HARMONIC AND QUASI-HARMONIC SPHERES, PART III. RECTIFIABLITY OF THE PARABOLIC DEFECT MEASURE AND GENERALIZED VARIFOLD FLOWS

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ABSTRACT. – We study weakly convergent sequences of suitable weak solutions of heat flows of harmonic maps or approximated harmonic maps. We prove a dimensional stratification for the space-time concentration measure and verify that the concentration measure, viewed as a generalized varifold, moves according to the generalized varifold flow rule which reduces to the Brakke's flow of varifold provided that the limiting harmonic map flow is suitable. We also establish an energy quantization for the density of the limiting varifold. © 2002 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Nous étudions des séquences faiblement convergentes de solutions faibles du flot de chaleur d'applications harmoniques (éventuellement approximées). Nous prouvons une stratification dimensionnelle pour la mesure de concentration de l'espace-temps et vérifions que la mesure de concentration, vue comme une varifold generalisée, est sujette à la règle du flot généralisé des varifolds qui se réduit à la règle du flot de Brakke pour autant que l'application harmonique soit adéquate. Nous établissons aussi une quantification de l'énergie pour la densité de de la varifold limite de la séquence. © 2002 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

This is the third part of our project initiated in Lin and Wang [31] on the study of the weakly convergent sequence of smooth (or certain classes of weak) solutions to the heat equation of harmonic maps or approximated harmonic maps (i.e. the negative gradient flow of the generalized Ginzburg–Landau functionals). The general situation for the

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heat flow of harmonic maps is as follows. Let $u_n(x, t): M \times R_+ \to N$ be a sequence of smooth solutions to the heat flow of harmonic maps from a m-dimensional compact smooth Riemannian manifold M (with possibly nonempty smooth boundary ∂M) into a compact smooth Riemannian manifold $N \subset R^k$ without boundary, namely,

$$\partial_t u_n - \Delta u_n = A(u_n)(Du_n, Du_n), \quad \text{in } M \times R_+ \qquad (1.0)$$

where $A(\cdot)(\cdot, \cdot)$ denotes the second fundamental form of N in R^k, such that $u_n(x, t)$ weakly converges to $u(x, t)$ in $H^1_{\text{loc}}(M \times R_+, R^k)$. By the Fatou's lemma, we may assume that

$$\frac{1}{2}|Du_n|^2(x, t)\, dx\, dt \to \frac{1}{2}|Du|^2(x, t)\, dx\, dt + \nu \qquad (1.1)$$

and

$$|\partial_t u_n|^2(x, t)\, dx\, dt \to |\partial_t u|^2(x, t)\, dx\, dt + \eta \qquad (1.2)$$

as convergence of Radon measures on $M \times R_+$ for some nonnegative Randon measures ν, η supported on the so-called energy concentration set $\Sigma \subset M \times R_+$ (see [31]). It is easy to check that $\nu = \nu_t\, dt$ for some nonnegative Radon measures ν_t, $t \in R_+$ (see, Lemma 2.5 below). The main result of [31] is (see Lin [23] for the static cass) to characterize the necessary and sufficient conditions, under which both ν and η vanish, in terms of the existence (or non-existence) of harmonic and quasi-harmonic spheres into N. In other words, the necessary and sufficient conditions for such weakly convergent sequences to be strongly convergent. As a consequence of such a characterization is a new proof of the classical theorem by Eells and Sampson [15] (without the nonpositive curvature condition on N) under a new set of necessary and sufficient conditions (cf. [31]). In general, we showed in [31] that, without any extra assumption on N, both ν and η are supported on the energy concentration set Σ, which is closed and has locally finite m-dimensional Hausdorff measure, $\mathcal{P}^m$, with respect to the parabolic metric on $M \times R_+$, and for $\mathcal{P}^m$ almost all such points in the energy concentration set Σ, one has the m-dimensional density of ν (with respect to the parabolic metric) strictly positive and finite. Moreover, for L^1 a.e. $t \in R_+$, ν_t has the $(m-2)$-dimensional density (with respect to the Euclidean metric on M) positive and finite for H^{m-2} a.e. $x \in M$. In fact, it was shown by Cheng [9] that for all $t \in R_+$, the support of ν_t has locally finite $(m-2)$-dimensional Hausdorff measure. It is not very hard to generalize the argument of Lin [23] to show that for L^1 a.e. $t \in R_+$, $\Sigma_t = \Sigma \cap \{t\}$ and the support of ν_t are $(m-2)$-rectifiable. Here we shall adopt a different and conceptually much easier approach in Section 4, namely the generalized varifold approach which is a natural extension of the classical varifold concept of Almgren [3,4] and Allard [6]. Roughly, we associate each u_n with a $(m-2)$-generalized varifold V_{u_n} on $M \times R_+$ and show that V_{u_n} converges to a $(m-2)$-generalized varifold $V = V_t\, dt$, V_t has its generalized mean curvature $H_t \in L^2_{\|V_t\|}(M, R^m)$ for L^1 a.e. $t \in R_+$, and then the extension of Allard's rectifiablity result from classical varifolds to generalized varifolds yields that $V_t L(\{x \in M: \Theta^{m-2}(\|V_t\|, x) > 0\})$ is $(m-2)$-rectifiable. This rectifiablity result of Σ_t was also proved in a recent paper of Li and Tian [29] for so-called strongly stationary weakly heat flow of harmonic maps which are weak solutions of the heat flow (1.0) with energy

monotonicity, energy inequality, and the small energy regularity properties, where they verified the condition of Preiss's rectifiablity theorem. As pointed out in [31], although all the analysis in the present article and in [31,32] are for smooth solutions of the heat equation of harmonic maps (or solutions to the gradient flow of the Ginzburg–Landau functionals), the facts that we need are exactly these three properties stated above (see also Section 7 below), therefore all the results of the present article and [31,32] remain to be true for the class of weak solutions to the heat flow of harmonic maps satisfying these three properties. For simplicity of descriptions, we will work for solutions to heat flows of the Ginzburg–Landau functionals only and state some analogous conclusions in Section 7 for this class of weak solutions of heat flows for harmonic maps.

One of the main results of the present paper is to show the pair $(u, v_t \, \mathrm{d}t)$ satisfies the so-called generalized varifold flow (see Definition 5.5 of Section 5 below for the definition), as in the recent very interesting work by Ambrosio and Soner [1]. Moreover, in the case that u is a suitable weak solution (e.g. u is smooth), i.e. satisfying the standard energy equality in both local and global forms, then $\{v_t\}_{t \geqslant 0}$ is a Brakke flow of (rectifiable) varifolds, i.e. flow by the mean curvature in the varifold sense defined by Brakke [8]. We point out that a weaker version of this fact was also shown by [29] where a factor $\frac{1}{2}$ is putted in front of the mean curvature square term of the energy inequality (5.7). To improve the factor $\frac{1}{2}$ to 1 and to establish that the flow is actually the Brakke flow is one of the most difficult analytical points in all such related analysis (see also discussions in [1], in particular §6 of [1]). To achieve such a goal, one method is to establish the local almost conformal property of the solution map restricted to the 2-dimensional plane orthogonal to the tangent plane of the energy concentration set Σ (see Section 5 below). We also establish in Theorem 6.7 the energy quantization result in dimension large than two, which extends the main result of our part II [32]. Note that, in the case $N = S^{k-1}$, Theorem 6.7 can be used to give an alternative proof of the improvement of $\frac{1}{2}$ to 1. However, the argument of Section 5 is independent of N. The energy quantization in dimension at least 3 was first established by Lin and Rivieré [27] for stationary harmonic maps into spheres, by Lin and Wang [28] for approximated harmonic maps, and by Lin and Rivieré [28] for Ginzburg–Landau vortices in R^3. Our result here can be viewed as parabolic version of [27,32].

Since we can treat smooth solutions to the heat flow of harmonic maps in almost the same way as that of the heat equation of the generalized Ginzburg–Landau functional. For simplicity, we will present our result in the context of solutions to the heat flow of generalized Ginzburg–Landau functional and make remarks concerning the heat flow of harmonic maps in Section 7. Now let us describe precisely the results and the structure of the present article.

For $\varepsilon > 0$, we consider the (generalized) Ginzburg–Landau functional

$$I_\varepsilon(u) = \int_M \left(\frac{1}{2}|Du|^2 + \frac{1}{\varepsilon^2} F(u) \right) \mathrm{d}x$$

where $F \in C^\infty(R^k, R)$ satisfies:

$$F(p) = \begin{cases} d^2(p, N), & \text{if } d(p, N) \leqslant \delta, \\ 4\delta^2, & \text{if } d(p, N) \geqslant 2\delta. \end{cases}$$

Here d denotes the Euclidean distance in R^k and $d(\cdot, N) = \inf\{d(\cdot, p): p \in N\}$. Note that $\delta > 0$ is chosen to be so small that $d^2(p, N)$ is smooth for $p \in N_{2\delta} \equiv \{p: d(p, N) \leqslant 2\delta\}$. Let $u_\varepsilon \in C^\infty(M \times R_+, R^k)$ be solutions to the heat equation

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = \frac{1}{\varepsilon^2} f(u_\varepsilon) \quad (x, t) \in M \times R_+ \tag{1.3}$$

$$u_\varepsilon(x, 0) = u_0(x), \quad x \in M, \tag{1.4}$$

where $f(u_\varepsilon) = -(DF)(u_\varepsilon)$ and $u_0 \in C^2(M, N)$ is a given map. We assume throughout this paper that if $N = S^{k-1}$ then $F(p) = \frac{1}{4}(1 - |p|^2)^2$ so that $f(p) = p(1 - |p|^2)$ and (1.4) becomes

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon(1 - |u_\varepsilon|^2). \tag{1.5}$$

It is easy to see that u_ε satisfies the following energy equality (see also Lemma 2.1 below):

$$\sup_{0 < t < \infty} \left(\int_0^t \int_M |\partial_t u_\varepsilon|^2 \, dx \, dt + I_\varepsilon(u_\varepsilon(\cdot, t)) \right) = \frac{1}{2} \int_M |Du_0|^2. \tag{1.6}$$

It follows from (1.6) that for any $\varepsilon \downarrow 0$ there exists a subsequence $\varepsilon_n \to 0$ such that $u_n \equiv u_{\varepsilon_n} \to u$ weakly in $H^1_{\text{loc}}(M \times R_+, R^k)$. Moreover, it was shown by Chen and Struwe [13] (for $\partial M = \emptyset$) and Chen and Lin [11] (for $\partial M \neq \emptyset$) that there exists a closed subset $\Sigma = \{(\Sigma_t, t): t > 0\} \subset M \times R_+$ (see the definition of Σ in Section 2 below) such that $u_{\varepsilon_n} \to u$ in $H^1_{\text{loc}} \cap C^1_{\text{loc}}(M \times R_+ \setminus \Sigma, R^k)$. In particular, $u \in C^\infty(M \times R_+ \setminus \Sigma, N)$ is a smooth solution to the harmonic map flow equation (1.0). Let

$$e_{\varepsilon_n}(u_n)(x, t) = \frac{1}{2}|Du_n|^2(x, t) + \frac{1}{\varepsilon_n^2} F(u_n)(x, t).$$

We will simply write $e(u_n)$ for $e_{\varepsilon_n}(u_n)$ through this paper as long as there is no ambiguity. We can assume that

$$e(u_n)(x, t) \, dx \, dt \to \frac{1}{2}|Du|^2(x, t) \, dx \, dt + \nu$$

and

$$|\partial_t u_n|^2(x, t) \, dx \, dt \to |\partial_t u|^2(x, t) \, dx \, dt + \eta$$

as convergence of Radon measures for two nonnegative Radon measures ν and η on $M \times R_+$.

The stratification for the singular set of area minimizing currents was studied by Federer [16], who introduced a powerful scheme called the Federer's dimension

reduction argument (see also Almgren [3] and the appendix of [33]). White [37] provided an abstract approach to obtain the stratification for a large class of variational problems varying from minimizing currents, energy minimizing harmonic maps, and mean curvature flows (see also [34] for the stratification for minimizing harmonic maps). Motivated by these stratification results, we carry out the stratification for the parabolic concentration set Σ in Section 3. The stratification Theorem 3.6 roughly says that the subset Σ_k, consisting of $z_0 \in \Sigma$ such that the $\mathcal{P}_\lambda$-invariant subspace of any tangent measure μ^0 of μ at z_0 has its dimension of vector space at most k, has Hausdorff dimension measured in the parabolic metric at most k, for $0 \leqslant k \leqslant m$.

In Section 4, we adopt the generalized $(m - 2)$-varifold concept, which is a natural and very useful extension of the classical varifold concept studied by Almgren [3,4] and Allard [6], to study the convergence problem in our case through a varifold approach. Once we associate a sequence of generalized $(m - 2)$-varifolds, V_{u_n}, for u_n's, we can consider the limiting generalized $(m - 2)$-varifold $V = V_t \, dt$ of V_{u_n}. In Section 4, we show that for L^1 a.e. $t > 0$, the first variation of V_t, δV_t, is a Radon measure which is absolutely continuous with respect to $\|V_t\|$ and its generalized measure curvature

$$H_t = \frac{\delta V_t}{\|V_t\|} \in L^2_{\|V_t\|}(M, R^m).$$

Therefore, $\Theta^{m-2}(\|V_t\|, \cdot)$ exists for H^{m-2} a.e. in M. We then show in Theorem 4.9 that $V_t \llcorner \{x \in M : 0 < \Theta^{m-2}(\|V_t\|, x) < \infty\}$ is a $(m - 2)$-rectifiable varifold. This, combined with the observation that for H^{m-2} a.e. $x \in \Sigma_t$ $\Theta^{m-2}(\|V_t\|, x)$ is positive and finite, shows that Σ_t is $(m - 2)$-rectifiable.

In Section 5, we continue our discussion from Section 4 and show, in Theorem 5.6, that the pair $(u, v_t \, dt)$ satisfies a generalized varifold flow defined as in Definition 5.5 of Section 5. As a consequence of this generalized varifold flow, we show that if the limiting map u is a suitable weak solution defined by Definition 5.9, which requires that u satisfies the energy equality (both locally and globally), then the defect measure $\{v_t\}_{t \geqslant 0}$ is in fact a Brakke flow, i.e.,

$$\overline{D}_t v_t(\phi) = \limsup_{s \to t} \frac{v_s(\phi) - v_t(\phi)}{s - t} \leqslant - \int_M \left(\phi |H_t|^2 - \langle (T_x \Sigma_t)^\perp D\phi, H_t \rangle \right) dv_t$$

for any $t \geqslant 0$ and $\phi \in C_0^2(M, R_+)$. One way to obtain the generalized varifold flow is to apply the energy quantization Theorem 6.1, which however is only proved for sphere targets at this stage. In Lemma 5.8 of Section 5, we provide another approach to improve the so-called factor $\frac{1}{2}$ to 1.

In Section 6, we consider the density function $\Theta^{m-2}(\|V_t\|, \cdot)$. Under the extra assumption that $N = S^{k-1} \subset R^k$, we show a quantization result for $\Theta^{m-2}(\|V_t\|, \cdot)$ in Theorem 6.1 and Theorem 6.7, which roughly says that for almost all $z_0 = (x_0, t_0) \in \Sigma$, $\Theta^{m-2}(\|V_{t_0}\|, x_0)$ is the sum of energies of finitely many harmonic S^2's. This type of result is obtained by estimating the norm of Du_n in the Lorentz spaces $L^{2,1}$ and $L^{2,\infty}$, which is a highly nontrivial observation in dimension large than two and largely owns its origin from Lin and Riviéré [27].

Section 2 of the paper is devoted to the collection of some necessary facts needed later, and in Section 7 we make a few remarks concerning either smooth solutions to the heat flow of harmonic maps or the weak solutions satisfying energy inequality, energy monotonicity inequality, and the ε_0-regularity.

2. Basic estimates

This section is devoted to establishing some necessary facts needed for later sections. First let us recall some useful notations from [11,13,35]. Let δ denote the parabolic metric on $M \times R_+$ defined by

$$\delta((x, t), (y, s)) = \max\{|x - y|, \sqrt{|t - s|}\}, \quad \forall (x, t), (y, s) \in M \times R_+.$$

For $0 \leqslant l \leqslant m$, let $\mathcal{P}^l$ denote the l-dimensional Hausdorff measure on $M \times R_+$ (or $R^m \times R_+$) with respect to the parabolic metric δ, and H^l denotes the l-dimensional Hausdorff measure on M (or R^m) with the Euclidean metric. For $z_0 = (x_0, t_0) \in R^m \times R_+$, let G_{z_0} denote the backward heat kernel:

$$G_{z_0}(x, t) = \left(4\pi(t_0 - t)\right)^{-\frac{m}{2}} \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right), \quad x \in R^m, \ 0 < t < t_0.$$

Let $i(M) > 0$ denote the injectivity radius of M. For $0 < R < \{\frac{\sqrt{t_0}}{2}, i(M)\}$, let

$$S_R(z_0) = M \times \{t = t_0 - R^2\},$$

$$T_R(z_0) = M \times \{t \in R_+ \colon t_0 - 4R^2 \leqslant t \leqslant t_0 - R^2\},$$

and

$$P_R(z_0) = B_R(x_0) \times [t_0 - R^2, t_0 + R^2],$$

where $B_R(x_0) \subset M$ denotes the geodesic ball with center x_0 and radius R.

For solutions $u_\varepsilon \in C^\infty(M \times R_+, R^k)$ to (1.3)–(1.4). Define two normalized energy quantities as follows.

$$\Psi(u_\varepsilon, z_0, R) = \int\limits_{T_R(z_0)} \eta^2(x) e(u_\varepsilon)(x, t) G_{z_0}(x, t) \, dx \, dt,$$

$$\Phi(u_\varepsilon, z_0, R) = R^2 \int\limits_{S_R(z_0)} \eta^2(x) e(u_\varepsilon)(x, t) G_{z_0}(x, t) \, dx,$$

for $0 < R < \min\{\frac{\sqrt{t_0}}{2}, i(M)\}$, here $\eta \in C_0^1(M)$ satisfies that $\eta = 1$ on $B_{r_0}(x_0)$, $\eta = 0$ outside $B_{2r_0}(x_0)$, and $|D\eta| \leqslant \frac{2}{r_0}$.

Now we recall the energy inequality, energy monotonicity inequality, and small energy regularity from [13].

LEMMA 2.1 (Energy equality). – *Let $u_\varepsilon \in C^\infty(M \times R_+, R^k)$ solve* (1.3)–(1.4). *Then we have, for any $\phi \in C_0^1(M, R_+)$ and $0 \leqslant t_1 \leqslant t_2 \leqslant \infty$,*

$$\int\limits_M e(u_\varepsilon)(x, t_1)\phi(x)\,dx - \int\limits_M e(u_\varepsilon)(x, t_2)\phi(x)\,dx$$

$$= \int\limits_{t_1}^{t_2} \int\limits_M \phi(x)|\partial_t u_\varepsilon|^2(x, t)\,dx\,dt + \int\limits_{t_1}^{t_2} \int\limits_M D\phi(x) Du_\varepsilon \cdot \partial_t u_\varepsilon\,dx\,dt. \qquad (2.1)$$

In particular,

$$\int\limits_M e(u_\varepsilon)(x, t_1)\,dx - \int\limits_M e(u_\varepsilon)(x, t_2)\,dx = \int\limits_{t_1}^{t_2} \int\limits_M |\partial_t u_\varepsilon|^2(x, t)\,dx\,dt. \qquad (2.2)$$

LEMMA 2.2 (Energy monotonicity inequality). – *Let $u_\varepsilon \in C^\infty(M \times R_+, R^k)$ solve* (1.3)–(1.4). *Then*

$$\Psi(u_\varepsilon, z_0, R_1) + c \int\limits_{R_1}^{R_2} r^{-1} \int\limits_{T_r(z_0)} \left(\eta^2 \frac{|(x - x_0) \cdot Du_\varepsilon + 2(t - t_0)\partial_t u_\varepsilon|^2}{|t_0 - t|} + \eta^2 \frac{F(u_\varepsilon)}{\varepsilon^2}\right) G_{z_0}$$

$$\leqslant e^{C(R_2 - R_1)}\Psi(u_\varepsilon, z_0, R_2) + CE_0(R_2 - R_1), \qquad (2.3)$$

$$\Phi(u_\varepsilon, z_0, R_1) \leqslant e^{C(R_2 - R_1)}\Phi(u_\varepsilon, z_0, R_2) + CE_0(R_2 - R_1), \qquad (2.4)$$

for $z_0 \in M \times R_+$, $0 < R_1 \leqslant R_2 < \min\{\frac{\sqrt{t_0}}{2}, i(M)\}$. Here $c, C > 0$ depend only on M, m, N, and $E_0 = \frac{1}{2}\int_M |Du_\varepsilon|^2(x, 0)$.

LEMMA 2.3 (ε_0-regularity). – *There exist $\varepsilon_0, \delta_0, C_0 > 0$ such that if, for some $0 < R \leqslant \min\{\frac{\sqrt{t_0}}{2}, i(M)\}$, $\Psi(u_\varepsilon, z_0, R) \leqslant \varepsilon_0^2$, then*

$$\sup_{z \in P_{\delta_0 R}(z_0)} e(u_\varepsilon)(z) \leqslant C_0(\delta_0 R)^{-2}. \qquad (2.5)$$

For $\varepsilon_n \downarrow 0$, we assume that $u_n \equiv u_{\varepsilon_n} \to u$ weakly in $H_{loc}^1(M \times R_+, R^k)$. Then there exist two nonnegative Radon measures ν, η on $M \times R_+$ such that

$$e(u_n)(x, t)\,dx\,dt \to \frac{1}{2}|Du|^2(x, t)\,dx\,dt + \nu \equiv \mu,$$

$$|\partial_t u_n|^2(x, t)\,dx\,dt \to |\partial_t u|^2(x, t)\,dx\,dt + \eta,$$

as convergence of Radon measures on $M \times R_+$. Moreover, if we define the concentration set

$$\Sigma = \bigcup_{0 < R < r_0} \left\{z \in M \times R_+: \lim_{n \to \infty} \int\limits_{T_R(z)} \eta^2(x)e(u_n)G_z \geqslant \varepsilon_0^2\right\},$$

where ε_0 is given by Lemma 2.3. Then the following facts are known:

FACT 2.4 ([13]). – Σ is closed and $\mathcal{P}^m(\Sigma \cap P_R(0)) \leqslant C_R < \infty$, for any $R < \infty$.

