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ON THE STABILITY OF MAPPINGS AND AN ANSWER TO A PROBLEM OF TH.M. RASSIAS

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Résumé

Le but principal de cet article est la démonstration d'un théorème concernant la stabilité HYERS-ULAM des applications ,qui donne une généralisation pour les résultats [1] et [3]. En plus, il repond à une problème posée par TH. M. RASSIAS en [3].

Abstract

The main purpose of this paper is to prove a theorem concerning the HYERS-ULAM stability of mappings, which gives a generalization of the results from [1] and [3]. It also answers a problem posed by TH.M.RASSIAS [3].

The question concerning the stability of mappings has been originally raised by S.M.ULAM [4]. The first answer was given in 1941 by D. H.HYERS (see [2] for a research survey of the development of the subject). In this paper we provide a generalization of a theorem of TH.M.RASSIAS [3] concerning the HYERS-ULAM stability of mappings and we also answer a problem that TH.M.RASSIAS posed in [3].

Theorem 1 *Let $(G,+)$ be an abelian group, k an integer, $k \geq 2$, $(X, \|\cdot\|)$ a BANACH space, $\varphi : G \times G \to [0,\infty)$ a mapping such that*

$$\phi_k(x,y) = \sum_{n=0}^{\infty} \frac{1}{k^{n+1}} \varphi(k^n x, k^n y) < \infty, \quad \forall x, y \in G \tag{1}$$

and $f : G \to X$ a mapping with the property

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y), \quad \forall x, y \in G. \tag{2}$$

Then there exists a unique additive mapping $T : G \to X$ such that

$$\|f(x) - T(x)\| \leq \sum_{m=1}^{k-1} \phi_k(x, mx), \quad \forall x \in G. \tag{3}$$

Proof. Setting $y = x$ and $y = 2x$ in relation (2) we obtain

$$\|f(2x) - 2f(x)\| \leq \varphi(x, x), \quad \forall x \in G \tag{4}$$

and respectively,

$$\|f(3x) - f(x) - f(2x)\| \leq \varphi(x, 2x), \quad \forall x \in G.$$

Using the triangle inequality and the last two relations, it follows:

$$\begin{aligned} \|f(3x) - 3f(x)\| &\leq \|f(3x) - f(x) - f(2x)\| + \|f(2x) - 2f(x)\| \leq \\ &\leq \varphi(x, 2x) + \varphi(x, x) = \sum_{m=1}^{2} \varphi(x, mx) \end{aligned}$$

hence

$$\|f(3x) - 3f(x)\| \leq \sum_{m=1}^{2} \varphi(x, mx). \tag{5}$$

We will prove by mathematical induction after k the following inequality:

$$\|f(kx) - kf(x)\| \leq \sum_{m=1}^{k-1} \varphi(x, mx). \tag{6}$$

Indeed, for $k = 2$ and $k = 3$ we have the relation (4) and, respectively (5). Suppose (6) true for k and let us prove it for $k + 1$. We replace y by kx in (2) and we obtain:

$$\|f((k + 1)x) - f(x) - f(kx)\| \leq \varphi(x, kx).$$

Hence, it follows

$$\begin{aligned} \|f((k + 1)x) - (k + 1)f(x)\| &\leq \|f((k + 1)x) - f(x) - f(kx)\| + \|f(kx) - kf(x)\| \leq \\ &\leq \varphi(x, kx) + \sum_{m=1}^{k-1} \varphi(x, mx) = \sum_{m=1}^{k} \varphi(x, mx), \end{aligned}$$

using (6) for the last inequality.

So, relation (6) is true for any $k \geq 2$, integer.

Dividing (6) by k we obtain:

$$\left\|\frac{f(kx)}{k} - f(x)\right\| \leq \sum_{m=1}^{k-1} \frac{1}{k} \varphi(x, mx). \tag{7}$$

We claim that

$$\left\|\frac{f(k^n x)}{k^n} - f(x)\right\| \leq \sum_{m=1}^{k-1} \sum_{p=0}^{n-1} \frac{1}{k^{p+1}} \varphi(k^p x, mk^p x), \quad \forall x \in G. \tag{8}$$

We see that for $n = 1$ we have (7). We suppose (8) true for n and we will prove it for $n+1$.We replace x by kx in (8) and we have

$$\left\|\frac{f(k^n \cdot kx)}{k^n} - f(kx)\right\| \leq \sum_{m=1}^{k-1}\sum_{p=0}^{n-1} \frac{1}{k^{p+1}}\varphi(k^p \cdot kx, mk^p \cdot kx), \quad \forall x \in G.$$

