ANNALES MATHÉMATIQUES BLAISE PASCAL

BERTIN DIARRA

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Annales mathématiques Blaise Pascal, tome 1, nº 1 (1994), p. 85-103
<http://www.numdam.org/item?id=AMBP_1994__1_1_85_0>

Ann. Math. Blaise Pascal, Vol. 1, N°1, 1994, pp. 85-103

P-ADIC CLIFFORD ALGEBRAS

Bertin DIARRA

In a previous paper [2], we gave the index of the standard quadratic form of rank n over the field of p-adic numbers. Here, we recover, as a consequence, the structure of the associated Clifford algebra.

The classification of all (equivalence classes of) quadratic forms over a p-adic field is well known (cf.[5]), with this classification, one is able to classify all p-adic Clifford algebras.

I - INTRODUCTION

Let K be a field of characteristic $\neq 2$ and E a vector space over K of finite dimension n. A mapping $q : E \to K$ is a *quadratic form* over E if there exists a bilinear symmetric form $f : E \times E \to K$ such that

$$q(x) = f(x,x) \quad \text{and} \quad f(x,y) = \frac{1}{2}[q(x+y) - q(x) - q(y)]$$

We assume that q is *regular*, that is f is non-degenerated.

An element $x \in E$ is *isotropic* if $q(x) = 0$. Let V be a subspace of E ; the orthogonal subspace of V is the set $V^{\perp} = \{y \in E / f(x,y) = 0 \text{ for all } x \in V\}$. The subspace V is called *totally isotropic* if $V \subset V^{\perp}$. It is well known (cf. for example [1]) that any totally isotropic subspace is contained in a maximal totally isotropic subspace. The maximal totally isotropic subspaces have the same dimension ν, called the *index* of q and $2\nu \leq n$. If $2\nu = n$, then (E, q) is called a *hyperbolic* space and for the case $n = 2$, one says hyberbolic plane. The index $\nu = 0$ iff $q(x) \neq 0$ for $x \neq 0$ i.e. (E, q) is *anisotropic*.

Let $E = K^n$ and $B = (e_1, \ldots, e_n)$ be the canonical basis of E; the standard quadratic form q_0 is the quadratic form associated to the bilinear form

$$< x, y > = \sum_{j=1}^{n} x_j y_j \quad ; \quad \text{where} \quad x = \sum_{j=1}^{n} x_j e_j \quad \text{and} \quad y = \sum_{j=1}^{n} y_j e_j \quad ;$$

hence $q_0(x) =< x, x > = \sum_{j=1}^{n} x_j^2$.

Let (E, q) be a quadratic space, possibly non regular ; an algebra $C \doteq C(E, q)$ over K, with unit 1, is said to be a *Clifford algebra* for (E, q) if

(i) There exists a one-to-one linear mapping $\rho : E \to C$ such that $\rho(x)^2 = q(x) \cdot 1$.

(ii) For every algebra A with unit 1 and linear mapping $\phi : E \to A$ satisfying $\phi(x)^2 = q(x) \cdot 1$, there exists an algebra homomorphism $\widetilde{\phi} : C \to A$ such that $\widetilde{\phi} \circ \rho = \phi$.

Clifford algebra exists and is unique up algebra isomorphism (cf. for instance [1] or [3]). For example , let $K < X_1, \cdots, X_n >$ be the free algebra with free system of generators $X_1, \cdots, X_n$ and I be the two-sided ideal of $K < X_1, \cdots, X_n >$ generated by $X_i X_j + X_j X_i - 2f(e_i, e_j) \cdot 1, 1 \leq i, j \leq n$, where $(e_1, \cdots, e_n)$ is an orthogonal basis of (E, q); then $C(E, q) = K < X_1, \cdots, X_n > /I$.

II - THE P-ADIC STANDARD QUADRATIC FORM q_0

II - 1 . The index of q_0

Let p be a prime number and $\mathbf{Q}_p$ be the p-adic field i.e. the completion of the field of rational numbers $\mathbf{Q}$ for the p-adic absolute value.

We denote by $[\alpha]$ the integral part of the real number α.

Proposition 1 [2]

The standard quadratic form $q_0(x) = \sum_{j=1}^{n} x_j^2$ *over* $E = \mathbf{Q}_p^n$ *has index*

(i) $\nu = \left[\frac{n}{2}\right]$ *if* $p \equiv 1 \pmod{4}$

(ii) $\nu = \left[\frac{n}{2}\right]$ *if* $p \equiv 3 \pmod{4}$ *and* $n \not\equiv 2 \pmod{4}$

(iii) $\nu = \left[\frac{n}{2}\right] - 1$ *if* $p \equiv 3 \pmod{4}$ *and* $n \equiv 2 \pmod{4}$

Proof :

1°) If $p \equiv 1$ (mod.4), it is well known that $i = \sqrt{-1} \in \mathbf{Q}_p$. Let $\nu = \left[\frac{n}{2}\right]$ and $\epsilon_j = i\ e_{2j-1} + e_{2j},\ 1 \leq j \leq \nu$, then $V = \bigoplus_{j=1}^{\nu} \mathbf{Q}_p \epsilon_j$ is a maximal totally isotropic subspace of $E = \mathbf{Q}_p^n$.

2°) $p \equiv 3$ (mod.4)

Therefore $i \notin \mathbf{Q}_p$ and if $n = 2$ the index of q_0 is 0.

If $n = 3$, applying Chevalley's theorem and Newton's method to $q_0(x) = x_1^2 + x_2^2 + x_3^2$ we find $a,\ b \in \mathbf{Q}_p,\ a \neq 0,\ b \neq 0$, such that $a^2 + b^2 + 1 = 0$. Therefore $\epsilon_1 = a\ e_1 + b\ e_2 + e_3$ is isotropic in $\mathbf{Q}_p^3$ and $\nu = \left[\frac{3}{2}\right] = 1$.

(a) For $n = 4m$, put $\epsilon_{2j-1} = a\ e_{4j-3} + b\ e_{4j-2} + e_{4j-1}$ and $\epsilon_{2j} = -b\ e_{4j-3} + a\ e_{4j-2} + e_{4j}$, $1 \leq j \leq m$. It is clear that $q_0(\epsilon_{2j-1}) = q_0(\epsilon_{2j}) = a^2 + b^2 + 1 = 0$ and $< \epsilon_{2j-1}, \epsilon_{2j} > = -ab + ab = 0$. Therefore $V = \bigoplus_{j=1}^{m} (\mathbf{Q}_p \epsilon_{2j-1} \oplus \mathbf{Q}_p \epsilon_{2j})$ is a totally isotropic subspace of $\mathbf{Q}_p^n$ and $\nu = 2m = \left[\frac{n}{2}\right]$.

If $n = 4m + 1$, with the same notations as above the subspace V is totally isotropic in $\mathbf{Q}_p^n$ and $\nu = 2m = \left[\frac{n}{2}\right]$.

On the other hand if $n = 4m + 3$ the subspaces $V = \bigoplus_{j=1}^{m} (\mathbf{Q}_p \epsilon_{2j-1} \oplus \mathbf{Q}_p \epsilon_{2j})$ and $\mathbf{Q}_p \epsilon_{2m+1}$ where $\epsilon_{2m+1} = a\ e_{4m+1} + b\ e_{4m+2} + e_{4m+3}$, are totally isotropic and orthogonal. Therefore $V_\circ = V \oplus \mathbf{Q}_p\ \epsilon_{2m+1}$ is totally isotropic and $\nu = 2m + 1 = \left[\frac{n}{2}\right]$.

(b) If $n = 4m + 2$, let $V = \bigoplus_{j=1}^{m} (\mathbf{Q}_p \epsilon_{2j-1} \oplus \mathbf{Q}_p \epsilon_{2j})$ be as obove. It is easy to verify that if $x \in \mathbf{Q}_p^n$ is isotropic and x is orthogonal to V then $x \in V$. Therefore V is a maximal totally isotropic subspace of $\mathbf{Q}_p^n$ and $\nu = 2m = \left[\frac{n}{2}\right] - 1$.