FACT 2.5 ([13]). – $u_n \to u$ strongly in $H^1_{\text{loc}} \cap C^1_{\text{loc}}(M \times R_+ \setminus \Sigma, R^k)$.

FACT 2.6 ([13]). – $u \in C^\infty(M \times R_+ \setminus \Sigma, N)$ is a weak solution of the heat flow of harmonic maps (1.0).

FACT 2.7 ([9]). – For any $t > 0$, let $\Sigma_t = \Sigma \cap \{t\}$. Then $H^{m-2}(\Sigma_t \cap K) \leqslant C_K < \infty$, for any compact $K \subset M$.

FACT 2.8 ([31]). – $\text{sing}(u) \cup \text{spt}(\nu) = \Sigma$, $\text{spt}\,\eta \subset \Sigma$.

FACT 2.9 ([31]). – For any $z \in M \times R_+$, $\int_{T_R(z)} \eta^2(x) G_z(x, t) \, d\mu(x, t)$ is monotonically nondecreasing with respect to R. Hence

$$\Theta^m(\mu, z) = \lim_{R \downarrow 0} \int_{T_R(z)} \eta^2(x) G_z(x, t) \, d\mu(x, t)$$

exists for all $x \in M \times R_+$ and is upper-semicontinuous function of z. In particular,

$$\Sigma = \{z \in M \times R_+ : \varepsilon_0^2 \leqslant \Theta^m(\mu, z) < \infty\}.$$

FACT 2.10 ([31]). – For $\mathcal{P}^m$ a.e. $z \in \Sigma$,

$$\Theta^m(u, z) = \lim_{R \downarrow 0} R^{-m} \int_{P_R(z)} |Du|^2(x, t) \, dx \, dt = 0, \quad \text{and} \quad \Theta^m(\nu, z) = \Theta^m(\mu, z).$$

FACT 2.11 ([31]). – u_n doesn't converge to u strongly in $H^1_{\text{loc}}(M \times R_+, R^k)$ if and only if $\mathcal{P}^m(\Sigma) > 0$ and there exists at least one harmonic S^2 in N.

Now, we add two more lemmas needed later.

LEMMA 2.12. – Under the same notations as above, we have

$$\lim_{n \to \infty} \int_{M \times [t, T]} \frac{1}{\varepsilon_n^2} F(u_n)(x, t) \, dx \, dt = 0, \quad \forall 0 < t < T < \infty. \tag{2.6}$$

Proof. – It follows from the Fact 2.5 that for any $\beta > 0$, we have

$$\lim_{n \to \infty} \int_{M \times [t, T] \setminus (\Sigma_t^T)_\beta} \frac{1}{\varepsilon_n^2} F(u_n)(x, t) \, dx \, dt = 0$$

where $\Sigma_t^T = \bigcup_{s=t}^T (\Sigma_s \times \{s\})$ and $(\Sigma_t^T)_\beta = \{z \in M \times [t, T] : \delta(z, \Sigma_t^T) \leqslant \beta\}$. It suffices to show that

$$\lim_{n \to \infty} \int_{(\Sigma_t^T)_\beta} \frac{1}{\varepsilon_n^2} F(u_n)(x, t) \, dx \, dt = O(\beta). \tag{2.7}$$

On the other hand, for any $z_0 \in \Sigma_t^T$, (2.3) gives

$$\Psi\left(\mu, z_0, \frac{\beta}{2}\right) + \lim_{n\to\infty} \int_{\frac{\beta}{2}}^{\beta} r^{-1} \int_{T_r(z_0)} \frac{1}{\varepsilon_n^2} F(u_n) G_{z_0} \leqslant e^{C\beta} \Psi(\mu, z_0, \beta) \qquad (2.8)$$

for sufficiently small $\beta > 0$. Moreover, since $\lim_{\beta\downarrow 0} \Psi(\mu, z_0, \beta)$ exists, we may assume that

$$\left| \Psi(\mu, z_0, \beta) - \Psi\left(\mu, z_0, \frac{\beta}{2}\right) \right| = O(\beta), \quad \forall \beta \ll 1.$$

Therefore we have

$$\int_{\frac{\beta}{2}}^{\beta} r^{-1} \lim_{n\to\infty} \int_{T_r(z_0)} \frac{1}{\varepsilon_n^2} F(u_n) G_{z_0} = O(\beta).$$

This, combined with the Fubini's theorem, implies

$$\lim_{n\to\infty} \int_{T_{\overline{\beta}}(z_0)} \frac{1}{\varepsilon_n^2} F(u_n) G_{z_0} = O(\beta)$$

for some $\overline{\beta} \in (\frac{\beta}{2}, \beta)$. In particular, we have

$$\lim_{n\to\infty} \int_{B_{\frac{\beta}{2}}(x_0)\times[t_0-\beta^2, t_0-\frac{\beta^2}{4}]} \frac{1}{\varepsilon_n^2} F(u_n) = O(\beta).$$

This, combined with a simple covering argument, implies (2.7). □

LEMMA 2.13. – *There exists a subsequence of $n' \to \infty$ such that*

$$e(u_{n'})(x, t)\, dx \to \mu_t, \quad \forall t > 0,$$

as convergences of Radon measures, for a family of nonnegative Radon measures $\{\mu_t\}_{t>0}$ on M. In particular, $\mu_t = \frac{1}{2}|Du|^2(x, t)\, dx + \nu_t$, $\mu = \mu_t\, dt$, and $\nu = \nu_t\, dt$.

Proof. – The idea here is similar to that of Brakke [8] (see also Ilmanen [20]). For completeness, we outline it here. Let $\phi \in C_0^2(M, R_+)$. Then (2.1) implies

$$\frac{d}{dt} \int_M \phi e(u_n) = -\int_M \phi |\partial_t u_n|^2 - \int_M D\phi Du_n \partial_t u_n$$

$$\leqslant C(\phi) \int_M \phi(x)|Du_n|^2(x, t)\, dx \leqslant C(\phi) E_0,$$

where $C(\phi) = \sup_{\phi > 0} \frac{|D\phi|^2}{2\phi} \leqslant \sup_{\phi > 0} |D^2\phi| > 0$ and $E_0 = E(u_n(\cdot, 0))$ denotes the energy of the initial data. Hence

$$\int_M \phi e(u_n) - C(\phi) E_0 t$$

is monotonically nonincreasing with respect to $t > 0$. Let $B \subset R_+$ be a countable dense subset. By the weak compactness of Radon measures with locally bounded masses, and a diagonal process, we can assume that

$$e(u_n)(x, t)\, dx \to \mu_t, \quad \forall t \in B.$$

Now, let $\{\phi_i\}_{i \geqslant 1}$ be a countable dense subset in $C_0^2(M, R_+)$. By the monotonicity of $\int_M \phi e(u_n) - C(\phi) E_0 t$, there exists a co-countable set $C \subset R_+$ such that for any $t \in C$ and $i \geqslant 1$, $\mu_s(\phi_i)$ is continuous at t as a function of $s \in B$. For any fixed $t \in C$, there exists a further subsequence $n_j \to \infty$ and a limit μ_t such that $\mu_{n_j} \to \mu_t$. Hence $\{\mu_s(\phi_i)\}_{s \in B \cup \{t\}}$ is continuous at t, for all $i \geqslant 1$. Hence μ_t is uniquely determined by μ_s for $s \in B$. Therefore, $\mu_n \to \mu_t$. Note that $R_+ \setminus C$ is countable, we can do another diagonal process to show the result on R_+. $\square$

3. Dimension reduction and stratification of the concentration set

In this section, we will establish the stratification result for the energy concentration set Σ. To do it, we consider the space, $T_z\mu$, of all tangent measures of μ at each $z \in \Sigma$. We show that for each $\mu^0 \in T_z\mu$, $\mu^0 L R_-^{m+1}$ is invariant under the parabolic dilation $\mathcal{P}_\lambda$ for all $\lambda > 0$. For each $\mu^0 \in T_z\mu$, we then associate a nonnegative integer d which is the dimension of the largest $\mathcal{P}_\lambda$-invariant subspace inside $M(\Theta^m(\mu^0)) = \{z \in R^{m+1}: \Theta^m(\mu^0, z) = \Theta^m(\mu^0, 0)\}$. Using this integer, we can stratify Σ accordingly. The proof of the stratification is based on an extension of the well-known Federer's dimension reduction argument [16,3], and [33]. We would also like to remark that a similar scheme has been carried out by White [37] in an abstract way, with applications to mean curvature flows.

For simplicity, we assume $M = R^m$ in this section. Note that the norm of the parabolic metric in R^{m+1} is

$$\|(x, t)\| = \max\{|x|, \sqrt{|t|}\}.$$

Define the parabolic dilation by

$$\mathcal{P}_{z_0, \lambda}(x, t) = \left(\frac{x - x_0}{\lambda}, \frac{t - t_0}{\lambda^2}\right)$$

for $z_0 = (x_0, t_0) \in R^{m+1}$ and $\lambda > 0$. Define the Euclidean dilation by

$$\mathcal{D}_{x_0, \lambda}(x) = \frac{x - x_0}{\lambda}, \quad x \in R^m$$

for $x_0 \in R^m$ and $\lambda > 0$. Denote $R_+^{m+1} = R^m \times R_+$ and $R_-^{m+1} = R^m \times R_-$. The Hausdorff dimension of a subset $S \in R^{m+1}$ is the Hausdorff dimension with respect to the parabolic metric δ. We write $\Sigma = \{(\Sigma_t, t): 0 < t < \infty\}$, here $\Sigma_t = \Sigma \cap \{t\}$. For $\varepsilon_n > 0$, let $u_n: R_+^{m+1} \to R^k$ solve (1.3)–(1.4), with $\varepsilon = \varepsilon_n$. Recall that (2.4) implies that for $0 < R_2 \leqslant R_1 < \frac{\sqrt{t_0}}{2}$,

$$\Phi(u_n, z_0, R_1) - \Phi(u_n, z_0, R_2)$$

$$\geqslant \frac{1}{2} \int\limits_{R_2}^{R_1} \int\limits_{R^m} \frac{|2(t_0 - t)\partial_t u_n - (x - x_0)Du_n|^2}{t_0 - t} G_{z_0} \, dz \qquad (3.1)$$

where $\Phi(u_n, z_0, R) = R^2 \int_{t=t_0 - R^2} e(u_n)(x, t) G_{z_0}(x, t) \, dx$. By (1.6) and Lemma 2.5, we can assume that

$$e(u_n)(x, t) \, dx \, dt \to \mu \equiv \mu_t \, dt$$

as convergence of Radon measures in R_+^{m+1}, for some nonnegative Radon measures $\{\mu_t\}_{t>0}$ on R^m. From (1.6) and (3.1), we have

$$\sup_{(x,t,r)\in R^m \times R_+ \times R_+} r^{-m} \mu(P_r(x, t)) < \infty \qquad (3.2)$$

and

$$\Theta^m(\mu, z_0) = \lim_{R\downarrow 0} R^2 \int\limits_{t=t_0 - R^2} G_{z_0} \, d\mu_{t_0}$$

exists for all $z_0 \in R_+^{m+1}$. Moreover, the Fact 2.9 implies that

$$\Sigma = \{z \in R_+^{m+1}: \varepsilon_0^2 \leqslant \Theta^m(\mu, z) < \infty\}.$$

For $z_0 \in \Sigma$ and $\lambda_i \downarrow 0$, we define the parabolic rescalings of μ, $\mathcal{P}_{z_0, \lambda_i^{-1}}(\mu)$, by

$$\mathcal{P}_{z_0, \lambda_i^{-1}}(\mu)(A) = \lambda_i^{-m} \mu(\mathcal{P}_{z_0, \lambda_i^{-1}}(A)), \quad \forall \text{ Borel } A \subset R^{m+1}.$$

Then it follows from (3.2) that we can find a subsequence $\lambda_{i'}$ of λ_i and a nonnegative Radon measure μ^0 on R^{m+1} such that

$$\mathcal{P}_{z_0, \lambda_{i'}^{-1}} \to \mu^0$$

as convergence of Radon measures on R^{m+1}.

DEFINITION 3.1. – *For any $z_0 \in \Sigma$, the tangent measure cone of μ at z_0, $T_{z_0}(\mu)$, consists of all nonnegative Radon measures on R^{m+1} obtained by*

$$T_{z_0}(\mu) = \{\mu^0: \text{ there exists a } r_i \downarrow 0 \text{ such that } \mathcal{P}_{z_0, \lambda_i^{-1}}(\mu) \to \mu^0\}.$$

Note that for any $z_0 \in \Sigma$ and $\mu^0 \in T_{z_0}(\mu)$, we have $\mu^0 = \mu_s^0 \, ds$ and for any $(x, t) \in R^{m+1}$

$$\Theta^m(\mu^0, (x, t), r) \equiv r^2 \int\limits_{s=t-r^2} G_{(x,t)}(y, s) \, d\mu_s^0(y)$$

is monotonically nondecreasing with respect to r. Hence

$$\Theta^m(\mu^0, (x, t)) = \lim_{r \downarrow 0} \Theta^m(\mu^0, (x, t), r)$$

exists for any $(x, t) \in R^{m+1}$ and is upper semicontinuous.

LEMMA 3.2. – *For any $z_0 \in \Sigma$ and $\mu^0 \in T_{z_0}(\mu)$. Then $\mu^0 L R_-^{m+1}$ is invariant under all parabolic dilations $\mathcal{P}_\lambda$, i.e.,*

$$\mathcal{P}_\lambda(\mu^0 L R_-^{m+1}) = \mu^0 L R_-^{m+1}. \tag{3.3}$$

Proof. – It follows from Lemma 2.5 that $\mu^0 = \mu_t^0 \, dt$. Therefore,

$$\begin{aligned}
\mathcal{P}_\lambda(\mu^0 L R_-^{m+1}) &= \mathcal{P}_\lambda(\{(\mu_t^0, t) \colon t \leqslant 0\}) \\
&= \{\mathcal{D}_\lambda((\mu_t^0), \lambda^2 t) \colon t \leqslant 0\} \\
&= \{\mathcal{D}_\lambda((\mu_{\frac{t}{\lambda^2}}^0), t) \colon t \leqslant 0\}.
\end{aligned}$$

Here $\mathcal{D}_\lambda(\mu_t^0)(A) = \lambda^{2-m} \mu_t^0(\lambda A)$ for any borel set $A \subset R^m$. Hence, it suffices to show that

$$\mathcal{D}_\lambda(\mu_{\frac{t}{\lambda^2}}^0) = \mu_t^0, \quad \forall t \leqslant 0. \tag{3.4}$$

Since $\lambda > 0$ is arbitrary, it suffices to prove (3.4) for $t = -1$, which is equivalent to

$$\lambda^{m-2} \int\limits_{R^m} \phi(\lambda x) G(\lambda x, -1) \, d\mu_{-\lambda^{-2}}^0(x) = \int\limits_{R^m} \phi(x) G(x, -1) \, d\mu_{-1}^0(x), \tag{3.5}$$

for any $\phi \in C_0^1(R^m)$ and $G = G_{(0,0)}$. On the other hand, we know that there exists $\lambda_n \downarrow 0$ such that $v_n(x, t) = u_n(x_0 + \lambda_n x, t_0 + \lambda_n^2 t)$ satisfies (1.3), with ε_n replaced by $\bar{\varepsilon}_n = \frac{\varepsilon_n}{\lambda_n}$, and

$$e(v_n)(x, t) \, dx \equiv e_{\bar{\varepsilon}_n}(v_n)(x, t) \, dx \to \mu_t^0, \quad \forall t \in R$$

as convergence of Radon measures on R^m. Then, for any $R > 0$,

$$\begin{aligned}
R^2 \int\limits_{t=-R^2} G \, d\mu_t^0 &= \lim_{n \to \infty} R^2 \int\limits_{t=-R^2} e(v_n)(x, t) G(x, t) \, dx \\
&= \lim_{\lambda_n \downarrow 0} (R\lambda_n)^2 \int\limits_{t=t_0-R^2\lambda_n^2} G_{z_0}(x, t) \, d\mu_t \\
&= \Theta^m(\mu, z_0).
\end{aligned}$$

This, combined with (3.1), implies that, for any $0 < r_1 < r_2 < \infty$,

$$\lim_{n\to\infty} \int\limits_{t=-r_1^2}^{t=-r_2^2} \int\limits_{R^m} |xDv_n + 2t\partial_t v_n|^2 G = 0. \tag{3.6}$$

Therefore, in oder to prove (3.5), it suffices to show

$$\lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\lambda^{m-2} \int\limits_{t=-1} \phi(\lambda x)G(\lambda x, -1)e(v_n)(x, -\lambda^{-2})\,\mathrm{d}x\right) = 0. \tag{3.7}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\lambda^{m-2} \int\limits_{t=-1} \phi(\lambda x)G(\lambda, x, -1)e(v_n)(x, -\lambda^{-2})\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\int\limits_{t=-1} \phi(x)G(x, t)e(v_n^\lambda)(x, t)\,\mathrm{d}x\right)$$

$$= -\int\limits_{t=-1} GD\phi Dv_n^\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}v_n^\lambda - \int\limits_{t=-1} \phi\left(\partial_t v_n^\lambda - \frac{1}{2}xDv_n^\lambda\right)\frac{\mathrm{d}}{\mathrm{d}\lambda}v_n^\lambda G$$

$$= -\frac{1}{2\lambda}\int\limits_{t=-\lambda^2} |yDv_n + 2t\partial_t v_n|^2 G\phi\left(\frac{y}{\lambda}\right) - \int\limits_{t=-\lambda^2} D\phi\left(\frac{y}{\lambda}\right)GDv_n(yDv_n + 2t\partial_t v_n).$$

Here $v_n^\lambda(x, t) = v_n(\lambda x, \lambda^2 t)$ and $e(v_n^\lambda)(x, t) = e_{\frac{\varepsilon_n}{\lambda}}(v_n^\lambda)(x, t)$. Hence, integrating the identity from 1 to λ and using (3.6), we see that (3.5) holds. $\square$

From Lemma 3.2, we see that for any $z_0 \in \Sigma$ and $\mu^0 \in T_{z_0}\mu$,

$$R^2 \int\limits_{t=-R^2} G\,\mathrm{d}\mu_t^0 = \Theta^m(\mu^0, 0) = \Theta^m(\mu, z_0), \quad \forall R > 0 \tag{3.8}$$

and, for any $z \in R_-^{m+1}$ and $\lambda > 0$,

$$\Theta^m(\mu^0, z) = \Theta^m(\mu_0, \mathcal{P}_\lambda(z)). \tag{3.9}$$

In general, we have

LEMMA 3.3. – *For $z_0 \in \Sigma$ and $\mu^0 \in T_{z_0}(\mu)$, we have*
(1) $\Theta^m(\mu^0, z) \leqslant \Theta^m(\mu^0, 0), \forall z \in R^{m+1}$.
(2) *If $z \in R^{m+1}$ satisfies $\Theta^m(\mu^0, z) = \Theta^m(\mu^0, 0)$, then*

$$\Theta^m(\mu^0, z + v) = \Theta^m(\mu^0, z + \mathcal{P}_\lambda v), \quad \forall \lambda > 0, v \in R_-^{m+1}. \tag{3.10}$$

Proof. – (1) For $\mu^0 \in T_{z_0}(\mu)$, there exists $r_i \downarrow 0$ such that $\mathcal{P}_{z_0, r_i}(\mu) \to \mu^0$. For any $r > 0$ and $z = (x, t) \in R^{m+1}$, we have

$$\Theta^m\left(\mu^0, z\right) \leqslant \Theta^m\left(\mu^0, z, r\right)$$
$$= \lim_{r_i \downarrow 0} \Theta^m\left(\mathcal{P}_{z_0, r_i}(\mu), z, r\right)$$
$$= \lim_{r_i \downarrow 0} \Theta^m\left(\mu, z_0 + \left(r_i x, r_i^2 t\right), r_i^2 r\right)$$
$$\leqslant \Theta^m(\mu, z_0) = \Theta^m\left(\mu^0, 0\right).$$

Here we have used the upper semicontinuity of $\Theta^m(\mu, \cdot, \cdot)$ in its last two variables.

