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On the spatially homogeneous Boltzmann equation

by

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ABSTRACT. – We consider the question of existence and uniqueness of solutions to the spatially homogeneous Boltzmann equation. The main result is that to any initial data with finite mass and energy, there exists a unique solution for which the same two quantities are conserved. We also prove that any solution which satisfies certain bounds on moments of order $s < 2$ must necessarily also have bounded energy.

A second part of the paper is devoted to the time discretization of the Boltzmann equation, the main results being estimates of the rate of convergence for the explicit and implicit Euler schemes.

Two auxiliary results are of independent interest: a sharpened form of the so called Povzner inequality, and a regularity result for an iterated gain term. © Elsevier, Paris

RÉSUMÉ. – Dans cet article nous nous intéressons aux problèmes d'existence et d'unicité pour l'équation de Boltzmann homogène. Nous montrons que pour toute donnée initiale de masse et d'énergie bornées il existe une unique solution qui conserve ces deux quantités. Nous montrons aussi que si une solution possède certains moments d'ordre $s < 2$ alors nécessairement elle a une énergie initiale bornée.

Dans un deuxième temps nous montrons que les schémas d'Euler explicite et implicite de discrétisation en temps de l'équation convergent et nous donnons des taux de convergence.

Pour établir ces résultats nous utilisons de nouvelles inégalités de Povzner, ainsi qu'un lemme de régularité pour le terme de gain itéré.

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1. INTRODUCTION

This paper deals with the Spatially Homogeneous Boltzmann equation

$$\begin{cases} \dfrac{\partial f}{\partial t}(t,v) = Q(f,f)(t,v) \quad \text{on } (0,+\infty)\times\mathbb{R}^3\,, \\ f(0,v) = f_0(v) \qquad \text{on } \mathbb{R}^3\,, \end{cases} \tag{1.1}$$

where $f(t,v)$ is a non negative function which describes the time evolution of the distribution of particles which move with velocity v. In the right hand side, $Q(f,f)$ is the so-called collision operator,

$$Q(f,f)(v) = \int_{\mathbb{R}^3}\int_{S^2} (f'f_1' - ff_1)B(\theta,|v-v_1|)\,d\omega dv_1\,. \tag{1.2}$$

Here $f = f(v)$, $f_1 = f(v_1)$, $f' = f(v')$ and $f_1' = f(v_1')$, and v' and v_1' are the velocities after the elastic collision of two particles which had the velocities v and v_1 before the encounter. One parameterization of these velocities is

$$\begin{cases} v' = \dfrac{v+v_1}{2} + \dfrac{|v-v_1|}{2}\,\omega\,, \\ v_1' = \dfrac{v+v_1}{2} - \dfrac{|v-v_1|}{2}\,\omega, \end{cases} \tag{1.3}$$

where ω is a unit vector of the sphere S^2 (see figure 1 below). In (1.2), θ is the angle between $v - v_1$ and $v' - v$. A different parameterization is given by

$$\begin{cases} v' = v + \big((v_1 - v)\cdot\Omega\big)\Omega\,, \\ v_1' = v_1 - \big((v_1 - v)\cdot\Omega\big)\Omega\,. \end{cases} \tag{1.4}$$

With this parameterization, the collision operator still is of the form (1.2) with $d\omega$ replaced by $d\Omega$, except that now the kernel B takes the form $2B(\theta,|v-v_1|)\cos(\theta)$. This takes into account also that the second parameterization implies a double covering of the sphere from the first parameterization.

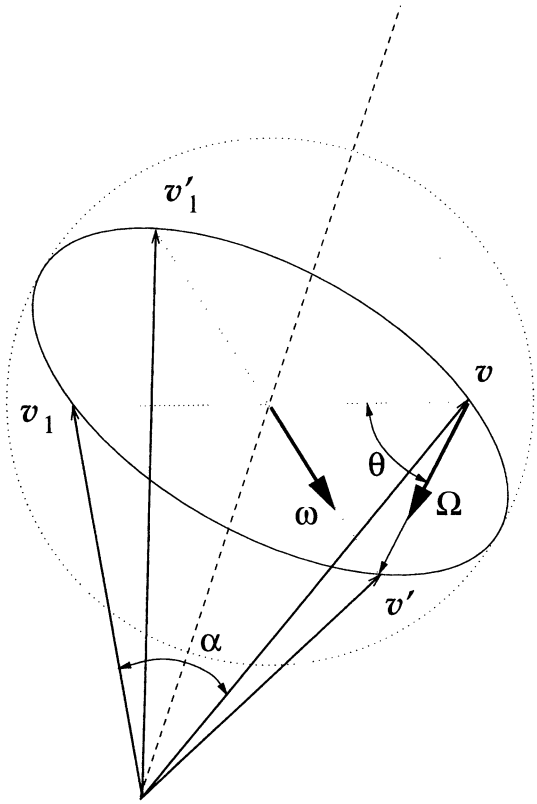

Figure 1. Geometry of a binary collision.

The precise form of the kernel B depends on the physical properties of the gas that is being studied. Here we consider the case of so-called hard potentials, and therefore

$$B(\theta, |v - v_1|) = b(\theta)\,|v - v_1|^{\beta}, \quad \text{with } \beta \in (0, 2]. \qquad (1.5)$$

In this case b is even and continuous in $]-\pi/2, \pi/2[$. Furthermore, we assume that b satisfies Grad's angular cut-off condition, namely that $b \in L^1(]-\pi/2, \pi/2[)$. This obviously holds for the elastic spheres, but if B is derived from the interaction by an inverse power law, the integrability condition does not hold unless b is truncated in some way.

The main concern of this paper is the proof of the existence of a unique solution to (1.1) with minimal assumptions on the initial data, and moreover of a convergence result for a suitable time discretization of the equation. The condition that we must impose on the initial data is $0 \leq f_0(v)(1+|v|^2) \in L^1(\mathbb{R}^3)$. No assumption of finite initial entropy is necessary, which is important since no control of entropy can be expected in the explicit Euler scheme.

A general reference for the Boltzmann equation is [3], or more recently [4], both of which give many further references, and many details on the development of the mathematical theory. The question of existence and uniqueness of solutions to the Boltzmann equation (1.1) was first addressed by Carleman [2], and the L^1-theory was developed by Arkeryd [1]. Elmroth [7] proved that all moments that initially are bounded remain bounded uniformly in time. Then Desvillettes [5] proved that if some moment of the initial data of order $s > 2$ is bounded, then all moments of the solution are bounded for any positive time. This result was extended by Wennberg [13], [14], [15] who proved that the result by Desvillettes holds also when only the energy of the initial data is bounded, and for very general cross-sections, also without the assumption of angular cutoff.

The first main result of this paper is the following:

THEOREM 1.1. – *Let $f_0(v)$ be in $L^1_2(\mathbb{R}^3)$. There exists a unique solution f in $C([0,+\infty); L^1_2(\mathbb{R}^3))$ of the Boltzmann equation (1.1) which conserves mass, momentum and energy; this solution also satisfies*

$$
\begin{aligned}
&i)\ \|f(t,\cdot)\|_{1,2+\beta} \leq \frac{\delta(t)}{t} \text{ with } \delta(t) \xrightarrow[t\to 0]{} 0 \\
&\qquad \text{and } \|f(t,\cdot)\|_{1,1+\beta} \in L^1_{loc}([0,+\infty)),\\
&ii)\ \forall s',\ \forall \bar{t} > 0\ \ \|f(t,\cdot)\|_{1,s'} \in L^\infty([\bar{t},+\infty)),\\
&iii)\ \text{if } f_0 \in L^1_s(\mathbb{R}^3) \text{ with } s > 2, \text{ then } \|f(t,\cdot)\|_{1,s} \in L^\infty([0,+\infty))\\
&\qquad \text{and } \|f(t,\cdot)\|_{1,s+\beta} \in L^1_{loc}([0,+\infty)).
\end{aligned}
$$

Here and below, $L^1_s(\mathbb{R}^3)$ denotes the space of all functions f such that

$$
\|f\|_{1,s} = \int_{\mathbb{R}^3} f(v)\,(1+|v|^s)\,dv
$$

is bounded.

