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ON UNIFORM EXPONENTIAL N-DICHOTOMY

M. MEGAN and D.R. LATCU

The problem of uniform exponential N -dichotomy of evolutionary processes in Banach spaces is discussed. Generalizations of the some well-known results of R. Datko, Z. Zabczyk, S. Rollewicz and A. Ichikawa are obtained. The results are applicable for a large class of nonlinear differential equations.

I - INTRODUCTION.

Let X be a real or complex Banach space with the norm $\|\cdot\|$. Let T be the set defined by

$$T = \{(t, t_0) : 0 \leq t_0 \leq t < \infty\}$$

Let $\Phi(t, t_0)$ with $(t, t_0) \in T$ be a family of operators with domain $X_{t_0} \subset X$.

Definition 1.1

The family $\Phi(t, t_0)$ with $(t, t_0) \in T$ is called an *evolutionary process* if :

- e_1) $\Phi(t, t_0)x_0 \in X_t$ for all (t, t_0) and $x_0 \in X_{t_0}$;
- e_2) $\Phi(t, t_1)\Phi(t_1, t_0)x_0 = \Phi(t, t_0)x_0$ for $(t, t_1), (t_1, t_0) \in T$ and $x_0 \in X_{t_0}$;
- e_3) $\Phi(t, t)x = x$ for all $t \geq 0$ and $x \in X_t$;
- e_4) for each $t_0 \geq 0$ and $x_0 \in X_{t_0}$ the function $t \mapsto \Phi(t, t_0)x_0$ is continuous on $[t_0, \infty)$;
- e_5) there is a positive nondecreasing function $\varphi : (0, \infty) \rightarrow (0, \infty)$ such that $\|\Phi(t, t_0)x_0\| \leq \varphi(t - t_0)\|x_0\|$ for all $(t, t_0) \in T$ and $x_0 \in X_{t_0}$.

Throughout in this paper for each $t_0 \geq 0$ we denote by

$$X_{t_0}^1 = \{x_0 \in X_{t_0} : \Phi(\cdot, t_0)x_0 \in L_{t_0}^\infty(X)\} \quad \text{and} \quad X_{t_0}^2 = X_{t_0}^2 = X_{t_0} \setminus X_{t_0}^1$$

where $L_{t_0}^\infty(X)$ is the Banach space of X -valued function f defined a.e. on $[t_0, \infty)$, such that f is strongly measurable and essentially bounded.

Remark 1.1 If $x_0 \in X_{t_0}^1$ and $t \geq t_0$ then $\Phi(t, t_0)x_0 \in X_t^1$.

Indeed, if $x_0 \in X_{t_0}^1$ then

$$\Phi(\cdot, t)\Phi(t, t_0)x_0 = \Phi(\cdot, t_0)x_0 \in L_{t_0}^\infty(X) \subset L_t^\infty(X)$$

and hence $\phi(t, t_0)x_0 \in X_t^1$.

Let \mathcal{N} be the set of strictly increasing real functions N defined on $[0, \infty]$ which satisfies :

$$\lim_{t \rightarrow 0} N(t) = 0 \quad \text{and} \quad N(t, t_0) \leq N(t)N(t_0)$$

for all $t, t_0 \geq 0$.

Remark 1.2. It is easy to see that if $N \in \mathcal{N}$ then

- i) $N(t) > 0$ for every $t > 0$;
- ii) $N(0) = 0$ and $N(1) \geq 1$;
- iii) $\lim_{t \rightarrow \infty} N(t) = \infty$.

