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Annales mathématiques Blaise Pascal, tome 2, n° 1 (1995), p. 267-273

http://www.numdam.org/item?id=AMBP_1995__2_1_267_0

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**A SCALAR FIELD FOR WHICH C -zero HAS
NO HAHN–BANACH PROPERTY**

W.H. Schikhof

Abstract. Let K be a field with a valuation $|\cdot|$ of infinite rank for which $(K, |\cdot|)$ is complete, separable and such that the value group is countable. We show that the Banach space c_0 consisting of all null sequences in K admits a closed subspace S and a continuous linear function $S \rightarrow K$ that cannot be extended to a continuous linear function $c_0 \rightarrow K$.

This surprising result is in contrast not only to Keller's example [2] of a separable inner product Banach space over K resembling classical Hilbert space, but also to the Hahn–Banach property of c_0 over a field with a rank 1 valuation [4, 3.16].

1991 Mathematics subject classification : 46S10.

PRELIMINARIES. (For details on valuations see e.g. [5]).

Let G be a multiplicatively written totally ordered abelian group. We require $G \neq \{1\}$ implying that G is torsion free. We add an element 0 such that $0 < g$ for all $g \in G$.

Let K be a field. A *valuation* (with *value group* G) is a surjection $|\cdot| : K \rightarrow G \cup \{0\}$ such that (i) $|x| = 0$ if and only if $x = 0$ (ii) $|xy| = |x| |y|$ (iii) $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in K$.

REMARK. Usually G is written additively (with reversed ordering and an element $+\infty$ adjoined) but here we prefer the multiplicative notation because of its analogy with the absolute value.

For $a \in K$, $\varepsilon \in G$ we write $B(a, \varepsilon) := \{x \in K : |x - a| \leq \varepsilon\}$. The topology induced naturally by the valuation makes K into a topological field. A subgroup H of G is called *convex* if $g, k \in H$, $g < h$ implies $\{t \in G : g \leq t \leq h\} \subset H$.

Henceforth we shall assume that

- (i) $K = (K, |\cdot|)$ is complete (i.e. each Cauchy net converges) and separable,

- (ii) G is countable.
- Then it follows that
- (iii) K is metrizable,
 - (iv) for every convex subgroup H of G , $H \neq \{1\}$, $H \neq G$ there exists a strictly decreasing sequence g_1, g_2, \dots in G such that $H = \bigcap_n \{g \in G : g_n^{-1} \leq g \leq g_n\}$.

We further assume that the valuation has infinite rank which in our case means that

- (v) G is the union of convex subgroups H_1, H_2, \dots for which

$$\{1\} \subsetneq H_1 \subsetneq H_2 \subsetneq \dots$$

We indicate an example of a field satisfying (i) – (v) ([3],2.1). Let, for each $i \in \mathbf{N}$, G_i be the ordered group \mathbf{Z} , written multiplicatively, with positive generator a_i . Let $G := \bigoplus_i G_i$, ordered antilexicographically. The following formulas define a valuation on $\mathbb{Q}(X_1, X_2, \dots)$, the field of rational functions in countably many variables over \mathbb{Q} .

$$|q| := \begin{cases} 0 & \text{if } q \in \mathbb{Q}, q = 0 \\ (1, 1, \dots) \in G & \text{if } q \in \mathbb{Q}, q \neq 0 \end{cases}$$

$$|X_i| := (b_{i1}, b_{i2}, \dots) \in G \quad (i \in \mathbf{N})$$

where $b_{ij} = a_i$ if $j = i$, $b_{ij} = 1$ if $j \neq i$. It is clear that the completion K of $\mathbb{Q}(X_1, X_2, \dots)$ with respect to the valuation $||$ satisfies (i), (ii) above.

Also (v) is true if we take $H_n = \bigoplus_{i=1}^n G_i = \{(b_1, b_2, \dots) \in G : b_i = 1 \text{ for } i > n\}$.

We finally note that this K also served as the scalar field of the Hilbert space constructed by Keller [2].