The existence theory in L^1 can be found already in Arkeryd's paper [1] from 1972, where two existence proofs are given under slightly stronger hypothesis on the initial data. In one of the proofs the assumption is that $f_0 \in L^1_s(\mathbb{R}^3)$, with $s \geq 2$, and $f_0 \log f_0 \in L^1(\mathbb{R}^3)$. The proof is based on a weak stability result and on the Povzner inequality, (see [1] and [7]).

In the other case, $f_0 \in L^1_4(\mathbb{R}^3)$, and the proof depends on a monotonicity argument. The solution constructed was known to satisfy ii) and iii), see [5], [7], [14].

Also here, two proofs are presented. The first one (Section 4) relies on a quite simple contraction argument and we show directly that a sequence of solutions of a approximated problem is strongly convergent, under the assumption that $f_0 \in L^1_s(\mathbb{R}^3)$, with $s > 2$. In the second one (Section 5), we prove that the stability result of Arkeryd [1] also holds in the case $f_0 \in L^1_2(\mathbb{R}^3)$. In the proof, we use a refined Povzner inequality and the regularity of the Q^+ term.

Several authors have investigated the question of uniqueness for the homogeneous Boltzmann equation, see [1], [6], [9], [13]. In Section 3, uniqueness is proven under the sole assumption that the solution conserves mass and energy. This is an improvement of the previous result in which it was assumed at least that iii) holds for some $s > 2$. The proof is based on a subtle use of the Povzner inequality which permits us to prove the estimate i). Then uniqueness follows by a general result known as Nagumo's uniqueness criterion.

The Povzner inequality is reversed when moments of order lower than 2 are considered. This essentially implies that one can estimate moments of the initial data in terms of moments at later times, and as a consequence of this we are able to prove that if $\|f(t,\cdot)\|_{1,s} \in L^1([0,T])$, where $s > \beta$, then $f_0 \in L^1_2$.

The Section 5 is devoted to our second main result. It is an application of the techniques introduced for Theorem 1.1 to the convergence of time discrete scheme for the Boltzmann equation.

THEOREM 1.2. – *Let $f_0(v)$ be in $L^1_s(\mathbb{R}^3)$, with $s \geq 2$. Then, the explicit and implicit Euler schemes constructed from the initial data f_0 converge to the unique solution of the Boltzmann equation, given by Theorem 1.1.*

Under stronger hypothesis on the moments of the initial data, we can also compute the convergence rate for these schemes. A different approach to discretization in time has recently been considered by Gabetta *et al.* [8].

Note that the convergence of the explicit Euler scheme in Theorem 1.2 gives the existence part of Theorem 1.1 . This method was first used by A.J. Povzner in [11], who proves that for a given $f_0 \in L^1_2(\mathbb{R}^3)$ there exists a measure solution f of Boltzmann equation (1), with not increassing energy, corresponding to the initiel data f_0. The combining use of Pozvner inequality (lemma 2.2) and of the regularity property of the gain term (lemma 2.1) allow us to improve Povzner existence result: the energy of f

is conserved and f is a measurable function. Of course, if we only assume that f_0 is a nonnegative measure with finite mass and energy, then the approximate solution constructed by the explicit Euler scheme converges to a measure solution of (1) which conseves mass and energy, and this solution is unique.

2. ESTIMATES OF THE COLLISION OPERATOR

This section contains two technical lemmas, which are related to the geometry of the velocities of two particles involved in a collision. The first one will be used in Section 5, in order to give local equi-integrability of solutions of an approximated problem, when we do not control the entropy. It is a new regularity result for the gain term Q^+, which is related to a previous result by P.L. Lions [10]. However, while the result by Lions relies on the deeper properties of the geometry, the one presented below is quite elementary.

LEMMA 2.1. – *Let Q^+ be the gain term with a kernel $B(|v-v_1|,\theta) = |v-v_1|^\beta b(\theta)$, where $b(\theta)$ is bounded. Let $f,g,h \in L^1_\beta(\mathbb{R}^3)$. Then $Q^+(Q^+(f,g),h)$ is (locally) uniformly integrable. More precisely, for each $\varepsilon > 0$ there is a $\delta > 0$ such that if A is a set with Lebesgue measure $\mu(A) < \delta$, then*

$$\int_A Q^+(Q^+(f,g),h)(v)\,dv < \varepsilon\,.$$

Here δ depends only on β, $b(\theta)$ and on the norm in L^1_β of f, g and h. In the case of hard spheres one can take $\delta = C(\|f\|_1\|g\|_1\|h\|_1)^{-3/2}\varepsilon^{3/2}$.

Proof. – The calculations are slightly more explicit in the case of hard spheres, and so we begin there. Recall that in this case $B(\theta,|v-v_1|) = |v-v_1|$. Let $\phi(v) \in L^\infty$; later it will be the indicator function of the set A. Here we change variables in the usual way, by letting $(v,v_1) \mapsto (v',v_1')$ (see [2,3]). Since $dv\,dv_1 = dv'\,dv_1'$, the iterated gain term can be written

$$\begin{aligned}
&\int_{\mathbb{R}^3} Q^+(Q^+(f,g),h)(v)\phi(v)dv \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} Q^+(f,g)(v)h(v_2)|v-v_2|\int_{S^2}\phi(v_2')d\omega_2\,dv_2\,dv\,, \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(v)g(v_1)h(v_2) \\
&\qquad\qquad \Big[|v-v_1|\int_{S^2}|v_1'-v_2|\int_{S^2}\phi(v_2'')d\omega_2 d\omega_1\Big]\,dv\,dv_1\,dv_2, \quad (2.1)
\end{aligned}$$

where (see figure 2)

$$v_2' = \frac{1}{2}\big(v + v_2 + |v - v_2|\,\omega_2\big) \qquad\qquad v_1' = \frac{1}{2}\big(v + v_1 + |v - v_1|\,\omega_1\big)$$

$$\begin{aligned} v_2'' &= \frac{1}{2}\big(v_1' + v_2 + |v_1' - v_2|\,\omega_2\big) \\ &= \frac{1}{4}\big(v + v_1 + 2v_2 + |v - v_1|\,\omega_1 + \big|v + v_1 - 2v_2 + |v - v_1|\,\omega_1\big|\,\omega_2\big)\,. \end{aligned}$$

Now let $S_{vv_1\varepsilon} = \{v_1' : \big||2v_1' - v - v_1| - |v - v_1|\big| < \varepsilon\}$, and let $\varepsilon\chi_\varepsilon(v_1')$ be the characteristic function of $S_{vv_1\varepsilon}$. That means that χ_ε approximates the surface measure on the sphere covered by v_1', and using the parametrization (1.4), the expression within brackets in (2.1) is the limit when $\varepsilon \to 0$ of

$$\frac{2}{|v - v_1|}\int_{\mathbb{R}^3} \chi_\varepsilon(v_1')|v_1' - v_2| \int_{S^2} \phi(v_2'')\cos\theta_2\, d\Omega_2 dv_1'\,. \tag{2.2}$$

This is an integral which is very similar to the gain term itself, and it is possible to carry out a change of variables just like when deriving the Carleman form of the gain term (see [2]): Let $r = |v_2'' - v_2|$, and denote by $E_{v_2''v_2}$ the plane that passes through v_2'' and is orthogonal to $v_2 - v_2''$ (in the usual Carleman representation of the gain term, one would have taken the plane through v_2 instead); $dE(v_1')$ denotes the surface measure on this plane. Then $dv_1' = dr\,dE(v_1')$, and in polar coordinates, $dv_2'' = r^2 dr\,d\Omega_2$.