Definition 1.2. Let $N \in \mathcal{N}$. The evolutionary process $\Phi(\cdot, \cdot)$ is said to be *uniformly exponentially N -dichotomic* (and we write u.e - N-d.) if there are $M_1, M_2, \nu_1, \nu_2 > 0$ such that for all $t \geq s \geq t_0 \geq 0$ and $x_1 \in X_{t_0}^1, x_2 \in X_{t_0}^2$ we have :

$$Nd_1) \quad N(\|\Phi(t, t_0)x_1\|) \geq M_1 e^{-\nu_1(t-s)}.N(\|\Phi(s, t_0)x_1\|), \text{ and}$$

$$Nd_2) \quad N(\|(t, t_0)x_2\|) \leq M_2 e^{\nu_2(t-s)}.N(\|\Phi(s, t_0)x_2\|).$$

Particulary, for $N(t) = t$, if $\Phi(\cdot, \cdot)$ is u.e-N-d. then $\Phi(\cdot, \cdot)$ is called an *uniform exponential dichotomic* (and we write u.e.d.) process. If $\Phi(\cdot, \cdot)$ is u.e-N-d. (respectively u.e.d.) and $X_{t_0}^1 = X_{t_0}$ for every $t_0 \leq 0$ then $\Phi(\cdot, \cdot)$ is called an *uniform exponential N -stable* (respectively uniform exponential stable) *process*.

Remark 1.3. $\Phi(\cdot, \cdot)$ is u.e-N-d. if and only if the inequalities (d_1) and (d_2) from Definition 1.2. hold for all $t \leq s + 1 > s \leq t_0 \leq 0$.

$$\begin{aligned} \text{Indeed, if } t_0 \geq s \geq t \geq s + 1, x_1 \in X_{t_0}^1 \text{ and } T_2 \in X_{t_0}^2 \text{ then} \\ N(\|\Phi(t, t_0)x_1\|) \leq N(\varphi(t-s)).N(\|\Phi(s, t_0)x_1\|) \geq N(\varphi(1))N(\|(s, t_0)x_1\|) \geq \\ \leq N(\varphi(1)).e^{\nu_1 - \nu_1(t-s)}.N(\|\Phi(s, t_0)x_1\|) \end{aligned}$$

and

$$\begin{aligned} M_2.e^{\nu_2}.N(\|\Phi(s, t_0)x_2\|) \leq N(\|\Phi(s+1, t_0)x_2\|) \leq \\ \leq N(\varphi(s+1-t)).N(\|\Phi(t, t_0)x_2\|) \leq \\ \leq N(\varphi(1)).e^{\nu_2 - \nu_2(t-s)}.N(\|\Phi(t, t_0)x_2\|). \end{aligned}$$

A necessary and sufficient condition for the uniform exponential stability of a linear evolutionary process in a Banach space has been proved by Datko in [1]. The extension of Datko's theorem for uniform exponential dichotomy has been obtained by Preda and Megan in [3].

The case of uniform exponential- N -stable processes has been considered by Ichikawa in [2]. The particular case when the process is a strongly continuous semigroup of bounded linear operators has been studied by Zabczyk in [5] and Rolewicz in [4].

In this paper we shall extend these results in two directions. First, we shall give a characterization of u.e.- N -dichotomy, which can be considered as a generalization of Datko's theorem. Second, we shall not assume the linearity and boundedness of the process $\Phi(.,.)$. The obtained results are applicable for a large class of nonlinear differential equations described in [2].

II - PRELIMINARY RESULTS

An useful characterization of the uniform exponential- N -dichotomy property is given by

Proposition 2.1

The evolutionary process $\Phi(.,.)$ is u.e.- N -d. if and only if there are two continuous functions $\varphi_1, \varphi_2 : [0, \infty] \rightarrow (0, \infty)$ with the properties :

$$Nd'_1) \quad N(\|\phi(t, t_0)x_1\|) \leq \varphi_1(t-s)N(\|\Phi(s, t_0)x_1\|)$$

$$Nd'_2) \quad N(\|\phi(t, t_0)x_2\|) \leq \varphi_2(t-s)N(\|\Phi(s, t_0)x_2\|)$$

$$Nd'_3) \quad \lim_{t \rightarrow \infty} \varphi_1(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \varphi_2(t) = \infty \quad \text{for all } t \geq s \geq t_0 \geq 0,$$

$$x_1 \in X_{t_0}^1 \quad \text{and} \quad x_2 \in X_{t_0}^2.$$

Proof.