(2) From the proof of (1), we see that if $\Theta^m(\mu^0, z) = \Theta^m(\mu^0, 0)$ then the inequalities are all equalities. In particular, $\Theta^m(\mu^0, z, r)$ is constant with respect to $r > 0$. Applying the argument of Lemma 3.2, we see that $\Theta^m(\mu^0, z + v) = \Theta^m(\mu^0, z + \mathcal{P}_\lambda(v))$ for any $v \in R_-^{m+1}$ and $\lambda > 0$. $\quad\square$

For any $z_0 \in \Sigma$ and $\mu^0 \in T_{z_0}(\mu)$, denote

$$M\left(\Theta^m\left(\mu^0, \cdot\right)\right) \equiv \left\{z \in R^{m+1}: \Theta^m\left(\mu^0, z\right) = \Theta^m\left(\mu^0, 0\right)\right\},$$

$$V\left(\Theta^m\left(\mu^0, \cdot\right)\right) \equiv M\left(\Theta^m\left(\mu^0, \cdot\right)\right) \cap \{t = 0\},$$

and

$$W\left(\Theta^m\left(\mu^0, \cdot\right)\right) \equiv \left\{x \in R^m: \Theta^m\left(\mu^0, (y, s)\right) = \Theta^m\left(\mu^0, (x + y, s)\right), \forall (y, s) \in R_-^{m+1}\right\}.$$

Then we have

PROPOSITION 3.4. – *For $z_0 \in \Sigma$ and $\mu^0 \in T_{z_0}(\mu)$, we have $V(\Theta^m(\mu^0, \cdot)) = W(\Theta^m(\mu^0, \cdot))$. In particular, both $V(\Theta^m(\mu^0, \cdot))$ and $W(\Theta^m(\mu^0, \cdot))$ are subspaces of R^m. Moreover, $M(\Theta^m(\mu^0, \cdot))$ is $V(\Theta^m(\mu^0, \cdot))$ or $V(\Theta^m(\mu^0, \cdot)) \times (-\infty, a]$ for some $0 \leqslant a \leqslant \infty$ and $\Theta^m(\mu^0, \cdot)$ is time-independent up to $t = a$.*

Proof. – It is clear that $W(\Theta^m(\mu^0, \cdot)) \subset V(\Theta^m(\mu^0, \cdot))$, $V(\Theta^m(\mu^0, \cdot))$ is closed under scalar multiplication, and $nW(\Theta^m(\mu^0, \cdot)) \subset W(\Theta^m(\mu^0, \cdot))$ for any integer n. For any $(x, 0) \neq (0, 0) \in V(\Theta^m(\mu^0, \cdot))$, $v \in R_-^{m+1}$, and $\lambda > 0$, we have

$$\Theta^m\left(\mu^0, (x, 0) + v\right) = \Theta^m\left(\mu^0, (x, 0) + \mathcal{P}_\lambda v\right)$$
$$= \Theta^m\left(\mu^0, \mathcal{P}_{\lambda^{-1}}\left((x, 0) + \mathcal{P}_\lambda v\right)\right)$$
$$= \Theta^m\left(\mu^0, \mathcal{P}_{\lambda^{-1}}(x, 0) + v\right)$$

so that

$$\Theta^m\left(\mu^0, (x, 0) + v\right) = \Theta^m\left(\mu^0, \mathcal{P}_{\lambda^{-1}}(x, 0) + v\right). \tag{3.11}$$

Note that $v - \mathcal{P}_{\lambda^{-1}}(x, 0) \in R_-^{m+1}$. Hence, replacing v by $v - \mathcal{P}_{\lambda^{-1}}(x, 0)$ gives

$$\Theta^m\left(\mu^0, (x, 0) + v - \mathcal{P}_{\lambda^{-1}}(x, 0)\right) = \Theta^m\left(\mu^0, v\right). \tag{3.12}$$

Taking λ into zero and using the upper semicontinuity of $\Theta^m(\mu^0, \cdot)$, we obtain, from (3.11) and (3.12),

$$\Theta^m\left(\mu^0, v\right) = \Theta^m\left(\mu^0, (x, 0) + v\right), \quad \forall v \in R_-^{m+1}.$$

This implies that $V(\Theta^m(\mu^0, \cdot)) \subset W(\Theta^m(\mu^0, \cdot))$. Hence $V(\Theta^m(\mu^0, \cdot)) = W(\Theta^m(\mu^0, \cdot))$ and is a subspace of R^m.

Suppose that $X = (x, t) \in M(\Theta^m(\mu^0, \cdot))$, with $t < 0$. Then, for any $Y = (y, s)$ with $s \leqslant t$ and $\lambda > 0$, we have

$$\Theta^m(\mu^0, \mathcal{P}_{\lambda^{-1}}(Y)) = \Theta^m(\mu^0, Y) = \Theta^m(\mu^0, X + \mathcal{P}_{\lambda^{-1}}(Y - X)).$$

Note that $s\lambda^{-2} \leqslant s \leqslant t$, for $\lambda < 1$. Hence replacing Y by $\mathcal{P}_\lambda(Y)$, we get

$$\Theta^m(\mu^0, Y) = \Theta^m(\mu^0, X + Y - \mathcal{P}_{\lambda^{-1}}(X)).$$

Taking λ into zero, we have $\Theta^m(\mu^0, Y) \leqslant \Theta^m(\mu^0, X + Y)$. By substituting Y by $Y + \mathcal{P}_{\lambda^{-1}}(X)$, we also get

$$\Theta^m(\mu^0, Y + \mathcal{P}_{\lambda^{-1}}(X)) = \Theta^m(\mu^0, X + Y)$$

this implies $\Theta^m(\mu^0, Y) \geqslant \Theta^m(\mu^0, X + Y)$. Therefore, we have

$$\Theta^m(\mu^0, Y) = \Theta^m(\mu^0, X + Y), \quad \forall X = (x, t), Y = (y, s), s \leqslant t < 0. \qquad (3.13)$$

This implies, by choosing $Y = (n - 1)X$,

$$\Theta^m(\mu^0, 0) = \Theta^m(\mu^0, (nx, nt)) = \Theta^m(\mu^0, \mathcal{P}_n(nx, nt))$$

$$= \Theta^m\left(\mu^0, \left(x, \frac{t}{n}\right)\right) \leqslant \Theta^m(\mu^0, (x, 0)).$$

Therefore, $(x, 0) \in V(\Theta^m(\mu^0, \cdot)) = W(\Theta^m(\mu^0, \cdot))$ and $(0, t) \in M(\Theta^m(\mu^0, \cdot))$. In particular, $\Theta^m(\mu^0, \cdot)$ is time-independent up to time $t = 0$.

If $X = (x, t) \in V(\Theta^m(\mu^0, \cdot))$, with $t > 0$, then we can prove similarly that $\Theta^m(\mu^0, \cdot)$ is time-independent up to time t. Let $t = a \geqslant 0$ be the maximal number such that $\Theta^m(\mu^0, \cdot)$ is time-independent up to $t = a$. Then one gets $M(\Theta^m(\mu^0, \cdot)) = V(\Theta^m(\mu^0, \cdot)) \times (-\infty, a]$. $\quad\square$

DEFINITION 3.5. – *For $z_0 \in \Sigma$ and $\mu^0 \in T_{z_0}(\Sigma)$, define $dim(\Theta^m(\mu^0, \cdot))$ by*

$$\dim(\Theta^m(\mu^0, \cdot)) = \begin{cases} \dim(V(\Theta^m(\mu^0, \cdot))) + 2, & \text{if } M(\Theta^m(\mu^0, \cdot)) = V(\Theta^m(\mu^0, \cdot)) \times R \\ \dim(V(\Theta^m(\mu^0, \cdot))), & \text{otherwise.} \end{cases}$$

Now we are ready to prove the main theorem of this section.

THEOREM 3.6. – *For $0 \leqslant k \leqslant m$, let*

$$\Sigma_k = \{z_0 \in \Sigma : \dim(\Theta^m(\mu^0, \cdot)) \leqslant k, \forall \mu^0 \in T_{z_0}(\mu)\}.$$

Then $\dim(\Sigma_k) \leqslant k$ for $0 \leqslant k \leqslant m$, and Σ_0 is discrete. In particular, $\Sigma = \Sigma_0 \cup (\Sigma_1 \setminus \Sigma_0) \cup \cdots \cup (\Sigma_m \setminus \Sigma_{m-1})$, and for $\mathcal{P}^m$ a.e. $z \in \Sigma$, there exists at least one $\mu^0 \in T_z(\mu)$ such that

$$\mu^0 = \Theta^m(\mu, z)(H^{m-2} \llcorner T_{m-2}) \times (L^1 \llcorner R).$$

Here $T_{m-2} \subset R^m$ is a $(m - 2)$-plane.

Proof of Theorem 3.6. – This is essentially an extension of the Federer's dimension reduction argument. It suffices to show that if $\mathcal{P}^d(\Sigma_k) > 0$ then $d \leqslant k$. Thus we only consider such a d. First, denote $A_{z,r} = \mathcal{P}_{z,r}(A)$ for $A \subset R^{m+1}$, $z \in R^{m+1}$, and $\lambda > 0$. Let

$$\mathcal{C} = \{V \times R, \text{ or } V: V \subset R^m \text{ a subspace of } \dim(V) \leqslant k-2\}$$
$$\cup \{V \times R_-: V \subset R^m \text{ a subspace of } \dim(V) \leqslant k\}.$$

Then we have

CLAIM 3.7. – *For any $z_0 \in \Sigma_k$ and $r > 0$ there exists $\eta = \eta(z, \varepsilon) > 0$ such that for any $\rho \in (0, \varepsilon)$*

$$\left(\{w \in P_\rho(z): \Theta^m(\mu, w) \geqslant \Theta^m(\mu, z) - \eta\}\right)_{z,\rho} \subset \varepsilon\text{-neighborhood of } C, \qquad (3.14)$$

for some $C \in \mathcal{C}$. Here ε-neighborhood is measured with respect to the parabolic metric δ.

For, otherwise, there exist $\varepsilon_0 > 0$, $z_0 \in \Sigma_k$, and $\rho_i, \eta_i \downarrow 0$ such that

$$B_i \equiv \{z \in P_1(0): \Theta^m(\mathcal{P}_{z_0,\rho_i}(\mu), z) \geqslant \Theta^m(\mu, z_0) - \eta_i\}$$
$$\not\subset \varepsilon_0\text{-neighborhood of any } C \in \mathcal{C}.$$

On the other hand, we may assume that $\mathcal{P}_{z_0,\rho_i}(\mu) \to \mu^0 \in T_{z_0}(\mu)$ and $B_i \to B \subset M(\Theta^m(\mu^0, \cdot)) = \{z \in R^{m+1}: \Theta^m(\mu^0, z) = \Theta^m(\mu, z_0)\}$. By Lemma 3.4, we know that, among the four possibilities of $M(\Theta^m(\mu^0, \cdot))$, only $M(\Theta^m(\mu^0, \cdot) = V(\Theta^m(\mu^0, \cdot)) \times (-\infty, a]$ for some $a > 0$ is not in $\mathcal{C}$. However, even for such a possibility, we can find $r_i \downarrow 0$ such that $\mathcal{P}_{r_i}(\mu^0) \to \mu^1$, and by the uppersemicontinuity

$$\Theta^m(\mu^1, w) = \Theta^m(\mu^1, 0) = \Theta^m(\mu^0, 0), \quad \forall w \in V(\Theta^m(\mu^0, \cdot)) \times R$$

this implies that $M(\Theta^m(\mu^0, \cdot)) \subset M(\Theta^m(\mu^1, \cdot)) \in \mathcal{C}$. Therefore we get the desired contradiction.

Now we proceed as follows. Let $\varepsilon_i \downarrow 0$ and decompose $\Sigma_k = \bigcup_{i,q \geqslant 1} \Sigma_{k,i,q}$, here $\Sigma_{k,i,q}$ denotes points where $\Theta^m(\mu, \cdot) \in ((q-1)\varepsilon_i, q\varepsilon_i)$ and Claim 3.7 holds with $\varepsilon = \varepsilon_i$. Therefore, for each i, there exists $q_i \geqslant 1$ such that $\mathcal{P}^d(\Sigma_{k,i,q_i}) > 0$. By the lower bound for the upper density (cf. [16]), we know that there exist $z_i \in \Sigma_{k,i,q_i}$ and $r_i \downarrow 0$ such that

$$\mathcal{P}^{d,\infty}\left((\Sigma_{k,i,q_i})_{z_i,r_i}\right) \geqslant 10^{-d}. \qquad (3.15)$$

Here $\mathcal{P}^{d,\infty}$ denotes the d-dimensional Hausdorff measure with size ∞. Moreover, for each $z \in (\Sigma_{k,i,q_i})_{z_i,r_i}$ there exists $C_z \in \mathcal{C}$ such that

$$\left((\Sigma)_{k,i,q_i}\right)_{z_i,r_i} - z \subset \varepsilon_i\text{-neighborhood of } C_z.$$

We may assume that $(\Sigma_{k,i,q_i})_{z_i,r_i} \to \Sigma_k^\infty$. Then we have

$$\Sigma_k^\infty - z \subset C_z, \quad \forall z \in \Sigma_k^\infty, \quad \text{and} \quad \mathcal{P}^{d,\infty}(\Sigma_k^\infty) \geqslant 10^{-d}. \qquad (3.16)$$

For any $C \in \mathcal{C}$, let $\Sigma_{k,j,C}^\infty = \{z \in \Sigma_k^\infty: \delta(C_z, C) \leqslant j^{-1}\}$. Then for each j there exists $C_j \in \mathcal{C}$ such that $\Sigma_{k,j}^\infty \equiv \Sigma_{k,j,C_j}^\infty$ has positive $\mathcal{P}^d$-measure. Therefore, there exist $z_j \in \Sigma_{k,j}^\infty$ and

$\rho_j \downarrow 0$ such that

$$\mathcal{P}^{d,\infty}\big((\Sigma_{k,j}^{\infty})_{z_j,\rho_j}\big) \geqslant 10^{-d}. \tag{3.17}$$

Assume that $C_j \to C_\infty \in \mathcal{C}$ and $(\Sigma_{k,j}^{\infty})_{z_j,\rho_j} \to \Sigma^\infty$. Then $\Sigma^\infty \subset C_\infty$, $\mathcal{P}^d(\Sigma^\infty) > 0$, and $\Sigma^\infty - z \subset C_\infty$ whenever $z \in \Sigma^\infty$. In particular, we have $\Sigma^\infty \subset C_\infty \cap (-C_\infty)$. But we note that if $C_\infty = V_\infty \times R_-$ for some vector subspace $V_\infty \subset R^m$ then $C_\infty \cap (-C_\infty) = V_\infty$. Hence $\mathcal{P}^d(V_\infty) > 0$ implies that $H^d(V_\infty) > 0$ so that $d \leqslant k$. For $C_\infty = V_\infty$ or $V_\infty \times R$, we see $C_\infty \cap (-C_\infty) = C_\infty$ so that $\mathcal{P}^d(\Sigma^\infty) > 0$ implies that $\mathcal{P}^d(C_\infty) > 0$ so that $d \leqslant k$ again. □

4. Generalized varifolds and rectifiablity of the concentration set

In this section, we first recall some of the classical theory of varifolds, which was studied by Almgren [3,4] and Allard [6] (see also Simon [33] for details), and at the same time we also recall the notion of generalized varifolds, which was remarked by Almgren [4] and recently adopted by Ambrosio and Soner [1] in their study of the dynamics of Ginzburg–Landau equations with complex values, and Lin [24] in the study of mapping problems.

For simplicity, we assume, throughout this section, that $M = R^m$. For $1 \leqslant l \leqslant m$, let $G_l(m)$ denote the standard Grassmann manifold of l-dimensional unoriented planes in R^m. For a bounded domain $\Omega \subset R^m$, recall a l-varifold in Ω is just a Radon measure in $\Omega \times G_l(m)$, and let $V_l(\Omega)$ denote all l-varifolds in Ω. The weight $\|V\|$ of $V \in V_l(\Omega)$ is $\pi_\#(V)$, where $\pi(x, A) = x : \Omega \times G_l(m) \to \Omega$. A set $E \subset R^m$ is called l-rectifiable if except a zero H^l measure subset E can be covered by countably many l-dimensional C^1 submanifolds of R^m. A $V \in V_l(\Omega)$ is said to be l-rectifiable varifold if there exist a l-rectifiable set $E \subset \Omega$ and a locally H^l integrable and a positive function θ such that $V = \delta_{T_x E} \theta H^l \llcorner E$ for H^l a.e. in Ω, here $T_x E$ denotes the tangent plane of E at x and $\delta_{T_x E}$ denotes the Dirac mass at $T_x E$. Let $RV_l(\Omega)$ denote all l-rectifiable varifolds.

Now, we recall the definition of generalized varifolds from [1].

DEFINITION 4.1. – *A l-dimensional generalized varifold V is a nonnegative Radon measure on $\Omega \times A_{l,m}$, where*

$$A_{l,m} = \big\{ A \in R^{m \times m} : A \text{ is symmetric, } \mathrm{trace}(A) = l, -lI_m \leqslant A \leqslant I_m \big\},$$

where I_m denotes the identity matrix of order m. The class of all generalized l-varifolds is denoted by $V_l^*(\Omega)$. Again, let $\|V\|$ denote the weight of $v \in V_l^*(\Omega)$. Since $G_l(m) \subset A_{l,m}$, we know that $V_l(\Omega) \subset V_l^*(\Omega)$.

DEFINITION 4.2. – *For any given $V \in V_l^*(\Omega)$, the first variation of V, δV, is a distribution on $C_0^1(\Omega, R^m)$ defined by*

$$\delta V(X) = - \int_{\Omega \times A_{l,m}} DX(x) : A \, dV(x, A), \quad \forall X \in C_0^1\big(\Omega, R^m\big). \tag{4.1}$$

Here $A : B = \sum_{ij} A_{ij} B_{ij}$ for $A, B \in R^{m \times m}$. V is called stationary if $\delta V = 0$.

Note that if δV is a Radon measure, that is, if

$$
\begin{aligned}
\|\delta V\|(G) &= \sup\{|\delta V(X)|\colon\ X \in C_0^1(\Omega, R^m),\ \|X\|_{L^\infty} \leqslant 1,\ \mathrm{spt}(X) \subset G\} \\
&\leqslant C(G) < \infty, \quad \forall G \Subset \Omega.
\end{aligned}
\tag{4.2}
$$

Then the Riesz Representation Theorem implies that

$$
\delta V(X) = \int_\Omega X(x)\beta(x)\,\mathrm{d}\|\delta V\|(x), \quad \forall X \in C_0^1(\Omega, R^m)
\tag{4.3}
$$

here β is a $\|\delta V\|$-measurable, S^{m-1}-valued function. If $\|\delta V\| \ll \|V\|$, then we have

$$
\delta V(X) = \int_\Omega \langle H(x), X(x) \rangle\,\mathrm{d}\|V\|(x), \quad \forall X \in C_0^1(\Omega, R^m),
\tag{4.4}
$$

where $H : \Omega \to R^m$ is a $\|V\|$-measurable function, which we call the generalized mean curvature of V.

Note also that the convergence of $V \in V_l^*(\Omega)$ is understood as weak convergence of Radon measures on $\Omega \times A_{l,m}$. Moreover, if $V_n \to V$, then $\delta V_n \to \delta V$ as distributions. In particular, if $\sup_n \|\delta V_n\|(A) < \infty$, then

$$
\|\delta V\|(A) \leqslant \liminf_{n\to\infty} \|\delta V_n\|(A) < \infty, \quad \text{for } A \subset \Omega.
\tag{4.5}
$$

To motivate the application of generalized varifolds to our problem, we give two examples.

Example 4.3. – For $u \in H^1(\Omega, R^k)$, we define $V_u(x) = \frac{1}{2}\delta_{A(u)(x)}|Du|^2(x)\,\mathrm{d}x$, where

$$
A(u)(x) = \begin{cases} I_m - 2\frac{Du \otimes Du}{|Du|^2}(x), & \text{if } |Du|(x) \neq 0, \\ I_{m-2}, & \text{if } |Du|(x) = 0. \end{cases}
\tag{4.6}
$$

Then it is clear that $A(u)(x) \in A_{m-2,m}$. Hence $V_u \in V_{m-2}^*(\Omega)$. In fact, for any Borel set $B \subset \Omega \times A_{m-2,m}$,

$$
V_u(B) = \frac{1}{2}\int_{\pi(B)} |Du|^2(x)\,\mathrm{d}x
$$

where $\pi(B) = \{x \in \Omega\colon\ (x, A_u(x)) \in B\}$. It is clear that δV_u is given by

$$
\begin{aligned}
\delta V_u(X) &= -\int_\Omega DX(x) : A\,\mathrm{d}V_u(x, A) \\
&= -\frac{1}{2}\int_\Omega DX(x) : A(u)(x)|Du|^2(x)\,\mathrm{d}x \\
&= -\frac{1}{2}\int_\Omega \left(|Du|^2(x)\mathrm{div}(X) - 2D_i u D_j u X_i^j\right)\mathrm{d}x.
\end{aligned}
$$

In particular, if $u \in H^1(\Omega, N)$ is a stationary harmonic map (see, Bethuel [7] and [23] for the definition and discussion), then we have $\delta V_u = 0$ so that V_u is a stationary generalized $(m-2)$-varifold in Ω.

Example 4.4. – For $\varepsilon_n \downarrow 0$, let $u_n \in H^1(\Omega, R^k)$ be critical points of $I_{\varepsilon_n}(\cdot)$, namely, u_n satisfies

$$\Delta u_n + \frac{1}{\varepsilon_n^2} f(u_n) = 0, \quad \text{in } \Omega. \tag{4.7}$$

Then it follows from [31] that

$$\delta V_{u_{\varepsilon_n}}(X) = -\int_\Omega \operatorname{div} X \frac{F(u_n)}{\varepsilon_n^2}, \quad \forall X \in C_0^1(\Omega, R^m). \tag{4.8}$$

In particular, if $\sup_n I_{\varepsilon_n}(u_n) < \infty$, then we may assume that $u_n \to u$ weakly in $H^1(\Omega, R^k)$ and

$$e(u_n)(x)\,dx \to \frac{1}{2}|Du|^2(x)\,dx + \nu$$

as convergence of Radon measures in Ω for some nonnegative Radon measure ν on Ω, and V_{u_n} weakly converges to a $V \in V_{m-2}^*(\Omega)$. Moreover, it follows from Lemma 2.4 that

$$\lim_{n \to \infty} \int_\Omega \frac{F(u_n)}{\varepsilon_n^2} = 0. \tag{4.9}$$

Hence $\delta V_{u_n}(X) \to 0$ for any $X \in C_0^1(\Omega, R^m)$ so that $\delta V = 0$ and V is stationary. In Corollary 4.10 below, we show that ν is a $(m-2)$-rectifiable Radon measure and $V = V_u + V_R$, where V_R is the $(m-2)$-rectifiable varifold, given by $V_R = \delta_{T_x \Sigma} \theta H^{m-2} L\Sigma$, here Σ is a closed $(m-2)$-rectifiable set, and $\nu = \theta H^{m-2} L\Sigma$.