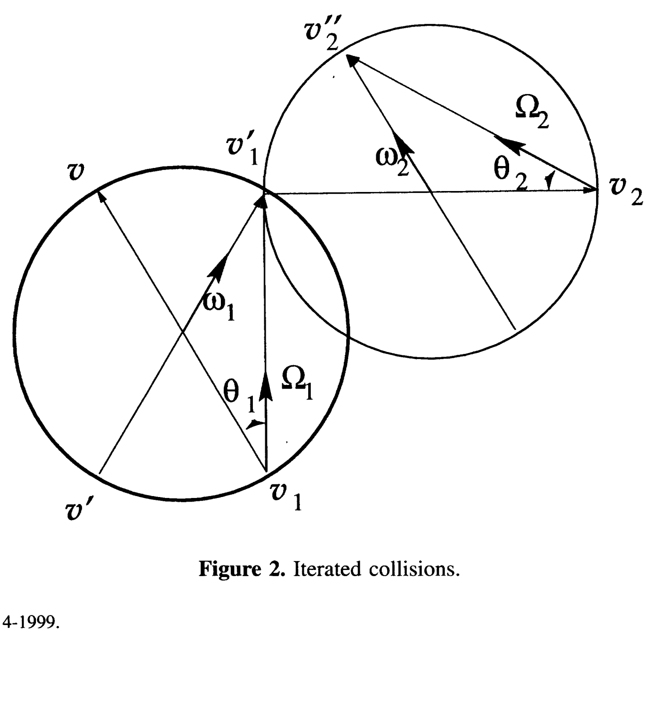

Figure 2. Iterated collisions.

That means that $d\Omega_2 dv_1' = |v_2'' - v_2|^{-2}\, dE(v_1')\, dv_2''$, and therefore (again taking into account that the Ω-parameterization of S^2 implies a double covering of the domain of integration), (2.2) equals

$$\frac{1}{|v - v_1|} \int_{\mathbb{R}^3} \phi(v_2'') \frac{1}{|v_2'' - v_2|} \int_{E_{v_2'' v_2}} \chi_\varepsilon(v_1')\, dE(v_1')\, dv_2'' . \tag{2.3}$$

The measure of the intersection between $E_{v_2'' v_2}$ and the thickened sphere $S_{vv_1\varepsilon}$ is approximately $\varepsilon|v - v_1|$ (or 0 when there is no intersection), and so the integral is bounded by

$$C \int_{\mathbb{R}^3} \phi(v_2'') \frac{1}{|v_2'' - v_2|}\, dv_2'' .$$

Then the estimate of the lemma follows by estimating separately the integral in $|v_2'' - v_2| < \delta^{1/3}$ and in the remaining part; the first part is bounded by $4\pi||\phi||_\infty \delta^{2/3}$ and the latter one is bounded by $||\phi||_1 \delta^{-1/3}$, and one concludes by taking ϕ to be the characteristic function of a set with Lebesgue measure smaller than δ.

In the more general case, the expression within square brackets in (2.1) is replaced by

$$\Lambda(v, v_1, v_2) \equiv \left[2|v - v_1|^\beta \int_{S^2} b(\theta_1)|v_1' - v_2|^\beta \int_{S^2} b(\theta_2)\phi(v_2'') \cos(\theta_2) d\Omega_2 d\omega_1 \right] .$$

For $|v - v_1| < \varepsilon_1$, this is bounded by $C\varepsilon_1^\beta(|v|^\beta + |v_1|^\beta + |v_2|^\beta)$, with a constant depending on $b(\theta)$ and on β. For $|v - v_1| \geq \varepsilon_1$, one can carry out the change of variables just as above, to obtain a bound of the form

$$\Lambda(v, v_1, v_2) < C|v - v_1|^{\beta - 1} \int_{\mathbb{R}^3} \phi(v_2'')|v_2'' - v_2|^{\beta - 2}\, dv_2'' .$$

Now if ϕ is the characteristic function of a set A of measure δ, then the same estimate as above can be carried out, and one gets, for arbitrary ε_1, ε_2,

$$\begin{aligned} &\int_A Q^+(Q^+(f, g), h)(v)\, dv \\ &\quad \leq C||f||_{L^1_\beta} ||g||_{L^1_\beta} ||h||_{L^1_\beta} \left(\varepsilon_1^\beta + \varepsilon_1^{\beta-1}\varepsilon_2^{\beta+1} + \varepsilon_1^{\beta-1}\varepsilon_2^{\beta-2}\delta \right), \end{aligned}$$

and one can conclude by choosing in turn ε_1, ε_2 and δ small enough to make each of the terms smaller than $\varepsilon/3$. □

Remark. – Here we have assumed that $b(\theta)$ is a bounded function, but again, a slightly more careful analysis shows that the result is true also in the locally bounded case.

The second result of this section is a sharpened form of the so-called Povzner inequality. This inequality relates the velocities before and after an elastic collision between two particles. In its original form, it was proven by Povzner [11], and more precise estimates were subsequently obtained by Elmroth [7].

LEMMA 2.2. – *Assume that $b(\theta)$ is a bounded function. For a given function Ψ let*

$$K(v,v_1)=\int_{S^2} b(\theta)\Big(\Psi(|v'|^2)+\Psi(|v_1'|^2)-\Psi(|v|^2)-\Psi(|v_1|^2)\Big)\,d\omega. \quad (2.4)$$

Then one can write $K(v,v_1)=G(v,v_1)-H(v,v_1)$, where

$$H(v,v_1)=8\pi\int_0^{\pi/2}\big(\bar{b}(\theta)+\bar{b}(\pi/2-\theta)\big)$$
$$\times\Big(\Psi(|v|^2)\cos^2\theta+\Psi(|v_1|^2)\sin^2\theta-\Psi(|v|^2\cos^2\theta+|v_1|^2\sin^2\theta)\Big)\,d\theta,$$

and $\bar{b}(\theta)=b(\theta)\cos\theta\sin\theta$. Let $\chi_1=1-\mathbb{1}_{\{|v|/2<|v_1|<2|v|\}}$ ($\mathbb{1}_A$ denotes the indicator function of the set A).

i) If $\Psi(x)=x^{1+\gamma}$ with $\gamma>0$, then

$$|G(v,v_1)|\le c_2\gamma(|v||v_1|)^{1+\gamma}$$

and $H(v,v_1)\ge c_1\gamma\big((|v|^2)^{1+\gamma}+(|v_1|^2)^{1+\gamma}\big)\chi_1(v,v_1)$.

ii) If $\Psi(x)=x^{1+\gamma}$ with $0<1+\gamma<1$, then

$$|G(v,v_1)|\le c_2|\gamma|(|v||v_1|)^{1+\gamma}$$

and $-H(v,v_1)\ge c_1|\gamma|\big((|v|^2)^{1+\gamma}+(|v_1|^2)^{1+\gamma}\big)\chi_1(v,v_1)$.

iii) Suppose that Ψ is a positive convex function that can be written $\Psi(x)=x\Phi(x)$, where Φ is concave, increasing to infinity, and such that for any $\varepsilon>0$ and any $\alpha\in]0,1[$, it satisfies $\big(\Phi(x)-\Phi(\alpha x)\big)x^{\varepsilon}\to\infty$ as $x\to\infty$. Then, for any $\varepsilon>0$,

$$|G(v,v_1)|\le c_2|v|\Phi(|v|^2)|v_1|\Phi(|v_1|^2)$$

and $H(v,v_1)\ge c_1\big(|v|^{2-\varepsilon}+|v_1|^{2-\varepsilon}\big)\chi_1(v,v_1)$.

If in addition there is a constant such that $\Phi'(x) \leq C/(1+x)$, *then* $G(v, v_1) \leq c_2|v||v_1|$.

Remark 1. – The constants in the lemma depend of Ψ and ε. Whenever necessary for clarity, $K(v, v_1)$ will be denoted $K_{1+\gamma}(v, v_1)$ or $K_\Psi(v, v_1)$, and similarly for G and H.

Remark 2. – The inequalities are monotonous in Ψ in the following sense: if $0 \leq \Psi_1 - \Psi_2$ is convex, then $H_{\Psi_2} - H_{\Psi_1} \geq 0$. Similarly, if $\Psi_1 - \Psi_2$ is concave, then the inequality is reversed. This is important in the application of the lemma, where unbounded convex or concave functions are replaced by truncated functions, and the result is obtained by a limit procedure. For Ψ convex (or concave) it is possible to construct a sequence of truncated functions Ψ_k in such a way that the difference of two subsequent functions is convex (or concave).