Proposition 2 : *Let $p = 2$.*

Let $n = 8m + s$, $0 \leq s \leq 7$.

The standard quadratic form $q_0(x) = \sum_{j=1}^{n} x_j^2$ over $E = \mathbf{Q}_2^n$ has index

(i) $\nu = 4m$ *if* $0 \leq s \leq 4$

(ii) $\nu = 4m + t$ *if* $s = 4 + t$, $1 \leq t \leq 3$

Proof :

1°) If $1 \leq n \leq 4$, then the index of q_0 is 0.

Indeed, this is clear when $n = 1$.

If $n = 2$, let $x = x_1e_1 + x_2e_2 \in \mathbf{Q}_2^2$ be isotropic and different from 0 i.e. $q_0(x) = x_1^2 + x_2^2 = 0$ and say $x_2 \neq 0$. Therefore $1 + a^2 = 0$ with $a = x_1x_2^{-1}$ and $v_2(a) = 0$ i.e. $a = 1 + 2^{\mu}a_0$, $\mu \geq 1$, $v_2(a_0) = 0$.

Then $1 + a^2 = 2 + 2^{\mu+1}a_0 + 2^{2\mu}a_0^2 = 0$ or $1 + 2^{\mu}a_0 + 2^{2\mu-1}a_0 = 0$; in other words $1 \equiv 0$ (mod.2) ; a contradiction.

In the same way, one shows that if $n = 3$ or 4 , the index of q_0 is 0.

2°) n = 5

Let $x_0 = 2e_1 + e_2 + e_3 + e_4 + e_5 \in \mathbf{Q}_2^5$, then $q_0(x_0) = 8$ and $\dfrac{\partial q_0}{\partial x_j}(x_0) = 2 \not\equiv 0$ (mod.4), $2 \leq j \leq 5$

By Newton's method there exists

$x = \sum_{j=1}^{5} x_je_j \in \mathbf{Q}_2^5$ such that $q_0(x) = 0$ with $x_1 \equiv 2$ (mod.8) , $x_j \equiv 1$ (mod.8), $2 \leq j \leq 5$.

Put $a = x_1x_5^{-1}$, $b = x_2x_5^{-1}$, $c = x_3x_5^{-1}$, $d = x_4x_5^{-1}$, then $a^2 + b^2 + c^2 + d^2 + 1 = 0$.

The two following elements of $\mathbf{Q}_2^5$

$$\epsilon_1 = a\, e_1 + b\, e_2 + c\, e_3 + d\, e_4 + e_5$$

$$\epsilon_1' = -a\, e_1 - b\, e_2 - c\, e_3 - d\, e_4 + e_5$$

are isotropic with $< \epsilon_1, \epsilon_1' >= 2$. Hence $H = \mathbf{Q}_2\epsilon_1 \oplus \mathbf{Q}_2\epsilon_1'$ is a hyperbolic plane in $\mathbf{Q}_2^5$. Let $U = H^{\perp}$ be the orthogonal subspace of H in $\mathbf{Q}_2^5$. The following three elements of $\mathbf{Q}_2^5$:

$$u_1 = b\, e_1 - a\, e_2 + d\, e_3 - c\, e_4$$

$$u_2 = e_1 - \frac{ac + bd}{c^2 + d^2}e_3 + \frac{bc - ad}{c^2 + d^2}e_4$$

$$u_3 = e_2 + \frac{ad - bc}{c^2 + d^2} e_3 - \frac{ac + bd}{c^2 + d^2} e_4$$

are elements of U, with

$$q_0(u_1) = -1, \; q_0(u_2) = -\frac{1}{c^2 + d^2} = q_0(u_3)$$

Furthermore $< u_i, u_j >= 0$ if $1 \leq i \neq j \leq 3$, and (u_1, u_2, u_3) is a basis of U.

For every $u = y_1 u_1 + y_2 u_2 + y_3 u_3 \in U$ we have $q_0(u) = y_1^2 q_0(u_1) + y_2^2 q_0(u_2) + y_3^2 q_0(u_3) = -\frac{c^2 y_1^2 + d^2 y_1^2 + y_2^2 + y_3^2}{c^2 + d^2}$ and $q_0(u) = 0$ iff $u = 0$ because the standard quadratic form of rank 4 is anisotropic. In other words (U, q_0) is anisotropic and $\mathbf{Q}_2^5 = H \perp U$ is a Witt decomposition of $(\mathbf{Q}_2^5, q_0)$. Hence the index of q_0 is 1.

3°) <u>n = 8m +s</u> , $0 \leq s \leq 4$.
Put , for $0 \leq j \leq m - 1$

$$(1) \quad \begin{cases} \epsilon_{j,1} = a\, e_{8j+1} + b\, e_{8j+2} + c\, e_{8j+3} + d\, e_{8j+4} + e_{8j+5} \\ \epsilon_{j,2} = -b\, e_{8j+1} + a\, e_{8j+2} + d\, e_{8j+3} - c\, e_{8j+4} + e_{8j+6} \\ \epsilon_{j,3} = -d\, e_{8j+1} + c\, e_{8j+2} - b\, e_{8j+3} + a\, e_{8j+4} + e_{8j+7} \\ \epsilon_{j,4} = c\, e_{8j+1} + d\, e_{8j+2} - a\, e_{8j+3} - b\, e_{8j+4} + e_{8j+8} \end{cases}$$

and

$$(2) \quad \begin{cases} \epsilon'_{j,1} = -a\, e_{8j+1} - b\, e_{8j+2} - c\, e_{8j+3} - d\, e_{8j+4} + e_{8j+5} \\ \epsilon'_{j,2} = b\, e_{8j+1} - a\, e_{8j+2} - d\, e_{8j+3} + c\, e_{8j+4} + e_{8j+6} \\ \epsilon'_{j,3} = d\, e_{8j+1} - c\, e_{8j+2} + b\, e_{8j+3} - a\, e_{8j+4} + e_{8j+7} \\ \epsilon'_{j,4} = -c\, e_{8j+1} - d\, e_{8j+2} + a\, e_{8j+3} + b\, e_{8j+4} + e_{8j+8} \end{cases}$$

A straightforward computation shows that $< \epsilon_{i,k}, \epsilon_{j,l} >= 0 =< \epsilon'_{i,k}, \epsilon'_{j,l} >$, $0 \leq i, j \leq m - 1$; $1 \leq k, \; l \leq 4$ and $< \epsilon_{j,l}, \epsilon'_{j,l} >= 2$; $0 \leq j \leq m - 1$; $1 \leq l \leq 4$. Furthermore $< \epsilon_{i,k}, \epsilon'_{j,l} >= 0$ if $(i, k) \neq (j, l)$.

Hence the subspaces $V = \bigoplus_{\substack{j=0 \\ 1 \leq l \leq 4}}^{m-1} \mathbf{Q}_2 \epsilon_{j,l}$ and $W = \bigoplus_{\substack{j=0 \\ 1 \leq l \leq 4}}^{m-1} \mathbf{Q}_2 \epsilon'_{j,l}$ are isotropic with $V \cap W = (0)$

Therefore $H = V \oplus W$ is a hyperbolic subspace of $E = \mathbf{Q}_2^{8m+s}$, with $dimV = dimW = 4m$.

But $E = E_m \perp E_s$ (orthogonal sum) where $E_m = \bigoplus_{j=1}^{8m} \mathbf{Q}_2 e_j$ and $E_s = \bigoplus_{k=1}^{s} \mathbf{Q}_2 e_{8m+k} \simeq \mathbf{Q}_2^s$.

If $s = 0$, we have $E = E_m = V \oplus W = H$ and (E, q_0) is a hyperbolic space with index $4m$.

If $1 \leq s \leq 4$; $E = E_m \perp E_s$ with $E_m = V \oplus W = H$. Since $1 \leq dim E_s = s \leq 4$, the standard quadratic space (E_s, q_0) is anisotropic. Consequently $E = (V \oplus W) \perp E_s$ is a Witt decomposition of E and the index of q_0 is $4m$.

