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C. PEREZ-GARCIA

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ON ISOMETRIES AND COMPACT OPERATORS

BETWEEN p-ADIC BANACH SPACES

C. Perez-Garcia *

Abstract. We study in this paper conditions under which two spherically complete non-archimedean Banach spaces are isomorphic (Theorem 2.5). As an application we describe (Corollary 3.2) the spherical completion of the closed subspaces of l^∞ constructed by the author (jointly with W.H. Schikhof) in [5].

Also, certain related questions concerning with the complementation of the space of compact operators are considered in this paper (Theorems 4.1 and 4.3). As a consequence (Corollary 4.5) we obtain extensions of some of the results proved by T. Kiyosawa in [3].

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1. PRELIMINARIES

Throughout this paper K is a non-archimedean valued field that is complete under the metric induced by the non-trivial valuation $|\ .\ |$, and $E, F, G...$ are non-archimedean Banach spaces over K. By $E \sim F$ we mean that E and F are *isomorphic*, i.e., there is a linear isometry from E onto F. We will denote the completed tensor product of E and F in the sense of [7], p.123 by $E\hat{\otimes}F$.

Let $X \subset E - \{0\}$ and let $t \in (0,1]$. X is called a *t- orthogonal* (orthogonal, when $t = 1$) *system* of E if

$$\| \sum_{i=1}^{m} \lambda_i x_i \| \geq t\ \ max_i \| \lambda_i x_i \|$$

for all $m \in N$, all $\lambda_1, \ldots, \lambda_m \in K$ and all $x_1, \ldots, x_m \in X$, where $x_i \neq x_j$ for $i \neq j$ (see [7], p.171). If in addition the closed linear hull of X is E itself, X is called a *(t-)orthogonal base* of E.

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A Banach space is called *spherically complete* if every sequence of closed balls

$$B(a_1, r_1) \supset B(a_2, r_2) \supset \dots$$

for which $r_1 > r_2 > \dots$ has a non-empty intersection. By $E^\vee$ we will denote the spherical completion of a Banach space E (see [7], p.148).

For a subset B of E, $[B]$ will denote the closed linear hull of B. B is called *compactoid* if for every $r > 0$ there exists a finite set S in E such that $B \subset coS + B(0, r)$, where coS denotes the absolutely convex hull of S.

$L(E, F)$ will denote the Banach space of all continuous linear maps (or operators) from E into F, endowed with the usual norm. The topological dual space of E is $E' = L(E, K)$. E is called *reflexive* if the canonical linear map from E into E'' is a surjective isometry.

By $C(E, F)$ we will denote the closed subspace of $L(E, F)$ consisting of all *compact operators* from E into F (i.e., the maps $T \in L(E, F)$ for which the image of the closed unit ball of E, is a compactoid subset of F).

We say that F is a *strict quotient* of E if there exists a $T \in L(E, F)$ such that $\| T \| \leq 1$ and for every $y \in F$ there is $x \in E$ for which $T(x) = y$ and $\| x \| = \| y \|$.

For unexplained terms and background we refer to [7].

2. ISOMETRIES AND STRICT QUOTIENTS

In the same spirit as Theorem 4.1 of [4], the main purpose of this section (Theorem 2.5) is study conditions under which we can assure that E and F are isomorphic, when E and F have orthogonal bases or are spherically complete.

First, we prove some preliminary results.

Lemma 2.1 : *Let X, Y be maximal orthogonal systems in E and F respectively. Suppose*

a) There exists a linear isometry from E into F,

or

b) There exists a strict quotient map from F onto E.

Then, $[X]$ is isomorphic to an orthocomplemented subspace of $[Y]$.

Proof : By assumption, there exists an orthogonal system Z in F such that $[X] \sim [Z]$. Let W be a maximal orthogonal system in F containing Z. Clearly $[Z]$ is orthocomplemented in $[W]$ and, since $[W] \sim [Y]$ ([7], 5.4), the conclusion follows.

