 in R the operator $E_0(A_0)\,|W-V|\,E_0(A_0)$ is bounded and*

$$\begin{aligned}&\|E_0(A_0)\,|W-V|\,E_0(A_0)\|_{2,2} \\ &\le \sup_{\xi\in A_0}\exp(t_0\xi)\, M_{2V}\exp\left(\frac{1}{2}t_0 b_{2V}\right)\left\|\exp\left(-\frac{1}{2}t_0K_0\right)|W-V|\right\|_\infty.\end{aligned} \tag{4.3}$$

(2) *A similar result is true for operators of the form* $E_1(A_1)\,|W-V|\,E_1(A_1)$, *where* A_1 *is a bounded interval in* R.
(3) *The Hamiltonians* $K_0\dot{+}V$ *and* $K_0\dot{+}W$ *are mutually subordinate.*

As before, in the sequel we write

$$V_0(t)=\exp\left(-t\left(K_0\dot{+}V\right)\right),\quad V_1(t)=\exp\left(-t\left(K_0\dot{+}W\right)\right),\quad \text{and}$$
$$D(t)=V_0(t)-V_1(t).$$

Proof. (1) We begin by noticing the identity

$$\begin{aligned}&E_0(A_0)\,|W-V|\,E_0(A_0)\\&=E_0(A_0)\exp\left(\frac{1}{2}t_0\left(K_0\dot{+}V\right)\right)V_0(t_0/2)\,|W-V|\,V_0(t_0/2)\exp\left(\frac{1}{2}t_0\left(K_0\dot{+}V\right)\right)E_0(A_0).\end{aligned}$$

Hence we see

$$\begin{aligned}&\|E_0(A_0)\,|W-V|\,E_0(A_0)\|_{2,2}\\&\le\left\|E_0(A_0)\exp\left(\frac{1}{2}t_0\left(K_0\dot{+}V\right)\right)\right\|_{2,2}^2\|V_0(t_0/2)\,|W-V|\,V_0(t_0/2)\|_{2,2}\\&=\left(\sup_{\xi\in A_0}\exp\left(\frac{1}{2}t_0\xi\right)\right)^2\|V_0(t_0/2)\,|W-V|\,V_0(t_0/2)\|_{2,2}\,.\end{aligned}$$

Next we shall estimate the L^2-L^2-norm of the operator

$$V_0(t_0/2)\,|W-V|\,V_0(t_0/2)=\exp\left(-\frac{1}{2}t_0\left(K_0\dot{+}V\right)\right)|W-V|\exp\left(-\frac{1}{2}t_0\left(K_0\dot{+}V\right)\right).$$

Therefore fix $f\ge 0$ in $L^2(E,m)$. Then, by Feynman-Kac formula and Cauchy-Schwartz' inequality, we have

$$\begin{aligned}&\left(\exp\left(-\frac{1}{2}t_0\left(K_0\dot{+}V\right)\right)f(x)\right)^2=\left(E_x\left[\exp\left(-\int_0^{t_0/2}V(X(s))ds\right)f(X(t_0/2))\right]\right)^2\\&\le E_x\left[\exp\left(-2\int_0^{t_0/2}V(X(s))ds\right)\right]E_x\left[f(X(t_0/2))^2\right]\\&\le M_{2V}\exp\left(\frac{1}{2}t_0b_{2V}\right)\left[\exp\left(-\frac{1}{2}t_0K_0\right)f^2\right](x).\end{aligned}\tag{4.4}$$