Now we start to discuss the generalized varifold flow, associated with solutions to (1.3)–(1.4). For any $\varepsilon_n \downarrow 0$, let $u_n \in C^\infty(\Omega \times R_+, R^k)$ solve (1.3) and satisfy (see also (1.6) and Lemma 2.1):

$$\sup_n \sup_{0 < t < \infty} \left(\int_0^t \int_\Omega |\partial_t u_n|^2 + \int_M e(u_n)(x, t)\,dx \right) \leqslant C < \infty. \tag{4.10}$$

For any such a u_n, we define a generalized $(m-2)$-varifold $V_n \in V_{m-2}^*(\Omega \times R_+)$ as follows

$$V_n(x, t, A) = \delta_{A(u_n)(x,t)}(A)\mu_t^n\,dt, \quad \forall(x, t, A) \in \Omega \times R_+ \times A_{m-2,m},$$

where $A(u_n)$ is defined by (4.6), and $\mu_t^n(x) = e(u_n)(x, t)\,dx$. Let $\pi_{(x,t)}: \Omega \times R_+ \times A_{m-2,m} \to \Omega \times R_+$ be the projection map at (x, t). Then we know that the weight $\|V_n\| = \pi_{(x,t)\#}(V_n) = \mu_t^n\,dt$. In particular, $\sup_n \|V_n\|(G) < \infty$ for any compact subset

$G \subset \Omega \times R_+$. Therefore, we may assume that there exists a generalized $(m-2)$-varifold $V \in V_{m-2}^*(\Omega \times R_+)$ such that

$$V_n \to V, \quad \|V_n\| = \mu_t^n \, dt \to \|V\| \tag{4.11}$$

as convergence of Radon measures. Moreover, by Lemma 2.5, we know that $\|V\| = \mu_t \, dt$. We can also represent $V = V_{x,t}\|V\| = V_{x,t}\mu_t \, dt$, where for each $(x,t) \in \Omega \times R_+$ $V_{x,t}$ is a probability measure on $A_{m-2,m}$. Note that for any compact subset $G \Subset \Omega \times R_+$, L^1 norm of $-Du_n\partial_t u_n$ on G is uniformly bounded. Hence we may assume that

$$-Du_n\partial_t u_n \, dx \, dt \to \sigma$$

as convergence of Radon measures on $\Omega \times R_+$, for some (signed) Radon measures σ on $\Omega \times R_+$. Since $-Du_n\partial_t u_n \, dx \, dt \ll e(u_n)(x,t) \, dx \, dt$, we have

$$\sigma \ll \mu_t \, dt = \|V\|. \tag{4.12}$$

By the Riesz Representation Theorem, we know that there exists a $H_t(x) \in L_{\|V\|}^1(\Omega \times, R^m)$ such that

$$\sigma(x,t) = H_t(x)\mu_t(x) \, dt. \tag{4.13}$$

Moreover, by the lower semicontinuity, we have

$$\int_0^\infty \int_\Omega |H_t(x)|^2 \, d\mu(x,t) \leqslant \liminf_{n \to \infty} \int_0^\infty \int_\Omega \left| \frac{Du_n\partial_t u_n}{e(u_n)} \right|^2 e(u_n) \, dx \, dt$$

$$\leqslant 2\liminf_{n \to \infty} \int_0^\infty \int_\Omega |\partial_t u_n|^2(x,t) \, dx \, dt < \infty. \tag{4.14}$$

Here we have used the Schwartz inequality in the last step.

LEMMA 4.5. – *For L^1 a.e. $t \in R_+$, $V_t = V_{x,t}\mu_t \in V_{m-2}^*(\Omega)$ has its first variation $\delta V_t \ll \mu_t$, and $\delta V_t = H_t\mu_t$ with $H_t \in L_{\mu_t}^2(\Omega, R^m)$.*

Proof. – For $Y \in C_0^1(\Omega, R^m)$, $\gamma \in C_0(R_+, R)$. Denote $V_t^n = V_{A(u_n)(x,t)}\mu_t^n \in V_{m-2}^*(\Omega)$. Then

$$\int_{R_+} \gamma(t)\delta V_t^n(Y) \, dt = -\int_{R_+} \gamma(t) \int_\Omega DY(x) : A \, dV_t^n(x,A)$$

$$= -\frac{1}{2}\int_{R_+} \gamma(t) \int_\Omega DY(x) : \left(|Du_n|^2 I_m - 2Du_n \otimes Du_n\right)(x,t) \, dx$$

$$- \int_{\Omega \times R_+} \gamma(t)DY(x) : A(u_n)(x,t)\frac{1}{\varepsilon_n^2}F(u_n)(x,t) \, dx \, dt = I + II.$$

For I, multiplying (1.3) by $Y(x)Du_n$ and integrating it by parts, we have

$$\int\limits_{R_+} \gamma(t)\,dt \int\limits_{\Omega} Y(x)Du_n\partial_t u_n$$

$$= \int\limits_{R_+} \gamma(t)\,dt \int\limits_{\Omega} \left(\Delta u_n + \frac{1}{\varepsilon_n^2}f(u_n)\right)Y(x)Du_n$$

$$= \int\limits_{R_+} \gamma(t)\,dt \int\limits_{\Omega} \Delta u_n Y(x)Du_n + \int\limits_{R_+} \gamma(t)\,dt \int\limits_{\Omega} D\left(\frac{F(u_n)}{\varepsilon_n^2}\right)Y(x)$$

$$= \int\limits_{R_+} \gamma(t)\,dt \int\limits_{\Omega} \left(u_{n,j}u_{n,l}Y^l\right)_j - YD\left(\frac{|Du_n|^2}{2}\right) - u_{n,j}u_{n,l}Y_j^l$$

$$\quad + \int\limits_{R_+} \gamma(t)\,dt \int\limits_{\Omega} \frac{F(u_n)}{\varepsilon_n^2}\mathrm{div}(Y)$$

$$= \int\limits_{R_+} \gamma(t)\,dt \int\limits_{\Omega} \left(\frac{1}{2}|Du_n|^2\mathrm{div}(Y) - u_{n,j}u_{n,l}Y_j^l\right) + \int\limits_{R_+} \gamma(t)\,dt \int\limits_{\Omega} \frac{F(u_n)}{\varepsilon_n^2}\mathrm{div}(Y).$$

Therefore,

$$\int\limits_{R_+} \gamma(t) \int\limits_{\Omega} \delta V_t^n(Y)\,dt = -\int\limits_{R_+} \gamma(t) \int\limits_{\Omega} YDu_n\partial_t u_n$$

$$\quad - \int\limits_{R_+} \gamma(t) \int\limits_{\{x\in\Omega:\ |Du_n|(x,t)\neq 0\}} 2\frac{u_{n,i}u_{n,j}Y_i^j}{|Du_n|^2}\frac{F(u_n)}{\varepsilon_n^2}.$$

By Lemma 2.4, we know that

$$\lim_{n\to\infty} \int\limits_{\Omega\times R_+} |\gamma(t)||DY(x)|\frac{F(u_n)}{\varepsilon_n^2}(x,t)\,dx\,dt = 0.$$

Therefore, by taking n into infinity, we have

$$\int\limits_{R_+} \gamma(t)\delta V_t(Y)\,dt = \lim_{n\to\infty} \int\limits_{R_+} \gamma(t)\,dt \int\limits_{\Omega} Y(x)(-Du_n\partial_t u_n)\,dx\,dt$$

$$= \int\limits_{R_+} \gamma(t)\,dt \int\limits_{\Omega} \langle H_t(x), Y(x)\rangle\,d\mu_t\,dt$$

so that for L^1 a.e. $t \in R_+$, $\delta V_t = H_t\mu_t$. □

For $V \in V_{m-2}^*(\Omega)$ and $x \in \Omega$, we define

$$\Theta^{m-2}(\|V\|, x) = \lim_{r\to 0} \frac{\|V\|(B_r(x))}{\alpha(m-2)r^{m-2}}. \tag{4.15}$$

Provided that the limit exists and $\alpha(m-2) = |B_1^{m-2}|$.

Now we state the monotonicity formula for generalized l-varifolds $V \in V_l^*(\Omega)$, whose first variation is a Radon measure. Note that the same formula was shown by Allard (Theorem 5.1 of [6]) for classical l-varifolds $V \in V_l(\Omega)$.

LEMMA 4.6. – *Suppose that $V \in V_l^*(\Omega)$ and $\|\delta V\|$ is a Radon measure on Ω. Then, for any $a \in \mathrm{spt}(\|V\|)$ and $0 < r < \mathrm{dist}(a, \partial\Omega)$,*

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-l}\|V\|(B_r(a))\right) = r^{-l-2}\int_{\partial B_r(a)}|S^\perp(x)|^2\,\mathrm{d}V(x,S)$$
$$- r^{-l-1}\lim_{\varepsilon\downarrow 0}\delta V\left(\theta_\varepsilon(|x|)x\right), \qquad (4.16)$$

where $\theta_\varepsilon(|x|) \in C_0^1(B_r(a))$ converges to the characteristic function of $B_r(a)$ as $\varepsilon \downarrow 0$ and $|S^\perp(x)|^2 = |x|^2 - |S(x)|^2$.

Proof. – The proof is exactly as same as that by Allard [5] for classical l-varifolds. For $\theta_\varepsilon(|x|)$ given by the lemma, one has

$$-\delta V\left(\theta_\varepsilon(|x|)x\right) = \int_{B_r(a)\times A_{l,m}}\theta'(|x|)\left(1 - \frac{|S^\perp(x)|^2}{|x|}\right)\mathrm{d}V(x,S) + l\|V\|(\theta_\varepsilon(|x|)).$$

This can easily seen to imply (4.16).   □

As a consequence, we obtain the existence of $\Theta^{m-2}(\|V_t\|, \cdot)$ for L^1 a.e. $t \in R_+$ as follows.

COROLLARY 4.7. – *Suppose that $\{V_t\}_{t>0}$ is the family of generalized $(m-2)$ varifolds obtained via (4.10)–(4.14). Then, for L^1 a.e. $t \in R_+$, there exists a set $E_t \subset \Omega$, with $H^{m-2}(E_t) = 0$, such that $\Theta^{m-2}(\|V_t\|, x)$ exists for any $x \in \Omega \setminus E_t$. Moreover, $\Theta^{m-2}(\|V_t\|, \cdot)$ is upper semicontinuous for $x \in \Omega \setminus E_t$.*

Proof. – It follows from Lemma 4.5 that for L^1 a.e. $t \in R_+$, $H_t \in L^2_{\|V_t\|}(\Omega, R^m)$, $\delta V_t = H_t\|V_t\|$, and $\lim_{n\to\infty}\int_\Omega |\partial_t u_n|^2(x,t) < \infty$. In particular,

$$\|\delta V_t\|\left(B_r(a)\right) \leqslant \lim_{n\to\infty}\int_{B_r(a)}|\partial_t u_n Du_n|$$
$$\leqslant 2\left(\|V_t\|(B_r(a))\right)^{\frac{1}{2}}\lim_{n\to\infty}\left(\int_{B_r(a)}|\partial_t u_n|^2\right)^{\frac{1}{2}}. \qquad (4.17)$$

Hence Lemma 4.6 implies,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{2-m}\|V_t\|(B_r(a))\right) \geqslant r^{-m-2}\int_{\partial B_r(a)}|S^\perp(x)|^2\,\mathrm{d}V_t(x,S)$$
$$- 2\left(r^{2-m}\|V_t\|(B_r(a))\right)^{\frac{1}{2}}\left(\lim_{n\to\infty}r^{2-m}\int_{B_r(a)}|\partial_t u_n|^2\right)^{\frac{1}{2}}$$

$$\geqslant r^{-m-2} \int_{\partial B_r(a)} |S^\perp(x)|^2 \, dV_t(x, S) - r^{2-m} \|V_t\| \big(B_r(a) \big)$$

$$- r^{2-m} \lim_{n \to \infty} \int_{B_r(a)} |\partial_t u_n|^2.$$

This implies that

$$\frac{d}{dr} \big(e^r r^{2-m} \|V_t\| (B_r(a)) \big) \geqslant r^{-m-2} \int_{\partial B_r(a)} |S^\perp(x)|^2 \, dV_t(x, S)$$

$$- r^{2-m} \lim_{n \to \infty} \int_{B_r(a)} |\partial_t u_n|^2. \tag{4.18}$$

If we let

$$E_t = \left\{ a \in \Omega \colon \lim_{r \to 0} r^{2.5-m} \lim_{n \to \infty} \int_{B_r(a)} |\partial_t u_n|^2 \geqslant 1 \right\}.$$

Then, for any $a \in \Omega \setminus E_t$, there exists $r_0 = r_0(a) > 0$ such that for any $0 < r \leqslant r_0$

$$r^{2-m} \lim_{n \to \infty} \int_{B_r(a)} |\partial_t u_n|^2 \leqslant 2r^{-\frac{1}{2}}.$$

Therefore, if we integrate (4.18) between $0 < r_1 \leqslant r_2 \leqslant r_0$, then we get

$$\big(e^{r_2} r_2^{2-m} \|V_t\| (B_{r_2}(a)) + \sqrt{r_2} \big) - \big(e^{r_1} r_1^{2-m} \|V_t\| (B_{r_1}(a)) + \sqrt{r_1} \big)$$

$$\geqslant \int_{r_1}^{r_2} r^{-m-2} \int_{\partial B_r(a)} |S^\perp(x)|^2 \, dV_t(x, S). \tag{4.19}$$

This implies that $\Theta^{m-2}(\|V_t\|, a)$ exists for all $a \in \Omega \setminus E_t$. Moreover it is upper semicontinuous for $a \in \Omega \setminus E_t$. Now we want to estimate the size of E_t as follows. In fact, a simple Vitali's covering argument implies that $H^{m-2.5}(E_t) < \infty$. In particular, $H^{m-2}(E_t) = 0$. This completes the proof. $\quad\square$

Note that, by (2.7) and (2.10), Σ_t has locally finite $(m - 2)$-dimensional Hausdorff measure for any $t > 0$. Now we have

LEMMA 4.8. – *For L^1 a.e. $t \in R_+$, there exists a subset $F_t \subset \Sigma_t$, with $H^{m-2}(F_t) = 0$, such that $\Theta^{m-2}(\|V_t\|, x) \geqslant \frac{\varepsilon_0^2}{2}$ for all $x \in \Sigma_t \setminus F_t$. Here ε_0 is given by Lemma 2.3.*

Proof. – Define

$$G = \left\{ z \in \Sigma \colon \lim_{r \downarrow 0} \lim_{n \to \infty} r^{2-m} \int_{P_r(z)} |\partial_t u_n|^2 \geqslant \varepsilon_0^5 \right\}.$$

Here Σ is the concentration set defined in Section 2 and ε_0 is given by Lemma 2.3. Then, by the Vitali's covering lemma, we have $\mathcal{P}^{m-2}(G) < \infty$. In particular, $\mathcal{P}^m(G) = 0$.

Therefore, for L^1 a.e. $t \in R_+$, $H^{m-2}(G_t) = 0$, here $G_t = G \cap \{t\} \subset \Sigma_t$. Let $F_t = G_t \cup (E_t \cap \Sigma_t)$, here E_t is given by Lemma 4.7. Then it is easy to see $H^{m-2}(F_t) = 0$. Now we want to show that for any $a \in \Sigma_t \setminus F_t$, $\Theta^{m-2}(\|V_t\|, a)$ is positive. In fact, there exists $r_a > 0$ such that

$$\lim_{n \to \infty} r^{2-m} \int_{P_r(a)} |\partial_t u_n|^2 < \varepsilon_0^5, \quad \forall 0 < r \leqslant r_a. \tag{4.20}$$

Since $a \in \Sigma_t$, it follows from [9] or [31] that for any $r \in (0, \frac{r_a}{2}]$

$$\lim_{n \to \infty} r^{2-m} \int_{B_{2r}(a)} e(u_n)(x, t - r^2) \geqslant \frac{\varepsilon_0^2}{2}.$$

On the other hand, Lemma 2.1 implies

$$\int_{B_r(a)} e(u_n)(x, t) \geqslant \int_{B_{2r}(a)} e(u_n)(x, t - r^2) - \int_{P_{2r}(a)} |\partial_t u_n|^2$$

$$- \left(r^{-2} \int_{P_{2r}(a)} |Du_n|^2 \right)^{\frac{1}{2}} \left(\int_{P_{2r}(a)} |\partial_t u_n|^2 \right)^{\frac{1}{2}}$$

$$\geqslant \int_{B_{2r}(a)} e(u_n)(x, t - r^2) - C\varepsilon_0^{2.5} r^{m-2} \geqslant \frac{\varepsilon_0^2}{4} r^{m-2}.$$

Here we have used the fact that $r^{-m} \int_{P_r(a)} |Du_n|^2 \leqslant C$. This implies that, for all $0 < r \leqslant r_a$, $\lim_{n \to \infty} r^{2-m} \int_{B_r(a)} e(u_n)(x, t) \geqslant \frac{\varepsilon_0^2}{4}$. Thus $\Theta^{m-2}(\|V_t\|, a) \geqslant \frac{\varepsilon_0^2}{4}$. $\quad\square$

THEOREM 4.9. – *Under the same notations as above. For L^1 a.e. $t \in R_+$, $V_t L(\Sigma_t \times A_{m-2,m})$ is a $(m-2)$-rectifiable varifold. In particular, Σ_t is a $(m-2)$-rectifiable set in Ω.*

Proof. – One can follow the proof of Theorem 5.5 of Allard [6]. Here we sketch a slightly different proof. First, it follows from Lemma 4.8 that for L^1 a.e. $t \in R_+$, there exists $G_t \subset \Sigma_t$, with $H^{m-2}(G_t) = 0$, such that $\Theta^{m-2}(\|V_t\|, x)$ is positive and finite for any $x \in \Sigma_t \setminus G_t$. We can also assume that $H^{m-2}(\Sigma_t \setminus G_t) > 0$ (otherwise, we have nothing to prove). Moreover, since $\Theta^{m-2}(\|V_t\|, \cdot)$ is upper semicontinuous on $\Sigma_t \setminus G_t$, $\Theta^{m-2}(\|V_t\|, x)$ is H^{m-2}-approximately continuous for H^{m-2} a.e. in $\Sigma_t \setminus G_t$. If we represent $V_t = V_{x,t} \|V_t\|$, with $V_{x,t}$ a probability measure on $A_{m-2,m}$, then $V_{x,t}$ is a H^{m-2}-measurable function with valued in the space of probability measures on $A_{m-2,m}$. It is well-known that $V_{x,t}$ is H^{m-2}-approximately continuous for H^{m-2} a.e. $x \in \Sigma_t$. Therefore, for H^{m-2} a.e. $x_0 \in \Sigma_t$, the following four properties hold:

$$\Theta^{*,m-2}(\Sigma_t, x_0) = \limsup_{r \downarrow 0} r^{2-m} H^{m-2}(\Sigma_t \cap B_r(x_0)) \geqslant 2^{-m-2}, \tag{4.21}$$

$$\Theta^{m-2}(\|V_t\|, \cdot) \text{ is } H^{m-2} \text{ approximately continuous at } x_0,$$

$V_{x,t}$ is H^{m-2} approximately continuous at x_0,

$$\lim_{r\downarrow 0} \frac{\int_{B_r(x_0)} |H_t| \, d\|V_t\|}{r^{m-2}} = |H_t(x_0)| \Theta^{m-2}(\|V_t\|, x_0) < \infty. \tag{4.22}$$

Based on these and the Geometric Lemma 2.4 of [23], we can assure that for any $r_i \downarrow 0$, there exists a subsequence $r_{i'} \downarrow 0$ such that

$$\mathcal{D}_{x_0,r_{i'}}(V_{x,t}) \to V_{x_0,t} H^{m-2} \mathsf{L} T \tag{4.23}$$

for some $(m-2)$-plane $T \subset R^m$, Now we want to show that T is independent of the choice of $\{i'\}$. In fact, by (4.22), we have

$$\lim_{i'\to\infty} \left\| \delta\big(\mathcal{D}_{x_0,r_{i'}}(V_{x,t})\big) \right\| = \lim_{r'\to\infty} r_{i'}^{3-m} (\mathcal{D}_{x_0,r_{i'}})_{\#} \|\delta V_{x,t}\| = 0.$$

Therefore

$$\delta\big(V_{x_0,t} H^{m-2} \mathsf{L} T\big) = 0$$

so that the constancy theorem for varifolds (see, Simon [33]) implies that $V_{x_0,t} = \delta_T$, i.e. the Dirac mass at T. In particular, T is unique. This proves that $V_t \mathsf{L}(\Sigma_t \times A_{m-2,m})$ is a $(m-2)$-rectifiable varifold. In particular, $\Sigma_t = \mathrm{spt}(\|V_t\|)$ is a H^{m-2}-rectifiable set. $\quad\square$

Finally, we derive some consequences of the Theorem 4.9. Let us first consider the critical points of the Ginzburg–Landau functional.