Theorem 2.2 : *If E and F have orthogonal bases (resp. E and F are spherically complete), then the following are equivalent.*

i) There exists a linear isometry from E into F.

ii) There exists a strict quotient map from F onto E.

iii) E is isomorphic to an orthocomplemented subspace of F.

Proof : First assume that E and F have orthogonal bases. Then, the result is a direct consequence of 2.1.

Now assume that E and F are spherically complete. The equivalence $i) \Leftrightarrow iii)$ follows from [7], 4.7. To prove $ii) \Rightarrow i)$, let X, Y be as in 2.1 with a linear isometry from $[X]$ into $[Y]$. By 4.7 and 4.42 of [7], this isometry extends to a linear isometry from $E = [X]^{\vee}$ into $F = [Y]^{\vee}$.

Corollary 2.3 : ([6], 2.1) *Let E,F be Banach spaces of countable type where K is spherically complete. Then, there exists a strict quotient map from F onto E iff E is isomorphic to an orthocomplemented subspace of F.*

Proof : Observe that if K is spherically complete, then every Banach space of countable type over K has an orthogonal base ([7], 5.5). Now, apply 2.2.

In the same line as 2.1 we have.

Lemma 2.4 : *Let X, Y be maximal orthogonal systems in E and F respectively. Suppose*

a) There exist linear isometries from E into F and from F into E,

or

b) There exist strict quotient maps from E onto F and from F onto E.

Then, $[X] \sim [Y]$.

Proof : Let $\Gamma = \{\mid \lambda \mid : \lambda \neq 0\}$ be the value group of K and let H be a system of representatives of $(0, \infty)/\Gamma$. Without loss of generality we can assume that all values of norms of elements of X and Y belong to H. For each $h \in H$, put

$$X_h = \{x \in X : \| x \| = h\}$$

$$Y_h = \{y \in Y : \| y \| = h\}.$$

It follows easily from 2.1 that there exist linear isometries from $[X_h]$ into $[Y_h]$ and from $[Y_h]$ into $[X_h]$. Applying 5.3 and Remark following 5.2 of [7] we deduce that the sets X_h and Y_h have the same cardinality for each $h \in H$. Hence, $[X] = \oplus_{h\in H}[X_h] \sim \oplus_{h\in H}[Y_h] = [Y]$.

We can now prove the main result of this section.

Theorem 2.5 : *If E and F have orthogonal bases (resp. E and F are spherically complete), then the following are equivalent.*

i) There exist linear isometries from E into F and from F into E.

ii) There exist strict quotient maps from E onto F and from F onto E.

iii) E is isomorphic to an orthocomplemented subspace of F and F is isomorphic to an orthocomplemented subspace of E.

iv) $E \sim F$.

Proof : The equivalences $i) \Leftrightarrow ii) \Leftrightarrow iii)$ are a direct consequence of 2.2.

When E and F have orthogonal bases, the rest follows directly from 2.4.

Now assume that E and F are spherically complete spaces satisfying i). Let X, Y be as in 2.4 and such that $[X] \sim [Y]$. By 4.7 and 4.42 of [7] we conclude that $E = [X]^\vee \sim [Y]^\vee = F$, and so iv) comes true.

3. TENSOR PRODUCT OF BANACH SPACES

It was showed in [5] that by forming tensor products we can construct in a simple way natural examples of non-reflexive and non-spherically complete closed subspaces of l^∞ (compare with [7], 4.J). More concretely, putting together the results proved in section 3 of [5] we obtain the following.

Theorem 3.1 : *For $n \in N$, $n \geq 2$, let $G_n = l^\infty \hat{\otimes} \overbrace{\cdots}^{n} \hat{\otimes} l^\infty$. Then, G_n is isomorphic to a non-reflexive closed subspace of l^∞ and such that*

G_n is spherically complete iff the valuation on K is discrete (1)

(however, it is well known that l^∞ is reflexive when K is not spherically complete ([7], 4.17), and l^∞ is spherically complete when K is spherically complete ([7], 4.A)).