Dividing this relation by k, it follows:

$$\left\|\frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(kx)}{k}\right\| \leq \sum_{m=1}^{k-1}\sum_{p=0}^{n-1} \frac{1}{k^{p+2}}\varphi(k^{p+1}x, mk^{p+1}x)$$

and further,

$$\begin{aligned}\left\|\frac{f(k^{n+1}x)}{k^{n+1}} - f(x)\right\| &\leq \left\|\frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(kx)}{k}\right\| + \left\|\frac{f(kx)}{k} - f(x)\right\| \leq \\ &\leq \sum_{m=1}^{k-1}\sum_{p=0}^{n-1} \frac{1}{k^{p+2}}\varphi(k^{p+1}x, mk^{p+1}x) + \sum_{m=1}^{k-1}\frac{1}{k}\varphi(x, mx) = \\ &= \sum_{m=1}^{k-1}\left[\sum_{p=1}^{n}\frac{1}{k^{p+1}}\varphi(k^p x, mk^p x) + \frac{1}{k}\varphi(x, mx)\right] = \\ &= \sum_{m=1}^{k-1}\sum_{p=0}^{n}\frac{1}{k^{p+1}}\varphi(k^p x, mk^p x), \quad \forall x \in G,\end{aligned}$$

so (8) is true for each $n \in \mathbb{N}^*$, by mathematical induction.

Then, for $0 < n_1 < n$, we have

$$\begin{aligned}\left\|\frac{f(k^n x)}{k^n} - \frac{f(k^{n_1}x)}{k^{n_1}}\right\| &= \frac{1}{k^{n_1}}\left\|\frac{f(k^{n-n_1}(k^{n_1}x))}{k^{n-n_1}} - f(k^{n_1}x)\right\| \leq \\ &\leq \frac{1}{k^{n_1}}\sum_{m=1}^{k-1}\sum_{p=0}^{n-n_1-1}\frac{1}{k^{p+1}}\varphi(k^{p+n_1}x, mk^{p+n_1}x) = \\ &= \sum_{m=1}^{k-1}\sum_{p=n_1}^{n-1}\frac{1}{k^{p+1}}\varphi(k^p x, mk^p x) \to 0 \text{ as } n_1 \to \infty.\end{aligned}$$

Therefore, the sequence $\left\{\frac{f(k^n x)}{k^n}\right\}_{n\in\mathbb{N}^*}$ is a fundamental sequence. Because X is a BANACH space it follows that there exists $\lim_{n\to\infty}\frac{f(k^n x)}{k^n}$, $\forall x \in G$, denoted by $T(x)$, so $T : G \to X$ and we claim that T is an additive mapping.

From (2) we have

$$\|f(k^n x + k^n y) - f(k^n x) - f(k^n y)\| \leq \varphi(k^n x, k^n y), \quad \forall x, y \in G.$$

Hence,

$$\|\frac{f(k^n(x+y))}{k^n} - \frac{f(k^n x)}{k^n} - \frac{f(k^n y)}{k^n}\| \leq \frac{1}{k^n}\varphi(k^n x, k^n y), \quad \forall x, y \in G.$$

Taking the limit as $n \to \infty$ we obtain:

$$\|T(x+y) - T(x) - T(y)\| \leq \lim_{n\to\infty} \frac{1}{k^n}\varphi(k^n x, k^n y) = 0$$

using the relation (1). This implies $T(x+y) = T(x) + T(y), \quad \forall x, y \in G.$

To prove that (3) holds, we take the limit as $n \to \infty$ in (8) and we obtain

$$\begin{aligned} \|T(x) - f(x)\| &\leq \lim_{n\to\infty} \sum_{m=1}^{k-1}\sum_{p=0}^{n-1} \frac{1}{k^{p+1}}\varphi(k^p x, mk^p x) = \\ &= \sum_{m=1}^{k-1}\sum_{p=0}^{\infty} \frac{1}{k^{p+1}}\varphi(k^p x, mk^p x) = \sum_{m=1}^{k-1} \phi_k(x, mx), \quad \forall x \in G. \end{aligned}$$