Proof. – For an integrable kernel $b(\theta)$, the four terms in (2.4) can be considered separately, and for the last two terms, the integration is trivial.

The sphere S^2 can then be parameterized by $\{(\theta, \varphi),\ -\pi \leq \varphi < \pi,\ 0 < \theta < \pi/2\}$, and $d\omega = 4 \sin\theta \cos\theta \, d\theta \, d\varphi$. The notation is described in figure 1. Let $r = |v|$, $r_1 = |v_1|$, $r' = |v'|$ and $r_1' = |v_1'|$. If $\tau = \sin\alpha$ denotes the sine of the angle between the vectors v and v_1, then

$$\begin{aligned} {r'}^2 &= r^2 \cos^2\theta + {r_1}^2 \sin^2\theta + 2\tau r r_1 \sin\theta \cos\theta \cos\varphi \\ &\equiv Y(\theta) + \tau Z(\theta) \cos\varphi \ , \\ {r_1'}^2 &= r^2 \sin^2\theta + {r_1}^2 \cos^2\theta - 2\tau r r_1 \sin\theta \cos\theta \cos\varphi \\ &= Y(\pi/2 - \theta) - \tau Z(\theta) \cos\varphi \ . \end{aligned}$$

Then the first term in (2.4) can be written

$$8 \int_0^{\pi/2} \bar{b}(\theta) \int_0^{\pi} \Psi\big(Y(\theta) + \tau Z(\theta) \cos\varphi\big) \, d\varphi \, d\theta. \tag{2.5}$$

The integral with respect to φ is (we omit the argument θ here)

$$\pi\Psi(Y) + \int_0^{\pi/2} \Big(\Psi(Y + \tau Z \cos\varphi) + \Psi(Y - \tau Z \cos\varphi) - 2\Psi(Y)\Big) \, d\varphi,$$

and by integrating partially twice one gets

$$\begin{aligned} &\pi\Psi(Y) + \tau Z \int_0^{\pi/2} \varphi \sin\varphi \Big(\Psi'(Y + \tau Z \cos\varphi) - \Psi'(Y - \tau Z \cos\varphi)\Big) \, d\varphi \\ &\quad = \pi\Psi(Y) + \tau^2 Z^2 \int_0^{\pi/2} (\sin\varphi - \varphi \cos\varphi) \sin\varphi \\ &\qquad \Big(\Psi''(Y + \tau Z \cos\varphi) + \Psi''(Y - \tau Z \cos\varphi)\Big) \, d\varphi. \end{aligned} \tag{2.6}$$

It remains to estimate the second of these terms, and to estimate

$$\begin{aligned}
&\pi \int_0^{\pi/2} \bar{b}(\theta)\Big(\Psi\big(Y(\theta)\big) + \Psi\big(Y(\pi/2-\theta)\big) - \Psi(r^2) - \Psi({r_1}^2)\Big)\, d\theta \\
&= \pi \int_0^{\pi/2} \big(\bar{b}(\theta) + \bar{b}(\pi/2-\theta)\big)\Big[\Psi\big(r^2\cos^2\theta + {r_1}^2\sin^2\theta\big) \\
&\qquad\qquad - \cos^2\theta\,\Psi(r^2) - \sin^2\theta\,\Psi({r_1}^2)\Big]\, d\theta.
\end{aligned} \tag{2.7}$$

In this expression, the integrand in the right hand side has a fixed sign, depending on whether Ψ is convex or concave, and this is the main idea behind the Povzner inequalities. Consider first $\Psi(z) = z^{1+\gamma}$. With $X = r^2\big(r^2 + {r_1}^2\big)^{-1}$, one can write the factor within brackets in (2.7) as

$$\begin{aligned}
&\big(r^2 + {r_1}^2\big)^{(1+\gamma)}\Big(\big(X\cos^2\theta + (1-X)\sin^2\theta\big)^{1+\gamma} \\
&\qquad - X^{1+\gamma}\cos^2\theta - (1-X)^{1+\gamma}\sin^2\theta\Big).
\end{aligned} \tag{2.8}$$

This term is non-negative if $\gamma < 0$ and non-positive if $\gamma > 0$, and it vanishes for all θ only if $r = r_1$. Moreover, the integrand is $\mathcal{O}(|\gamma|)$, uniformly in $\varepsilon < \theta < \pi/2 - \varepsilon$ and $\max(r/r_1, r_1/r) \geq 2$, and this yields the estimate of H in i) and ii).

The estimate of G in i) and ii) can be obtained by an estimate of the integrand in the second part of (2.6); this is

$$\gamma(1-\gamma)Y^{\gamma-1}(\sin\varphi - \varphi\cos\varphi)\sin\varphi\Big(\big(1+\tau Z_0\cos\varphi\big)^{\gamma-1} + \big(1-\tau Z_0\cos\varphi\big)^{\gamma-1}\Big),$$

where $Z_0 = Z/Y$. It is integrable for all $\gamma > -3/2$, uniformly in $Z_0 \in [-1, 1]$, and gives a contribution of the size

$$|\gamma|Z^2 Y^{\gamma-1} = |\gamma|Z^{1+\gamma}(Z/Y)^{1-\gamma} = |\gamma|Y^{1+\gamma}(Z/Y)^2.$$

Hence, if $\gamma \leq 1$, one can estimate G directly by $C|\gamma|(rr_1)^{1+\gamma}$, as in the lemma. On the other hand, if $\gamma > 1$, then for (Z/Y) sufficiently small, this term is dominated by (2.8), and for larger values of (Z/Y), the estimate holds just like for $\gamma \leq 1$.

Next we turn to the case of more slowly growing $\Psi(z) = z\Phi(z)$ as in iii), and we begin by considering the φ-integral in (2.5). Since Φ is concave, and since $Y + \tau Z\cos\varphi$ is non-negative, the integrand is smaller than

$$Y\Phi(Y) + \tau Z\cos\varphi\Phi(Y) + Y\Phi'(Y)\tau Z\cos\varphi + \tau^2\cos^2(\varphi)Z^2\Phi'(Y).$$

After integration in φ only two terms remain, and again because Φ is concave, and since $|Z(\theta)| \le Y(\theta)$,

$$|Z|\Phi'(Y) \le Y\Phi'(Y) \le \Phi(Y),$$

and $Z\Phi'(Y)$ can even be estimated by a constant if Φ is growing more slowly than logarithmically ($\Phi'(x) \le C(1+x)^{-1}$). It follows that the integral in (2.5) is bounded by

$$\pi \int_0^{\pi/2} \bar{b}(\theta)\Big(2Y\Phi(Y) + Z\Phi(Y)\Big)\,d\theta,$$

where the second term in the integrand could be replaced by *Const* Z for a sufficiently slowly growing function Φ. Hence the estimate of the G_Ψ is ready, and the estimate of H_Ψ follows after collecting the terms not involving φ, just as in the previous case. Since $\Psi(z) = z\Phi(z)$, we have $Y(\theta) + Y(\pi/2-\theta) - \Psi(r^2) - \Psi({r_1}^2) \le 0$. Moreover $Y(\theta)+Y(\pi/2-\theta)$ takes its minimum at $\theta = \pi/4$, and is increasing with $|\theta - \pi/4|$. Hence the integral is bounded from above by

$$\begin{aligned}-8\int_{\sin^{-1}(1/\sqrt{3})}^{\cos^{-1}(1/\sqrt{3})} \bar{b}(\theta)d\theta\,\Big(&r^2\Phi(r^2) + {r_1}^2\Phi({r_1}^2)\\ &-\Big(\frac{1}{3}r^2+\frac{2}{3}{r_1}^2\Big)\Phi\Big(\frac{1}{3}r^2+\frac{2}{3}{r_1}^2\Big) - \Big(\frac{2}{3}r^2+\frac{1}{3}{r_1}^2\Big)\Phi\Big(\frac{1}{3}r^2+\frac{1}{3}{r_1}^2\Big)\Big).\end{aligned}$$

For $r_1 > 2r$, this is bounded from above by

$$-C{r_1}^2\big(\Phi({r_1}^2) - \Phi(3{r_1}^2/4)\big) + Cr^2\Phi({r_1}^2)\,.$$

From the hypothesis on Φ, $\big(\Phi({r_1}^2) - \Phi(3{r_1}^2/4)\big)/{r_1}^{-\varepsilon} \to \infty$ as $r_1 \to \infty$ and thus gives the negative term in iii), and in the second term, $r^2 \le rr_1/2$, which means that this term can be included in the previous estimate of G.

