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EXPONENTIAL-BESSEL PARTIAL DIFFERENTIAL EQUATION AND FOX'S H-FUNCTION

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ABSTRACT : In this paper, we present and solve a two dimensional Exponential-Bessel partial differential equation, and obtain a particular solution of it involving Fox's H−function.

1. - INTRODUCTION . The object of this paper is to formulate a two dimensional Exponential-Bessel partial differential equation and obtain its double series solution. We further present a particular solution of our Exponential-Bessel equation involving Fox's H−function. It is interesting to note that the particular solution also yields a new two dimensional series expansion for Fox's H-function involving exponential functions and Bessel functions.

The H-function introduced by Fox [5, p. 408], will be represented as follows :

(1.1) $$H_{p,q}^{m,n}\left[z \,\middle|\, {}^{(a_1,e_1),\ldots,(a_p,e_p)}_{(b_1,f_1),\ldots,(b_q,f_q)} \right] \equiv H_{p,q}^{m,n}\left[z \,\middle|\, {}^{(a_p,e_p)}_{(b_q,f_q)} \right].$$

In what follows for sake of brevity :

$$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \equiv A, \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \equiv B.$$

The following formulae are required in the proof :

The integral [2, p. 704, (2.2)] :

$$(1.2)\qquad \int_0^{\Pi} cos2ux(sin\frac{x}{2})^{-2w_1}\, H_{p,q}^{m,n}\left[z\,(sin\frac{x}{2})^{-2h} \mid {}^{(a_p,e_p)}_{(b_q,f_q)} \right] dx$$

$$= \sqrt{(\Pi)}H_{p+2,q+2}^{m+1,n+1}\left[z \,\Big|^{(1-w_1-2u,h),(a_p,e_p),(1-w_1+2u,h)}_{(1/2-w_1,h),(b_q,f_q),(1-w_1,h)} \right],$$

where $h > 0, \sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j \equiv A \leq 0, \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{1} e_j + \sum_{j=1}^{m} e_j - \sum_{j=m+1}^{q} f_j \equiv B > 0,$
$|argz| < 1/2B\Pi, Re(1-2w_1) - 2h \max_{1\leq j\leq n} [Re(a_j-1)/e_j] > 0.$

The integral [7, p. 94, (2.2)] :

$$(1.3)\qquad \int_0^{\infty} y^{w_2-1} siny J_v(y) H_{p,q}^{m,n}\left[zy^{2k} \mid {}^{(a_p,e_p)}_{(b_q,f_q)} \right] dy$$

$$= 2^{w_2-1}\sqrt{\Pi}\, H_{p+4,q+1}^{m+1,n+1}\left[2^{2k}z \;\middle|\; \begin{matrix} \left(\frac{1-w_2-v}{2},k\right), & (a_p,e_p), & \left(1+\frac{v-w_2}{2},k\right) \\ \left(1-\frac{v+w_2}{2},k\right), & \left(\frac{1+v-w_2}{2},k\right) & \\ (\frac{1}{2}-w_2,2k), & (b_q,f_q) & \end{matrix} \right],$$

where $k > 0, A \leq 0, B > 0, |argz| < 1/2B\Pi, Re(w_2+v) + 2k \min_{1\leq j\leq m} [Reb_j/f_j] > 0.$

The orthogonality property of the Bessel functions [6, p. 291, (6)] :

$$(1.4)\qquad \int_0^{\infty} x^{-1} J_{a+2n+1}(x)\, J_{a+2m+1}(x) dx$$

$$= \begin{cases} 0, \text{ if } \; m \neq n\,; \\ (4n+2a+2)^{-1}\,, \text{if } m = n\,,\; Rea+m+n > -1. \end{cases}$$

The following orthogonality property :

$$(1.5)\qquad \int_0^{\Pi} e^{2imx} cos2nx dx = \begin{cases} 0, m \neq n \\ \Pi/2, m = n \neq 0 \\ \Pi, m = n = 0. \end{cases}$$

2. TWO DIMENSIONAL EXPONENTIAL-BESSEL PARTIAL DIFFERENTIAL EQUATION

Let us consider

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} + y^2 u, \tag{2.1}$$

where $u \equiv u(x,y,t)$ and $u(x,y,0) = f(x,y)$.