4°) n = 8m + 4 +t , $1 \leq t \leq 3$.

a) n = 8m + 5

With the same notations as above, we have $E = E_m \perp E_5$ where $E_5 = \bigoplus_{k=1}^{5} \mathbf{Q}_2 e_{8m+k} \simeq \mathbf{Q}_2^5$.

Let us write , as for $n = 5$,

$$(3) \quad \begin{cases} \epsilon_{4m+1} = a\, e_{8m+1} + b\, e_{8m+2} + c\, e_{8m+3} + d\, e_{8m+4} + e_{8m+5} \\ \epsilon'_{4m+1} = -a\, e_{8m+1} - b\, e_{8m+2} - c\, e_{8m+3} - d\, e_{8m+4} + e_{8m+5} \end{cases}$$

and

$$(4) \quad \begin{cases} u_{m+1} = b\, e_{8m+1} - a\, e_{8m+2} + d\, e_{8m+3} - c\, e_{8m+4} \\ u_{m+2} = e_{8m+1} - \frac{ac+bd}{c^2+d^2} e_{8m+3} + \frac{bc-ad}{c^2+d^2} e_{8m+4} \\ u_{m+3} = e_{8m+2} + \frac{ad-bc}{c^2+d^2} e_{8m+3} - \frac{ac+bd}{c^2+d^2} e_{8m+4} \end{cases}$$

The subspace $U_5 = \bigoplus_{h=1}^{3} \mathbf{Q}_2 u_{m+h}$ of E_5 is anisotropic. On the other hand, $q_0(\epsilon_{4m+1}) = 0 = q_0(\epsilon'_{4m+1})$; $< \epsilon_{4m+1}, \epsilon'_{4m+1} >= 2$ and $\epsilon_{4m+1}, \epsilon'_{4m+1}$ are orthogonal to U_5 . Therefore $V_0 = V \oplus \mathbf{Q}_2 \epsilon_{4m+1}$ and $W_0 = W \oplus \mathbf{Q}_2 \epsilon'_{4m+1}$ are isotropic subspaces of E and $E = (V_0 \oplus W_0) \perp U_5$ is a Witt decomposotion of E. Hence the index of q_0 is $dim\, V_0 = dim\, W_0 = 4m+1$.

(b) n = 8 m+6.

As before, we have $E = E_m \perp E_6$ where $E_6 = \bigoplus_{k=1}^{6} \mathbf{Q}_2 e_{8m+k} \supset E_5$; hence ϵ_{4m+1} and $\epsilon'_{4m+1} \in E_6$.

Let us put

$$(5)\quad \begin{cases} \epsilon_{4m+2} = -b\, e_{8m+1} + a\, e_{8m+2} + d\, e_{8m+3} - c\, e_{8m+4} + e_{8m+6} \\ \epsilon'_{4m+2} = b\, e_{8m+1} - a\, e_{8m+2} - d\, e_{8m+3} + c\, e_{8m+4} + e_{8m+6} \end{cases}$$

and

$$(6)\quad \begin{cases} \omega_{m+1} = e_{8m+1} + \frac{bd-ac}{c^2+d^2} e_{8m+3} - \frac{ad+bc}{c^2+d^2} e_{8m+4} \\ \omega_{m+2} = e_{8m+2} - \frac{bc+ad}{c^2+d^2} e_{8m+3} + \frac{ac-bd}{c^2+d^2} e_{8m+4} \end{cases}$$

The subspace $U_6 = \mathbf{Q}_2\omega_{m+1} \oplus \mathbf{Q}_2\omega_{m+2}$ of E_6 is anisotropic. Moreover, $q_0(\epsilon_{4m+2}) = 0 = q_0(\epsilon'_{4m+2})$; $< \epsilon_{4m+2}, \epsilon'_{4m+2} >= 2$ and ϵ_{4m+2}, ϵ'_{4m+2} are orthogonal to U_6. Therefore $V_1 = V_0 \oplus \mathbf{Q}_2\epsilon_{4m+2}$ and $W_1 = W_0 \oplus \mathbf{Q}_2\epsilon'_{4m+2}$ are isotropic subspaces of E and $E = (V_1 \oplus W_1)\perp U_6$ is a Witt decomposition of E. Hence the index of q_0 is $dim\ V_1 = dim\ W_1 = 4m+2$.

(c) n= 8m+7.

We have $E = E_m \perp E_7$, where $E_7 = \bigoplus_{k=1}^{7} \mathbf{Q}_2 e_{8m+k} \supset E_6$.

Let us write

$$(7)\quad \begin{cases} \epsilon_{4m+3} = -d\, e_{8m+1} + c\, e_{8m+2} - b\, e_{8m+3} + a\, e_{8m+4} + e_{8m+7} \\ \epsilon'_{4m+3} = d\, e_{8m+1} - c\, e_{8m+2} + b\, e_{8m+3} - a\, e_{8m+4} + e_{8m+7} \end{cases}$$

and

$$(8)\quad u_m = c\, e_{8m+1} + d\, e_{8m+2} - a\, e_{8m+3} - b\, e_{8m+4}$$

The subspace $U_7 = \mathbf{Q}_2 u_m$ of E_7 is anisotropic. Furthermore $q_0(\epsilon_{4m+3}) = 0 = q_0(\epsilon'_{4m+3})$; $< \epsilon_{4m+3}, \epsilon'_{4m+3} >= 2$ and ϵ_{4m+3}, ϵ'_{4m+3} are orthogonal to U_7. Therefore $V_2 = V_1 \oplus \mathbf{Q}_2\epsilon_{4m+2}$ and $W_2 = W_1 \oplus \mathbf{Q}_2\epsilon_{4m+2}$ are isotropic subspaces of E and $E = (V_2 \oplus W_2)\perp U_7$ is a Witt decomposition of E. Hence the index of q_0 is $dimV_2 = dimW_2 = 4m+3$.

Remark

Let K be a non formally real field . The level of K is the least integer s such that $-1 = \sum_{j=1}^{s} a_j^2$ where $a_j \in K, a_j \neq 0$. It is well known that $s = 2^r,\ r \geq 0$ (c f. [3] or [4]).

The level of a p-adic field is 1 if $p \equiv 1$ (mod 4) ; 2 *if* $p \equiv 3($ mod 4) *and* 4 *if* $p = 2$.

If the level of a field K is 1 (resp. 2, resp.4) *then the index of the standard quadratic form over* K^n *is given by* Proposition 1 - (i) [resp. Prop.1 - (ii) - (iii) , resp. Prop.2].

More generally let K be a field of level $s = 2^r, r \geq 0$. If we write for any integer $n, n = m2^{r+1} + a$ where $0 \leq a \leq 2^{r+1} - 1$; then the index of the standard quadratic form over K^n is

(i) $\nu = m2^r$ if $0 \leq a \leq 2^r$

(ii) $\nu = m2^r + t$ if $a = 2^r + t, \quad 1 \leq t \leq 2^r - 1$.

II - 2 The Clifford algebra $C(\mathbf{Q}_p^n, q_0)$

The following results can be deduced from a general setting (cf. [3] p. 128-129). Here we establish them by using the computation of the index of q_0 made in II-1.

Let us recall that if E is a vector space over a field K then the exterior algebra $\wedge(E)$ is the Clifford algebra associated to the null quadratic from over E.

On the other hand , let (E, q) be a regular quadratic space over K. If $E = V \oplus W$ is a hyperbolic space (V and W being maximal totally isotropic subspaces) , it is well known that the Clifford algebra $C(E, q)$ is isomorphic to *End*$(\wedge(V))$, the space of linear endomorphisms of the vector space $\wedge(V)$. Furthermore the subalgebra of the even elements of $C(E, q)$, say $C_+(E, q)$ is isomorphic to *End* $(\wedge_+(V)) \times$ *End*$(\wedge_-(V))$ where $\wedge_+(V)$ (*resp.* $\wedge_-(V)$) is the subspace of the even (resp. odd) elementts of $\wedge(V)$.

Generally, if $E = (V \oplus W) \perp U$ is a Witt decomposition of E, then $C(E, q) \simeq End(\wedge(V)) \otimes_2 C(U, q)$, the tensor product of $\mathbf{Z}/_{2\mathbf{Z}}$ - graded algebras (cf. for example [1]).