The necessity is obvious from Definition 1.2 for $\varphi_1(t) = M_1 e^{-\nu_1 t}$ and $\varphi_2(t) = M_2 e^{\nu_2 t}$.

The sufficiency. From (Nd'_3) it follows that there are $s_1, s_2 > 0$ such that $\varphi_1(s_1) < 1$ and $\varphi_2(s_2) > 1$. Then for all $t \geq s \geq t_0$ there are two natural numbers n_1 and n_2 such that

$$t - s = n_1 s_1 + r_1 = n_2 s_2 + r_2, \quad \text{where } r_1 \in [0, s_2].$$

From (e_5) and (Nd'_1) it results that if $t \geq s \geq t_0 \geq 0$ and $x_1 \in X_{t_0}^1$ then

$$N(\|\Phi(t, t_0)x_1\|) \leq N(\varphi(r_1))N(\|\Phi(s + n_1 s_1, t_0)x_1\|) \leq N(\varphi(s_1))\varphi_1(s_1)$$

$$.N(\|\Phi(s, t_0)x_1\|) \leq M_1 e^{-\nu_1(t-s)}.N(\|\Phi(s, t_0)x_1\|)$$

$$\text{where } M_1 = N(\varphi(s_1))e^{\nu_1 s_1} = \frac{N(\varphi(s_1))}{\varphi_1(x_1)} \quad \text{and} \quad \nu_1 = -\frac{\ln \varphi_1(s_1)}{x_1}.$$

Similary, if $t \geq s \geq t_0 \geq 0$ and $x_2 \in X_t^2$ then

$$N(\|\Phi(t, t_0)x_2\|) \geq \varphi_2(r_2)N(\|(s + n_2 s_2, t_0)x_2\|) \geq \varphi_2(r_2)\varphi_2(s_2)^n.$$

$$N(\|\Phi(s, t_0)x_2\|) \geq m_2 e^{\nu_2 n_2 s_2}.M(\|\Phi(s, t_0)x_2\|)$$

$$M_2 e^{\nu_2(t-s)}N(\|\Phi(s, t_0)x_2\|),$$

$$\text{where } m_2 = \inf_{0 \leq t \leq s_2} \varphi_2(t), \quad M_2 = \frac{m_2}{\varphi_2(s_2)} \quad \text{and} \quad \nu_2 = \frac{\ln \varphi_2(s_2)}{s_2}.$$

In virtue of Definition 1.2 it follows that $\Phi(.,.)$ is u.e-N-d.

Corollary 2.1.

The evolutionary process $\Phi(.,.)$ is u.e.d. if and only if there are two continuous functions $\varphi_1, \varphi_2 : (0, \infty) \rightarrow (0, \infty)$ with the properties :

$$d'_1) \quad \|\Phi(t, t_0)x_1\| \leq \varphi_1(t-s).\|\Phi(s, t_0)x_1\|,$$

$$d'_2) \quad \|(t, t_0)x_2\| \geq \varphi_2(t-s).\|\phi(s, t_0)x_2\|,$$

$$Nd'_3) \quad \lim_{t \rightarrow \infty} \varphi_1(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \varphi_2(t) = \infty.$$

for all $t \geq s \geq t_0 \geq 0$, $x_1 \in X_{t_0}^1$ and $x_2 \in S_{t_0}^2$.

Proof. : Is obvious from Proposition 2.1 for $N(t) = t$.

The relation between u. e-N-d. and u.e.d. properties is given by

Proposition 2.2 :

The evolutionary process $\Phi(.,.)$ is u.e.d. if and only if there is $N \in \mathcal{N}$ such that $\Phi(.,.)$ is u.e-N-d.

Proof :

The necessity is obvious from Definition 1.2.