From (4.4) we get

$$\begin{aligned}
&\left\langle \exp\left(-\frac{1}{2}t_0\left(K_0\dot{+}V\right)\right)|W-V|\exp\left(-\frac{1}{2}t_0\left(K_0\dot{+}V\right)\right)f,f\right\rangle \\
&=\left\langle |W-V|\exp\left(-\frac{1}{2}t_0\left(K_0\dot{+}V\right)\right)f,\exp\left(-\frac{1}{2}t_0\left(K_0\dot{+}V\right)\right)f\right\rangle \\
&\le M_{2V}\exp\left(\frac{1}{2}t_0b_{2V}\right)\int|W(x)-V(x)|\left[\exp\left(-\frac{1}{2}t_0K_0\right)f^2\right](x)dx \\
&=M_{2V}\exp\left(\frac{1}{2}t_0b_{2V}\right)\left\langle |W-V|,\exp\left(-\frac{1}{2}t_0K_0\right)f^2\right\rangle \\
&=M_{2V}\exp\left(\frac{1}{2}t_0b_{2V}\right)\left\langle \exp\left(-\frac{1}{2}t_0K_0\right)|W-V|,f^2\right\rangle \\
&\le M_{2V}\exp\left(\frac{1}{2}t_0b_{2V}\right)\left\|\exp\left(-\frac{t_0}{2}K_0\right)|W-V|\right\|_\infty \|f\|_2^2. \qquad (4.5)
\end{aligned}$$

From (4.5) inequality (4.3) in Theorem 4.1. readily follows.

(2) The proof of this assertion is exactly the same as the one for assertion (1).

(3) From the arguments in (4.5) it will follow that the operator

$$V_0(t_0/2)\,|W-V|\,V_1(t_0/2)$$

is bounded. If a is larger than the bottom of the spectrum of $K_0\dot{+}W$ and of $K_0\dot{+}V$ it will follow that the operator $V_0(t_0/2)\left(aI+K_0\dot{+}W\right)V_0(t_0/2)$. Put $H_0=K_0\dot{+}V$ and $H_1=K_0\dot{+}W$. Put $f_0(\lambda)=\exp\left(t_0\,|a+\lambda|/2\right)$ and $f_1(\lambda)=1+\sqrt{|a+\lambda|}$. So the Hamiltonian H_1 is subordinate to H_0. Upon interchanging the roles V and W in (4.5) we also see that H_0 is subordinate to H_1.

A relevant result reads as follows.

4.2. Theorem. *Let V and W be (Kato-Feller) potentials for which*

$$\limsup_{t\downarrow 0}\left\|\int_0^t \exp\left(-\sigma K_0\right)V_-d\sigma\right\|_\infty<\frac{1}{2}\quad\text{and for which}$$
$$\limsup_{t\downarrow 0}\left\|\int_0^t \exp\left(-\sigma K_0\right)W_-d\sigma\right\|_\infty<\frac{1}{2}$$

as well. Suppose

$$\left\|\exp\left(-\frac{t_0}{2}K_0\right)|W-V|\right\|_\infty<\infty, \qquad (4.6)$$

or suppose that

$$\limsup_{a\downarrow 0}\sup_{x\in E}E_x\left[\left(\int_0^a\left(W(X(u))-V(X(u))\right)du\right)^2\right]=0. \qquad (4.7)$$

Then the following assertions are equivalent:

(i) *For all bounded intervals* A_0 *and* A_1 *the operator* $E_0(A_0)(W-V)E_1(A_1)$ *is compact;*

(ii) *The operator* $V_0(t_0/2)(W-V)V_1(t_0/2)$ *is compact;*

(iii) *The operator* $D(t_0)$ *is compact.*

Here $\{E_0(\xi) : \xi \in R\}$ is the spectral resolution of $K_0 \dot{+} V$ and $\{E_1(\xi) : \xi \in R\}$ that of $K_0 \dot{+} W$.

Remark 1. In the presence of (4.6), item (3) in Theorem 4.1. implies that the Hamiltonians $K_0 \dot{+} V$ and $K_0 \dot{+} W$ are mutually subordinate.

Remark 2. Suppose that $W - V$ satisfies

$$\lim_{t \downarrow 0} \sup_{x \in E} \int_0^t [\exp(-sK_0) |W - V|](x) = 0.$$

From the Markov property it follows that condition (4.7) is satisfied.

Proof. The assertions (i) and (ii) are always equivalent.