If $dimE = n$, then $dimC(E, q) = 2^n = dim \wedge (E)$.

If $a, b \in K^*$, we denote by $\left(\frac{a,b}{K}\right)$ the associated quaternion algebra : i.e. the algebra generated by i, j with $i^2 = a$; $j^2 = b$; $ij = -ji$. Also $\left(\frac{a,b}{K}\right)$ is the Clifford algebra of the rank 2 quadratic form $q(x) = ax_1^2 + bx_2^2$.

Let us write $M(n, K)$ the algebra of the $n \times n$ matrices with coefficients in K.

Theorem 1 : $p \equiv 1$ (mod. 4)

(i) *If* $n = 2m$, *then* $C(\mathbf{Q}_p^n, q_0) \simeq M(2^m, \mathbf{Q}_p)$

(ii) *If* $n = 2m+1$, *then* $C(\mathbf{Q}_p^n, q_0) \simeq M(2^m, \mathbf{Q}_p) \oplus M(2^m, \mathbf{Q}_p)$

Proof

Indeed, if $n = 2m$, then $(\mathbf{Q}_p^n, q_0)$ is a hyperbolic space.

It follows that $C(\mathbf{Q}_p^n, q_0) \simeq End(\wedge(\mathbf{Q}_p^m))$.

And, if $n = 2m+1$, we have a Witt decomposition $\mathbf{Q}_p^n = (V \oplus W) \perp U$ where $U = \mathbf{Q}_p e_n$.

It follows that $C(U, q_0) \simeq \mathbf{Q}_p \oplus \mathbf{Q}_p$ which gives (ii)

Theorem 2 : $p \equiv 3$ (mod. 4)

(i) *If* $n = 4m$, *then* $C(\mathbf{Q}_p^n, q_0) \simeq M(2^{2m}, \mathbf{Q}_p)$

(ii) *If* $n = 4m+1$, *then* $C(\mathbf{Q}_p^n, q_0) \simeq M(2^{2m}, \mathbf{Q}_p) \oplus M(2^{2m}, \mathbf{Q}_p)$

(iii) *If* $n = 4m+2$, *then* $C(\mathbf{Q}_p^n, q_0) \simeq M(2^{2m+1}, \mathbf{Q}_p)$

(iv) *If* $n = 4m+3$, *then* $C(\mathbf{Q}_p^n, q_0) \simeq M(2^{2m+1}, \mathbf{Q}_p[i])$

with $i = \sqrt{-1}$.

Proof :

The case (i) is evident , since $\mathbf{Q}_p^{2m}$ is a hyperbolic space.

If $\underline{n = 4m+1}$, we have a Witt decomposition $\mathbf{Q}_p^n = (V \oplus W) \perp U$ where $U = \mathbf{Q}_p u$ with $u = a\ e_{4m-3} + b\ e_{4m-2} + e_{4m-1} - e_{4m+1}$ and $q_0(u) = a^2 + b^2 + 1 + 1 = 1$. It follows that $C(U, q_0) \simeq \mathbf{Q}_p \oplus \mathbf{Q}_p$, which gives (ii).

If $\underline{n = 4m+2}$, we have a Witt decomposition $\mathbf{Q}_p^n = (V \oplus W) \perp U$ where

$$U = \mathbf{Q}_p u_1 \oplus \mathbf{Q}_p u_2 \text{ and } u_1 = a\ e_{4m-3} + b\ e_{4m-2} + e_{4m-1} + a\ e_{4m+1} + b\ e_{4m+2}$$
$$u_2 = -b\ e_{4m-3} + a\ e_{4m-2} + e_{4m} - b\ e_{4m+1} + a\ e_{4m+2}$$

Furthermore $< u_1, u_2 >= 0$, $q_0(u_1) = -1 = q_0(u_2)$ and $C(U, q_0) \simeq \left(\frac{-1,-1}{\mathbf{Q}_p}\right)$. This quaternion algebra contains an element z with $N(z) = a^2 + b^2 + 1 = 0$. Hence $\left(\frac{-1,-1}{\mathbf{Q}_p}\right) \simeq M(2, \mathbf{Q}_p)$ and finally we have $C(\mathbf{Q}_p^n, q_0) \simeq M(2^{2m}, \mathbf{Q}_p) \otimes_2 M(2, \mathbf{Q}_p) \simeq M(2^{2m+1}, \mathbf{Q}_p)$.

If $\underline{n = 4m+3}$, we have a Witt decomposition $\mathbf{Q}_p^n = (V \oplus W) \perp U$ where $U = \mathbf{Q}_p u$, with $u = -b\ e_{4m+1} + a\ e_{4m+2}$ and $q_0(u) = b^2 + a^2 = -1$. Hence $C(U, q_0) \simeq \mathbf{Q}_p[i]$, because $u^2 = q_0(u) = -1$.

We conclude that $C(\mathbf{Q}_p^n, q_0) \simeq M(2^{2m+1}, \mathbf{Q}_p[i])$.

In the proof of the forecoming theorem, one needs the following lemma

Lemma :

Let K be a field (char. $\neq 2$), $c, d \in K^*$ such that $c^2 + d^2 \neq 0$.

If $\sigma = \dfrac{1}{c^2+d^2}$, then $\left(\dfrac{-\sigma, -\sigma}{K}\right) \simeq \left(\dfrac{-1,-1}{K}\right)$.

If the two-rank quadratic forms $q_1(x) = -\sigma x_1^2 - \sigma x_2^2$ and $q_2(x) = -x_1^2 - x_2^2$ are equivalent, then their Clifford algebras are isomorphic. But, putting $x_1 = cx_1' + dx_2'$ and $x_2 = dx_1' - cx_2'$, we have $q_1(u(x')) = -\sigma(cx_1' + dx_2')^2 - \sigma(dx_1' - cx_2')^2 = -\sigma(c^2+d^2)(x_1'^2 + x_2'^2) = q_2(x')$. Hence q_1 and q_2 are equivalent and the lemma is proved.

Remark

The quaternion algebra $\left(\dfrac{-1,-1}{\mathbf{Q}_2}\right) = \mathbf{H}_2$ is a skew field.

Indeed, for any $z \in \mathbf{H}_2 = \left(\dfrac{-1,-1}{\mathbf{Q}_2}\right)$, $z \neq 0$, the norm of z is $N(z) = x_0^2+x_1^2+x_2^2+x_3^2 \neq 0$ (the standard quadratic form of rank 4 over $\mathbf{Q}_2$ is anisotropic).

Theorem 3 : $p = 2$

The Clifford algebra $C(\mathbf{Q}_2^n, q_0)$ is isomorphic to :

(0) $End(\wedge(\mathbf{Q}_2^{4m})) \simeq M(2^{4m}, \mathbf{Q}_2)$, *if* $n = 8m$

(1) $M(2^{4m}, \mathbf{Q}_2) \oplus M(2^{4m}, \mathbf{Q}_2)$, *if* $n = 8m+1$

(2) $M(2^{4m+1}, \mathbf{Q}_2)$, *if* $n = 8m+2$

(3) $M(2^{4m+1}, \mathbf{Q}_2[i])$, *with* $i = \sqrt{-1}$, *if* $n = 8m+3$

(4) $M(2^{4m+1}, \mathbf{H}_2)$, *if* $n = 8m+4$

(5) $M(2^{4m+1}, \mathbf{H}_2) \oplus M(2^{4m+1}, \mathbf{H}_2)$, *if* $n = 8m+5$

(6) $M(2^{4m+2}, \mathbf{H}_2)$, *if* $n = 8m+6$

(7) $M(2^{4m+3}, \mathbf{Q}_2[i])$, *if* $n = 8m+7$

Proof

According to the proof od Proposition 2, if $n = 8m+s, 0 \leq s \leq 7$, then $\mathbf{Q}_2^n = (V \oplus W) \perp E_s$ where V and W are totally isotropic subspaces of dimension $4m$, and $(E_s, q_0) \simeq$