COROLLARY 4.10 (Continuation of Example 4.4). – *Under the same assumptions as in Example 4.4. There exist a closed $(m-2)$-rectifiable set $\Sigma \subset \Omega$ and a H^{m-2}-measurable function $\varepsilon_0^2 \leqslant \theta < \infty$ on Ω such that*
(1) $\nu(x) = \theta(x) H^{m-2} \mathsf{L} \Sigma$ for H^{m-2} a.e. $x \in \Sigma$, and

$$V_{u_n} \to V \equiv V_u + V(\Sigma, \theta) \tag{4.24}$$

as convergences of generalized $(m-2)$-varifolds on Ω, here

$$V(\Sigma, \theta) = \delta_{T_x \Sigma} \theta H^{m-2} \mathsf{L} \Sigma.$$

Moreover, V is stationary, i.e., for any $Y \in C_0^1(\Omega, R^m)$,

$$\int_\Omega \frac{1}{2} |Du|^2 \mathrm{div}(Y) - \sum_{1 \leqslant ij \leqslant m} u_i u_j Y_i^j + \int_\Sigma \mathrm{div}_\Sigma(Y) \theta \, dH^{m-2} = 0. \tag{4.25}$$

(2) If, in addition, $N = S^{k-1}$. Then

$$\theta(x) = \sum_{i=1}^{l_x} E(\phi_j, S^2), \quad \text{for } H^{m-2} \text{ a.e. } x \in \Sigma. \tag{4.26}$$

Here $1 \leqslant l_x < \infty$ and $\phi_j : S^2 \to S^{k-1}$ are nontrivial harmonic maps for $1 \leqslant j \leqslant l_x$.

(3) *If, in addition, $N = S^2$. Then $\theta(x) = 4\pi n_x$ for some positive integer n_x, for H^{m-2} a.e. $x \in \Sigma$. In particular, $\frac{1}{4\pi} V(\Sigma, \theta)$ is an integral $(m-2)$-varifold.*

Proof. – It follows from the static versions of Lemmas 2.1–2.3 for u_n that the concentration set is given by

$$\Sigma = \{x \in \Omega: \varepsilon_0^2 \leqslant \Theta^{m-2}(\|V\|, x) < \infty\} = \{x \in \Omega: \Theta^{m-2}(\|V\|, x) > 0\}.$$

Moreover, as in Example 4.4, $\delta V = 0$. Therefore, Theorem 4.9 implies

$$V \mathbf{L} \{(x, A): \Theta^{m-2}(\|V\|, x) > 0, A \in A_{m-2,m}\} = \delta_{T_x \Sigma} \Theta^{m-2}(\|V\|, x) H^{m-2} \mathbf{L} \Sigma$$

is a $(m-2)$-rectifiable varifold. In particular, Σ is a $(m-2)$-rectifiable set. Moreover, since $u_n \to u$ in $C^1_{\text{loc}}(\Omega \setminus \Sigma, R^k)$, we have

$$V \mathbf{L} (\Omega \setminus \Sigma) \times A_{m-2,m} = \frac{1}{2} \delta_{A(u)} |Du|^2(x) \, dx.$$

Therefore, we obtain (4.24) and (4.25). This proves (1).

The conclusion of (2) comes from the Theorem B of [32] (one can also see Section 6 below). Part (3) follows from (2) and the fact that any nontrivial harmonic map from S^2 to S^2 has energy equal to $4\pi n$ for some positive integer n.   □

Recall that a stationary harmonic map $u \in H^1(\Omega, N)$ is a weakly harmonic map, which satisfies

$$\int_\Omega |Du|^2 \text{div}(X) - 2 \sum_{1 \leqslant ij \leqslant m} u_i u_j X_j^i = 0, \quad \forall X \in C_0^1(\Omega, R^m). \qquad (4.27)$$

By quoting the result by Lin and Riviere [27], we can obtain

COROLLARY 4.11. – *Let $\{u_n\} \subset H^1(\Omega, N)$ be stationary harmonic maps. Assume that $u_n \to u$ weakly in $H^1(\Omega, N)$, $\frac{1}{2}|Du_n|^2(x) \, dx \to \frac{1}{2}|Du|^2(x) \, dx + \nu$ for some nonnegative Radon measure ν on Ω, and $V_{u_n} \to V$ as convergence of generalized $(m-2)$-varifolds. Then*

(1) *There exist a $(m-2)$-rectifiable close set $\Sigma \subset \Omega$ and a H^{m-2} measurable function $\varepsilon_0^2 \leqslant \theta < \infty$ on Ω such that $\nu(x) = \theta(x) H^{m-2} \mathbf{L} \Sigma(x)$ for H^{m-2} a.e. $x \in \Sigma$.*

(2) *$V = V_u + V(\Sigma, \theta)$ and is stationary, i.e., for any $Y \in C_0^1(\Omega, R^m)$,*

$$\int_\Omega \frac{1}{2}|Du|^2 \text{div} \, Y - \sum_{1 \leqslant ij \leqslant m} u_i u_j Y_i^j + \int_\Sigma \text{div}_\Sigma \, Y \theta \, dH^{m-2} = 0. \qquad (4.28)$$

(3) *If, in addition, $N = S^{k-1}$. Then $\theta(x) = \sum_{i=1}^{l_x} E(\phi_j, S^2)$ for H^{m-2} a.e. $x \in \Sigma$, here $1 \leqslant l_x < \infty$ and $\phi_j: S^2 \to S^{k-1}$ is a nontrivial harmonic map. Furthermore, if $k = 3$, then $\theta(x) = 4\pi n_x$ for some positive integer n_x, for H^{m-2} a.e. $x \in \Sigma$. In particular, $\frac{1}{4\pi} V(\Sigma, \theta)$ is an integral $(m-2)$-varifold.*

5. Generalized varifold flows and the Brakke flow

In this section, we will prove that the limiting pair $(u, \nu_t \, dt)$ satisfies the generalized varifold flow, which will be defined below. The generalized varifold flow implies that $\{\nu_t\}_{t \geqslant 0}$ is a Brakke flow of $(m-2)$-rectifiable varifolds, under the extra assumption that u is in the class of "suitably weak solutions" to the heat flow of harmonic maps, which requires that the energy equality (2.1) holds. Similar notion of suitably weak solutions to the Navie–Stokes equations was introduced by Cafferalli, Nirenberg and Kohn [12]. A stronger class of weak solutions behaving like parabolic stationary harmonic maps was introduced by Chen, Li and Lin [10] and Feldman [17].

We will use the same notations in Section 4 throughout this section. We first apply Theorem 4.9 to express the varifold V_t for L^1 a.e. $t > 0$.

LEMMA 5.1. – *For L^1 a.e. $t > 0$, we have*

$$V_t = \frac{1}{2} \delta_{A(u(\cdot,t))} |Du|^2 (x,t) \, dx + V\left(\Sigma_t, \Theta^{m-2}(\|V_t\|, \cdot)\right). \tag{5.1}$$

Proof. – It follows from Lemmas 4.5–4.8 that for L^1 a.e. $t > 0$, $\delta V_t = H_t \|V_t\|$, $H_t \in L^2_{\|V_t\|}(\Omega, R^m)$, $\varepsilon_0^2 \leqslant \Theta^{m-2}(\|V_t\|, x) < \infty$ for H^{m-2} a.e. $x \in \Sigma_t$, here $\Sigma_t = \Sigma \cap \{t\}$ and Σ is defined by Section 2. Therefore, Theorem 4.9 implies that $V_t \mathsf{L}(\Sigma_t \times A_{m-2,m})$ is a $(m-2)$-varifold and

$$V_t \mathsf{L} \Sigma_t = \delta_{T_x \Sigma_t} \Theta^{m-2}(\|V_t\|, x) H^{m-2} \mathsf{L} \Sigma_t = V\left(\Sigma_t, \Theta^{m-2}(\|V_t\|, \cdot)\right).$$

Since, on $\Omega \setminus \Sigma_t$, we have $u_n \to u$ in C^1_{loc} so that $V_{u_n(\cdot,t)} \to V_{u(\cdot,t)}$ on $\Omega \setminus \Sigma_t$. Therefore,

$$V_t \mathsf{L}(\Omega \setminus \Sigma_t) = \frac{1}{2} \delta_{A(u(\cdot,t))} |Du|^2 (x,t) \, dx.$$

Combining these two facts, we obtain (5.1).   □

The next Lemma shows that generically $H_t(x) \in (T_x \Sigma_t)^\perp$.

LEMMA 5.2. – *For L^1 a.e. $t > 0$, we have*

$$H_t(x) \perp T_x \Sigma_t, \quad \text{for } H^{m-2} \text{ a.e. } x \in \Sigma_t. \tag{5.2}$$

Proof. – This can be proved by the Young measure method. Let M^{mk} denote the set of $m \times k$ matrices and consider Radon measures W_n on $\Omega \times R_+ \times M^{mk}$ by

$$W_n(x,t,A) = \delta_{\frac{Du_n}{|Du_n|}(x,t)}(A) e(u_n)(x,t) \, dx \, dt.$$

Define $\phi : M^{mk} \to A_{m-2,m}$ by $\phi(A) = I_m - 2A^t A$. Then we see that $\phi_\#(W_n) = V_{u_n}$, here V_{u_n} is the generalized $(m-2)$-varifold on $\Omega \times R_+ \times A_{m-2,m}$ defined in Section 4. Since we can assume that $V_{u_n} \to V = V_{x,t} \mu_t \, dt$ and $W_n \to W = W_{x,t} \mu_t \, dt$ for some probability measures $V_{x,t}$ on $A_{m-2,m}$, and $W_{x,t}$ on M^{mk}, we then have $V_{x,t} = \phi_\#(W_{x,t})$. Since, for

L^1 a.e. $t > 0$, Lemma 5.1 holds. Therefore, for L^1 a.e. $t > 0$, $V_{x,t} = \delta_{T_x \Sigma_t}$ for $x \in \Sigma_t$. In particular, for H^{m-2} a.e. $x \in \Sigma_t$,

$$\int_{M^{mk}} \left(I_m - 2A^t A \right) dW_{x,t}(A) = \int_{A_{m-2,m}} A \, dV_{x,t}(A) = T_x \Sigma_t.$$

For any unit vector $e \in T_x \Sigma_t$, we then have

$$1 = \langle T_x \Sigma_t(e), e \rangle$$
$$= \left\langle e, \int_{M^{mk}} \left(I_m - 2A^t A \right) dW_{x,t}(A)(e) \right\rangle$$
$$= 1 - 2 \int_{M^{mk}} |A(e)|^2 \, dW_{x,t}(A).$$

Hence, for H^{m-2} a.e. $x \in \Sigma_t$, $|A(e)| = 0$ for $W_{x,t}$ a.e. $A \in M^{mk}$. This implies that for H^{m-2} a.e. $x \in \Sigma_t$, the support of $W_{x,t}$ is contained in $E(A) \equiv \{A = (A_1, \ldots, A_k)^t : \text{span}\{A_1, \ldots, A_k\} \subset (T_x \Sigma_t)^\perp\}$. Note also that if we define $Z_n = \delta_{\frac{Du_n}{|Du_n|}} \partial_t u_n Du_n \, dx \, dt$, then $Z_n \ll W_n$. Therefore, if we assume that $Z_n \to Z$ on $\Omega \times R_+ \times M^{mk}$, then $Z \ll W$ and there exists a vector valued function $Z_{x,t}$ on M^{mk} such that $Z = Z_{x,t} W_{x,t} \mu_t \, dt$. Since $(\pi_{x,t})_\# Z_n = \partial_t u_n Du_n \, dx \, dt \to -H_t(x) \mu_t \, dt$, we have

$$-H_t(x) = \int_{M^{mk}} Z_{x,t}(A) \, dW_{x,t}(A).$$

We now claim that for H^{m-2} a.e. $x \in \Sigma_t$, $Z_{x,t}(A) \in \text{spt}(W_{x,t})$, which clearly implies $H_t(x) \in (T_x \Sigma_t)^\perp$. In fact, since $\partial_t u_n Du_n \in E(\frac{Du_n}{|Du_n|})$, we have

$$\int_{M^{mk}} \text{dist}\left(A, \frac{dZ_n}{d\|Z_n\|} \right) d\|Z_n\|(A) = 0$$

taking n into infinity and by the lower semicontinuity, this gives

$$\int_{M^{mk}} \text{dist}\left(A, \frac{dZ}{d\|Z\|} \right) d\|Z\| = 0.$$

This also implies that for $W_{x,t}$ a.e. $A \in M^{mk}$, $Z_{x,t}(A) \in \text{spt}(W_{x,t})$.   □

Now we prove an energy inequality for the limiting Radon measures ν, η. Assume that

$$e(u_n)(x,t) \, dx \to \frac{1}{2} |Du|^2(x,t) \, dx + \nu_t,$$

$$|\partial_t u_n|^2(x,t) \, dx \, dt \to |\partial_t u|^2(x,t) \, dx \, dt + \eta$$

for some Radon measures $\{\nu_t\}_{t>0}$ on Ω and η on $\Omega \times R_+$. Then

PROPOSITION 5.3 (Energy inequality). – *Under the same notations as above. We have, for any $0 < t_1 < t_2 < \infty$ and $\phi \in C_0^1(\Omega, R_+)$,*

$$\int_\Omega \frac{1}{2}\phi(x)|Du|^2(x, t_2) + v_{t_2}(\phi) - \int_\Omega \frac{1}{2}\phi(x)|Du|^2(x, t_1) - v_{t_1}(\phi)$$

$$\leqslant - \int_{\Omega \times [t_1, t_2]} \left(|\partial_t u|^2 \phi + D\phi \, \partial_t u \, Du\right)$$

$$- \int_{\Sigma_{t_1}^{t_2}} \left(\phi \, d\eta - \left\langle (T_x(\Sigma_t))^\perp D\phi, H_t \right\rangle d\|V_t\|\right) dt. \qquad (5.3)$$

Here $v_t(\phi) = \int_\Omega \phi(x) \, dv_t(x)$, $\Sigma_{t_1}^{t_2} = \Sigma \cap R^m \times [t_1, t_2]$, and $(T_x \Sigma_t)^\perp$ denotes the normal space of Σ_t at x.

Proof. – By taking n into infinity in the equality (2.1), we have

$$\int_\Omega \frac{1}{2}\phi(x)|Du|^2(x, t_2) + v_{t_2}(\phi) - \int_\Omega \frac{1}{2}\phi(x)|Du|^2(x, t_1) - v_{t_1}(\phi)$$

$$= - \int_{\Omega \times [t_1, t_2]} \phi|\partial_t u|^2 - \int_{\Sigma_{t_1}^{t_2}} \phi \, d\eta + \int_{\Omega \times [t_1, t_2]} \langle D\phi, H_t \rangle d\|V_t\| \, dt. \qquad (5.4)$$

Sine $\partial_t u_n Du_n \to \partial_t u Du$ strongly in $L_{\text{loc}}^2(\Omega \times R_+ \setminus \Sigma)$, $H_t \, d\mu_t = -\partial_t u Du \, dx$ on $\Omega \times R_+ \setminus \Sigma$. Therefore, by Lemma 5.2, we have

$$\int_{\Omega \times [t_1, t_2]} \langle D\phi, H_t \rangle d\|V_t\| \, dt = - \int_{\Omega \times [t_1, t_2]} D\phi \, \partial_t u Du + \int_{\Sigma_{t_1}^{t_2}} \langle D\phi, H_t \rangle d\mu_t$$

$$= - \int_{\Omega \times [t_1, t_2]} D\phi \, \partial_t u Du + \int_{\Sigma_{t_1}^{t_2}} \langle (T_x \Sigma_t)^\perp D\phi, H_t \rangle d\mu_t.$$

This gives (5.3). □

COROLLARY 5.4. – *Under the same notations as above. We have, for any $0 < t_1 < t_2 < \infty$ and $\phi \in C_0^1(\Omega, R_+)$,*

$$\int_\Omega \frac{1}{2}\phi(x)|Du|^2(x, t_2) + v_{t_2}(\phi) - \int_\Omega \frac{1}{2}\phi(x)|Du|^2(x, t_1) - v_{t_1}(\phi)$$

$$\leqslant - \int_{\Omega \times [t_1, t_2]} \left(|\partial_t u|^2 \phi + D\phi \, \partial_t u Du\right)$$

$$- \int_{\Sigma_{t_1}^{t_2}} \left(\frac{1}{2}|H_t|^2 \phi - \left\langle (T_x(\Sigma_t))^\perp D\phi, H_t \right\rangle\right) d\|V_t\| \, dt. \qquad (5.5)$$

Proof. – It suffices to prove

$$\int_{\Sigma_{t_1}^{t_2}} \phi \, d\eta \geqslant \frac{1}{2} \int_{\Sigma_{t_1}^{t_2}} |H_t|^2 \phi \, d\nu_t \, dt. \tag{5.6}$$

To see it, we note that, for $\mathcal{P}^m$ a.e. $z_0 = (x_0, t_0) \in \Sigma$, by the Schwartz inequality, we have

$$|H_{t_0}(x_0)|^2 \leqslant \lim_{r \downarrow 0} \lim_{n \to \infty} \left(\frac{|\int_{P_r(z_0)} \partial_t u_n \, D u_n|}{\int_{P_r(z_0)} e(u_n)} \right)^2$$

$$\leqslant 2 \lim_{r \downarrow 0} \lim_{n \downarrow \infty} \frac{\int_{P_r(z_0)} |\partial_t u_n|^2}{\int_{P_r(z_0)} e(u_n)}.$$

Hence

$$|H_{t_0}(x_0)|^2 \, d\mu(z_0) \leqslant 2 \lim_{r \downarrow 0} \lim_{n \to \infty} \int_{P_r(z_0)} |\partial_t u_n|^2$$

$$= 2 \lim_{r \to 0} \left(\int_{P_r(z_0)} |\partial_t u|^2 + \eta\big(P_r(z_0)\big) \right)$$

$$\leqslant 2 \, d\eta(z_0).$$

This gives (5.6). □

Now we give the definition of generalized varifold flow for a pair $(v, \eta_t \, dt)$.

DEFINITION 5.5. – *Let* $v \in H^1_{\mathrm{loc}}(\Omega \times R_+, N) \cap L^\infty(R_+, H^1(\Omega, N))$ *and* $\{\eta_t\}_{t \geqslant 0}$ *be nonnegative Radon measures on* Ω. *We say that the pair* $(v, \eta_t \, dt)$ *is a generalized varifold flow, if the following holds*

(1) v *is a weak solution to the heat equation of harmonic maps, i.e. satisfies* (1.0) *in the sense of distribution.*

(2) *For* L^1 *a.e.* $t \in R_+$, $\eta_t = \|V_t\|$ *for some* $(m-2)$-*rectifiable varifold* $V_t \in V_{m-2}(\Omega)$, $\delta V_t = H_t \|V_t\|$, *and* $H_t \in L^2_{\|V_t\|}(\Omega, R^m)$.

(3) *For any* $0 \leqslant s \leqslant t < \infty$ *and* $\phi \in C^1_0(\Omega, R_+)$, *we have*

$$\int_\Omega \frac{1}{2} |Dv|^2(x, t)\phi(x) + \eta_t(\phi) - \int_\Omega \frac{1}{2} |Dv|^2(x, s)\phi(x) - \eta_s(\phi)$$

$$\leqslant - \int_s^t \int_\Omega \big(|\partial_t v|^2 \phi + D\phi \, \partial_t v \, Dv \big)$$

$$- \int_s^t \int_{A_t} \big(\phi(x)|H_t(x)|^2 - \langle (T_x A_t)^\perp D\phi, H_t(x) \rangle \big) \, d\eta_t \, dt. \tag{5.7}$$

Here $A_t = \mathrm{spt}(\eta_t)$.

One of the main theorem of this section is to show that the limiting pair $(u, \nu_t \, dt)$ obtained from the limiting process of sequences of solutions to the heat flow of the Ginzburg–Landau functional, i.e. solutions to (1.3)–(1.4), is a generalized varifold flow.

THEOREM 5.6. – *Under the same notations as above. The limiting pair $(u, v_t \, dt)$ is a generalized varifold flow.*

Proof. – By comparing (5.5) and (5.7), we know that in order to show the pair $(u, v_t \, dt)$ is a generalized varifold flow, it suffices to improve the $\frac{1}{2}$ factor in front of the term $\int_s^t \int_{\Sigma_t} \phi(x)|H_t(x)|^2$ of (5.5) to 1. In other word, we need to prove

LEMMA 5.7. – *Under the same notations as above.*

$$\int\limits_{\Sigma_t^s} \phi(x)|H_t(x)|^2 \, dv_t(x) \, dt \leqslant \int\limits_{\Sigma_t^s} \phi(x) \, d\eta(x, t) \tag{5.8}$$

for any $0 < t \leqslant s < \infty$ and $\phi \in C_0^1(\Omega, R_+)$.

Before we prove Lemma 5.7, we would like to remark that (5.8) also follows from the energy quantization Theorem 6.1 of Section 6 below, which is only proved at present under the assumption that $N = S^{k-1}$ however. Here we present a different proof of it, which is valid for all manifold N.

LEMMA 5.8. – *For $\mathcal{P}^m$ a.e. $z = (x, t) \in \Sigma$, we have*

$$\lim_{r \downarrow 0} \lim_{n \to \infty} r^{-m} \int\limits_{P_r(z)} \left(|D_x u_n|^2 - |D_y u_n|^2\right) dx \, dt = 0, \tag{5.9}$$

$$\lim_{r \downarrow 0} \lim_{n \to \infty} r^{-m} \int\limits_{P_r(z)} D_x u_n D_y u_n \, dx \, dt = 0. \tag{5.10}$$

Here (x, y) is the coordinate function of $(T_x \Sigma_t)^\perp$.