3. UNIQUENESS

This section is devoted to the proof of the uniqueness result in Theorem 1.1. More precisely, we prove the following:

THEOREM 1.1′. – *Under the assumptions of Theorem 1.1, there exists at most one solution f in $C([0,+\infty); L^1(\mathbb{R}^3))$ of the Boltzmann equation (1.1) which conserves mass, energy and satisfies $\|f(t,\cdot)\|_{1+\beta} \in L^1_{loc}([0,+\infty))$.*

Proof. – The proof is carried out in four steps. Since initially only the energy, $\int_{\mathbb{R}^3} f(v)(1+|v|^2)\,dv$ is bounded, the main problem lies in controlling the behavior of $\int_{\mathbb{R}^3} f(t,v)(1+|v|^{2+\beta})\,dv$, which appears in the collision term. In the first two steps it is shown that for any solution, and any positive t, all moments of f are bounded. The existence of a solution that satisfies this property has been known previously (see [5] and [14]), but the proof here shows that this is a consequence of the conservation of energy and not on the way in which a solution is constructed. The third step gives an estimate of the blow-up of $\|f(t,\cdot)\|_{1,2+\beta}$ as $t \to 0$. This estimate is strong enough to prove the uniqueness result; this is done in Step 4.

Step 1. – We establish that there exists a convex function $\Psi(r)$ such that $\Psi(r)/r \to \infty$, which satisfies the conditions for iii) of Lemma 2.2, and such that for some constants C, C_1 and C_2, and for all $t \geq 0$

$$C_1 \leq \int_{\mathbb{R}^3} f(t,v)\,\Psi(|v|^2)\,dv \leq C_2\,, \tag{3.1}$$

$$\int_0^t \int_{\mathbb{R}^3} f(\tau,v)\,|v|^{2+\beta/2}\,dvd\tau \leq C(1+t)\,. \tag{3.2}$$

That a function Ψ exists such that (3.1) holds for the initial data f_0 is established in the Appendix. We define for all $n \in \mathbb{N}$ the first degree polynomial $p_n(x) = a_n x + b_n = \Psi'(n)x + \Psi(n) - n\,\Psi'(n)$ and the convex approximation of Ψ with linear growth,

$$\Psi_n(x) = \begin{cases} \Psi(x) & \text{if } x \leq n\,, \\ p_n(x) & \text{if } x \geq n\,. \end{cases}$$

The sequence $\{\Psi_n\}_{n\geq 1}$ increases pointwise to Ψ, and for all n, $\Psi_{n+1} - \Psi_n$ is convex. Using the conservation of mass and energy and the fact that the function $\Psi_n - p_n$ has compact support, we can compute

$$\begin{aligned}
&\int_{\mathbb{R}^3} (f(t,v) - f_0(v))\,\Psi_n(|v|^2)\,dv \\
&= \int_{\mathbb{R}^3} (f(t,v) - f_0(v))\,(\Psi_n(|v|^2) - p_n(|v|^2))\,dv \\
&= \int_{\mathbb{R}^3} \int_0^t Q(f,f)(v)\,(\Psi_n(|v|^2) - p_n(|v|^2))\,dvd\tau \\
&= \frac{1}{2} \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f f_1\, B\,(\Psi_n' + \Psi_{n\,1}' - \Psi_n - \Psi_{n\,1})\,d\omega dv_1 dv d\tau.
\end{aligned} \tag{3.3}$$

At this point we can use iii) from Lemma 2.2. With the notation from Section 2, we have $K_{\Psi_n} = G_{\Psi_n} - H_{\Psi_n}$, and for a sufficiently slowly growing function Ψ, the estimate $|G_{\Psi_n}(v,v_1)| \le C\,|v|\,|v_1|$ holds independently of n; without loss of generality one can assume that this is the case. The lemma implies that $H_{\Psi_n}(v,v_1)$ is non-negative and because of the convexity of $\Psi_{n+1}-\Psi_n$, it is pointwise increasing with n, and converges pointwise to H_Ψ. Then

$$\int_{\mathbb{R}^3} f(t,v)\,\Psi_n(|v|^2)\,dv + \frac{1}{2}\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f f_1\,|v-v_1|^\beta\, H_{\Psi_n} dv_1 dv d\tau =$$
$$= \int_{\mathbb{R}^3} f_0(v)\,\Psi_n(|v|^2)\,dv + \frac{1}{2}\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f f_1\,|v-v_1|^\beta\, G_{\Psi_n} dv_1 dv d\tau, \quad (3.4)$$

and we can pass to the limit in the right hand side thanks to Lebesgue's theorem because $f\in L^1_{loc}([0,+\infty);L^1_{\beta+1}(\mathbb{R}^3))$ and in the left hand side using Fatou's lemma. We obtain

$$\int_{\mathbb{R}^3} f(t,v)\,\Psi(|v|^2)\,dv + \frac{1}{2}\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f f_1\,|v-v_1|^\beta\, H_\Psi dv_1 dv d\tau \le$$
$$\le \int_{\mathbb{R}^3} f_0(v)\,\Psi(|v|^2)\,dv + C_2\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f f_1\,|v-v_1|^\beta\, |v|\,|v_1| dv_1 dv d\tau\,.$$

But from Lemma 2.2 we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f f_1\,|v-v_1|^\beta\, H_\Psi dv_1 dv\\
&\quad\ge \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f f_1\,|v-v_1|^\beta\, c_1\big(|v|^{2-\beta/2}+|v_1|^{2-\beta/2}\big)\\
&\qquad (1-\mathbb{1}_{|v|/2\le|v_1|\le 2|v|})\,dv_1 dv\\
&\quad\ge C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f f_1\,|v|^{2+\beta/2}\,dv_1 dv\\
&\qquad - c_2''\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f f_1(|v||v_1|)^{1+\beta/4}\mathbb{1}_{|v|/2\le|v_1|\le 2|v|}\,dv_1 dv.
\end{aligned}$$

Expressing this with the notation $Y_s(t)=\int_{\mathbb{R}^3} f(t,v)\,|v|^s\,dv$, which gives

$$\int_{\mathbb{R}^3} f(t,v)\,\Psi(|v|^2)\,dv + \frac{c_1}{2}\int_0^t Y_0(\tau)\,Y_{2+\beta/2}(\tau)\,d\tau \le$$
$$\le \int_{\mathbb{R}^3} f_0(v)\,\Psi(|v|^2)\,dv + 2\,C_2\int_0^t Y_1\,Y_{\beta+1}\,d\tau + c_2''\int_0^t \big(Y_{1+\beta/4}\big)^2\,d\tau.$$

Therefore using the conservation of mass and energy we bound $Y_{2-\beta/2}$, $Y_{1-\beta/2}$ and Y_1 by $\|f_0\|_{1,2}$, and using Young's inequality we bound

$\int_{\mathbb{R}^3} f\,\Psi(|v|^2)\,dv$, Y_β, $Y_{1+\beta}$ and $Y_{1+\beta/4}$ by $\varepsilon\, Y_{2+\beta/2} + C_\varepsilon\, Y_0$. Taking ε small enough we then obtain

$$\int_{\mathbb{R}^3} f(t,v)\,\Psi(|v|^2)\,dv + \frac{c_1}{4}\,Y_0 \int_0^t \int_{\mathbb{R}^3} f(\tau,v)\left(|v|^{2+\beta/2} + \Psi(|v|^2)\right) dv d\tau \leq$$
$$\leq \int_{\mathbb{R}^3} f_0(v)\,\Psi(|v|^2)\,dv + C\,\|f_0\|_{1,2}\,\|f_0\|_1\,t\,, \tag{3.5}$$

which prove the uper bound in (3.1), at least locally in time. The lower bound in (3.1) just comes from the inequality $\Psi(|v|^2) \geq 1$ and the conservation of mass.