$(\mathbf{Q}_2^s, q_0)$. It follows that $C(\mathbf{Q}_2^n, q_0) \simeq End(\wedge(\mathbf{Q}_p^{4m})) \otimes_2 C(\mathbf{Q}_2^s, q_0)$. It is easy to see that $C(\mathbf{Q}_2, q_0) \simeq \mathbf{Q}_2 \oplus \mathbf{Q}_2$; $C(\mathbf{Q}_2^2, q_0) \simeq \left(\frac{1,1}{\mathbf{Q}_2}\right) \simeq M(2, \mathbf{Q}_2)$ and $C(\mathbf{Q}_2^3, q_0) \simeq M(2, \mathbf{Q}_2[i])$.
If $s = 4$, the subalgebra, generated by e_1e_2, e_2e_4 and e_1e_4 , is isomorphic to $\left(\frac{-1,-1}{\mathbf{Q}_2}\right) = \mathbf{H}_2$. Hence $C(\mathbf{Q}_2^4, q_0) \simeq M(2, \mathbf{H}_2)$.
If $s = 5$, then $\mathbf{Q}_2^5 = F \perp U$, where F is a hyperbolic plane and U a three-dimensional anisotropic subspace, with orthogonal basis (u_1, u_2, u_3) satisfying $q_0(u_1) = -1, q_0(u_2) = -\sigma = q_0(u_3)$. $\left(\sigma = \frac{1}{c^2+d^2}\right.$ and $a, b, c, d \in \mathbf{Q}_2$ such that $\left. a^2 + b^2 + c^2 + d^2 + 1 = 0\right)$.
Therefore $C_+(U, q_0) \simeq \left(\frac{-\sigma,-\sigma}{\mathbf{Q}_2}\right) \simeq \mathbf{H}_2$; C_+ stands for the even subalgebra. But in $C(U, q_0), (u_1u_2u_3)^2 = \sigma^2$ is a square in $\mathbf{Q}_2$; therefore $C(U, q_0) \simeq \mathbf{H}_2 \oplus \mathbf{H}_2$. Furthermore $C(\mathbf{Q}_2^5, q_0) \simeq C(F, q_0) \otimes_2 C(U, q_0) \simeq M(2, \mathbf{H}_2) \oplus M(2, \mathbf{H}_2)$, because $C(F, q_0) \simeq M(2, \mathbf{Q}_2)$.
If $s = 6$, then $\mathbf{Q}_2^6 = F \perp U$, where F is a hyperbolic space of dimension 4 and U a two-dimensional anisotropic subspace with an orthogonal basis (u_1, u_2) satisfying $q(u_1) = -\sigma = q(u_2)$. Therefore $C(U, q_0) \simeq \left(\frac{-\sigma,-\sigma}{\mathbf{Q}_2}\right) \simeq \mathbf{H}_2$. And consequently $C(\mathbf{Q}_2^6, q_0) \simeq C(F, q_0) \otimes_2 C(U, q_0) \simeq M(2^2, \mathbf{H}_2)$.
If $s = 7$, then $\mathbf{Q}_2^7 = F \perp U$, where F is a hyperbolic space of dimension 6 and $U = \mathbf{Q}_2 u$, with $q_0(u) = -1$. Hence $C(U, q_0) \simeq \mathbf{Q}_2[i]$ and $C(\mathbf{Q}_2^7, q_0) \simeq M(2^3, \mathbf{Q}_2[i])$.
One deduces the isomorphisms of the therorem from $C(\mathbf{Q}_2^n, q_0) \simeq M(2^{4m}, \mathbf{Q}_2) \otimes_2 C(\mathbf{Q}_2^s, q_0)$.

N.B : A classical way to prove the above theorems is based on the isomorphisms

$$C(K^{n+2}, q_0) \simeq C(K^n, -q_0) \otimes C(K^2, q_0)$$

and $C(K^{n+2}, -q_0) \simeq C(K^n, q_0) \otimes C(K^2, -q_0)$

which give first 8-periodicity, etc ...

($-q_0$ is the opposite of the standard quadratic form q_0)

III - THE FAMILIES OF P-ADIC CLIFFORD ALGEBRAS

III-1. Equivalent classes of the p-adic quadratic forms

Let $a, b \in \mathbf{Q}_p^* = \mathbf{Q}_p \setminus \{0\}$. The Hilbert symbol (a, b) is defined by $(a, b) = 1$ if the quadratic form of rank 3, $q'(x) = x_0^2 - ax_1^2 - bx_2^2$ is isotropic $(a, b) = -1$ otherwise.

N.B. $(a, b) = 1$ iff $\left(\frac{a, b}{\mathbf{Q}_p}\right) \simeq M(2, \mathbf{Q}_p)$.

Let E be a vector space over $\mathbf{Q}_p$ of dimension n. Let us consider a regular quadratic form q over E. If $(e_j)_{1 \leq j \leq n}$ is an orthogonal basis of E and $a_j = q(e_j)$; then the discriminant $d(q)$ of q is equal to $a_1 \ldots a_n$ in the group $M_p = \mathbf{Q}_p^* / \mathbf{Q}_p^{*2}$. Let $\epsilon(q) = \prod_{1 \leq i < j \leq n} (a_i, a_j)$.

Theorem A

(i) *The p-adic regular quadratic forms q and q' of rank n are equivalent iff $d(q) = d(q')$ and $\epsilon(q) = \epsilon(q')$.*

(ii) *Let $d \in M_p$ and $\epsilon = \pm 1$. There exists a p-adic regular quadratic form q such that $d(q) = d$ and $\epsilon(q) = \epsilon$ iff*

(a) $n = 1$ *and* $\epsilon = 1$

(b) $n = 2$ *and* $(d, \epsilon) \neq (-1, -1)$

(c) $n \geq 3$

Proof : cf. [5]

According to that proof of Theorem A, one can give, explicitily, representatives of the equivalence classes of p-adic regular quadratic forms.

Let us recall that $M_2 = \{\pm 1, \pm 2, \pm 5, \pm 10\}$ and $M_p = \{1, p, \omega, \omega p\}$ if $p \neq 2$, where ω is a unit such that $\left(\frac{\omega}{p}\right) = -1$; $\left(\frac{-}{p}\right) =$ the Legendre symbol. Furthermore $-1 = 1$ in M_p if $p \equiv 1$ (mod.4) and $M_p = \{1, p, -1, -p\}$ if $p \equiv 3$ (mod.4).

We are content ourself here, with the primes p *different from* 2. Then a complete set of representatives of the equivalent classes of regular p-adic quadratic forms is obtained as follows.

(a) $\underline{n=1}$

Then $q^a(x) = ax^2, a \in M_p$; and the Clifford algebras $C(\mathbf{Q}_p, q^a)$ are isomorphic respectively to $\mathbf{Q}_p \oplus \mathbf{Q}_p$, $\mathbf{Q}_p[\sqrt{p}]$, $\mathbf{Q}_p[\sqrt{\omega}]$ and $\mathbf{Q}_p[\sqrt{\omega p}]$.

(b) $\underline{n=2}$

Then we have over $\mathbf{Q}_p^2$ (with $\omega = -1$ if $p \equiv 3$ (mod.4))

$q_0(x) = x_1^2 + x_2^2$		$q_4(x) = p\ x_1^2 + \omega\ p\ x_2^2$ if $p \equiv 1$ (mod.4)
$q_1(x) = x_1^2 + p\ x_2^2$	(resp.	$q_4(x) = p\ x_1^2 +\ p\ x_2^2$ if $p \equiv 3$ (mod.4))
$q_2(x) = \omega x_1^2 + \omega\ p\ x_2^2$		$q_5(x) = x_1^2 + \omega\ p\ x_2^2$
$q_3(x) = x_1^2 + \omega\ x_2^2$		$q_6(x) = p\ x_1^2 + \omega\ x_2^2$

Furthermore $\epsilon(q_\ell) = 1$ if $\ell = 0, 1, 3, 5$ and $\epsilon(q_\ell) = -1$ if $\ell = 2, 4, 6$.

N.B : If $p = 2$, then for $n = 2$, one has
8 regular quadratic forms q such that $\epsilon(q) = 1$
and 7 regular quadratic forms q such that $\epsilon(q) = -1$.