In this note a *norm* on a K -vector space E is a map $|| \cdot || : E \rightarrow G \cup \{0\}$ (we do not need the more usual notion of norm with extended range) satisfying (i) $||x|| = 0$ if and only if $x = 0$ (ii) $||\lambda x|| = |\lambda| ||x||$ (iii) $||x + y|| \leq \max(||x||, ||y||)$ for all $x, y \in E, \lambda \in K$. The *dual space* E' is the space of all continuous linear functions $E \rightarrow K$. $E = (E, || \cdot ||)$ is called a *K -Banach space* if E is complete.

Let E_1, E_2, \dots be K -Banach spaces. By $\bigoplus_n E_n$ we mean the space consisting of all sequences $x = (x_n)_{n \in \mathbf{N}}$, where $x_n \in E_n$ for each n , for which $\lim_{n \rightarrow \infty} x_n = 0$ normed by $x \mapsto \max_n ||x_n||$. As usual, we denote $\bigoplus_n K$ by c_0 . It follows easily that $\bigoplus_n E_n$ is a K -Banach space. If each E_n is separable then so is $\bigoplus_n E_n$.

Let E, F be K -Banach spaces. We say that a surjective linear continuous map $\pi : E \rightarrow F$ is a *quotient map* (and F is a *quotient of* E) if $||y|| = \inf\{||x|| : x \in E, \pi(x) = y\}$ for each $y \in F$.

The following Lemma is a non-archimedean version of the 'second half of the Open Mapping Theorem'. Although the proof is not materially different from the classical one we include it to convince the reader of this very fact.

Denote the 'closed unit ball' $\{x \in E : \|x\| \leq 1\}$ of a K -Banach space E by B_E , and the closure of a set $X \subset E$ by \overline{X} .

LEMMA. *Let E, F be K -Banach spaces, let $\pi : E \rightarrow F$ be a continuous linear map. Suppose $\pi(B_E)$ is a zero neighbourhood in F . Then $\pi(B_E) = \overline{\pi(B_E)}$ (and in particular $\pi(B_E)$ is a zero neighbourhood).*

Proof. There exist $\lambda_0, \lambda_1, \dots \in K$ such that $\lambda_0 = 1$, $|\lambda_0| > |\lambda_1| > \dots$, $\lim_{n \rightarrow \infty} \lambda_n = 0$ and $\lambda_1 B_F \subset \overline{\pi(B_E)}$. Let $y \in \overline{\pi(B_E)}$; we construct an $x \in B_E$ with $\pi(x) = y$. There exists an $x_0 \in B_E$ with $\|y - \pi(x_0)\| \leq |\lambda_1|^2$. Then

$$y = \pi(\lambda_0 x_0) + \lambda_1 y_1 \text{ where } y_1 \in \lambda_1 B_F \subset \overline{\pi(B_E)}.$$

Next, we can take an $x_1 \in B_E$ with $\|y_1 - \pi(x_1)\| \leq |\lambda_2|$ so

$$y = \pi(\lambda_0 x_0) + \pi(\lambda_1 x_1) + \lambda_2 y_2 \text{ where } y_2 \in \lambda_1 B_F \subset \overline{\pi(B_E)}.$$

Similarly we can find an $x_2 \in B_E$ with $\|y_2 - \pi(x_2)\| \leq |\lambda_3|$. Then

$$y = \pi(\lambda_0 x_0) + \pi(\lambda_1 x_1) + \pi(\lambda_2 x_2) + \lambda_3 y_3 \text{ where } y_3 \in \lambda_2 B_F \subset \overline{\pi(B_E)}$$

etc. Inductively we obtain $x_1, x_2, \dots \in B_E$ such that $y = \sum_{n=0}^{\infty} \pi(\lambda_n x_n)$. By completeness $x := \sum_{n=0}^{\infty} \lambda_n x_n$ exists in E , $\|x\| \leq \max_n \|\lambda_n x_n\| \leq 1$ and $\pi(x) = y$.