Step 2. – Next, using Povzner's inequality once more, we prove that for all $\bar{t} > 0$, $s' > 2$,

$$\sup_{t \geq \bar{t}} \int_{\mathbb{R}^3} f(t,v)|v|^{s'}\,dv \leq C_{\bar{t},s'}\,. \tag{3.6}$$

We proceed like in L. Desvillettes [5]. From (3.5) we can deduce that there exists $t_0 > 0$, as small as we wish, such that

$$\int_{\mathbb{R}^3} f(t_0,v)\,|v|^{2+\beta/2}\,dv < +\infty\,.$$

Then we start from t_0 and we take a sequence Ψ_n which approximates $\Psi(r) = r^{1+\beta/4}$. As in step 1, we obtain (3.4), with $G_{\Psi_n}(v,v_1) \leq c_2\left(|v|^{1+\beta/2}\,|v_1| + |v_1|^{1+\beta/2}\,|v|\right)$, independently of n, and $H_{\Psi_n}(v,v_1)$ increases pointwise to $H_{2+\beta/2}$. Then we can pass to the limit as before, and we get

$$\int_{\mathbb{R}^3} f(t,v)\,|v|^{2+\beta/2}\,dv + \frac{1}{2}\int_{t_0}^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f f_1\,|v-v_1|^\beta\,H_{2+\beta/2} dv_1 dv d\tau \leq$$
$$\leq \int_{\mathbb{R}^3} f(t_0,v)\,|v|^{2+\beta/2}\,dv + C\,\|f_0\|_{1,2}\int_{t_0}^t \|f(\tau,.)\|_{1,2+\beta/2}\,d\tau\,.$$

Using the estimate $K_{2+\beta/2}(v,v_1) \geq c_1\,|v|^{2+\beta/2} - c_2'|v|\,|v_1|^{1+\beta/2}$ for the left hand side, and using Young's inequality to kill the dominant term in the right hand side we get, for all $t \geq t_0$

$$\int_{\mathbb{R}^3} f(t,v)\,|v|^{2+\beta/2}\,dv + \frac{c_1}{4}\,Y_0 \int_{t_0}^t \int_{\mathbb{R}^3} f(\tau,v)\left(|v|^{2+\beta/2} + |v|^{2+3\beta/2}\right) dv d\tau$$
$$\leq C(1+t).$$

Then, by an inductive argument it follows that $f \in L^\infty_{loc}([\bar{t},+\infty); L^1_s(\mathbb{R}^3))$, $\forall s > 2$, $\forall \bar{t} > 0$, and those $f \in C^1([\bar{t},+\infty); L^1_s(\mathbb{R}^3))$, $\forall s > 2$, since f solves Boltzmann equation (1.1). In order to obtain (3.6), from what the uniform upper bound in (3.1) follows, we make similar calculations to thus previously executed and we obtain the following differentialble inequality (where now, every term makes sense)

$$\frac{d}{dt} Y_s \leq C - s - c_s Y_s \quad \text{on } [\bar{t}, +\infty)$$

and we deduce $Y_s(t) \leq C_s/c_s + Y_s(\bar{t})$ for every $t \geq \bar{t}$.

Step 3. – In this step we show that there is a function $\delta(t) \to 0$ as $t \to 0$, such that near $t = 0$

$$\int_{\mathbb{R}^3} f(v,t)|v|^{2+\beta}\, dv \leq \frac{\delta(t)}{t}. \tag{3.7}$$

The preceding step implies that f belongs to $C^1(\,]0,+\infty[\,; L^1_{2+2\beta}(\mathbb{R}^3))$ and

$$\frac{d}{dt} Y_{2+\beta} \leq C - c\, Y_{2+2\beta} \quad \text{on }]0,+\infty[\,. \tag{3.8}$$

Here it is important to note that the constants c and C depend only on the mass and energy of the initial data f_0, and on the kernel B but they do not depend on the time. The first idea is to use Jensen's inequality to estimate $Y_{2+2\beta}$ in terms of $Y_{2+\beta}$. Because of the lower bound in (3.1), one can write

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^3} f(v,t)\Psi(|v|^2)\, dv\Big)^{-1} \int_{\mathbb{R}^3} f(v,t)|v|^{2+\beta}\, dv \\
&\quad = \Big(\int_{\mathbb{R}^3} f(v,t)\Psi(|v|^2)\, dv\Big)^{-1} \int_{\mathbb{R}^3} f(v,t)\Psi(|v|^2)\, \frac{|v|^{2+\beta}}{\Psi(|v|^2)}\, dv\,.
\end{aligned}$$

The function $\Psi(z) = z\,\Phi(z)$ is convex by construction, and we verify without difficulty that $\Upsilon(z) \equiv z^2\,\Phi(z)$ is a convex function and satisfies, for all $v \in \mathbb{R}^3$,

$$\Psi(|v|^2)\, \Upsilon\Big(\frac{|v|^{2+\beta}}{\Psi(|v|^2)}\Big) \leq |v|^{2+2\beta}.$$

Then Jensen's inequality implies that

$$C_2\, \Upsilon\big(Y_{2+\beta}/C_1\big) \leq Y_{2+2\beta}\,,$$

where C_1 and C_2 are the constants from (3.1), and we obtain a differential inequality for $Y_{2+\beta}(t)$. If $Y_{2+\beta}(t)$ does not explode in $t = 0$, this means that $\|f_0\|_{1,2+\beta} < +\infty$ and there is nothing to prove. Hence we can choose an interval $]0,\bar{t}]$ so small that the lower order terms in the right hand side of (3.8) are dominated by the negative term, and thus

$$\frac{d}{dt}Y_{2+\beta} \leq -\frac{C_2}{2}\,\Upsilon(Y_{2+\beta}/C_1) \quad \text{on }]0,\bar{t}]. \tag{3.9}$$

Now consider

$$\Gamma(y) = \int_y^{\infty} \Upsilon(z/C_1)^{-1}\,dz\,.$$

Since Φ does not decrease, $\Psi(y)\,\Gamma(y) \leq 1$, and

$$\frac{d}{dt}\Gamma(Y_{2+\beta}(t)) = -\frac{1}{\Upsilon(Y_{2+\beta}(t)/C_1)}\frac{d}{dt}Y_{2+\beta}(t) \geq \frac{C_2}{2}\,,$$

which in turn implies that $\Gamma(Y_{2+\beta}(t)) \geq \frac{C_2}{2}\,t$ for $t \leq \bar{t}$, and hence that $\Psi(Y_{2+\beta}) \leq 2/(C_2\,t)$. Since $t\,\Psi^{-1}(2/(C_2\,t)) \equiv \delta(t)$ tends to 0 when t tends to 0, we get (3.6).

Step 4. – Next we turn to the uniqueness. The uniqueness result in [1] is directly related to the construction of a solution by means of a monotonous sequence. The calculation here follows more closely [9] or [6] (but a similar calculation can be found also in [1], in the proof of energy conservation, which in turn is essential for the proof of uniqueness).