(iii) $\Rightarrow$ (i). By virtue of Theorem 3.3. we see that the compactness of $D(t_0)$ implies the compactness of the operator $V_0(t_0/2)(W-V)V_1(t_0/2)$. But then it is easy to see that all operators of the form $E_0(A_0)(W-V)E_1(A_1)$, where A_0 and A_1 are bounded intervals in R are compact.

(i) $\Rightarrow$ (iii). Let A_0 and A_1 be bounded intervals in R. Since

$$E_0(A_0)D(t_0)E_1(A_1) = \mathcal{D}(t_0)E_0(A_0)(W-V)E_1)A_1)$$

we infer from assertion (i) that the operator $E_0(A_0)D(t_0)E_1(A_1)$ is compact. Next let f_0 and $f_1(\lambda)$ be functions which make H_0 subordinate to H_1. For n and m in N and $-\gamma_1$ the bottom of the spectrum of $K_0 \dot{+} W$, we then have

$$\begin{aligned}
&\|E_0(n,\infty)V_1(t_0)\|_{2,2} \\
&\le \|E_0(n,\infty)V_1(t_0)E_1[-\gamma_1, m]\|_{2,2} + \|E_0(n,\infty)V_1(t_0)E_1(m,\infty)\|_{2,2} \\
&\le \|E_0(n,\infty)E_1[-\gamma_1, m]\|_{2,2} \|V_1(t_0)\|_{2,2} + \|V_1(t_0)E_1(m,\infty)\|_{2,2} \\
&\le \left\|E_0(n,\infty)f_0(H_0)^{-1}f_0(H_0)f_1(H_1)^{-1}f(H_1)E_1[-\gamma_1, m]\right\|_{2,2} \|V_1(t_0)\|_{2,2} \\
&\qquad + \|V_1(t_0)E_1(m,\infty)\|_{2,2} \\
&\le \sup_{\lambda > n} \frac{1}{f_0(\lambda} \left\|f_0(H_0)f_1(H_1)^{-1}\right\|_{2,2} \sup_{-\gamma_1 \le \lambda \le m} f(\lambda)e^{\gamma_1 t_0} + e^{-mt_0}.
\end{aligned}$$

Hence we obtain $\lim_{n\to\infty} \|E_0(n,\infty)V_1(t_0)\|_{2,2} = 0$. A similar argument implies $\lim_{n\to\infty} \|V_0(t_0)E_1(n,\infty)\|_{2,2} = 0$. Consequently, for $-\gamma_j$ the bottom of the spectrum of H_j ($j = 0, 1$),

$$\lim_{n,m\to\infty} \|E_0[-\gamma_0, n]D(t_0)E_1[-\gamma_1, m] - D(t_0)\|_{2,2} = 0.$$

Since we already know that operators of the form $E_0(A_0)D(t_0E_1(A_1)$, where A_0 and A_1 are bounded intervals are compact, assertion (ii) follows.

(iii) $\Rightarrow$ (ii). This implication immediately follows from Theorem 3.3.

(ii) $\Rightarrow$ (iii). From Theorem 3.3. it follows that we only need to show that (see (3.12)):

$$\lim_{a\downarrow 0} \|V_0(a)\mathcal{D}(t_0)(W-V)V_1(a) - \mathcal{D}(t_0)(W-V)\|_{2,2} = 0. \tag{4.8}$$

Since $\mathcal{D}(t_0)(W-V) = V_0(t_0) - V_1(t_0)$, (4.8) is true whenever the following limits are zero:

$$\lim_{a\downarrow 0} \|V_0(a)V_0(t_0)V_1(a) - V_0(t_0)\|_{2,2} = \lim_{a\downarrow 0} \|V_0(a)V_1(t_0)V_1(a) - V_1(t_0)\|_{2,2} = 0. \tag{4.9}$$

Since

$$\lim_{a\downarrow 0} \|V_0(2a)V_0(t_0) - V_0(t_0)\|_{2,2} = \lim_{a\downarrow 0} \|V_1(2a)V_1(t_0) - V_1(t_0)\|_{2,2} = 0,$$

the identities in (4.9) are true whenever we prove the equalities:

$$\lim_{a\downarrow 0} \|V_1(a)V_0(t_0-a) - V_0(t_0)\|_{2,2} = \lim_{a\downarrow 0} \|V_0(a)V_1(t_0-a) - V_1(t_0)\|_{2,2} = 0, \tag{4.10}$$