Proof. – First note from the proof of Theorem 6.1 of Section 6 below that for $\mathcal{P}^m$ a.e. $z_0 = (x_0, t_0) \in \Sigma$, the following properties hold:

$$\lim_{r \downarrow 0} \lim_{n \to \infty} r^{-m} \int\limits_{P_r(z_0)} e(u_n) = \Theta^{m-2}(\|V_{t_0}\|, x_0), \tag{5.11}$$

$$\lim_{r \downarrow 0} r^{2-m} \int\limits_{P_r(z_0)} |\partial_t u_n|^2 = 0, \tag{5.12}$$

$$\lim_{r \downarrow 0} \left(r^{2-m} \int\limits_{P_r(z_0)} |\partial_t u|^2 + r^{-m} \int\limits_{P_r(z_0)} |Du|^2 \right) = 0, \tag{5.13}$$

$$\lim_{r \downarrow 0} \lim_{n \to \infty} r^{-m} \int\limits_{P_r(z_0)} |D_z u_n|^2 = 0, \tag{5.14}$$

where z is any vector in the $(m - 2)$-plane $T_{x_0} \Sigma_{t_0} \subset R^m$.

For any such $z_0 = (x_0, t_0)$, we identify $T_{x_0} \Sigma_{t_0} = \{(0,0)\} \times R^{m-2}$ and write the coordinate of R^m as $R^m = \{(x, y, z) \colon (x, y) \in R^2, z \in R^{m-2}\}$.

For $r_n \downarrow 0$, let $v_n(x, t) = u_n(x_0 + r_n x, t_0 + r_n^2 t) \colon P_2(0) \to R^k$. Then we know that v_n solves (1.3), with ε_n replaced by $\overline{\varepsilon}_n = \frac{\varepsilon_n}{r_n} \to 0$, and

$$v_n(x, y, z, t) \to \text{constant in } C^1_{\text{loc}}(R^{m+1} \setminus R^{m-2} \times R, R^k),$$

$$e(v_n)(x, y, z, t)\, dx\, dy\, dz\, dt \to \Theta^{m-2}(\|V_{t_0}\|, x_0)(H^{m-2}\mathsf{L}R^{m-2}) \times (L^1\mathsf{L}R), \quad (5.15)$$

$$(|D_x v_n|^2 - |D_y v_n|^2)(x, y, z, t)\, dx\, dy\, dz\, dt \to \alpha(z, t) H^{m-2}\mathsf{L}R^{m-2} \times L^1\mathsf{L}R, \quad (5.16)$$

$$D_x v_n D_y v_n(x, y, z, t)\, dx\, dy\, dz\, dt \to \beta(z, t) H^{m-2}\mathsf{L}R^{m-2} \times L^1\mathsf{L}R \quad (5.17)$$

as convergences of Radon measures on $P_2(0)$, for some measurable functions α, β on $R^{m-2} \times R$. Observe that (5.9) and (5.10) are equivalent to

$$\int\limits_{B_1^{m-2} \times (-1,1)} \alpha(z, t)\, dz\, dt = \int\limits_{B_1^{m-2} \times (-1,1)} \beta(z, t)\, dz\, dt = 0. \quad (5.18)$$

In order to prove (5.18), we need the Pohozaev identity for v_n as follows. For $X \in C_0^1(B_2^m, R^m)$, multiplying the equations of v_n by $X(x, y, z) D v_n(x, y, z)$, we get

$$\int\limits_{P_2(0)} \partial_t v_n D v_n X = \int\limits_{P_2(0)} e(v_n) \text{div}(X) - \sum_{i,j} v_{n,i} v_{n,j} X_i^j. \quad (5.19)$$

Note that (5.12) implies

$$\lim_{n \to \infty} \int\limits_{P_2} \partial_t v_n D v_n X = 0$$

and Lemma 2.4 implies

$$\int\limits_{P_2(0)} e(v_n) \text{div}(X) = \int\limits_{P_2(0)} \frac{1}{2} |D v_n|^2 \text{div}(X) + \mathrm{O}(n^{-1}).$$

Hence, for $X = (X^1, X^2, \ldots, X^m) \in C_0^1(B_2^m, R^m)$, we have

$$\int\limits_{P_2(0)} \frac{1}{2} |D v_n|^2 \text{div}(X) = \int\limits_{P_2(0)} \frac{1}{2} |D v_n|^2 (X_x^1 + X_y^2) + \sum_{j=3}^m \int\limits_{P_2(0)} \frac{1}{2} |D v_n|^2 X_j^j$$

$$- \sum_{3 \leqslant ij \leqslant m} \int\limits_{P_2(0)} v_{n,i} v_{n,j} X_i^j - \sum_{3 \leqslant i \leqslant m} \int\limits_{P_2(0)} v_{n,i} (v_{n,x} X_i^1 + v_{n,y} X_i^2)$$

$$- \sum_{3 \leqslant j \leqslant m} \int\limits_{P_2(0)} v_{n,j} (v_{n,x} X_x^j + v_{n,y} X_y^j)$$

$$- \int_{P_2(0)} \left(|v_{n,x}|^2 X_x^1 + |v_{n,y}|^2 X_y^2 \right) + v_{n,x} v_{n,y} \left(X_x^2 + X_y^1 \right) + \mathrm{O}(n^{-1})$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + \mathrm{O}(n^{-1}).$$

It follows from (5.14) that

$$I_3 + I_4 + I_5 = \mathrm{O}(n^{-1}).$$

Hence

$$\int_{P_2(0)} \frac{1}{2} |Dv_n|^2 \mathrm{div}(X)$$

$$= \int_{P_2(0)} \frac{1}{2} |Dv_n|^2 \left(X_x^1 + X_y^2 \right) + \sum_{j=3}^{m} \int_{P_2(0)} \frac{1}{2} |Dv_n|^2 X_j^j$$

$$- \int_{P_2(0)} \left(|v_{n,x}|^2 X_x^1 + |v_{n,y}|^2 X_y^2 \right) + v_{n,x} v_{n,y} \left(X_x^2 + X_y^1 \right) + \mathrm{O}(n^{-1}). \quad (5.20)$$

By choosing $X = (0, 0, X^3, \ldots, X^m)$, we then get

$$\int_{P_2(0)} \frac{1}{2} |Dv_n|^2 \mathrm{div}(X) = \sum_{j=3}^{m} \int_{P_2(0)} \frac{1}{2} |Dv_n|^2 X_j^j + \mathrm{O}(n^{-1}) \quad (5.21)$$

this, combined with (5.19) and (5.20), implies

$$\int_{P_2(0)} \frac{1}{2} \left(|v_{n,x}|^2 + |v_{n,y}|^2 \right) \left(X_x^1 + X_y^2 \right)$$

$$= \int_{P_2(0)} \left(|v_{n,x}|^2 X_x^1 + |v_{n,y}|^2 X_y^2 \right) + v_{n,x} v_{n,y} \left(X_x^2 + X_y^1 \right) + \mathrm{O}(n^{-1}). \quad (5.22)$$

In particular, one has

$$\int_{P_2(0)} \left(|D_x v_n|^2 - |D_y v_n|^2 \right) \left(X_y^2 - X_x^1 \right) - 2 \int_{P_2(0)} D_x v_n D_y v_n \left(X_x^2 + X_y^1 \right) = \mathrm{O}(n^{-1}).$$

Therefore, we get

$$\int_{B_2^{m-2} \times (-4,4)} \left(\alpha(z, t) \left(X_y^2 - X_x^1 \right) - 2\beta(z, t) \left(X_x^2 + X_y^1 \right) \right) = 0 \quad (5.23)$$

for any $X^1, X^2 \in C_0^1(B_2^m)$. Now choosing $X^1(x, y, z) = x\phi(x, y, z)$ and $X^2 = 0$ for suitable cut-off function $\phi \in C_0^1(B_2^m)$, one can see

$$\int_{B_1^{m-2} \times (-1,1)} \alpha(z, t) \, \mathrm{d}z \, \mathrm{d}t = 0.$$

Similarly, by choosing $X^2(x, y, z) = y\phi(x, y, z)$ and $X^1 = 0$, we obtain

$$\int\limits_{B_1^{m-2} \times (-1,1)} \beta(z, t) \, dz \, dt = 0.$$

This completes the proof of Lemma 5.8. $\square$

Proof of Lemma 5.7. – Note that Lemma 5.8 guarantees that for $\mathcal{P}^m$ a.e. $z_0 = (x_0, t_0) \in \Sigma$

$$\int\limits_{P_r(z_0)} |f_n|^2 = \int\limits_{P_r(z_0)} |g_n|^2 = 1 + O(r, n^{-1}), \qquad \int\limits_{P_r(z_0)} f_n \cdot g_n = O(r, n^{-1}). \qquad (5.24)$$

Here

$$f_n = \frac{\sqrt{2} D_x u_n}{(\int_{P_r(z_0)} (|D_x u_n|^2 + |D_y u_n|^2))^{\frac{1}{2}}}, \qquad g_n = \frac{\sqrt{2} D_y u_n}{(\int_{P_r(z_0)} (|D_x u_n|^2 + |D_y u_n|^2))^{\frac{1}{2}}}.$$

Therefore, applying the Parseval's inequality, we have

$$\lim_{r \downarrow 0} \lim_{n \to \infty} \int\limits_{P_r(z_0)} |\partial_t u_n|^2 \geqslant \lim_{r \downarrow 0} \lim_{n \to \infty} \left(\int\limits_{P_r(z_0)} \partial_t u_n f_n \right)^2 + \left(\int\limits_{P_r(z_0)} \partial_t u_n g_n \right)^2$$

substituting f_n and g_n into the inequality and using the fact that

$$\lim_{r \downarrow 0} \lim_{n \to \infty} r^{-m} \int\limits_{P_r(z_0)} |D_z u_n|^2 = 0$$

we have

$$\lim_{r \downarrow 0} \lim_{n \to \infty} \frac{(\int_{P_r(z_0)} \partial_t u_n D u_n)^2}{\int_{P_r(z_0)} |D u_n|^2} \leqslant \lim_{r \downarrow 0} \lim_{n \to \infty} \frac{1}{2} \int\limits_{P_r(z_0)} |\partial_t u_n|^2. \qquad (5.25)$$

On the other hand, we know that for $\mathcal{P}^m$ a.e. $z_0 = (x_0, t_0) \in \Sigma$,

$$|H_{t_0}(x_0)|^2 \, d\mu_t(z_0) \leqslant \lim_{r \downarrow 0} \lim_{n \to \infty} 2 \frac{|\int_{P_r(z_0)} \partial_t u_n D u_n|^2}{\int_{P_r(z_0)} |D u_n|^2}.$$

Therefore, we have

$$|H_{t_0}(x_0)|^2 \, d\mu(x_0, t_0) \leqslant \lim_{r \downarrow 0} \lim_{n \downarrow \infty} \int\limits_{P_r(z_0)} |\partial_t u_n|^2 \, dx \, dt$$

$$= \lim_{r \downarrow 0} \int\limits_{P_r(z_0)} |\partial_t u|^2 + \eta(P_r(z_0)) = \lim_{r \downarrow 0} \eta(P_r(z_0)).$$

This clearly implies

$$\int\limits_{\Sigma_t^s} \phi(x)|H_t(x)|^2 \, \mathrm{d}\mu_t(x) \, \mathrm{d}t \leqslant \int\limits_{\Sigma_t^s} \phi(x) \, \mathrm{d}\eta(x, t), \quad \forall \phi \in C_0^1(\Omega, R_+). \quad \square$$

Once Lemma 5.7 is proved, we see easily that Theorem 5.6 is proved as well. Now we introduce the notion of suitably weak solution to the heat equation of harmonic maps.

DEFINITION 5.9. – *A map* $u \in H_{\mathrm{loc}}^1(\Omega \times R_+, N) \cap L^\infty(R_+, H^1(\Omega, N))$ *is a suitably weak solution to the heat equation of harmonic maps, if:*
(1) *It is a weak solution to the heat equation of harmonic maps* (1.0).
(2) *It satisfies the energy conservation law as follows. For any* $0 \leqslant t_1 < t_2 < \infty$ *and* $\phi \in C_0^1(\Omega, R_+)$,

$$\int\limits_\Omega \frac{1}{2}|Du|^2(x, t_2)\phi(x) - \int\limits_\Omega \frac{1}{2}|Du|^2(x, t_1)\phi(x)$$

$$= -\int\limits_{t_1}^{t_2} \int\limits_\Omega \left(|\partial_t u|^2(x, t)\phi(x) + D\phi \partial_t u \, Du \right). \tag{5.26}$$

We would like to remark that it is easy to check that any smooth solution to (1.0) is a suitably weak solution.
A direct consequence of Theorem 5.6 is

COROLLARY 5.10. – *Under the same assumptions as in Theorem* 5.6. *If, in addition, that the weak limiting map* $u \in H_{\mathrm{loc}}^1(\Omega \times R_+, N)$ *is a suitably weak solution to* (1.0). *Then, the defect measures* $\{v_t\}_{t \geqslant 0}$ *satisfies: for any* $0 \leqslant s \leqslant t < \infty$ *and* $\phi \in C_0^1(\Omega, R_+)$,

$$v_t(\phi) - v_s(\phi) \leqslant -\int\limits_s^t \int\limits_{\Sigma_t} \left(\phi(x)|H_t(x)|^2 - \langle (T_x \Sigma_t)^\perp D\phi(x), H_t(x) \rangle \right) \mathrm{d}v_t(x) \, \mathrm{d}t. \tag{5.27}$$

Now we want to show that (5.27) actually implies that $\{v_t\}_{t \geqslant 0}$ is a Brakke flow. First, let us recall the definition of Brakke flow given by Illmann [20], which is slightly stronger than the original definition by Brakke [8].

DEFINITION 5.11 ([8,20]). – *Let* v *be a Radon measure in* Ω *and* $\phi \in C_0^2(\Omega, R_+)$, *we set*

$$\mathcal{B}(v, \phi) = -\int\limits_\Omega \left(\phi|H|^2 - \langle (T_x v)^\perp D\phi, H \rangle \right) \mathrm{d}v$$

provided that the following three conditions hold
   (1) $v = \|V\|$ *in* $\{\phi > 0\}$ *for some* $V \in RV_{m-2}(\Omega)$,
   (2) $\delta V = H\|V\|$ *in* $\{\phi > 0\}$,
   (3) $H \in L_{\|V\|}^2(\{\phi > 0\}, R^m)$.
Otherwise, we set $\mathcal{B}(v, \phi) = -\infty$.

DEFINITION 5.12 ([8,20]). – *Let $\{\mu_t\}_{t\geqslant 0}$ be Radon measures on Ω. We say that $\{\mu_t\}_{t\geqslant 0}$ is a Brakke flow if*

$$\overline{D}_t\mu_t(\phi) \equiv \limsup_{s\to t}\frac{\mu_s(\phi) - \mu_t(\phi)}{s - t} \leqslant \mathcal{B}(\mu_t, \phi) \tag{5.28}$$

for all $t \geqslant 0$ and $\phi \in C_0^2(\Omega, R_+)$.

THEOREM 5.13. – *Under the same assumption as Theorem 5.6. If, in addition, that $u \in H^1_{\text{loc}}(\Omega \times R_+, N)$ is a suitably weak solution to (1.0). Then $\{v_t\}_{t\geqslant 0}$ is a Brakke flow.*

Proof. – First it follows from Section 4 and Lemma 5.2 that for L^1 a.e. $t \in R_+$, we have (a): $v_t = \|V_t\|$ for some $V_t \in RV_{m-2}(\Omega)$; (b): $\Theta^{m-2}(\|V_t\|, x) \geqslant \frac{\varepsilon_0^2}{4}$ for H^{m-2} a.e. $x \in \Sigma_t$; (c): $\delta V_t = H_t\|V_t\|$ with $H_t \in L^2_{\|V_t\|}(\Omega, R^m)$; (d): $H_t(x) \perp T_x\|V_t\|$ for H^{m-2} a.e. $x \in \Sigma_t$. Now we argue that (a)–(d) and (5.27) are sufficient to show (5.28) for $\{v_t\}_{t\geqslant 0}$. To see it, let us check the upper right derivative $\overline{D}_+$ of v_t for $t \geqslant 0$, the proof for lower right derivative is similar for $t > 0$. Let

$$L = \limsup_{s\downarrow t} -\frac{1}{s-t}\int\limits_t^s\int\limits_\Omega (\phi|H_t|^2 - \langle D\phi, H_t\rangle)\,\mathrm{d}v_t\,\mathrm{d}t.$$

Note that (5.27) implies $L \geqslant \overline{D}_+v_t(\phi)$. If $L = -\infty$, then $\overline{D}_+v_t(\phi) = -\infty$ so that (5.28) holds automatically. Hence we assume that $L > -\infty$ and $\overline{D}_+v_t(\phi) > -\infty$. Let $s_i \downarrow t$ be such that

$$\lim_{i\to\infty} -\frac{1}{s_i-t}\int\limits_t^{s_i}\int\limits_\Omega (\phi|H_t|^2 - \langle D\phi, H_t\rangle)\,\mathrm{d}v_t\,\mathrm{d}t = L \tag{5.29}$$

and $t_i \in (t, s_i)$ be such that (a)–(d) hold at t_i and

$$\int\limits_\Omega (\phi|H_{t_i}|^2 - \langle (T_x\Sigma_{t_i})^\perp D\phi, H_{t_i}\rangle)\,\mathrm{d}v_{t_i} \leqslant -L + \mathrm{O}(i^{-1}). \tag{5.30}$$

By the compactness theorem of Allard [6], we may assume that $V_{t_i} \to V$ in $\{\phi > 0\} \times G_{m-2,m}$ for some $V \in RV_{m-2}(\Omega)$. Moreover, by the result of Ilmanen [20] (cf. also Lemma 2.5 of Section 1), we know that $\|V\| = v_t$. There exists a $H \in L^2_{\|V\|}(\Omega, R^m)$ such that $\delta V = H\|V\| = Hv_t$ and

$$\int\limits_\Omega (\phi|H|^2 - \langle T_x\Sigma_t, H\rangle)\,\mathrm{d}v_t$$

$$\leqslant \liminf_{i\to\infty} \int\limits_\Omega (\phi|H_{t_i}|^2 - \langle (T_x\Sigma_{t_i})^\perp D\phi, H_{t_i}\rangle)\,\mathrm{d}v_{t_i} = -L.$$

Therefore

$$\overline{D}_+ \nu_t(\phi) \leqslant L \leqslant - \int_\Omega \left(\phi |H|^2 - \langle T_x \Sigma_t, H \rangle \right) \mathrm{d}\nu_t = \mathcal{B}(\nu_t, \phi). \qquad \square$$

We end this section with the following remark.

Remark 5.14. – (1) It follows from Proposition 5.3 of Ambrosio and Soner [2] that the Brakke flow is also a distance solution to the mean curvature flow. Therefore, under the condition that u is a suitable weak solution, Theorem 5.13 implies that $\{\nu_t\}_{t \geqslant 0}$ is a distance solution to the mean curvature flow.

(2) Under the assumption that u is a suitable weak solution. If $\nu_0 = \alpha H^{m-2} \mathrm{L} \Gamma_0$ for some $\alpha > 0$ and a closed $(m - 2)$-dimensional Riemannian manifold Γ_0. Let $\{\Gamma_t\}_{t \in [0, T)}$ is the smooth mean curvature flow. Then there exists a nonincreasing function $\alpha : [0, T) \to [0, \alpha]$ such that $\nu_t = \alpha(t) H^{m-2} \mathrm{L} \Gamma_t$ for $t \in [0, T)$ (see Proposition 4.5 of [1], and also [25,22]).

6. Energy quantization of the energy density function

Throughout this section, we assume that $N = S^{k-1} \subset R^k$ and $m \geqslant 3$. We will show that, for $\mathcal{P}^m$ a.e. $z_0 = (x_0, t_0) \in \Sigma$, the density function $\Theta^{m-2}(\|V_{t_0}\|, x_0)$ is the finite sum of energies of harmonic S^2's (i.e., nontrivial harmonic maps from S^2). In the static case, this type of quantization result was first obtained by Lin and Riviseré [27] for stationary harmonic maps, and then Lin and Wang [32] for critical points for Ginzburg–Landau functionals. Our results here can be viewed as the parabolic extension of that of [27, 32]. For $m = 2$, this type of quantization result is called as energy identity or bubbling phenomena by people (see, [34] and [32] references therein). Let's consider the heat flow of the Ginzburg–Landau functional here, the corresponding result for the heat flow of harmonic maps is treated in Section 7.

The main theorem of this section is

THEOREM 6.1. – *For $\mathcal{P}^m$ a.e. $z_0 = (x_0, t_0) \in \Sigma$,*

$$\Theta^{m-2}(\|V_{t_0}\|, x_0) = \sum_{i=1}^{l_{z_0}} E(\phi_i, S^2) \qquad (6.1)$$

for some $1 \leqslant l_{z_0} < \infty$, here $\phi_i : S^2 \to S^{k-1}$ $(1 \leqslant i \leqslant l_{z_0})$ are nontrivial harmonic maps.