Looking at (1) the following question arises in a natural way. Describe the spherical completion of G_n when the valuation on K is dense. Theorem 2.5 contains the key to the answer. In fact, we have.

Corollary 3.2 : ([4], 4.5.3) *For each $n \geq 2$, $(G_n)^\vee \sim (l^\infty)^\vee$. In particular, if K is spherically complete then $(G_n)^\vee \sim l^\infty$.*

Proof : There exist linear isometries from l^∞ into G_n and from G_n into l^∞. Now the conclusion follows from 2.5 and [7], 4.42.

Now, we are going to show that by taking tensor products we can also construct natural examples of Banach spaces for which Theorems 2.2 and 2.5 are false.

It is very easy to verify that the following generalization of 3.7 in [5] remains true.

Theorem 3.3 : *Suppose that E contains an orthogonal sequence $v_1, v_2, \ldots$ such that $\| v_1 \| > \| v_2 \| > \ldots$ and $lim_n \| v_n \| = 1$. Then, $l^\infty \hat{\otimes} E$ is not spherically complete.*

Also, the following slight extension of 3.1 will be very useful to our purpose.

Proposition 3.4 : *Let $s_1, s_2, \ldots$ be a sequence in $(0, \infty)$ such that $s_1 > s_2 > \ldots$ and $lim_n s_n = 1$. Make $s : N \longrightarrow \{s_1, s_2, \ldots\} \subset (0, \infty)$ such that $\{m \in N : s(m) = s_n\}$ is an infinite set for all $n \in N$. Then, $E := l^\infty \hat{\otimes} l^\infty(N, s)$ is isomorphic to a non-reflexive and non-spherically complete closed subspace of $F := l^\infty(N, s)$.*

Proof : If $\{e_i : i \in N\}$ is the canonical orthogonal base of c_0 and $c_0(N, 1/s)$, then $\{e_i \otimes e_j : i, j \in N\}$ is an orthogonal base of $c_0 \hat{\otimes} c_0(N, 1/s)$ ([7], 4.30) and so

$$c_0 \hat{\otimes} c_0(N, 1/s) \sim c_0(N, 1/s). \tag{2}$$

Further, it follows from [7], 3.Q.ii) and 4.34 that there exists a linear isometry from $l^\infty \hat{\otimes} l^\infty(N,s)$ into $(c_0 \hat{\otimes} c_0(N,1/s))'$. Applying (2) we deduce the existence of a linear isometry from E into F.

Clearly if K is spherically complete, E is not reflexive ([7], 4.16). If K is not spherically complete the non-reflexivity of E follows, by 3.Q.ii) and 4.22.ii) of [7], like in the proof of 3.6.iii) in [5].

Finally observe that, by 3.3, E is not spherically complete.

This is enough material to give our example.

Example 3.5 : Let E,F be as in 3.4. Then, there exist linear isometries from E into F and from F into E, but E is not isomorphic to an orthocomplemented subspace of F.

Indeed, since $K \hat{\otimes} l^\infty(N,s) \sim l^\infty(N,s)$ ([7], 4.P.iii)), we clearly have a linear isometry from F into E. Also, by 3.4, there is a linear isometry from E into F.

Suppose that

E is isomorphic to an orthocomplemented subspace of F. (*)

We derive a contradiction.

First assume that K is spherically complete. Since F is spherically complete ([7], 4.A), and by (*) E is isomorphic to a quotient of F, we deduce that E is also spherically complete ([7], 4.2), in contrast to 3.4.

Now assume that K is not spherically complete. By (*), there is a Banach space G such that $F \sim E \oplus G$ and since $E \not\sim F$ (recall that F is reflexive ([7], 4.22.ii)) whereas E is not reflexive (3.4)), we have $G \neq \{0\}$.