The sufficiency. Suppose that there is $N \in \mathcal{N}$ such that $\Phi(.,.)$ satisfies the condition (Nd_1) and (Nd_2) from Definition 1.2.

Let $s_1, s_2, s_3 > 0$ such that $M_1 N(2) < e^{\nu_1 s_1}$, $N(2) < M_2 e^{\nu_2 s_2}$ and $N(s_3) < M_2$. If $t \geq s \geq t_0 \geq 0$ then there are two natural numbers n_1 and n_2 such that $t - s = n_1 s_1 + r_1 = n_2 s_2 + r_2$, where $r_1 \in (0, s_1)$ and $r_2 \in (0, s_2)$. Then for $s \geq t_0 \geq 0$ and $x_1 \in X_{t_0}^1$ we have

$$\begin{aligned} N(\|\Phi(s, t_0)x_1\|) &\geq \frac{e^{\nu_1 s_1}}{M_1} N(\|\Phi(s_1 + s, t_0)x_1\|) \\ &\geq N(2)N(\|\Phi(s + s_1, t_0)x_1\|) \geq N(2\|\Phi(s + s_1, t_0)x_1\|) \end{aligned}$$

and hence (because N is nondecreasing)

$$\begin{aligned} \|\Phi(s, t_0)x_1\| &\geq 2\|(s + s_1, t_0)x_1\| \text{ and (by induction)} \\ \|\Phi(s, t_0)x_1\| &\geq 2^n \|\Phi(s + ns_1, t_0)x_1\| \text{ for every natural number } n. \end{aligned}$$

Therefore for $t \geq s \geq t_0 \geq 0$ and $x_1 \in X_{t_0}^1$ we obtain that

$$\|\phi(t, t_0)x_1\| \leq \varphi(r_1) \|\Phi(s + n_1 s_1, t_0)x_1\| \leq \frac{\varphi(s_1)}{2^{n_1}} \|\Phi(s, t_0)x_1\|$$

and hence

$$(2.1) \quad \|\phi(t, t_0)x_1\| \leq \varphi_1(t - s) \|\Phi(s, t_0)x_1\| \text{ for } t \geq s \geq t_0 \text{ and } x_1 \in X_{t_0}^1, \text{ where } \varphi_1(u) = \frac{\varphi(s_1)}{2^{u/s_1}}.$$

On the other hand, for $s \geq t_0 \geq 0$ and $x_2 \in X_{t_0}^2$ we have

$$\begin{aligned} \|\phi(t, t_0)x_2\| &\geq M_2 e^{\nu_2 s_2} N(\|\Phi(s, t_0)x_2\|) \geq N(2)N(\|\phi(s, t_0)x_2\|) \\ &\geq N(2\|\Phi(s, t_0)x_2\|) \end{aligned}$$

and hence

$$\begin{aligned} \|\phi(s + s_2, t_0)x_2\| &\geq 2\|\phi(s, t_0)x_2\| \text{ and (by induction)} \\ \|\phi(s + ns_2, t_0)x_2\| &\geq 2^n \|\phi(s, t_0)x_2\| \text{ for all } s \geq t_0 \geq 0, x_2 \in X_{t_0}^2 \text{ and every natural} \\ &\text{number } n. \end{aligned}$$

hence, if $t \geq s \geq t_0 \geq 0$ and $x_2 \in X_{t_0}^2$ then

$$\begin{aligned} N(\|\Phi(t, t_0)x_2\|) &= N(\|\Phi(s + n_2 s_2 + r_2, t_0)x_2\|) M_2 e^{\nu_2 r_2} N(\|\Phi(s + n_2 s_2, t_0)x_2\|) \\ &\geq N(s_3 \|\Phi(s + n_2 s_2, t_0)x_2\|), \end{aligned}$$

which implies

$$\|\Phi(t, t_0)x_2\| \geq s_3 \|\Phi(s + n_2 s_2, t_0)x_2\| \geq 2^{n_2} s_3 \|\Phi(s, t_0)x_2\|$$

and hence

$$(2.2) \quad \|\Phi(t, t_0)x_2\| \geq \varphi_2(t - s) \|\Phi(s, t_0)x_2\| \text{ for } t \geq s \geq 0 \text{ and } x_2 \in X_{t_0}^2, \text{ where } \varphi_2(u) = \frac{s_3}{2} 2^{u/s_2}.$$

From (2.1), (2.2) and Corollary 2.1 it follows that $\Phi(.,.)$ is u.e.d.