Proof. – Let us first list all the necessary facts needed, which can be found from Sections 3, 5, and [31]. The following properties hold: For L^1 a.e $t_0 \in R_+$,

$$\lim_{n \to \infty} \int_\Omega |\partial_t u_n|^2 (x, t_0) \, \mathrm{d}x < \infty, \qquad (6.2)$$

$$\lim_{r \downarrow 0} \lim_{n \to \infty} r^{2-m} \int_{B_r(x)} |\partial_t u_n|^2 < \infty, \quad \text{for } H^{m-2} \text{ a.e. } x \in \Sigma_{t_0} \qquad (6.3)$$

and, for $\mathcal{P}^m$ a.e. $z_0 = (x_0, t_0) \in \Sigma$,

$$\varepsilon_0^2 \leqslant \Theta^{m-2}(\|V_{t_0}\|, x_0) < \infty, \tag{6.4}$$

$$\lim_{r \downarrow 0} \lim_{n \to \infty} r^{2-m} \int\limits_{P_r(z_0)} |\partial_t u_n|^2 = 0, \tag{6.5}$$

$\Theta^m(\mu, z_0)$ is $\mathcal{P}^m$ approximately continuous at z_0, $\tag{6.6}$

$$\lim_{r \downarrow 0} \lim_{n \to \infty} r^{-m} \int\limits_{P_r(z_0)} |D_T u_n|^2 = 0, \quad \forall T \in T_{x_0} \Sigma_{t_0}, \tag{6.7}$$

$$\lim_{r \downarrow 0} r^{-m} \int\limits_{P_r(z_0)} |Du|^2 + r^2 |\partial_t u|^2 = 0, \tag{6.8}$$

and, for L^1 a.e. $t_0 \in R_+$, we have, for H^{m-2} a.e. $x_0 \in \Sigma_{t_0}$,

$\Theta^{m-2}(\|V_{t_0}\|, x_0)$ is H^{m-2} approximately continuous at x_0. $\tag{6.9}$

Let us now pick up a $z_0 = (x_0, t_0) \in \Sigma$ such that (6.2)–(6.9) all hold. Moreover, we may assume that $T_{x_0} \Sigma_{t_0} = \{(0, 0)\} \times R^{m-2} = \{(0, 0, Y) : Y \in R^{m-2}\}$, and write $x = (X, Y) \in R^2 \times R^{m-2}$ for $x \in R^m$. For any $r_n \downarrow 0$, define the rescaling maps $v_n : P_2(0) \to R^k$ by letting $v_n(x, t) = u_n(x_0 + r_n x, t_0 + r_n^2 t)$. Then, we have,

$$\lim_{n \to \infty} \int\limits_{P_1(0)} |D_Y v_n|^2 + |\partial_t v_n|^2 = 0,$$

$$v_n \to \text{ constant weakly in } H^1(P_2(0), R^k), \tag{6.10}$$

$$e(v_n)(X, Y, t) \, dX \, dY \, dt \to \overline{v}_t \, dt$$

as convergence of Radon measures on $P_2(0)$. Similar to [31], we have

CLAIM 6.2. – $\overline{v}_t \, dt = \Theta^{m-2}(\|V_{t_0}\|, x_0) H^{m-2} \mathsf{L} R^{m-2} \times L^1 \mathsf{L} R$, on $P_1(0)$.

To see this, let $\phi \in C_0^1(B_1^2, R_+)$ and define $f_n, g_n, h_n : R^{m-2} \times R \to R_+$ by

$$f_n(Y, t) = \int\limits_{B_1^2} e(v_n)(X, Y, t)\phi(X) \, dX, \qquad g_n(Y, t) = \int\limits_{B_1^2} |\partial_t v_n|^2(X, Y, t) \, dX$$

and

$$h_n(Y, t) = \int\limits_{B_1^2} |D_Y v_n|^2(X, Y, t) \, dX.$$

Then (6.10) implies

$$\lim_{n\to\infty} \int\limits_{B_1^{m-2}\times(-1,1)} g_n(Y,t) + h_n(Y,t)\,\mathrm{d}Y\,\mathrm{d}t = 0. \tag{6.11}$$

For $1 \leqslant j \leqslant m-2$, Y_j and t-derivative of f_n are as follows.

$$\frac{\mathrm{d}}{\mathrm{d}Y_j} f_n(Y,t) = \int\limits_{B_1^2} \left(D_X v_n D_{XY_j}^2 v_n - \frac{1}{\varepsilon_n^2} f(v_n) D_{Y_j} v_n \right)\phi + \int\limits_{B_1^2} D_{Y_i} v_n D_{Y_i Y_j}^2 v_n \phi$$

$$= -\int\limits_{B_1^2} \phi\left(\Delta_X v_n + \frac{1}{\varepsilon_n^2} f(v_n) \right) D_{Y_j} v_n$$

$$\quad - \int\limits_{B_1^2} D_X\phi D_X v_n D_{Y_j} v_n + \int\limits_{B_1^2} D_{Y_i} v_n D_{Y_i Y_j}^2 v_n \phi$$

$$= -\int\limits_{B_1^2} \phi \partial_t v_n D_{Y_j} v_n - \int\limits_{B_1^2} D_X\phi D_X v_n D_{Y_j} v_n + \frac{\mathrm{d}}{\mathrm{d}Y_i} \int\limits_{B_1^2} \phi D_{Y_i} v_n D_{Y_j} v_n$$

$$= f_n^{1,j} + \mathrm{div}_{(Y,t)} f_n^{2,j}. \tag{6.12}$$

Here

$$f_n^{1,j}(Y,t) = -\int\limits_{B_1^2} (D_X\phi D_X v_n D_{Y_j} v_n + \phi \partial_t v_n D_{Y_j} v_n)$$

and

$$f_n^{2,j}(Y,t) = \left(\int\limits_{B_1^2} \phi D_{Y_1} v_n D_{Y_j} v_n, \dots, \int\limits_{B_1^2} \phi D_{Y_{m-2}} v_n D_{Y_j} v_n, 0 \right).$$

$$\frac{\mathrm{d}}{\mathrm{d}t} g_n(Y,t) = -\int\limits_{B_1^2} \left(\Delta_X v_n + \frac{1}{\varepsilon_n^2} f(v_n) \right) \partial_t v_n \phi$$

$$\quad - \int\limits_{B_1^2} D_X\phi D_X v_n \partial_t v_n + \int\limits_{B_2^2} \phi D_{Y_i} v_n D_{Y_i}(\partial_t v_n)$$

$$= -\int\limits_{B_1^2} |\partial_t v_n|^2 \phi - \int\limits_{B_1^2} D_X v_n D_X\phi \partial_t v_n + \frac{\mathrm{d}}{\mathrm{d}Y_i} \int\limits_{B_1^2} \phi D_{Y_i} v_n \partial_t v_n$$

$$= g_n^1 + \mathrm{div}_{(Y,t)} g_n^2. \tag{6.13}$$

Here

$$g_n^1(Y,t) = -\int\limits_{B_1^2} (|\partial_t v_n|^2 \phi + D_X v_n D_X\phi \partial_t v_n)$$

and

$$g_n^2(Y, t) = \left(\int_{B_1^2} \phi D_{Y_1} v_n \partial_t v_n, \ldots, \int_{B_1^2} \phi D_{Y_{m-2}} v_n \partial_t v_n, 0 \right).$$

Note that (6.10) implies

$$\lim_{n \to \infty} \sum_{i=1}^{2} (\|f_n^i\|_{L^1(B_1^{m-2} \times (-1,1))} + \|g_n^i\|_{L^1(B_1^{m-2} \times (-1,1))}) = 0. \tag{6.14}$$

Based on (6.12), (6.13), and (6.14), we can apply the Allard's strong constancy Lemma [5] as in [23] or [31] to conclude the Claim 6.2. Moreover, one has

$$\lim_{n \to \infty} \|f_n(Y, t) - \Theta^{m-2}(\|V_{t_0}\|, x_0)\|_{L^1(B_1^{m-2} \times (-1,1))} = 0. \tag{6.15}$$

Therefore, for any $\delta > 0$, there exists $E_\delta \subset B_1^{m-2} \times (-1, 1)$ with $|E_\delta| \geqslant 1 - \delta$ such that

$$\lim_{n \to \infty} \sup_{(Y,t) \in E_\delta} |f_n(Y, t) - \Theta^{m-2}(\|V_{t_0}\|, x_0)| = 0. \tag{6.16}$$

In order to prove that $\Theta^{m-2}(\|V_{t_0}\|, x_0)$ is the sum of energies of finitely many harmonic S^2's, it suffices to prove that $f_n(Y, t)$ converges to the sum of energies of finitely many harmonic S^2's, for $(Y, t) \in E_\delta$. Now we define the local Hardy–Littlewood maximal function for a function $f \in L^1(B_1^{m-2} \times (-1, 1))$, with respect to the parabolic metric in $R^{m-2} \times R$, as follows

$$M(f)(Y, t) = \sup \left\{ r^{-m} \int_{P_r(Y,t)} (f)(z, s) \, dy \, ds, \ P_r(Y, t) \subset B_1^{m-2} \times (-1, 1) \right\}.$$

Then the weak $(1, 1)$ estimates implies that there exists $F_\delta^n \subset B_1^{m-2} \times (-1, 1)$, with $|F_\delta^n| \geqslant 1 - \delta$, such that

$$\lim_{n \to \infty} M(g_n + h_n)(Y, t) = 0, \quad \forall (Y, t) \in F_\delta^n, \tag{6.17}$$

$$\lim_{n \to \infty} M(f_n)(Y, t) \leqslant C \Theta^{m-2}(\|V_{t_0}\|, x_0), \quad \forall (Y, t) \in F_\delta^n, \tag{6.18}$$

$$\lim_{n \to \infty} M(p_n)(Y, t) = 0, \quad \forall (Y, t) \in F_\delta^n, \tag{6.19}$$

where

$$p_n(Y, t) = \int_{B_1^2} \frac{1}{\bar{\varepsilon}_n^2} F(v_n)(X, Y, t) \, dX$$

here $\bar{\varepsilon}_n = \frac{\varepsilon_n}{r_n} \to 0$. Now we try to prove that for any $(Y, t) \in E_\delta \cap F_\delta^n$

$$\lim_{n \to \infty} f_n(Y, t) = \sum_{j=1}^{l} E(\phi_j, S^2) \tag{6.20}$$

for some $1 \leqslant l < \infty$, here $\phi_j : S^2 \to S^{k-1}$ $(1 \leqslant j \leqslant l)$ are nontrivial harmonic maps.

Step 1. First Bubble.

This step has been carried out in [31]. Here we give a slightly different proof. For any given $(Y_n, t_n) \in E_\delta \cap F_\delta^n$, let $X_n \in B_{\frac{1}{2}}^2$ and $\delta_n > 0$ be such that

$$\int_{B_{\delta_n}^2} e(v_n)(X, Y_n, t_n)\, dX = \frac{\varepsilon_0^2}{C(m)} = \max\left\{ \int_{B_{\delta_n}^2(\overline{X})} e(v_n)(X, Y_n, t_n)\, dX : \overline{X} \in B_{\frac{1}{2}}^2 \right\}. \quad (6.21)$$

Here $\varepsilon_0 > 0$ is given by Lemma 2.3 and $C(m) > 0$ is a large constant to be chosen. It is not difficult to see that $X_n \to 0$ and $\delta_n \to 0$ (cf. [23] and [31]). Moreover, as in [23] and [31], we can apply (6.12)–(6.14) to get, for any $X \in B_{\frac{1}{2}}^2$,

$$(2\delta_n)^{-m} \int_{B_{2\delta_n}^2(X) \times B_{2\delta_n}^{m-2}(Y_n) \times (t_n - 4\delta_n^2, t_n + 4\delta_n^2)} e(v_n)(X, Y, t)\, dX\, dY\, dt \leqslant \varepsilon_0^2 \quad (6.22)$$

and

$$\delta_n^{-m} \int_{B_{\delta_n}^2(X) \times B_{\delta_n}^{m-2}(Y_n) \times (t_n - \delta_n^2, t_n + \delta_n^2)} e(v_n)(X, Y, t)\, dX\, dY\, dt \geqslant \frac{\varepsilon_0^2}{2}. \quad (6.23)$$

We let

$$w_n(X, Y, t) = v_n\left(X_n + \delta_n X, Y_n + \delta_n Y, t_n + \delta_n^2 t\right).$$

Then Lemma 2.3 implies

$$w_n \to w \quad \text{in } C_{\text{loc}}^1\left(R^2 \times B_2^{m-2} \times (-4, 4), R^k\right).$$

Moreover, (6.14) implies that $\partial_t w = D_Y w = 0$ so that $w(X, Y, t) = w(X) : R^2 \to S^{k-1}$ is a harmonic map with positive and finite energy, which can be lifted to a nontrivial harmonic map from S^2 to S^{k-1}, named as ϕ_1. By repeating all the possible blowing-up at different points and scales, we can get

$$\Theta^{m-2}(\|V_{t_0}\|, x_0) = \lim_{n \to \infty} f_n(Y_n, t_n) \geqslant \sum_{j=1}^l E\left(\phi_j, S^2\right) \quad (6.24)$$

for some $l = l_{z_0} \leqslant \Theta^{m-2}(\|V_{t_0}\|, x_0)/\varepsilon_0^2$, and some nontrivial harmonic maps $\phi_j : S^2 \to S^{k-1}$ $(1 \leqslant j \leqslant l)$.

Step 2. (6.24) is an equality.

To achieve this, it suffices to show that there is no energy concentration over the neck regions between two bubbles at the same point. This step is very similar to that of [32]. The idea is to use the interpolation between $L^{2,1}$ and $L^{2,\infty}$ norms of Dv_n over the neck region, which has been recently explored by Lin and Riviere [27] in the context of stationary harmonic maps in higher dimensions, and [28,32] in the context of critical points of Ginzburg–Landau functionals. For completeness, we sketch it here. First, we

observe that by an induction argument on l it suffices to show that (6.24) is an equality for $l = 1$ (cf. [14] for $m = 2$ and [27,32] for $m \geqslant 3$ for the induction argument).

CLAIM 6.3. – *For any $\varepsilon > 0$ and sufficiently large $R > 0$, we have*

$$\int\limits_{B_{2r}^2(X_n) \setminus B_r^2(X_n)} e(v_n)(X, Y_n, t_n) \, \mathrm{d}X \leqslant \varepsilon^2, \quad \forall R\delta_n < r < \frac{1}{2}. \tag{6.25}$$

For, otherwise, one can argue exactly as in [32] to conclude that one can rescale v_n suitably to get a second bubble, which would contradict with $l = 1$.

Now one can apply the Allard's strong constancy lemma (cf. [5] and [32]) and Lemma 2.3 to conclude that

$$e(v_n)(X, Y, t) \leqslant \frac{C\varepsilon^2}{|X - X_n|^2 + |Y - Y_n|^2 + |t - t_n|^2} \tag{6.26}$$

for $2R\delta_n \leqslant |X - X_n| \leqslant \frac{1}{4}$, $|Y - Y_n| \leqslant \frac{|X - X_n|}{2}$, $|t - t_n| \leqslant \frac{|X - X_n|^2}{4}$. In particular, we have

$$e(v_n)(X, Y, t) \leqslant \frac{C\varepsilon^2}{|X - X_n|^2} \tag{6.27}$$

for $2R\delta_n \leqslant |X - X_n| \leqslant \frac{1}{4}$, $|Y - Y_n| \leqslant R\delta_n$, $|t - t_n| \leqslant R^2\delta_n^2$. Hence, if we let $w_n(X, Y, t) = v_n(X_n + \delta_n X, Y_n + \delta_n Y, t_n + \delta_n^2 t)$, then we have

$$e(w_n)(X, Y, t) \leqslant \frac{C\varepsilon^2}{|X|^2}, \quad \forall 2R \leqslant |X| \leqslant \frac{1}{4\delta_n}, |Y| \leqslant R, |t| \leqslant R^2. \tag{6.28}$$

This implies that $Dw_n(\cdot, Y, t) \in L^{2,\infty}(B_{(4\delta_n)^{-1}}^2 \setminus B_{2R}^2)$ for any $(Y, t) \in B_R^{m-2} \times (-R^2, R^2)$, and

$$\sup_{(Y,t) \in B_R^{m-2} \times (-R^2, R^2)} \|Dw_n(\cdot, Y, t)\|_{L^{2,\infty}(B_{(4\delta_n)^{-1}}^2 \setminus B_{2R}^2)} \leqslant C\varepsilon. \tag{6.29}$$

Here $L^{2,\infty}$ denotes the Lorentz space with index $(2, \infty)$ (see Ziemer [38] for the definition). Now we try to estimate the $L^{2,1}$ norm of $D\frac{w_n}{|w_n|}(\cdot, Y, t)$ over $B_{(4\delta_n)^{-1}}^2$.

CLAIM 6.4. – *For $\mathcal{P}^m$ a.e. $(Y, t) \in B_R^{m-2} \times (-R^2, R^2)$, $D\frac{w_n}{|w_n|}(\cdot, Y, t) \in L^{2,1}(B_{(4\delta_n)^{-1}}^2)$. Moreover,*

$$\int\limits_{B_R^{m-2} \times (-R^2, R^2)} \left\| D\frac{w_n}{|w_n|}(\cdot, Y, t) \right\|_{L^{2,1}(B_{(4\delta_n)^{-1}}^2)} \mathrm{d}Y \, \mathrm{d}t$$

$$\leqslant C\delta_n^{-m} \int\limits_{B_{\frac{1}{2}}^2(X_n) \times B_{R\delta_n}^{m-2}(Y_n) \times (t_n - R^2\delta_n^2, t_n + R^2\delta_n^2)} |Dv_n|^2 + |\partial_t v_n|^2 \leqslant C. \tag{6.30}$$

Proof. – It is very similar to the proof of Theorem B in [32]. Here we only sketch the outline. For any $t \in (-R^2, R^2)$, denote $\omega_n(X, Y) = w_n(X, Y, t) : B_{(2\delta_n)^{-1}}^2 \times B_{2R}^{m-2} \to R^k$.

Then we have

$$\Delta \omega_n + \frac{1}{\varepsilon_n^2}\left(1 - |\omega_n|^2\right)\omega_n = l_n$$

where $l_n(X, Y) = \partial_t w_n(X, Y, t)$.

For $1 \leqslant i, j \leqslant k$, let α_n^{ij} be the 1-forms defined by $\alpha_n^{ij} = \mathrm{d}\omega_n^i \omega_n^j - \omega_n^i \, \mathrm{d}\omega_n^j$. Then

$$\begin{aligned}
\mathrm{d}^* \alpha_n^{ij} &= \Delta \omega_n^i \omega_n^j - \Delta \omega_n^j \omega_n^i \\
&= l_n^i \omega_n^j - l_n^j \omega_n^i \equiv H_n^{ij},
\end{aligned} \tag{6.31}$$

$$\Delta \alpha_n^{ij} = \mathrm{d}H_n^{ij} + 2\mathrm{d}^*\left(\mathrm{d}\omega_n^i \wedge \mathrm{d}\omega_n^j\right). \tag{6.32}$$

Now, let $\tilde{\omega}_n : R^m \to R^k$ be an extension of ω_n such that

$$\|D\tilde{\omega}_n\|_{L^2(R^m)} \leqslant C\|D\omega_n\|_{L^2(B^2_{(2\delta_n)^{-1}} \times B^{m-2}_{2R})} \tag{6.33}$$

and $\overline{H}_n^{ij} : R^m \to R$ be an extension of H_n^{ij} such that $\overline{H}_n^{ij} = 0$ outside $B^2_{(2\delta_n)^{-1}} \times B^{m-2}_{2R}$. Let $F_n^{ij} \in H^1(R^m, \wedge^2(R^m))$ solve

$$\Delta F_n^{ij} = 2 \, \mathrm{d}\tilde{\omega}_n^i \wedge \mathrm{d}\tilde{\omega}_n^j. \tag{6.34}$$

Then, we know from [32] that $F_n^{ij} \in W^{2,1}(R^m, \wedge^2(R^m))$ and

$$\begin{aligned}
\left\|D^2 F_n^{ij}\right\|_{L^1(R^m)} &\leqslant C\left\|\mathrm{d}\tilde{\omega}_n^i \wedge \mathrm{d}\tilde{\omega}_n^j\right\|_{\mathcal{H}^1(R^m)} \\
&\leqslant C\|D\tilde{\omega}_n\|_{L^2(R^m)}^2 \\
&\leqslant C\|D\omega_n\|_{L^2(B^2_{(2\delta_n)^{-1}} \times B^{m-2}_{2R})}^2,
\end{aligned} \tag{6.35}$$

where $\mathcal{H}^1$ denotes the Hardy space of R^m. Let $G_n^{ij} \in H^1(R^m, R)$ solve

$$\Delta G_n^{ij} = \overline{H}_n^{ij}. \tag{6.36}$$

Then, we have that $D^2 G_n^{ij} \in L^2(R^m)$ and

$$\left\|D^2 G_n^{ij}\right\|_{L^2(R^m)} \leqslant C\left\|\overline{H}_n^{ij}\right\|_{L^2(R^m)} \leqslant C\|\partial_t w_n\|_{L^2(B^2_{(2\delta_n)^{-1}} \times B^{m-2}_{2R})}. \tag{6.37}$$

In particular, we have, by the Hölder inequality,

$$\begin{aligned}
\left\|D^2 G_n^{ij}\right\|_{L^1(B^2_{(2\delta_n)^{-1}} \times B^{m-2}_{2R})} &\leqslant C\left\|D^2 G_n^{ij}\right\|_{L^2(B^2_{(2\delta_n)^{-1}} \times B^{m-2}_{2R})}\left(R^{m-2}\delta_n^{-2}\right)^{\frac{1}{2}} \\
&\leqslant C\left(R^{m-2}\delta_n^{-2}\right)^{\frac{1}{2}}\|\partial_t w_n\|_{L^2(B^2_{(2\delta_n)^{-1}} \times B^{m-2}_{2R})}.
\end{aligned} \tag{6.38}$$