Since the conditions on V and W are interchangeable, it suffices to prove that

$$\lim_{a\downarrow 0} \|V_0(a)V_1(t_0-a) - V_1(t_0)\|_{2,2} = 0.$$

Put $e_V(t) = \exp\left(-\int_0^t V(X(u))du\right)$, put $V_s = (1-s)V + sW$, and fix $f \in L^2(E,m)$. From the Feynman-Kac formula together with the Markov property we obtain

$$\begin{aligned}
&[V_0(a)V_1(t_0-a)f](x) - [V_1(t_0)f](x) \\
&= E_x\Big[\int_0^a (V(X(u)) - W(X(u)))\,du \int_0^1 e_{(1-s)V+sW}(t)ds\, e_V(t_0-a)\circ\vartheta_a f(X(t_0))\Big].
\end{aligned}$$

So from Cauchy-Schwarz' inequality together with another appeal to the Markov property it follows that

$$\begin{aligned}
&|[V_0(a)V_1(t_0-a)f](x) - [V_1(t_0)f](x)|^2 \\
&= E_x\left[\left(\int_0^a (V(X(u)) - W(X(u)))\,du\right)^2\right] \\
&\quad E_x\left[\int_0^1 ds_1 \int_0^1 ds_2 e_{(2-s_1-s_2)V+(s_1+s_2)W}(a)\exp\left(-2\int_a^{t_0} V(X(u))du\right)|f(X(t_0))|^2\right] \\
&= E_x\left[\left(\int_0^a (V(X(u)) - W(X(u)))\,du\right)^2\right] \qquad (4.11)\\
&\quad \int_0^1 ds_1 \int_0^1 ds_2 \left[\exp\left(-a\left(K_0\dot{+}2V_{(s_1+s_2)/2}\right)\right)\exp\left(-(t_0-a)\left(K_0\dot{+}2W\right)\right)|f|^2\right](x).
\end{aligned}$$

From (4.11) we infer

$$\begin{aligned}
&\|V_0(a)V_1(t_0a) - V_1(t_0)\|_{2,2}^2 \\
&\leq \sup_{x\in E} E_x\left[\left(\int_0^a (V(X(u)) - W(X(u)))\,du\right)^2\right] \\
&\quad \left\|\int_0^1 ds_1 \int_0^1 ds_2 \exp\left(-a\left(K_0\dot{+}2V_{(s_1+s_2)/2}\right)\right)\exp\left(-(t_0-a)\left(K_0\dot{+}2W\right)\right)\right\|_{1,1} \\
&= \sup_{x\in E} E_x\left[\left(\int_0^a (V(X(u)) - W(X(u)))\,du\right)^2\right] \\
&\quad \left\|\exp\left(-(t_0-a)\left(K_0\dot{+}2W\right)\right)\int_0^1 ds_1 \int_0^1 ds_2 \exp\left(-a\left(K_0\dot{+}2V_{(s_1 s_2)/2}\right)\right)\right\|_{\infty,\infty} \\
&\leq \sup_{x\in E} E_x\left[\left(\int_0^a (V(X(u)) - W(X(u)))\,du\right)^2\right] \\
&\quad \sup_{0<b<t_0}\sup_{x\in E}\left[\exp\left(-(t_0-b)\left(K_0\dot{+}2W\right)\right)1\right](x) \\
&\quad \sup_{0<b<t_0}\sup_{x\in E}\left[\int_0^1 ds_1\int_0^1 ds_2 \exp\left(-b\left(K_0\dot{+}(2-s_1-s_2)V + (s_1+s_2)W\right)\right)1\right](x).
\end{aligned}$$

From Khas'minskii's lemma and our assumptions we see that (4.10) is true. So assertion (iii) follows.

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