Supposing now that there exists another additive mapping $T_1 : G \to X$, with the property (3). Then

$$\begin{aligned} \|T_1(x) - T(x)\| &= \|\frac{T_1(k^n x)}{k^n} - \frac{T(k^n x)}{k^n}\| \leq \\ &\leq \frac{1}{k^n}(\|T_1(k^n x) - f(k^n x)\| + \|f(k^n x) - T(k^n x)\|) \leq \\ &\leq \frac{2}{k^n}\sum_{m=1}^{k-1} \phi_k(k^n x, mk^n x) = \frac{2}{k^n}\sum_{m=1}^{k-1}\sum_{p=0}^{\infty} \frac{1}{k^{p+1}}\varphi(k^{p+n}x, mk^{p+n}x) = \\ &= 2\sum_{m=1}^{k-1}\sum_{p=n}^{\infty} \frac{1}{k^{p+1}}\varphi(k^p x, mk^p x). \end{aligned}$$

Thus, $\lim_{n\to\infty} \|T_1(x) - T(x)\| = 0$, for any $x \in G$, which implies $T_1(x) = T(x), \quad \forall x \in G.$

Q.E.D.

REMARKS:

1.For $k = 2$ we obtain for the first relation:

$$\phi_2(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}\varphi(2^n x, 2^n y) < \infty, \quad \forall x, y \in G$$

and for the third relation

$$\|f(x)-T(x)\| \leq \phi_2(x,x), \quad \forall x \in G$$

which is the main theorem from [1].

2.If we take $\varphi(x,y)=\theta(\|x\|^p+\|y\|^p)$ with $\theta \geq 0$ and $p \in [0,1)$ we have

$$\begin{aligned}\phi_k(x,y) &= \sum_{n=0}^{\infty}\frac{1}{k^{n+1}}\theta(k^{np}(\|x\|^p+\|y\|^p)) = \\ &= \frac{\theta}{k}(\|x\|^p+\|y\|^p)\sum_{n=0}^{\infty}k^{n(p-1)} = \\ &= \frac{\theta}{k}(\|x\|^p+\|y\|^p)\cdot\frac{k}{k-k^p}.\end{aligned}$$

Then $\phi_k(x,mx)=\frac{k\theta}{k-k^p}\cdot\frac{1}{k}\cdot\|x\|^p(1+m^p)$ and

$$\begin{aligned}\sum_{m=1}^{k-1}\phi_k(x,mx) &= \frac{k\theta}{k-k^p}\|x\|^p\frac{1}{k}\sum_{m=1}^{k-1}(1+m^p) = \\ &= \frac{k\theta}{k-k^p}\|x\|^p\frac{1}{k}(k+\sum_{m=2}^{k-1}m^p) = \frac{k\theta}{k-k^p}\|x\|^p s(k,p).\end{aligned}$$

where $s(k,p)=1+\frac{1}{k}\sum_{m=2}^{k-1}m^p$ which implies the theorem of TH.M.RASSIAS proved in [3] .

We prove that the best possible value of k is 2. Set

$$R(p)=\frac{2}{2-2^p} \text{ and } Q(k,p)=\frac{k\cdot s(k,p)}{k-k^p}, k>2.$$

We prove that

$$R(p) < Q(k,p) \text{ for all } k \geq 3. \tag{9}$$

The verification of (9) follows by mathematical induction on k .

The case $k=3$ is true, because

$$Q(3,p)-R(p)=\frac{2\cdot 3^p-2^p-4^p}{(2-2^p)(3-3^p)}>0,$$

where we use the JENSEN inequality for the concave function $f:(0,\infty)\to \mathbb{R}$, $f(x)=x^p, p\in[0,1)$:

$$\left(\frac{x_1+x_2}{2}\right)^p > \frac{x_1^p+x_2^p}{2} \quad \text{for } x_1,x_2 \in (0,\infty) \tag{10}$$

with $x_1 = 2, x_2 = 4$.

Assume now that (9) is true and we prove that

$$Q(k+1,p) > R(p). \tag{11}$$

We have from (9)

$$\begin{aligned} Q(k+1) - R(p) &> R(p)\frac{k-k^p}{k+1-(k+1)^p} + \frac{k^p+1}{k+1-(k+1)^p} - \frac{2}{2-2^p} = \\ &= \frac{2(k+1)^p - 2^p - k^p \cdot 2^p}{(2-2^p)(k+1-(k+1)^p)} > 0, \end{aligned}$$

where we use the inequality (10) with $x_1 = 2k, x_2 = 2$.

Thus, (9) is proved.

This last result gives an answer to a problem that was posed by TH.M. RASSIAS in 1991.

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