3 - THE MAIN RESULTS.

The following theorem is an extension of Datko's theorem ([1]) to the general case of uniform exponential- N -dichtomy.

Theorem 3.1.

The evolutionary process $\Phi(.,.)$ is u.e- N -d. if and only if there are $M, m > 0$ such that

$$(Nd''_1) \quad \int_t^\infty N(\|\Phi(s, t_0)x_1\|)ds \leq M.N(\|\Phi(t, t_0)x_1\|),$$

$$(Nd''_2) \quad \int_{t_0}^t N(\|\Phi(s, t_0)x_2\|)ds \leq M.N(\|\Phi(t, t_0)x_2\|),$$

$$(Nd''_3) \quad N(\|\Phi(t+1, t_0)x_2\|) \leq m.N(\|\Phi(t, t_0)x_2\|)$$

for all $t \geq t_0 \geq 0, x_1 \in X_{t_0}^1$ and $x_2 \in X_{t_0}^2$.

Proof. The necessity is simply verified. Now we prove the sufficiency part.

Let $s \geq t_0 \geq 0, x_1 \in X_{t_0}^1$ and $\frac{1}{M_0} = \int_0^1 \frac{dt}{\psi(t)}$, where $\psi = N.\varphi$.

If $t \geq s+1$ then

$$\begin{aligned} \frac{N(\|\Phi(t, t_0)x_1\|)}{M_0} &= \int_0^1 \frac{N(\|\Phi(t, t_0)x_1\|)}{\psi(r)} dr \leq \int_s^1 \frac{N(\|\Phi(t, t_0)x_1\|)}{\psi(t-v)} dr \leq \\ &\leq \int_s^t N(\|\Phi(v, t_0)x_1\|)dv \leq \int_s^\infty N(\|\Phi(t, t_0)x_1\|)dv \leq M.N(\|\Phi(s, t_0)x_1\|) \end{aligned}$$

and hence

$$N(\|\Phi(t, t_0)x_1\|) \leq M.M_0 N(\|\Phi(s, t_0)x_1\|), \text{ for all } t \geq s+1 \geq t_0 \geq 0 \text{ and } x_1 \in X_{t_0}^1.$$

Therefore

$$\begin{aligned} (t-s-1)N(\|\Phi(t, t_0)x_1\|) &= \int_s^{t-1} N(\|\Phi(t, t_0)x_1\|)ds \leq M.M_0 \int_s^\infty N(\|\Phi(t, t_0)x_1\|)dv \leq \\ &\leq M^2.M_0.N(\|\Phi(s, t_0)x_1\|), \end{aligned}$$

which implies

$$(3.1) \quad N(\|\Phi(t, t_0)x_1\|) \leq \varphi_1(t-s)N(\|\Phi(s, t_0)x_1\|),$$

for all $t \geq s + 1 \geq s \geq t_0 \geq 0$ and $x_1 \in X_{t_0}^1$, where

$$\varphi_1(v) = \frac{M.M_0(1+M)}{1+v}$$

Let $t_0 \geq 0, x_2 \in X_{t_0}^2$ and $s \geq t_0 + 1$. Then

$$\begin{aligned} \frac{N(\|\Phi(s, t_0)x_2\|)}{M_0} &\leq N(\|\Phi(s, t_0)x_2\|) \int_{t_0}^s \frac{dv}{\psi(s-v)} \leq \int_{t_0}^s N(\|\Phi(v, t_0)x_2\|)dv \leq \\ &\leq \int_{t_0}^t N(\|\Phi(v, t_0)x_2\|)dv \leq M.N(\|\Phi(t, t_0)x_2\|) \end{aligned}$$

and hence

$$N(\|\Phi(t, t_0)x_2\|) \geq \frac{N(\|\Phi(s, t_0)x_2\|)}{M.M_0} \text{ for all } t \geq s \geq t_0 + 1 \text{ and } x_2 \in X_{t_0}^2.$$