(c) $\underline{n=3}$

If (e_1, e_2, e_3) is the canonical basis of $\mathbf{Q}_p^3$, then

- $q'_\ell(x) = q_\ell(x_1e_1 + x_2e_2) + x_3^2,\ 0 \le \ell \le 6$

and

- $q'_7(x) = p\ x_1^2 + \omega x_2^2 + \omega\ p\ x_3^2 = q_6(x_1e_1 + x_2e_2) + \omega\ p\ x_3^2$ if $p \equiv 1$ (mod.4)

resp.

- $q'_7(x) = p\ x_1^2 - x_2^2 + p\ x_3^2 = q_6(x_1e_1 + x_2e_2) + p\ x_3^2$ if $p \equiv 3$ (mod.4)

Furthermore $d(q'_\ell) = d(q_\ell), \epsilon(q'_\ell) = \epsilon(q_\ell),\ 0 \le \ell \le 6$ and $d(q'_7) = -1, \epsilon(q'_7) = -1$.

(d) $\underline{n \ge 4}$

Let $(e_j)_{1\le j\le n}$ be the canonical basis of $\mathbf{Q}_p^n$, then

- $q''_\ell(x) = q_\ell(x_1e_1 + x_2e_2) + \sum_{j=3}^{n} x_j^2,\ 0 \le \ell \le 6.$

In other words $q''_\ell(x) = q_\ell(x_1e_1 + x_2e_2) + q_0\Big(\sum_{j=3}^{n} x_je_j\Big)$

i.e.$(\mathbf{Q}_p^n, q''_\ell) \simeq (\mathbf{Q}_p^2, q_\ell)\perp(\mathbf{Q}_p^{n-2}, q_0),\ 0 \le \ell \le 6$

and

- $q_7''(x) = q_7'\Big(\sum_{j=1}^{3} x_j e_j\Big) + \sum_{j=4}^{n} x_j^2 = q_7'\Big(\sum_{j=1}^{3} x_j e_j\Big) + q_0\Big(\sum_{j=4}^{n} x_j e_j\Big)$
  i.e.$(\mathbf{Q}_p^n, q_7'') \simeq (\mathbf{Q}_p^3, q_7') \perp (\mathbf{Q}_p^{n-3}, q_0)$

N.B : $p = 2$

If $n = 3$, then the classes of regular quadratic forms have 15 representative forms q' with $\epsilon(q') = 1$,resp.$\epsilon(q') = -1$ and $d(q') \neq -1$, obtained from corresponding representative quadratic forms of ranks 2 by adding the rank 1 form x_3^2. The other representative form is $q_{15}'(x) = -x_1^2 - x_2^2 - x_3^2$ with $\epsilon(q_{15}') = -1$ and $d(q_{15}') = -1$.

And if $n \geq 4$, one proceeds as above.

III - 2 The p-adic Clifford algebras

With the above notations , we have the following concrete propositions

Proposition 3 : $p \neq 2$

(i) $C(\mathbf{Q}_p^2, q_\ell) \simeq M(2, \mathbf{Q}_p)$ *if* $\ell = 0, 1, 3, 5.$

(ii) $C(\mathbf{Q}_p^2, q_\ell) \simeq \left(\frac{p, \omega}{\mathbf{Q}_p}\right) = \mathbf{H}_p =$ *the p-adic quaternion field , if* $\ell = 2, 4, 6.$

Proof

(i) Indeed, if $\ell = 0, 1, 3, 5$; then $\epsilon(q_\ell) = 1$. Therefore $C(\mathbf{Q}_p^2, q_\ell) \simeq M(2, \mathbf{Q}_p)$.

(ii) If $\ell = 2, 4, 6$ then the Clifford algebras $C(\mathbf{Q}_p^2, q_\ell)$ are isomorphic to the quaternion algebras with norm respectively ,

$N_2(z) = x_0^2 - \omega\, x_1^2 - \omega\, p\, x_2^2 + \omega^2\, p\, x_3^2$;

$N_4(z) = x_0^2 - p\, x_1^2 - \omega\, p\, x_2^2 + \omega\, p^2\, x_3^2$ if $p \equiv 1$ (mod.4) ;

(resp. $N_4(z) = x_0^2 - p\, x_1^2 - p\, x_2^2 + p^2 x_3^2$ if $p \equiv 3$ (mod. 4))

and $N_6(z) = x_0^2 - p\, x_1^2 - \omega\, x_2^2 + \omega\, p x_3^2$.

It is easily seen that these quadratic forms are anisotropic and equivalent. Therefore $C(\mathbf{Q}_p^2, q_2) \simeq C(\mathbf{Q}_p^2, q_4) \simeq C(\mathbf{Q}_p^2, q_6) \simeq \left(\frac{p, \omega}{\mathbf{Q}_p}\right) = \mathbf{H}_p$ is a skew fied. Hence $\mathbf{H}_p$ is the unique quaternion field over $\mathbf{Q}_p$ (according isomorphism). This result obtained directly here is a general result for local fields (cf. [3]).

Proposition 4 : $p \equiv 1$ (mod. 4)

The Clifford algebra $C(\mathbf{Q}_p^3, q'_\ell)$ *is isomorphic to*

(i)	$M(2,\mathbf{Q}_p) \oplus M(2,\mathbf{Q}_p)$	*if*	$\ell = 0$
(ii)	$M(2,\mathbf{Q}_p[\sqrt{p}])$	*if*	$\ell = 1,2$
(iii)	$M(2,\mathbf{Q}_p[\sqrt{\omega}])$	*if*	$\ell = 3,4$
(iv)	$M(2,\mathbf{Q}_p[\sqrt{\omega p}])$	*if*	$\ell = 5,6$
(v)	$\mathbf{H}_p \oplus \mathbf{H}_p$	*if*	$\ell = 7$

Similarly we have

Proposition 4' : $p \equiv 3$ (mod.4)

The Clifford algebra $C(Q_p^3, q'_\ell)$ *is isomorphic to*

(i)	$M(2,\mathbf{Q}_p[i])$	*if*	$\ell = 0,4$
(ii)	$M(2,\mathbf{Q}_p[\sqrt{-p}\,])$	*if*	$\ell = 1,2$
(iii)	$M(2,\mathbf{Q}_p) \oplus M(2,\mathbf{Q}_p)$	*if*	$\ell = 3$
(iv)	$M(2,\mathbf{Q}_p[\sqrt{p}])$	*if*	$\ell = 5,6$
(v)	$\mathbf{H}_p \oplus \mathbf{H}_p$	*if*	$\ell = 7$

Proof of Propositions 4 and 4'

Let us recall that if (E,q) is a regular quadratic space over a field K with $n = dim E$ odd, then $C(E,q) \simeq Z \otimes C_+(E,q)$, where Z is the centre of $C(E,q)$ and $C_+(E,q)$ the subalgebra of even elements. Furthermore, if $(e_1,\dots,e_n)$ is an orthogonal basis of (E,q) then $u = e_1 \dots e_n$ is such that $u^2 = (-1)^{[\frac{n}{2}]} d(q)$ and $Z = K[u]$.

In particular for $n = 3$ and $q(x) = \alpha\, x_1^2 + \beta\, x_2^2 + \gamma\, x_3^2$, we have $e_1^2 = \alpha, e_2^2 = \beta$, $e_3^2 = \gamma; u^2 = -\alpha\beta\gamma = \delta \neq 0$ and $C_+(E,q) =< 1, e_1e_2, e_1e_3, e_2e_3 >=$ subspace generated by $1,\dots,e_2e_3$. Put $E_1 = e_1e_2, E_2 = e_1e_3, E_3 = -\alpha e_2e_3$, hence $C_+(E,q) =< 1, E_1, E_2, E_3 >$ with $E_1^2 = -\alpha\beta, E_2^2 = -\alpha\gamma, E_1E_2 = E_3 = -E_2E_1$. Therefore $C_+(E,q) \simeq \left(\frac{-\alpha\beta,-\alpha\gamma}{K}\right)$.