§1. A SEPARABLE BANACH SPACE WITHOUT THE HAHN-BANACH PROPERTY

As a stepping stone we construct a separable K -Banach space E having no Hahn-Banach property (Theorem 1.5). Let us first observe that continuous maps between K -Banach spaces are bounded. This is a simple consequence of our assumption that the range of the norm is $G \cup \{0\}$, enabling us to normalize each nonzero vector by a suitable scalar multiplication. (If one allows norms to have a more general range, continuous linear maps are not always bounded [1]). For our purpose we just need the following Lemma.

LEMMA 1.1. *For a K -Banach space E , every $f \in E'$ is bounded i.e. there exists a $c \in G$ such that $|f(x)| \leq c\|x\|$ for all $x \in E$.*

The next Lemma tells us that K is not spherically complete.

LEMMA 1.2. *Let $\lambda_1, \lambda_2, \dots \in K$ be such that $|\lambda_1| > |\lambda_2| > \dots$ is bounded below in G . Then there exist $a_1, a_2, \dots \in K$ such that $B(a_1, |\lambda_1|), B(a_2, |\lambda_2|), \dots$ form a nest with empty intersection.*

Proof. Let $\{b_1, b_2, \dots\}$ be dense in K . The equivalence relation $|x - y| \leq |\lambda_1|$ divides K into balls, so there is one, say $B(a_1, |\lambda_1|)$, that does not contain b_1 . Similarly, the ball $B(a_1, |\lambda_1|)$ is divided into (at least two) balls of radius $|\lambda_2|$ so that we can find one, say $B_2(a_2, |\lambda_2|)$, that does not contain b_2 etc.. Inductively we arrive at a nest $B(a_1, |\lambda_1|) \supset B(a_2, |\lambda_2|) \supset \dots$ where $b_n \notin B(a_n, |\lambda_n|)$ for each n . Then $B := \bigcap_n B(a_n, |\lambda_n|)$ contains none of the b_n , so it has empty interior. On the other hand, if $B \neq \emptyset$ it would contain a ball with radius $|\lambda|$ where $\lambda \in K, 0 < |\lambda| \leq |\lambda_n|$ for each n . We conclude that $B = \emptyset$.

LEMMA 1.3. *Let $\lambda_1, \lambda_2, \dots \in K$ be such that $H := \{s \in G : |\lambda_n|^{-1} \leq s \leq |\lambda_n| \text{ for all } n\}$ is a convex subgroup, $H \neq \{1\}$. If $a_1, a_2, \dots \in K$ are such that $B(a_1, |\lambda_1|), B(a_2, |\lambda_2|), \dots$ is a nest with empty intersection then so is for each $c \in H, c \geq 1, B(a_1, c|\lambda_1|), B(a_2, c|\lambda_2|), \dots$*

Proof. For each $n, a_{n+1} \in B(a_n, |\lambda_n|) \subset B(a_n, c|\lambda_n|)$ so the balls $B(a_n, c|\lambda_n|)$ ($n \in \mathbb{N}$) form a nest. To complete the proof we show that any $x \in \bigcap_n B(a_n, c|\lambda_n|)$ is also in

$$\bigcap_n B(a_n, |\lambda_n|).$$

To this end it suffices to show that $x \in B(a_1, |\lambda_1|)$. Suppose $|x - a_1| > |\lambda_1|$. Since $|a_m - a_1| \leq |\lambda_1|$ we have, for all $m, |x - a_m| = |x - a_1|$ so there is an $\alpha \in K$ such that

$$|\alpha| = |x - a_m| \leq c|\lambda_m| \quad (m \in \mathbb{N}).$$

Also $|\lambda_m|^{-1} \leq 1 \leq c^{-1}|\lambda_1| < c^{-1}|\alpha|$ so that $|\lambda_m|^{-1} \leq c^{-1}|\alpha| \leq |\lambda_m|$ for each m i.e., $c^{-1}|\alpha| \in H$. Then $|\alpha| \in H$ conflicting $|\alpha| > |\lambda_1|$.