To solve (2.1), we assume that (2.1) has a solution of the form :

$$u(x,y,t) = e^{4cr^2t+(v+2s+1)^2t}X(ix)Y(y). \tag{2.2}$$

The substitution of (2.2) into (2.1) yields :

$$-c\Big[X" + 4r^2X\Big]Y + X\Big[y^2Y" + yY' + \{y^2 - (v+2s+1)^2\}Y\Big] = 0. \tag{2.3}$$

We see that $X" + 4r^2X = 0$ has a solution $X = e^{2rix}$ and $y^2Y" + yY' + \{y^2 - (v+2s+1)^2\}Y = 0$ is Bessel equation [1, p. 200, (6.25)], with solution $Y = J_{v+2s+1}(y)$. Therefore the soluiton of (2.1) is of the form :

$$u(x,y,t) = e^{4cr^2t+(v+2s+1)^2t}e^{2rix}J_{v+2s+1}(y). \tag{2.4}$$

In view of the principle of superposition, the general solution of (2.1) is given by

$$u(x,y,t) = \sum_{r=-\infty}^{\infty}\sum_{s=0}^{\infty} A_{r,s}e^{4cr^2t+(v+2s+1)^2t+2rix}J_{v+2s+1}(y). \tag{2.5}$$

In (2.5), putting $t = 0$, we get

$$f(x,y) = \sum_{r=-\infty}^{\infty}\sum_{s=0}^{\infty} A_{r,s}e^{2rix}J_{v+2s+1}(y). \tag{2.6}$$

Multiplying both sides of (2.6) by $y^{-1}cos2uxJ_{v+2w+1}(y)$, integrating with respect to y from 0 to ∞ and with respect to x from 0 to Π, then using (1.4) and (1.5), the Fourier Exponential-Bessel coefficients are given by

$$A_{r,s} = \frac{4}{\Pi}(v+2s+1) \times \int_0^{\Pi}\int_0^{\infty} f(x,y)y^{-1}cos2rxJ_{v+2s+1}(y)dydx. \tag{2.7}$$

In view of the theory of double and multiple Fourier series given by Carslaw and Jaeger [3, pp. 180-183], and many other references, such as Erdélyi [4, pp. 64-65]etc..., the double series (2.6) is convergent, provided the function $f(x,y)$ is defined in the region $0 < x < \pi, 0 < y < \infty$. In brief, the double series (2.6) converges, if the double integral on the right hand side of (2.7) exists.

In the subsequent section, we take $f(x,y)$ as Fox's H-function and present another method to obtain Fourier exponential-Bessel coefficients $A_{r,s}$.

3. PARTICULAR SOLUTION INVOLVING FOX'S H-FUNCTION

The particular solution to be obtained is

$$u(x,y,t) = 2^{W_2+1} \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} e^{4cr^2t+(v+2s+1)^2t+2rix}(v+2+s+1)j_{v+2s+1}(y) \tag{3.1}$$

$$\times H^{m+2,n+2}_{p+6,q+3}\left[2^{2k}z \,\middle|\, \begin{matrix} (1-w_1-2r,h), \left(-\frac{w_2+v+2s}{2}, k\right), (a_p, e_p), \\ (1-w_1+2r,h), \left(1+\frac{v+2s+1-w_2}{2}, k\right), \\ \\ \left(1+\frac{v+2s+1+w_2}{2}, k\right), \left(1+\frac{v+2s-w_2}{2}, k\right) \\ (\frac{1}{2}-w,h), (\frac{1}{2}-w_2,k); (b_q,f_q), (1-w_1,h) \end{matrix} \right],$$

valid under the conditions of (1.2), (1.3) and (1;4).

Proof. Let

$$f(x,y) = \left(sin\frac{x}{2}\right)^{-2w_1} y^{w_2} siny H^{m,n}_{p,q}\left[z\left(sin\frac{x}{2}\right)^{-2h} y^{2k} \,\middle|\, \begin{matrix} (a_p, e_p) \\ \\ (b_q, f_q) \end{matrix} \right] \tag{3.2}$$

$$= \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} A_{r,s} e^{2irx} J_{v+2s+1}(y).$$

Equation (3.2) is valid, since $f(x,y)$ is defined in the region $0 < x < \Pi, 0 < y < \infty$.

Multiplying both sides of (3.2) by $y^{-1}J_{v+2w+1}(y)$ and integrating with respect to y from 0 to ∞, then using (1.3) and (1;4). Now multiplying both sides of the resulting expression by $cos2ux$ and integrating with respect to x from 0 to Π, then using (1.2) and (1.5), we obtain the value of $A_{r,s}$. Substituting this value of $A_{r,s}$ in (2.5), the expansion (3.1) is obtained.

NOTE 1 : The value of $A_{0,s}$ is one-half the value of $A_{r,s}$.

NOTE 2 : If we put $t = 0$ in (3.1), it reduces to a new two dimensional series expansion for Fox's H-function involving exponential functions and Bessel functions.

Since on specializing the parameters Fox's H-function yields almost all special functions appearing in applied mathematics and physical sciences. Therefore, the result (3.1) presented in this paper is of a general character and hence may encompass several cases of interest.

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