Thus let f and g be two solutions corresponding to the same initial data f_0. Let $F(t,v) = |f(t,v) - g(t,v)|$ and $G(v,t) = f(t,v) + g(t,v)$. Then F and G satisfy the same estimates as do f and g, and in addition $F(0,v) = 0$. Moreover

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} |f(t,v) - g(t,v)|\,dv \\
&\qquad = \int_{\mathbb{R}^3} \operatorname{sgn}\,(f(t,v) - g(t,v))\Big(Q(f,f)(t,v) - Q(g,g)(t,v)\Big)\,dv \\
&\qquad = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} \operatorname{sgn}\,(f-g)\big((f'-g')\,(f_1'+g_1') + (f'+g')\,(f_1'-g_1') \\
&\qquad\quad - (f-g)\,(f_1+g_1) - (f+g)\,(f_1-g_1)\big)\,B\,d\omega dv_1 dv \\
&\qquad \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} (f(t,v)+g(t,v))\,|f(t,v_1) - g(t,v_1)|\,B\,d\omega dv_1 dv\,,
\end{aligned}$$

i.e.,

$$\frac{d}{dt}\int_{\mathbb{R}^3} F(t,v)\,dv \leq \|b\|_1 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} F(t,v)|v-v_1|^{\beta} G(t,v_1)\,dv\,dv_1. \tag{3.10}$$

Similarly

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} F(v,t)(1+|v|^2)\,dv \\
&= \int_{\mathbb{R}^3} \mathrm{sgn}\,(f(v,t)-g(v,t))\Big(Q(f,f)(t,v)-Q(g,g)(t,v)\Big)\,(1+|v|^2)\,dv \\
&\le \|b\|_1 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} F(t,v_1)|v-v_1|^\beta G(t,v)(1+|v|^2)\,dv\,dv_1. \qquad (3.11)
\end{aligned}
$$

The estimate (3.11) (and also (3.10)) comes from the invariance of the integrals under the change of variables $(v,v_1)\to(v',v_1')$, and the fact that $|v|^2+|v_1|^2 = |v'|^2+|v_1'|^2$.

With the notation $X_1(t)=\int_{\mathbb{R}^3} F(t,v)\,dv$ and $X_2(t)=\int_{\mathbb{R}^3} F(t,v)(1+|v|^2)\,dv$, and because $\beta\in(0,2]$ we obtain the inequalities

$$
\begin{aligned}
&\frac{d}{dt}X_1(t) \le C\Big(X_2(t)\int_{\mathbb{R}^3} G(t,v)\,dv + X_1(t)\int_{\mathbb{R}^3} G(t,v)|v|^\beta\,dv\Big), \qquad (3.12)\\
&\frac{d}{dt}X_2(t) \le C\Big(X_2(t)\int_{\mathbb{R}^3} G(t,v)(1+|v|^2)\,dv + X_1(t) \\
&\qquad\qquad \times\int_{\mathbb{R}^3} G(v,t)(1+|v|^{2+\beta})\,dv\Big). \qquad (3.13)
\end{aligned}
$$

By the estimates on F and G one knows that $X_i, i=1,2$ are bounded and that $X_i(0)=0$. First, it follows from (3.12) that $X_1\le C\,t$, and, since the integral in the right hand side of (3.13) is bounded by $\delta(t)/t$, that $X_2\le C\,t$. Next, using these previous bounds and succesively equations (3.12) and (3.13), we get that $X_1\le C\,t^2$ and $X_2\le C\,t^2$. Thus X_2 is continuous, right differentiable at $t=0$, and $X_2'(0)=0$. Now, by integrating (3.13) one sees that for t sufficiently small

$$
\begin{aligned}
X_2(t) &\le 2\,C\int_0^t \|G(\tau,.)\|_{1,2+\beta}\,X_2(\tau)\,d\tau \\
&\le 2\,C\int_0^t \frac{\delta(\tau)}{\tau}X_2(\tau)\,d\tau \le \int_0^t \frac{X_2(\tau)}{\tau}\,d\tau. \qquad (3.14)
\end{aligned}
$$

These are exactly the conditions for Nagumo's uniqueness criterion, which now implies that $X_2\equiv 0$, i.e. that $f\equiv g$ in the considered interval. But this estimate is needed only in an arbitrarily small interval, since for any positive time, all moments of the solutions f are bounded, and hence uniqueness follows directly by Gronwall lemma in (3.14), see [1]. To prove Nagumo's result one takes $Z(t) = \int_0^t X_2(\tau)/\tau \, d\tau$, and using (3.13) one sees that $Z(t)/t$ is non-increasing. Since it is non-negative and zero at $t = 0$ it must be identically zero, which implies the same for X_2. This concludes the proof of the uniqueness theorem. $\square$

Remark 1. – When $\beta \in (0, 1]$ the condition $\|f(t, \cdot)\|_{1,1+\beta} \in L^1([0, T])$ holds automatically because of the conservation of mass and energy. In fact, when $1 < \beta < 2$, we prove in Theorem 4.3 that this condition is also a consequence of the boundedness of mass and energy. But, when $\beta = 2$, we need to do this additional hypothesis to prove the uniqueness, (or at least, that $\|f(t, \cdot)\|_{1,\beta+\varepsilon} \in L^1_{loc}([0, +\infty))$ and use Theorem 4.3). Nevertheless, the solutions built in the sections 4 and 5 satisfy always $\|f(t, \cdot)\|_{1,2+\beta-\nu} \in L^1([0, T])$, for all $\nu > 0$ and $T > 0$. Furthermore, we only use in the present proof that the energy is noincreasing (or least, that the energy at any time is smaller than the initial energy). As a matter of fact, under this weaker asumption we get an inequality in (3.3) instead of an equality (remind that $a_n = \Psi'(n) \geq 0$) and the sequel of the proof is unchanged. But, it not really improve the asumption of the Theorem 3.1 since in this case the energy is automatically conserved thanks to Theorem 4.3.

Remark 2. – When $\beta > 2$ we do not know if Theorem 1.1′ still holds, because (3.14) fails, but we can proof bounds i), ii) and iii) of Theorem 1.1.

To get (3.1) we proceed in the same way that in Step 1, using now the bound by below $H_\Psi(v, v_1) \geq c_1 \, |v|^{3/2} - c_2' \, |v|^{3/4} \, |v_1|^{3/4}$.

In Step 2 we take $t_0 > 0$ such that $\int_{\mathbb{R}^3} f(t_0, v) \, |v|^{\beta+1} \, dv < +\infty$ and we define $\Psi(r) = r^{s/2}$ for $s \in]2, 4[$. We obtain (3.3) with $G_{\Psi_n}(v, v_1) \leq C \, |v|^{s/2} \, |v_1|^{s/2}$ and H_{Ψ_n} increases pointwise to $H_s \geq 0$, satisfying

$$H_s(v, v_1) \geq c_1 \, |v|^s - c_2' \, |v|^{s/2} \, |v|^{s/2}. \tag{3.15}$$

Since $f \in C([0, +\infty); L^1_2(\mathbb{R}^3)) \cap L^1_{loc}(0, +\infty; L^1_{\beta+s/2}(\mathbb{R}^3))$ we can pass to the limit in equation (3.4) and we get

$$\begin{aligned} Y_s(t) &+ \frac{1}{2} \int_{t_0}^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f f_1 \, |v - v_1|^\beta H \, dv_1 dv d\tau \\ &\leq Y_s(t_0) + C_2 \int_{t_0}^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f f_1 \, |v - v_1|^\beta \, |v|^{s/2} \, |v_1|^{s/2} \, dv_1 dv d\tau. \end{aligned}$$

We deduce that $\int_{\mathbb{R}^3} f(t, v) \, |v|^s \, dv \in L^\infty_{loc}([t_0, +\infty))$ and we can use (3.15) to obtain

$$Y_s(t) + \frac{c_1}{2} Y_0 \int_{t_0}^t Y_{\beta+s} \, d\tau \leq Y_s(t_0) + \int_{t_0}^t \left\{ Y_{\beta+s/2} \, Y_{s/2} + Y_\beta \, Y_s \right\} d\tau.$$

The term $Y_{\beta+s/2}\,Y_{s/2}$ can be bounded thanksto Young's inequality by $\|f_0\|_{1,2}\big(\varepsilon\,Y_{\beta+s}+C_\varepsilon\,Y_0\big)$. For the last term we use Holder's inequality:

$$Y_s \le Y_2^{\beta/(\beta+s-2)}\,Y_{s+\beta}^{(s-2)/(s+\beta-2)} \quad\text{and}\quad Y_\beta \le Y_2^{s/(\beta+s-2)}\,Y_{s+\beta}^{(\beta-2)/(s+\beta-2)}$$

then

$$Y_s Y_\beta \le Y_2^{(\beta+s)/(\beta+s-2)}\,Y_{s+\beta}^{(s+\beta-4)/(s+\beta-2)} \le \varepsilon\,Y_2\,Y_\beta + C_\varepsilon\,Y_2^2.$$

Taking ε small enough it follows

$$Y_s(t)+\frac{c_1}{4}\,Y_0\int_{t_0}^{t}\{Y_s(\tau)+Y_{\beta+s}(\tau)\}\,d\tau \le Y_s(t_0)+C_2\,\|f_0\|_{1,2}^2\,t.$$

Again, (3.5) is proved by induction. When $Y_s(0)<+\infty$ we can choose $t_0=0$ and we get iii) of Theorem 1.1.