LEMMA 1.4. (Compare [4], p. 68) *For every convex subgroup H of $G, H \neq \{1\}, H \neq G$ there exist $\lambda_1, \lambda_2, \dots, a_1, a_2, \dots \in K$ as in Lemma 1.3.*

The formula

$$\|(\lambda, \mu)\| = \lim_{m \rightarrow \infty} |\lambda - \mu a_m|$$

defines a norm on K^2 . The function $f : (\lambda, 0) \mapsto \lambda$ ($\lambda \in K$) satisfies $|f(x)| \leq \|x\|$ ($x \in K \times \{0\}$) but for any extension $\tilde{f} \in (K^2)'$ we have $|\tilde{f}(x)| \leq c\|x\|$ ($x \in K^2$) for no $c \in H$.

Proof. Choose, for each $n, \lambda_n \in K$ such that $|\lambda_n| = g_n$ where g_n is as in (iv) of the Preliminaries. For the existence of a_1, a_2, \dots combine Lemmas 1.2 and 1.3. After observing that for each $\xi \in K$ the sequence $m \mapsto |\xi - a_m|$ is eventually constant the proof that $\| \cdot \|$ is a norm becomes straightforward. Obviously $\|(\lambda, 0)\| = |\lambda| = |f(\lambda, 0)|$ for each $\lambda \in K$. Now let $\tilde{f} \in (K^2)'$ be an extension of f and suppose

$$(*) \quad |\tilde{f}(x)| \leq c\|x\| \quad (x \in K^2)$$

for some $c \in H$. Then $c \geq 1$. Writing $\alpha := -\tilde{f}(0, 1)$ formula (*) becomes

$$|\lambda - \mu\alpha| \leq c \lim_{m \rightarrow \infty} |\lambda - \mu a_m| \quad (\lambda, \mu \in K),$$

implying

$$|\lambda - \alpha| \leq c \lim_{m \rightarrow \infty} |\lambda - a_m| \quad (\lambda \in K),$$

whence

$$|a_n - \alpha| \leq c \lim_{m \rightarrow \infty} |a_n - a_m| \quad (n \in \mathbf{N})$$

But $|a_n - a_m| \leq |\lambda_n|$ for $m \geq n$ so that $|a_n - \alpha| \leq c|\lambda_n|$ for all $n \in \mathbf{N}$, in other words $\alpha \in \bigcap_n B(a_n, c|\lambda_n|) = \emptyset$, a contradiction.

THEOREM 1.5. *There exist a separable K -Banach space E , a subspace D of E and an element $f \in D'$ that cannot be extended to an element of E' .*

Proof. Let G be the union of the convex subgroups $\{1\} \subsetneq H_1 \subsetneq H_2 \subsetneq \dots$. By applying Lemma 1.4 to $H := H_n$ we can find for each $n \in \mathbf{N}$ a norm $\|\cdot\|_n$ on K^2 such that for any linear extension \tilde{f}_n of $f_n : (\lambda, 0) \mapsto \lambda$ ($\lambda \in K$) we have

$$(*) \quad |\tilde{f}_n(x)| \leq c\|x\|_n \quad (x \in K^2) \quad \text{for no } c \in H_n.$$

Now set $E := \bigoplus_n (K^2, \|\cdot\|_n)$ and denote its norm by $\|\cdot\|$. E is separable. Let D be the subspace of all $(\lambda_n, \mu_n)_{n \in \mathbf{N}}$ for which $\mu_n = 0$ for all n and define $f : D \rightarrow K$ by the formula

$$f((\lambda_n, 0)_{n \in \mathbf{N}}) = \sum_{n=1}^{\infty} f_n((\lambda_n, 0)) = \sum_{n=1}^{\infty} \lambda_n.$$

Then $|f(x)| \leq \|x\|$ ($x \in D$) so $f \in D'$. However, if $\tilde{f} \in E'$ is an extension then by Lemma 1.1 there exists a $c \in G$ for which $|\tilde{f}(x)| \leq c\|x\|$ ($x \in E$). Then $c \in H_n$ for some n and the restriction of \tilde{f} to $(K^2, \|\cdot\|_n)$ is an extension of f_n for which $|\tilde{f}(x)| \leq c\|x\|_n$ ($x \in (K^2, \|\cdot\|_n)$) conflicting (*). It follows that F has no continuous linear extension $E \rightarrow K$.