Consequently (1) if $\delta \in K^{*2}$, then $Z \simeq K \oplus K$ and $C(E,q) \simeq \left(\frac{-\alpha\beta,-\alpha\gamma}{K}\right) \oplus \left(\frac{-\alpha\beta,-\alpha\gamma}{K}\right)$

(2) if $\delta \notin K^{*2}$, then $Z = K[u]$ is a field and $C(E,q) \simeq \left(\frac{-\alpha\beta,-\alpha\gamma}{K[u]}\right)$.

Applying these remarks to Propositions 4 and 4', one finds the desired isomorphisms. For example if $p \equiv 1$ (mod. 4) and $\ell = 2$, then $\delta = -\omega^2 p = (i\omega)^2 p$ and $Z = \mathbf{Q}_p[\sqrt{p}]$, hence

$C(\mathbf{Q}_p^3, q_2') \simeq \left(\frac{-\omega^2 p, -\omega}{\mathbf{Q}_p[\sqrt{p}]}\right) = \left(\frac{p, \omega}{\mathbf{Q}_p[\sqrt{p}]}\right) \simeq M(2, \mathbf{Q}_p[\sqrt{p}]) : \tilde{q}(v) = p\ x^2 + \omega y^2$ represents 1 over $\mathbf{Q}_p[\sqrt{p}]$. Also if $\ell = 7$, then $\delta = -p^2\omega^2 = (i\omega p)^2$, hence $Z \simeq \mathbf{Q}_p \oplus \mathbf{Q}_p$ and since $\left(\frac{-p\omega, -\omega p^2}{\mathbf{Q}_p}\right) \simeq \left(\frac{p\omega, \omega}{\mathbf{Q}_p}\right) \simeq \mathbf{H}_p$ we have $C(\mathbf{Q}_p^3, q_7') \simeq \mathbf{H}_p \oplus \mathbf{H}_p$.

In the case $p \equiv 3(\text{mod. } 4)$, for example if $\ell = 0 (resp. \ell = 3)$ we have $\delta = -1$ $(resp. = 1)$ and $Z \simeq \mathbf{Q}_p[i], (resp. Z \simeq \mathbf{Q}_p \oplus \mathbf{Q}_p)$. Hence $C(\mathbf{Q}_p^3, q_0') \simeq \left(\frac{-1,-1}{\mathbf{Q}_p[i]}\right) \simeq M(2, \mathbf{Q}_p[i]), \left(resp. C(\mathbf{Q}_p^3, q_3') \simeq \left(\frac{-1,-1}{\mathbf{Q}_p}\right) \oplus \left(\frac{-1,1}{\mathbf{Q}_p}\right) \simeq M(2, \mathbf{Q}_p) \oplus M(2, \mathbf{Q}_p)\right)$.

The other verifications are left to the reader.

Lemma 2 : $p \neq 2$

$C(\mathbf{Q}_p^4, q_7'') \simeq M(2, \mathbf{H}_p)$. □

Indeed, since $q_7'' = p\ x_1^2 + \omega\ x_2^2 + \omega'\ p\ x_3^2 + x_4^2$ where $\omega' = \omega$ if $p \equiv 1$ (mod.4) and $\omega = -1, \omega' = 1$ if $p \equiv 3$ (mod. 4) ; we have $C(\mathbf{Q}_7^4, q_7'') \simeq \left(\frac{p, \omega}{\mathbf{Q}_p}\right) \otimes_2 \left(\frac{\omega' p, 1}{\mathbf{Q}_p}\right) \simeq \mathbf{H}_p \otimes_2 M(2, \mathbf{Q}_p) \simeq M(2, \mathbf{H}_p)$.

Theorem 4 : $p \equiv 1$ (mod. 4) ; $n \geq 4$

1°) *If $n = 2m$, then the Clifford algebra $C(\mathbf{Q}_p^n, q_\ell'')$ is isomorphic to*

(i) $M(2^m, \mathbf{Q}_p)$ *if* $\ell = 0, 1, 3, 5$

(ii) $M(2^{m-1}, \mathbf{H}_p)$ *if* $\ell = 2, 4, 6, 7$

2°) *If $n = 2m + 1$, then the Clifford algebra $C(\mathbf{Q}_p^n, q_\ell'')$ is isomorphic to*

(i) $M(2^m, \mathbf{Q}_p) \oplus M(2^m, \mathbf{Q}_p)$ *if* $\ell = 0$

(ii) $M(2^m, \mathbf{Q}_p[\sqrt{\tau}])$ *if* $\ell = 1, 2, 3, 4, 5, 6$

with $\tau = p$ $(resp. \omega, resp. \omega p)$ for $\ell = 1, 2$ $(resp. \ell = 3, 4$; $resp. 5, 6)$.

(iii) $M(2^{m-1}, \mathbf{H}_p) \oplus M(2^{m-1}, \mathbf{H}_p)$ *if* $\ell = 7$

Proof :

1°) <u>n = 2m</u>

Notice that $C(\mathbf{Q}_p^n, q_\ell'') \simeq C(\mathbf{Q}_p^{n-2}, q_0) \otimes_2 C(\mathbf{Q}_p^2, q_\ell), 0 \leq \ell \leq 6$. But by Proposition 3, we have $C(\mathbf{Q}_p^2, q_\ell) \simeq M(2, \mathbf{Q}_p)$ if $\ell = 0, 1, 3, 5$ and $C(\mathbf{Q}_p^2, q_\ell) \simeq \mathbf{H}_p$ if $\ell =$

2, 4, 6. Since $C(\mathbf{Q}_p^{n-2}, q_0) \simeq M(2^{m-1}, \mathbf{Q}_p)$ by Theorem 1 - (i) - , we have $C(\mathbf{Q}_p^n, q_\ell'') \simeq M(2^{m-1}, \mathbf{Q}_p) \otimes_2 M(2, \mathbf{Q}_p) \simeq M(2^m, \mathbf{Q}_p)$ if $\ell = 0, 1, 3, 5$ and $C(\mathbf{Q}_p^n, q_\ell'') \simeq M(2^{m-1}, \mathbf{Q}_p) \otimes_2 \mathbf{H}_p \simeq M(2^{m-1}, \mathbf{H}_p)$ if $\ell = 2, 4, 6$.

For $\ell = 7$, applying Lemma 2 and Theorem 1 - (i) - we obtain $C(\mathbf{Q}_p^n, q_\ell'') \simeq C(\mathbf{Q}_p^{n-4}, q_0) \otimes_2 C(\mathbf{Q}_p^4, q_\ell'') \simeq M(2^{m-2}, \mathbf{Q}_p) \otimes_2 M(2, \mathbf{H}_p) \simeq M(2^{m-1}, \mathbf{H}_p)$.

2°) <u>n = 2m+1</u>

If $1 \leq \ell \leq 6$, then we have $C(\mathbf{Q}_p^n, q_\ell'') \simeq C(\mathbf{Q}_p^{n-3}, q_0) \otimes_2 C(\mathbf{Q}_p^3, q_\ell') \simeq M(2^{m-1}, \mathbf{Q}_p) \otimes_2 C(Q_p^3, q_\ell')$.

Applying Proposition 4, we obtain the isomorphism $C(\mathbf{Q}_p^n, q_\ell'') \simeq M(2^m, \mathbf{Q}_p[\sqrt{\tau}\ ])$ as claimed.

The case $\ell = 0$ is Theorem 1 - (ii) -

If $\ell = 7$, then $C(\mathbf{Q}_p^3, q_7') \simeq \mathbf{H}_p \oplus \mathbf{H}_p$ and $C(\mathbf{Q}_p^n, q_7'') \simeq M(2^{m-1}, \mathbf{Q}_p) \otimes_2 (\mathbf{H}_p \oplus \mathbf{H}_p) \simeq M(2^{m-1}, \mathbf{H}_p) \oplus M(2^{m-1}, \mathbf{H}_p)$.

Theorem 5 : $p \equiv 3(mod.4)$; $n \geq 4$

The Clifford algebra $C(\mathbf{Q}_p^n, q_\ell'')$ is isomorphic to the following matrix algebra or direct sum of two matrix algebras.