We know that E is isomorphic to the Banach space of all compactoid sequences on $l^\infty(N,s)$ endowed with the supremum norm ([7], 4.R.v)). On the other hand, applying 3.Q and 4.R.i) of [7] and (2) we obtain that

$$\times_n l^\infty(N,s) \sim (\oplus_n c_0(N,1/s))' \sim (c_0 \hat{\otimes} c_0(N,1/s))' \sim F.$$

Taking account these facts in conjuction with the fact that

$$(\times_n l^\infty(N,s) / \oplus_n l^\infty(N,s))' = \{0\}$$

([7], 4,1, and 4.3), we derive that $G' = \{0\}$, a contradiction.

Remark 3.6 : Let E,F be the Banach spaces considered in 3.4 and 3.5.

1. F is isomorphic to an orthocomplemented subspace of E .

Indeed, observe that $l^\infty(N,s) \sim K \hat{\otimes} l^\infty(N,s)$ and K is isomorphic to an orthocomplemented subspace of l^∞.

2. Suppose that K is not spherically complete. Although $E \not\sim F$ we have $E' \sim F'$.

Indeed, by 4.22.ii) of [7], $F' \sim c_0(N,1/s)$. Further, like in 3.6.ii) of [5], we can prove that $E' \sim (l^\infty)' \hat{\otimes} (l^\infty(N,s))' \sim c_0 \hat{\otimes} c_0(N,1/s)$. By (2), $E' \sim c_0(N,1/s)$.

3. Similarly to 3.2 we can prove that $E^\vee \sim F^\vee$.

4. COMPLEMENTATION OF THE SPACE OF COMPACT OPERATORS

It follows from [7], 4.41 that
$$l^\infty \hat{\otimes} F \sim C(c_0, F) \tag{3}$$
for every Banach space F over K. Then, applying 3.3 we obtain.

Theorem 4.1 : *Suppose that K is spherically complete and the valuation on K is dense. Let F be an infinite-dimensional and spherically complete Banach space over K. Then, $C(c_0, F)$ is not orthocomplemented in $L(c_0, F)$.*

Proof : F contains an orthogonal sequence $v_1, v_2, \dots$ such that $\| v_1 \| > \| v_2 \| > \dots$ and $lim_n \| v_n \| = 1$ ([7], 5.5). By 3.3 and (3), $C(c_0, F)$ is not spherically complete. Hence, since $L(c_0, F)$ is spherically complete ([7], 4.5), $C(c_0, F)$ is not isomorphic to a quotient of $L(c_0, F)$ ([7], 4.2), and we are done.

Remark 4.2 : Theorem 4.1 does not remain true when the valuation on K is discrete.
Indeed, by (1) and (3), $C(c_0, l^\infty)$ is spherically complete and so it is orthocomplemented in $L(c_0, l^\infty)$ ([7], 4.7).

However, for non-spherically complete fields one verifies.

Theorem 4.3 : *Let F be an infinite-dimensional Banach space over a non-spherically complete field K. Then, $C(c_0, F)$ is not complemented in $L(c_0, F)$.*

Proof : If $Q : L(c_0, F) \longrightarrow C(c_0, F)$ is a continuous linear projection, we define the following operators :

- $f : l^\infty \longrightarrow L(c_0, F)$ by

$$f(\xi)(x) = \sum_n \xi_n x_n y_n \qquad (\xi = (\xi_n) \in l^\infty, x = (x_n) \in c_0)$$

where $y_1, y_2, \dots$ is an infinite t-orthogonal sequence in F $(0 < t < 1)$ with $1 \leq \| y_n \| \leq 2$ for all n (the existence of a such sequence follows by [7], 3.16).