If $t \geq s + 1 \geq s \geq t_0$ then (by preceding inequality and Nd_3'')

$$\begin{aligned} N(\|\Phi(y, t_0)x_2\|) &\geq \frac{N(\|\Phi(s+1, t_0)x_2\|)}{M.M_0} \geq \frac{mN(\|\Phi(s, t_0)x_2\|)}{M.M_0} \geq \\ &\geq \frac{N(\|\Phi(s, t_0)x_2\|)}{M_2} \end{aligned}$$

for all $x_2 \in X_{t_0}^2$, where $\frac{1}{M_2} = \min\{\frac{1}{M.M_0}, \frac{m}{M.M_0}\}$.

Therefore

$$\begin{aligned} (t-s-1)N(\|\Phi(t, t_0)x_2\|) &\leq M_2 \int_{s+1}^t N(\|\Phi(v, t_0)x_2\|)dv \leq M_2 \int_{t_0}^t N(\|\Phi(v, t_0)x_1\|)dv \\ &M.M_2N(\|\Phi(t, t_0)x_2\|), \end{aligned}$$

which implies

$$(3.2) \quad N(\|\Phi(t, t_0)x_2\|) \geq \varphi_2(t-s)N(\|\Phi(s, t_0)x_2\|)$$

for all $t \geq s + 1 \geq s \geq t_0 \geq 0$ and $x_2 \in X_{t_0}^2$, where $\varphi_2 = \frac{v+1}{M_2.(M+1)}$.

From (3.1), (3.2) and Proposition 2.1 it follows that $\Phi(.,.)$ is u.e-N-d.

As a particular case we obtain

Corollary 3.1

The evolutionary process $\Phi(.,.)$ is u. e. d. if and only if there are two positive constants M and m such that

$$(d_1'') \quad \int_t^\infty \|\Phi(s, t_0)x_1\| ds \leq M \cdot \|\Phi(t, t_0)x_1\|,$$

$$(d_2'') \quad \int_{t_0}^t \|\Phi(s, t_0)x_2\| ds \leq M \cdot \|\Phi(t, t_0)x_2\|,$$

$$(d_3'') \quad \|\Phi(t+1, t_0)x_2\| ds \geq m \|\Phi(t, t_0)x_2\|,$$

for all $t \geq t_0 \geq 0$, $x_1 \in X_{t_0}^1$ and $x_2 \in X_{t_0}^2$.

Proof. Is obvious from Theorem 3.1 for $N(t) = t$.

Remark 3.1 Corollary 3.1 is a nonlinear version of Theorem 3.2 from [3]. It is an extension of Theorem 2.1 from [2] from the general case of uniform exponential dichotomy.

Remark 3.2. Corollary 3.1 remains valid if the power 1 from (d_1'') and (d_2'') is replaced by any $p \in (1, \infty)$, i.e. the inequalities (d_1'') and (d_2'') can be replaced, respectively, by

$$(d_1''') \quad \int_t^\infty \|\Phi(s, t_0)x_1\|^p ds \leq M \cdot \|\Phi(t, t_0)x_1\|^p$$

and

$$(d_2''') \quad \int_{t_0}^t \|\Phi(s, t_0)x_2\|^p ds \leq M \cdot \|\Phi(t, t_0)x_2\|^p.$$

The proof follows almost verbatim from those given in the case $p = 1$ for $N(t) = t$.

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