1°) <u>n = 4m</u>

(i) $M(2^{2m}, \mathbf{Q}_p)$ *if* $\ell = 0, 1, 3, 5$

(ii) $M(2^{2m-1}, \mathbf{H}_p)$ *if* $\ell = 2, 4, 6, 7$

2°) <u>n = 4m+1</u>

(i) $M(2^{2m}, \mathbf{Q}_p) \oplus M(2^{2m}, \mathbf{Q}_p)$ *if* $\ell = 0, 4$

(ii) $M(2^{2m}, \mathbf{Q}_p[\sqrt{\tau}])$ *if* $\ell = 1, 2, 3, 5, 6, 7$

with $\tau = p\ (resp. -1, res. -p)$ *for* $\ell = 1, 2\ (resp. \ell = 3, 7, resp. \ell = 5, 6)$.

3°) <u>n = 4m+2</u>

(i) $M(2^{2m+1}, \mathbf{Q}_p)$ *if* $\ell = 0, 1, 3, 5$

(ii) $M(2^{2m}, \mathbf{H}_p)$ *if* $\ell = 2, 4, 6, 7$

4°) n = 4m+3

(i) $M(2^{2m+1}, \mathbf{Q}_p) \oplus M(2^{2m+1}, \mathbf{Q}_p)$ *if* $\ell = 3$

(ii) $M(2^{2m+1}, \mathbf{Q}_p[\sqrt{\tau}])$ *if* $\ell = 0,1,2,4,5,6,$

with $\tau = -1$ $(resp. -p, res.p)$ *for* $\ell = 0,4$ $(resp.\ \ \ell = 1,2, resp.\ \ \ell = 5,6)$.

(iii) $M(2^{2m}, \mathbf{H}_p) \oplus M(2^{2m}, \mathbf{H}_p)$ *if* $\ell = 7$.

Proof :

1°) n = 4m

As in Lemma 2, it is readily seen that $C(\mathbf{Q}_p^4, q_\ell'') \simeq M(2^2, \mathbf{Q}_p)$ if $\ell = 0,1,3,5$ and $C(\mathbf{Q}_p^4, q_\ell'') \simeq M(2, \mathbf{H}_p)$ if $\ell = 2,4,6,7$.

If $n = 4m,\ m \geq 2$, we have $C(\mathbf{Q}_p^n, q_\ell'') \simeq C(\mathbf{Q}_p^{n-4}, q_0) \otimes_2 C(\mathbf{Q}_p^4, q_\ell'')$. But Theorem 3 - (i) - gives $C(\mathbf{Q}_p^{n-4}, q_0) \simeq M(2^{2m-2}, \mathbf{Q}_p)$. Therefore $C(\mathbf{Q}_p^n, q_\ell'') \simeq M(2^{2m-2}, \mathbf{Q}_p) \otimes_2 M(2^2, \mathbf{Q}_p) \simeq M(2^{2m}, \mathbf{Q}_p)$ if $\ell = 0,1,3,5$ and $C(\mathbf{Q}_p^n, q_\ell'') \simeq M(2^{2m-2}, \mathbf{Q}_p) \otimes_2 M(2^2, \mathbf{H}_p) \simeq M(2^{2m-1}, \mathbf{H}_p)$ if $\ell = 2,4,6,7$.

2°) n = 4m+1

With notations used in the proof of Propositions 4 and 4' we have $C(\mathbf{Q}_p^n, q_\ell'') \simeq Z \otimes C_+(\mathbf{Q}_p^n, q_\ell'')$ and $Z = \mathbf{Q}_p[u]$ where $u^2 = d(q_\ell'')$. Hence Z is isomorphic to $\mathbf{Q}_p \oplus \mathbf{Q}_p$ if $\ell = 0,4$; *resp.* $\mathbf{Q}_p[\sqrt{p}\,]$ if $\ell = 1,2$; *resp.* $\mathbf{Q}_p[\sqrt{-1}\,]$ if $\ell = 3,7$; *resp.* $\mathbf{Q}_p[\sqrt{-p}\,]$ if $\ell = 5,6$. On the other hand $C_+(\mathbf{Q}_p^n, q_\ell'') \simeq C_+(\mathbf{Q}_p \cdot x_n, x_n^2) \otimes C(\mathbf{Q}_p^{n-1}, -q_\ell'') \simeq C(\mathbf{Q}_p^{n-1}, -q_\ell'') \simeq M(2^{2m}, \mathbf{Q}_p)$. Hence $C(\mathbf{Q}_p^n, q_\ell'') \simeq Z \otimes M(2^{2m}, \mathbf{Q}_p)$ which proves the isomorphisms.

3°) n = 4m+2

Since n - 2 = 4m, we obtain $C(\mathbf{Q}_p^n, q_\ell'') \simeq C(\mathbf{Q}_p^{4m}, q_0) \otimes_2 C(\mathbf{Q}_p^2, q_\ell)$.

By Theorem 2 - (i) - one has $C(\mathbf{Q}_p^{4m}, q_0) \simeq M(2^{2m}, \mathbf{Q}_p)$ and by Proposition 3, $C(\mathbf{Q}_p^2, q_\ell) \simeq M(2, \mathbf{Q}_p)$ if $\ell = 0,1,3,5$ and $C(\mathbf{Q}_p^2, q_\ell) \simeq \mathbf{H}_p$ if $\ell = 2,4,6$. It follows that $C(\mathbf{Q}_p^n, q_\ell'') \simeq M(2^{2m+1}, Q_p)$ if $\ell = 0,1,3,5$ and $C(\mathbf{Q}_p^n, q_\ell'') \simeq M(2^{2m}, \mathbf{H}_p)$ if $\ell = 2,4,6$.

For the case $\ell = 7$, since $n - 4 = 4(m-1) + 2$ we have $C(\mathbf{Q}_p^n, q_7'') \simeq C(\mathbf{Q}_p^{n-4}, q_0) \otimes_2 C(\mathbf{Q}_p^4, q_\ell'')$. By theorem 2 - (iii) -, $C(\mathbf{Q}_p^{n-4}, q_0) \simeq M(2^{2m+1}, \mathbf{Q}_p)$ and by Lemma 2, $C(\mathbf{Q}_p^4, q_7'') \simeq M(2, \mathbf{H}_p)$. Hence $C(\mathbf{Q}_p^n, q_7'') \simeq M(2^{2m}, \mathbf{H}_p)$.

Notice that in 1°) and 3°) the exponent of 2 is $\frac{n}{2}$.

4°) <u>n = 4m+3</u>

Here, n-3 = 4m and $C(\mathbf{Q}_p^n, q_\ell'') \simeq C(\mathbf{Q}_p^{4m}, q_0) \otimes_2 C(\mathbf{Q}_p^3, q_\ell')$. But $C(\mathbf{Q}_p^{4m}, q_0) \simeq M(2^{2m}, \mathbf{Q}_p)$ and by Proposition 4' , $C(\mathbf{Q}_p^3, q_\ell')$ is isomorphic to $M(2, \mathbf{Q}_p) \oplus M(2, \mathbf{Q}_p)$ if $\ell = 3$, *resp.* $\mathbf{H}_p \oplus \mathbf{H}_p$ if $\ell = 7$, *resp.* $M(2, \mathbf{Q}_p[\sqrt{\tau}\,])$ if $\ell = 0,1,2,4,5,6$ with $\tau = -1$ for $\ell = 0,4$; $\tau = -p$ for $1,2$ and $\tau = p$ for $\ell = 5,6$.

Taking tensor product we obtain the desired isomorphisms.

Remark :

As for $C(\mathbf{Q}_p^n, q_0)$, for the other Clifford algebras $C(\mathbf{Q}_p^n, q_\ell'')$ we have 2-periodicity when $p \equiv 1$ (mod. 4) and 4-periodicity when $p \equiv 3$ (mod. 4).

N.B. When $p = 2$, in the same way one can give as obove the table of the 2-adic Clifford algebras.

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Université Blaise Pascal, Complexe Scientifique des Cézeaux, Laboratoire de Mathématiques Pures, 63177 AUBIERE Cedex.

manuscrit reçu en juin 1993