- $g : l^\infty \longrightarrow L(c_0, c_0^\vee)$ by

$$g(\xi)(x) = (S \circ f(\xi))(x) \qquad (\xi \in l^\infty, x \in c_0)$$

where $S : F \longrightarrow c_0^\vee$ is a continuous linear map whose restriction to $[\{y_1, y_2, \dots\}]$ is an homeomorphism (to see that a such map exists apply [7], 3.16.ii) and 4.8).

- $g_1 : l^\infty \longrightarrow C(c_0, c_0^\vee)$ by

$$g_1(\xi)(x) = (S \circ (Q[f(\xi)])(x) \qquad (\xi \in l^\infty, x \in c_0).$$

Then, $g = g_1$ on c_0 and therefore $g = g_1$ on l^∞ (recall that c_0 is weakly dense in l^∞, [7], 4.15). Hence, $f(\xi) \in C(c_0, F)$ for all $\xi \in l^\infty$. However, taking $\xi \in l^\infty$ with $\xi_n = 1$

for all n, we have that $f(\xi)(e_n) = y_n$ for all n and so $\{f(\xi)(e_1), f(\xi)(e_2), \ldots\}$ is not a compactoid subset of F ([7], 4.37), a contradiction.

Remark 4.4 : For a simpler proof of 4.3 when F is a polar space see [2], 3.3.i).

As a consequence of 4.3 we derive the following extension of Corollary 15 in [3].

Corollary 4.5 : *Suppose that K is not spherically complete. Let F be an infinite-dimensional Banach space such that every closed subspace of countable type of F is complemented (e.g. when F has a base, [7], 3.18). Then, the following are equivalent.*

i) C(E,F) = L(E,F).

ii) C(E,F) is complemented in L(E,F).

iii) There exists an infinite-dimensional Banach space G such that C(E,G) = L(E,G).

iv) There exists an infinite-dimensional Banach space G such that C(E,G) is complemented in L(E,G).

v) E does not contain a complemented closed subspace linearly homeomorphic to c_0.

Proof : Clearly $i) \Rightarrow ii) \Rightarrow iv)$ and $i) \Rightarrow iii) \Rightarrow iv)$.

$iv) \Rightarrow v)$ Suppose that there exists a continuous linear projection $Q : L(E,G) \longrightarrow C(E,G)$ and E contains a closed subspace H linearly homeonorphic to c_0 with continuous linear projection $P : E \longrightarrow H$. We define $S : L(H,G) \longrightarrow C(H,G)$ by

$$S(T)(x) = (Q(T \circ P))(x) \qquad (T \in L(H,G), x \in H).$$

Then, S is a continuous linear projection, and by 4.3 we derive a contradiction.

$v) \Rightarrow i)$ It follows from Theorem 8, iii) $\Rightarrow$ i) of [1] that $L(E, c_0) = C(E, c_0)$ (throughout her paper the spherical completeness of K is assumed, however this part holds without this assumption). Suppose there exists $T \in L(E,F) - C(E,F)$. Then, $T(E)$ contains an infinite-dimensional subspace M of countable type that is closed in F ([7], 4.40). If $P : T(E) \longrightarrow M$ is a continuous linear projection from $T(E)$ onto M, then $P \circ T \in L(E,M)$ is a surjective compact map, a contradiction ([7], 4.40).

Remark 4.6 :

1. If the valuation on K is discrete, 4.3 is false. (compare with 4.2).
Indeed, every closed subspace of a Banach space over K is complemented in that space ([7], 4.14).

2. Looking at 4.1 and 4.3, the following questions arise in a natural way.
PROBLEMS.
1. Is 4.1 true when F is not spherically complete?.
2. Is 4.3 true when K is spherically complete and the valuation on K is dense?. (Observe that an affirmative answer to Problem 2 implies an affirmative answer to Problem 1).

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C. Perez-Garcia
Departamento de Matematicas
Facultad de Ciencias
Universidad de Cantabria
Avda. de los Castros s/n
39071 Santander
SPAIN