Note that

$$\alpha_n^{ij} = \mathrm{d}G_n^{ij} + 2\mathrm{d}^*\left(F_n^{ij}\right) + K_n^{ij},$$

where K_n^{ij} is a harmonic 1-form with $j^*(\alpha_n^{ij} - 2\mathrm{d}^* F_n^{ij} - \mathrm{d} G_n^{ij}) = 0$, and $j : \partial(B_{(2\delta_n)^{-1}}^2 \times B_{\frac{3R}{2}}^{m-2}) \to R^m$ denotes the inclusion map. By choosing $R > 0$ suitably and using the Fubini's theorem, we may assume that

$$\left\| \alpha_n^{ij} \right\|_{L^1(\partial(B_{(2\delta_n)^{-1}}^2 \times B_{\frac{3R}{2}}^{m-2}))} \leqslant C R^{-1} \| D\omega_n \|_{L^1(B_{\delta_n^{-1}}^2 \times B_{2R}^{m-2})},$$

$$\left\| |D^2 G_n^{ij}| + |D^2 F_n^{ij}| \right\|_{L^1(\partial(B_{(2\delta_n)^{-1}}^2 \times B_{\frac{3R}{2}}^{m-2}))}$$

$$\leqslant C R^{-1} \left\| |D^2 G_n^{ij}| + |D^2 F_n^{ij}| \right\|_{L^1((B_{\delta_n^{-1}}^2 \times B_{2R}^{m-2}))}.$$

Therefore, by the well-known estimate on harmonic functions, we have

$$\left\| DK_n^{ij} \right\|_{L^1(B_{(4\delta_n)^{-1}}^2 \times B_R^{m-2})}$$

$$\leqslant C \left(R^{m-2} \delta_n^{-2} \right)^{\frac{1}{2}} \left(\int_{B_{\delta_n^{-1}}^2 \times B_R^{m-2}} \left(|Dw_n|^2 + |\partial_t w_n|^2 \right) \mathrm{d}X \, \mathrm{d}Y \right)^{\frac{1}{2}}. \qquad (6.39)$$

Hence it follows from the embedding result, $W^{1,1}(R^2) \subset L^{2,1}(R^2)$ (cf. Heléin [19]), that we have, for H^{m-2} a.e. $Y \in B_R^{m-2}$, that $\alpha_n^{ij}(\cdot, Y) \in L^{2,1}(B_{(4\delta_n)^{-1}}^2)$ and

$$\left\| \alpha_n^{ij}(\cdot, Y) \right\|_{L^{2,1}(B_{(4\delta_n)^{-1}}^2)} \leqslant C \left\| D\alpha_n^{ij} \right\|_{W^{1,1}(B_{(4\delta_n)^{-1}}^2)}$$

$$\leqslant C \left\| |D^2 G_n^{ij}| + |D^2 F_n^{ij}| + |DK_n^{ij}| \right\|_{L^1(B_{(4\delta_n)^{-1}}^2)}.$$

Therefore

$$\int_{B_R^{m-2}} \left\| \alpha_n^{ij}(\cdot, Y) \right\|_{L^{2,1}(B_{(4\delta_n)^{-1}}^2)} \mathrm{d}Y$$

$$\leqslant C R^{m-2} \delta_n^{-2} \int_{B_{\delta_n^{-1}}^2 \times B_{2R}^{m-2}} \left(|Dw_n|^2 + |\partial_t w_n|^2 \right)(X, Y, t) \, \mathrm{d}X \, \mathrm{d}Y. \qquad (6.40)$$

Hence, by the duality between $L^{2,1}$ and $L^{2,\infty}$ and (6.25), we obtain

$$\int_{(B_{(4\delta_n)^{-1}}^2 \setminus B_{2R}^2) \times B_R^{m-2}} |\alpha_n^{ij}|^2 (X, Y) \, \mathrm{d}X \, \mathrm{d}Y$$

$$\leqslant \int_{B_R^{m-2}} \left\| \alpha_n^{ij}(\cdot, Y) \right\|_{L^{2,1}(B_{(4\delta_n)^{-1}}^2)} \left\| \alpha_n^{ij}(\cdot, Y) \right\|_{L^{2,\infty}(B_{(4\delta_n)^{-1}}^2 \setminus B_{2R}^2)} \mathrm{d}Y$$

$$\leqslant \sup_{Y \in B_R^{m-2}} \left\| \alpha_n^{ij}(\cdot, Y) \right\|_{L^{2,\infty}(B_{(4\delta_n)^{-1}}^2 \setminus B_{2R}^2)} \int_{B_R^{m-2}} \left\| \alpha_n^{ij}(\cdot, Y) \right\|_{L^{2,1}(B_{(4\delta_n)^{-1}}^2)} \mathrm{d}Y$$

$$\leqslant C\varepsilon R^{m-2}\delta_n^{-2} \int\limits_{B_{\delta_n^{-1}}^2 \times B_R^{m-2}} \big(|Dw_n|^2 + |\partial_t w_n|^2\big)(X, Y, t)\, dX\, dY. \tag{6.41}$$

Observe that $\sum_{ij} |d\omega_n^i \omega_n^j - \omega_n^i\, d\omega_n^j|^2 = |\omega_n|^2 |D\frac{\omega_n}{|\omega_n|}|^2$ and $|\omega_n| \geqslant \frac{1}{2}$ on $B_{\delta_n^{-1}}^2 \times B_{2R}^{m-2}$. This, plus integration over $t \in (-R^2, R^2)$, yields

$$\int\limits_{(B_{(4\delta_n)^{-1}}^2 \setminus B_{2R}^2) \times B_R^{m-2} \times (-R^2, R^2)} \left| D\frac{w_n}{|w_n|} \right|^2 (X, Y, t)\, dX\, dY\, dt$$

$$\leqslant C\varepsilon R^{m-2}\delta_n^{-2} \int\limits_{B_{\delta_n^{-1}}^2 \times B_R^{m-2} \times (-R^2, R^2)} \big(|Dw_n|^2 + |\partial_t w_n|^2\big). \tag{6.42}$$

Finally, we need to control the L^2 norm of $D|w_n|$. To do it, write $w_n = \rho_n \theta_n$, with $\rho_n \geqslant \frac{1}{2}$ and θ_n valued in S^{k-1}, then one has

$$\Delta \rho_n + \varepsilon_n^{-2}(1 - \rho_n^2)\rho_n - \rho_n |D\theta_n|^2 = \partial_t w_n \theta_n. \tag{6.43}$$

Multiplying both side by $(1 - \rho_n)$ and integrating it over $(B_{(4\delta_n)^{-1}}^2 \setminus B_{2R}^2) \times B_R^{m-2} \times (-R^2, R^2)$, we obtain

$$\int\limits_{(B_{(4\delta_n)^{-1}}^2 \setminus B_{2R}^2) \times B_R^{m-2} \times (-R^2, R^2)} |D\rho_n|^2$$

$$\leqslant C\varepsilon_n^{-2} \int\limits_{B_{\delta_n^{-1}}^2 \times B_R^{m-2} \times (-R^2, R^2)} \big(1 - \rho_n^2\big)^2$$

$$+ C \int\limits_{(B_{(4\delta_n)^{-1}}^2 \setminus B_{2R}^2) \times B_R^{m-2} \times (-R^2, R^2)} \left(\left| D\frac{w_n}{|w_n|} \right|^2 + |\partial_t w_n|^2 \right) + \text{boundary terms}$$

$$\leqslant C\varepsilon + \mathrm{O}(n^{-1}).$$

Here we have used (6.17)–(6.19) to show that the boundary term converges to zero. In particular, we get

$$R^{-m} \int\limits_{(B_{(4\delta_n)^{-1}}^2 \setminus B_{2R}^2) \times B_R^{m-2} \times (-R^2, R^2)} |Dw_n|^2(X, Y, t)\, dX\, dY\, dt \leqslant C\varepsilon. \tag{6.44}$$

This, combines with the Allard's strong constancy lemma (see, [23,32]), implies

$$\int\limits_{B_{(4\delta_n)^{-1}}^2(X_n) \setminus B_{2R}^2(X_n)} |Dw_n|^2(X, Y_n, t_n)\, dx \leqslant C\varepsilon. \tag{6.45}$$

This finishes the proof of Step 2. Therefore the proof of Theorem 6.1 is complete.   □

Now we discuss the quantization result at time infinity for sequences of solutions to (1.5)–(1.6). Let $u_n \in C^\infty(\Omega \times R_+, R^k)$ solve (1.5) and (1.6). By adopting the same notations as Section 4, we know that for $V_{u_n(t)} = \delta_{A(u_n)(\cdot,t)} e(u_n)(x,t)\,dx$,

$$\delta V_{u_n(t)}(X) = \int_\Omega \partial_t u_n(x,t) Du_n(x,t) X(x)\,dx$$

$$- 2\int_\Omega DX : \frac{Du_n \otimes Du_n}{|Du_n|^2} \frac{F(u_n)}{\varepsilon_n^2}(x,t)\,dx \qquad (6.46)$$

for any $X \in C_0^1(\Omega, R^m)$. Note also that, by (1.6) and Lemma 2.4, we can find $t_n \uparrow \infty$ such that

$$\lim_{n\uparrow\infty} \int_{t_n-1}^{t_n+1} \int_\Omega |\partial_t u_n|^2\,dx\,dt + \int_\Omega |\partial_t u_n|^2(x,t_n)\,dx = 0, \qquad (6.47)$$

$$\lim_{n\uparrow\infty} \int_\Omega \frac{1}{\varepsilon_n^2} F(u_n)(x,t_n)\,dx = 0. \qquad (6.48)$$

We may assume that $u_n(t_n) \to u_\infty$ weakly in $H^1(\Omega, R^m)$,

$$e(u_n)(x,t_n)\,dx \to \mu_\infty \equiv \frac{1}{2}|Du_\infty|^2(x)\,dx + \nu_\infty$$

for some nonnegative Radon measure ν_∞ on Ω. Moreover, $V_{u_n(t_n)} \to V_\infty$ in $V_{m-2}^*(\Omega)$ so that $\|V_\infty\| = \mu_\infty$. It follows from (6.46)–(6.48) that $\delta V_\infty = 0$. Therefore, (4.16) of Lemma 4.6 implies, for all $a \in \mathrm{spt}(\|V_\infty\|)$ and $0 < r \leqslant R < \mathrm{dist}(a, \partial\Omega)$,

$$R^{2-m}\|V_\infty\|(B_R(a)) - r^{2-m}\|V_\infty\|(B_r(a))$$

$$\geqslant \int_{B_R(a)\setminus B_r(a)} |y-a|^{-m-4}|S^\perp(y)|^2\,dV(y,S). \qquad (6.49)$$

In particular, $\Theta^{m-2}(\|V_\infty\|, x)$ exists for all $x \in \mathrm{spt}(\|V_\infty\|)$. Now define

$$\Sigma_\infty^1 = \{x \in \Omega : \Theta^{m-2}(\|V_\infty\|, x) \geqslant \varepsilon_1^2\},$$

$$\Sigma_\infty^2 = \left\{x \in \Omega : \lim_{r\downarrow 0} \lim_{n\uparrow\infty} r^{2-m} \int_{P_r(x,t_n)} |\partial_t u_n|^2(z)\,dz > 0\right\}.$$

Then, by (6.47) and the Vitali's covering argument, we have

$$H^{m-2}(\Sigma_\infty^1) < \infty, \qquad H^{m-2}(\Sigma_\infty^2) = 0.$$

Now we need a slice-type ε_0-regularity result.

CLAIM 6.5. – *There exist $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for any $x \in \Omega$ if*

$$r^{2-m} \int_{B_r(x)} e(u_n)(x, t_n) \, \mathrm{d}x \leqslant \varepsilon_1^2, \qquad r^{2-m} \int_{P_r(x, t_n)} |\partial_t u_n|^2(z) \, \mathrm{d}z \leqslant \varepsilon_1^4. \tag{6.50}$$

Then

$$(\delta_1 r)^2 \sup_{B_{\delta_1 r}(x)} e(u_n)(x, t_n) \leqslant C\varepsilon_1^2. \tag{6.51}$$

Proof. – It follows from Lemma 2.1 that, for any $t_n - r^2 \leqslant t \leqslant t_n$

$$r^{2-m} \int_{B_r(x)} e(u_n)(x, t_n) - r^{2-m} \int_{B_{\frac{r}{2}}(x)} e(u_n)(x, t)$$

$$\geqslant -r^{2-m} \int_{P_r(x, t_n)} |\partial_t u_n|^2(z) \, \mathrm{d}z - C\left(r^{-m} \int_{P_r(x, t_n)} |Du_n|^2\right)^{\frac{1}{2}} \left(r^{2-m} \int_{P_r(x, t_n)} |\partial_t u_n|^2\right)^{\frac{1}{2}}$$

$$\geqslant -C\varepsilon_1^2.$$

Here we have used (3.2). Therefore, we have, for all $t \in [t_n - r^2, t_n]$

$$\left(\frac{r}{2}\right)^{2-m} \int_{B_{\frac{r}{2}}(x)} e(u_n)(x, t) \, \mathrm{d}x \leqslant C\varepsilon_1^2$$

so that we have

$$\left(\frac{r}{2}\right)^{-m} \int_{P_{\frac{r}{2}}(x, t_n)} e(u_n)(z) \, \mathrm{d}z \leqslant C\varepsilon_1^2.$$

Therefore, by choosing ε_1 sufficiently small and applying Lemma 2.3, we obtain (6.51).

CLAIM 6.6. – $\Sigma_\infty = \Sigma_\infty^1 \cup \Sigma_\infty^2$ *is closed and has finite H^{m-2} measure, and $u_n \to u_\infty$ in $C^1_{\mathrm{loc}}(\Omega \setminus \Sigma_\infty, R^k)$.*

Proof. – For any $x_0 \in \Omega \setminus \Sigma_\infty$, there exist $r_0 > 0$ and $n_0 \gg 1$ such that for $n \geqslant n_0$

$$r_0^{2-m} \int_{B_{r_0}(x_0)} e(u_n)(x, t_n) \, \mathrm{d}x \leqslant r_0^{2-m} \|V_\infty\|(B_{r_0}(x_0)) + \varepsilon_1^2 \leqslant 2\varepsilon_1^2,$$

$$r_0^{2-m} \int_{P_{r_0}(x_0, t_n)} |\partial_t u_n|^2(z) \, \mathrm{d}z \leqslant \varepsilon_1^4.$$

Therefore, Claim 6.5 implies that for $n \geqslant n_0$, $\sup_{B_{\delta_1 r_0}(x_0)} e(u_n)(x, t_n) \leqslant C\varepsilon_1^2$. Hence $B_{\delta_1 r_0}(x_0) \cap \Sigma_\infty = \emptyset$ so that Σ_∞ is closed and $u_n \to u_\infty$ in $C^1_{\mathrm{loc}}(\Omega \setminus \Sigma, R^k)$. Note also that this and (6.47) imply that u_∞ is a weakly harmonic map whose singular set is contained in Σ_∞. Now we are ready to state the energy quantization theorem at time infinity.

THEOREM 6.7. – *Under the same notations as above. We have* (1) Σ_∞ *is a closed* $(m-2)$-*rectifiable set.* (2) *If, in addition, that* $N = S^{k-1}$. *Then, for* H^{m-2} *a.e.* $x \in \Sigma_\infty$, *there exist* $1 \leqslant l_x \leqslant \frac{E_0}{\varepsilon_0^2}$ *and* l_x-*many harmonic* S^2's, $\{\phi_j\}_{j=1}^{l_x}$, *such that*

$$\Theta^{m-2}(\|V_\infty\|, x) = \sum_{j=1}^{l_x} E(\phi_j, S^2). \tag{6.52}$$

Proof. – (1) follows from the fact that V_∞ is stationary and Theorem 4.9. (2) It is very similar to that of Theorem 6.1. One can also view it as a L^2-perturbation argument of that of [32]. The only difference we need to make is to replace (6.7) by the following: for H^{m-2} a.e. $x_0 \in \Sigma_\infty$,

$$\lim_{r \downarrow 0} \lim_{n \uparrow \infty} r^{2-m} \int_{B_r(x_0)} |D_T u_n|^2(x)\, \mathrm{d}x = 0, \tag{6.53}$$

for all $T \in T_{x_0} \Sigma_\infty$. (6.53) follows from the H^{m-2}-approximate continuity of $\Theta^{m-2}(\|V_\infty\|, \cdot)$ at x_0 and the monotonicity inequality (6.49) (one can see Lemma 2.4 of [23] or §3 of [32]). Then one can follows lines by lines of the proof of theorem 6.1 to show (6.52).   □

7. Final remarks

In this section, we consider the class $\mathcal{A}$ consisting all of the weak solutions $u \in H^1_{\mathrm{loc}}(\Omega \times R_+, N) \cap L^\infty(R_+, H^1(\Omega, N))$ to the heat equation of harmonic maps (1.0), which satisfy (1) the Pohozaev identity: (cf. also (5.19))

$$\int_{\Omega \times R_+} \partial_t u\, Du\, X(x) = \int_{\Omega \times R_+} \frac{1}{2}|Du|^2 \mathrm{div}(X) - \sum_{1 \leqslant ij \leqslant m} u_i u_j X^i_j \tag{7.1}$$

for any $X \in C_0^1(\Omega, R^m)$; (2) the energy inequality (2.1); (3) the ε_0-regularity Lemma 2.3. Note that the class of weak solutions satisfying both (7.1) and (2.1) was introduced by Feldman [17], which was shown to satisfy Lemma 2.3 for $N = S^{k-1}$ by [10] and [17] independently. Since the partial regularity was not proven for general manifold N yet, we henceforth add the property (3) in the definition of the class $\mathcal{A}$. The goal of this section is to point out that all the results from Sections 3–6 are remaining to be true for the class $\mathcal{A}$, and the proofs are almost the same or slightly easier.

As calculated in [17], any $u \in \mathcal{A}$ satisfies the energy monotonicity inequality (2.3) and (2.4) with $e(u_\varepsilon)$ replaced by $\frac{1}{2}|Du|^2$ and $F(u_\varepsilon)$ replaced by 0. Now suppose that $\{u_n\} \subset \mathcal{A}$ satisfy the same initial value $u(x, 0) = u_0(x)$ for a $u_0 \in C^1(\Omega, N)$

$$\sup_{0 < t < \infty} \left(\int_0^t \int_\Omega |\partial_t u_n|^2 + E(u_n(\cdot, t)) \right) \leqslant E(u_0). \tag{7.2}$$

Then, as before, one assumes

$$\frac{1}{2}|Du_n|^2(x,t)\,dx\,dt \to \frac{1}{2}|Du|^2(x,t)\,dx\,dt + \nu \equiv \mu,$$

$$|\partial_t u_n|^2(x,t)\,dx\,dt \to |\partial_t u|^2(x,t)\,dx\,dt + \eta$$

for two nonnegative Radon measures $\nu = \nu_t\,dt$ and η on $\Omega \times R_+$. If we define Σ as in Section 2, with $e(u_n)$ replaced by $\frac{1}{2}|Du_n|^2$, then Facts 2.4–2.11 all remain to be true.

For $z_0 \in \Sigma$, one can consider the tangent cone measure space, $T_{z_0}(\mu)$, the same way as in Section 3 and Lemma 3.2–Proposition 3.4 remain to hold. In particular, one can define, exactly as same as Definition 3.5, $\dim(\Theta^m(\mu^0, \cdot))$ for any $\mu^0 \in T_{z_0}(\mu)$. Therefore, we can obtain the same stratification for Σ as in Theorem 3.6, namely

THEOREM 7.1. – *For any sequence $u_n \subset \mathcal{A}$ as above. Let*

$$\Sigma_k = \left\{ z_0 \in \Sigma \colon \dim \Theta^m(\mu^0, \cdot) \leqslant k, \forall \mu^0 \in T_{z_0}(\mu) \right\} \quad for\ 0 \leqslant k \leqslant m.$$

Then $\dim(\Sigma_k) \leqslant k$ for $0 \leqslant k \leqslant m$ and Σ_0 is discrete.

One can also associate a generalized $(m-2)$-varifold V_{u_n} for each u_n as in Section 4. If we let V denote the generalized varifold limit of V_{u_n}, then all the results from Section 4 remain to be true for V. In particular,

THEOREM 7.2. – *For L^1 a.e. $t \in R_+$, $V_t\llcorner(\Sigma_t \times A_{m-2,m})$ is a $(m-2)$-rectifiable varifold. In particular, Σ_t is a $(m-2)$-rectifiable set.*

For the generalized varifold flow, all the results from Section 5 remain true for $\mathcal{A}$. For example, we have

THEOREM 7.3. – *Under the same notations as above. If, in addition, u is a suitable weak solution to the heat equation of harmonic maps. Then $\{\nu_t\}_{t \geqslant 0}$ is a Brakke flow.*

Finally, we can prove an energy quantization for the density function of V_t as follows. The proof is similar to that of Theorem 6.1 and in fact is slightly easier. One can also modify the proof of [27] to show the following result.

THEOREM 7.4. – *If, in addition, that $N = S^{k-1}$. Then, for $\mathcal{P}^m$ a.e $z_0 \in \Sigma$,*

$$\Theta^{m-2}(\|V_{t_0}\|, x_0) = \sum_{j=1}^{l_{x_0}} E(\phi_j, S^2) \tag{7.3}$$

for some $1 \leqslant l_{x_0} < \infty$ and $\{\phi\}_{j=1}^{l_{x_0}}$ harmonic S^2's.

Remark 7.5. – We conjecture that Theorems 6.1, 6.7, and 7.4 are true for any Riemannian manifold N.

Remark 7.6. – We believe that the concentration set Σ is also $(m-1)$-rectifiable set with respect to the Euclidean metric on $M \times R_+$. The stratification Theorem 3.6 may be useful to attack this problem.

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