§2. THE SPACE c_0 DOES NOT HAVE THE HAHN-BANACH PROPERTY

THEOREM 2.1. *Every separable K -Banach space E is a quotient of c_0 .*

Proof. The set $\{y \in E : \|y\| = 1\}$ is separable; let $\{y_1, y_1, \dots\}$ be a dense subset. Define $\pi : c_0 \rightarrow E$ by the formula

$$\pi((\xi_1, \xi_2, \dots)) = \sum_{i=1}^{\infty} \xi_i y_i.$$

Then clearly $\|\pi(x)\| \leq \|x\|$ for all $x \in c_0$. For each $\lambda \in K, 0 < |\lambda| \leq 1$ the set $\{\lambda y_1, \lambda y_2, \dots\}$ is dense in $\{x \in E : \|x\| = |\lambda|\}$. It follows that $\pi(B_{c_0})$ is dense in B_E . By the Lemma of the Preliminaries we may conclude that $\pi(B_{c_0}) = B_E$ and also $\pi(\lambda B_{c_0}) = \lambda B_E$ for each $\lambda \in K$. Now let $y \in E, \|y\| = |\lambda|$. Then there exists an $x \in c_0$ with $\pi(x) = y$ and $\|x\| \leq |\lambda|$. But also $|\lambda| = \|y\| = \|\pi(x)\| \leq \|x\|$, so $\|x\| = |\lambda|$. It follows that π is a quotient map.

REMARK. We even proved that E is a *strict* quotient of c_0 i.e., $\|y\| = \min\{\|x\| : x \in c_0, \pi(x) = y\}$ for each $y \in E$.

THEOREM 2.2. *There exist a (closed) subspace S of c_0 and a $g \in S'$ that cannot be extended to an element of c'_0 .*

Proof. Let E, D, f be as in Theorem 1.5 and let $\pi : c_0 \rightarrow E$ be the strict quotient map of Theorem 2.1. Set $S := \pi^{-1}(D)$. The function $g := f \circ \pi$ is in S' . If $\tilde{g} \in c'_0$ were an extension of g then the unique map \tilde{f} making the diagram

$$\begin{array}{ccc}
 c_0 & \xrightarrow{\pi} & E \\
 \tilde{g} \searrow & & \swarrow \tilde{f} \\
 & K &
 \end{array}$$

commute, is a linear extension of f . But \tilde{f} is also continuous (by Lemma 1.1 there is a $c \in G$ such that $|\tilde{g}(z)| \leq c\|z\|$ for all $z \in c_0$). Let $y \in E$, choose an $x \in c_0$ with $\pi(x) = y, \|x\| = \|y\|$. Then $|\tilde{f}(y)| = |\tilde{f}(\pi(x))| = |\tilde{g}(x)| \leq c\|x\| = c\|y\|$, a contradiction by Lemma 1.5.

Some further questions (recall that we assumed throughout that norm values are in $G \cup \{0\}$ and that $(K, | \cdot |)$ satisfies (i)–(v) of the Preliminaries) :

1. Does there exist an infinite-dimensional K -Banach space having the Hahn-Banach property?
2. Does there exist a separable K -Banach space whose dual does not separate points (resp. is trivial)?
3. Does there exist a separable K -Banach space without a Schauder base? In particular, does the space E of Theorem 1.5 have a Schauder base?
4. It is not very hard to see that any Banach space over a spherically complete valued field $(L, | \cdot |)$ has the Hahn-Banach property. Now let $(L, | \cdot |)$ be a field with a complete valuation of infinite rank that is not spherically complete. Does it follow that c_0 does not have the Hahn-Banach property?

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