Remark 3. – Let just present an alternative simpler proof of Theorem 3.1, where Step 3 and 4 are modified. From (3.8) and Holder's inequality $Y_{2+\beta}\le Y_2^{1/2}\,Y_{2+2\beta}^{1/2}$ one gets on a small interval $]0,\bar t]$ the differentiable inequality

$$\frac{d}{dt}Y_{2+\beta}\le -\frac{1}{C_0}\,Y_{2+\beta}^2 \quad\text{on }]0,\bar t],$$

which clearly implies $Y_{2+\beta}(t)\le\frac{C_0}{t}$ on $]0,\bar t]$. This bound is weaker than (3.8) but is strong enough to conclude in Step 4. Indeed, using the trick described after formula (3.13) one gets by an iterative argument, that, for all $n\in\mathbb{N}$, $\frac{X_2(t)}{t^m}$ is continuous, right differential at $t=0$, and $\big(\frac{X_2(t)}{t^m}\big)_{t=0}=0$, and so do $Z(t)$. Then, taking $m\ge C_0$ and using (3.14) one proves that $\frac{Z(t)}{t^m}$ is non-increasing, thus is identically zero.

4. EXISTENCE

In this section we deal with the problem of the existence of solution to the Boltzmann equation. Essentially we prove the existence part of Theorem 1.1 in a slightly less general case because we assume that $f_0\in L^1_s$ with $s>2$. We prove the following.

THEOREM 1.2'. – *Let $f_0(v)$ be in $L^1_s(\mathbb{R}^3)$, with $s>2$. There exists a solution f in $C([0,+\infty[;\,L^1_2(\mathbb{R}^3))$ of the Boltzmann equation (1.1) which conserves mass, momentum and energy.*

The proof that we present here holds only for initial data f_0 in $L^1_s(\mathbb{R}^3)$, with $s > 2$, but, using an argument of weak compactness in $L^1_2(\mathbb{R}^3)$ we will prove the existence result only assuming that f_0 belongs to $L^1_2(\mathbb{R}^3)$, as in Theorem 1.1. This is done in Section 5.

For every integer n we introduce the truncated cross section $B_n(z,\theta) = b(\theta)\,(|z| \wedge n)^\beta$, where $a \wedge b = \min(a,b)$, and we denote by Q_n the kernel associated to B_n. Then, we consider the solution $f_n \in C([0,+\infty[\,; L^1_2(\mathbb{R}^3))$ of the truncated Boltzmann equation

$$\begin{cases} \dfrac{\partial f_n}{\partial t} = Q_n(f_n, f_n)\,, \\ f_n(0,v) = f_0(v)\,, \end{cases} \tag{4.1}$$

for which one can prove existence and uniqueness by a contraction argument, see [1]. We proceed by showing that $\{f_n\}_n$ is a Cauchy sequence in L^1_s.

THEOREM 4.1. – *Let $f_0(v)$ be in $L^1_s(\mathbb{R}^3)$, with $s > 2$. Then for all fixed $T > 0$, there exists a constant C_T such that*

$$\sup_{[0,T]} \|f_m - f_n\|_{1,2} \le \frac{C_T}{n^{s-2}}, \quad \textit{for all } m \ge n\,. \tag{4.2}$$

With this theorem at hand, it is not difficult to pass to the limit in equation (4.1) and to show that the limit f is a solution of (1.1). This proves the existence part of Theorem 1.1 for this particular case. Furthermore, one obtains the estimate

$$\sup_{[0,T]} \|f - f_n\|_{1,2} \le \frac{C_T}{n^{s-2}}\,. \tag{4.3}$$

For the proof of Theorem 4.1 we need a technical lemma.

LEMMA 4.2. – *There exist constants M and C, depending only on f_0, $\|b\|_1$, s and β, such that for all $n \ge M$ the solutions f_n satisfy*

$$\sup_{t\ge 0} \|f_n\|_{1,s} \le C\,, \quad \textit{and} \quad \int_0^t \|f_n(\tau,\cdot)\,(|.| \wedge n)^\beta\|_{1,s}\, d\tau \le C(1+t)\,. \tag{4.4}$$

Proof of Lemma 4.2. – In order to simplify the notation, we denote by g the solution f_n and set $Y_{s'} = \int_{\mathbb{R}^3} g(v)|v|^{s'}\, dv$. First, if $2 < s \le 3$, we set $\gamma = \min(s-\beta, 1)$. We have

$$\begin{aligned} \frac{d}{dt} Y_s &= \int_{\mathbb{R}^3} Q_n(g,g)\, |v|^s\, dv \\ &= \frac{1}{2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g\, g_1\, (|v - v_1| \wedge n)^\beta\, K_s(v, v_1)\, dv_1 dv, \end{aligned}$$

with the notation from Lemma 2.2. Then the Povzner inequality and Young's inequality imply that

$$K_s(v, v_1) \leq C_1 \left(|v|^{s-\gamma}|v_1|^\gamma + |v|^\gamma|v_1|^{s-\gamma}\right) - c_1 \, |v|^s,$$

and using the inequalities $(|v - v_1| \wedge n)^\beta \leq 4(|v_1|^\beta + (|v| \wedge n)^\beta)$ and $(|v-v_1|\wedge n)^\beta \geq \frac{1}{4}(|v|\wedge n)^\beta - |v_1|^\beta$, one obtains the differentiable inequality

$$\begin{aligned} \frac{d}{dt} Y_s \leq C_2 \Big(Y_{s-\gamma} \, Y_{\gamma+\beta} + Y_\gamma \int_{\mathbb{R}^3} g \, |v|^{s-\gamma} (|v| \wedge n)^\beta \, dv \Big) \\ + c_2 \, Y_\beta \, Y_s - \frac{c_1}{4} \, Y_0 \int_{\mathbb{R}^3} g \, |v|^s (|v| \wedge n)^\beta \, dv. \end{aligned}$$

Then, for all $\varepsilon > 0$, there exists C_ε such that

$$\int_{\mathbb{R}^3} g \, |v|^{s-\gamma} (1 + (|v| \wedge n)^\beta) \, dv \leq C_\varepsilon Y_0 + \varepsilon \int_{\mathbb{R}^3} g \, |v|^s (|v| \wedge n)^\beta \, dv,$$

and for all $n \geq M$

$$|v|^s \leq \frac{1}{M^\beta} \, |v|^s \, (|v| \wedge n)^\beta + M^s,$$

and we obtain

$$\frac{d}{dt} Y_s + \Big(\frac{c_1 \, Y_0}{8} - C_3 \, (Y_0 + Y_2) \left(\varepsilon + \frac{1}{M^\beta} \right) \Big) \Big(\int_{\mathbb{R}^3} g \, |v|^s (|v| \wedge n)^\beta \, dv + Y_s \Big) \leq C_{\varepsilon, M}.$$

We conclude by using Gronwall's lemma, taking ε small enough and M large enough.

Next, if $s > 3$, we take $\gamma = 1$ and we perform the same calculations taking in mind the fact that we already know that $\sup_{t\geq 0} Y_3(t)$ is bounded.□

Proof of Theorem 4.1. – Let f_n and f_m be two solutions of the equation (4.1) corresponding to n and m respectively. Similarly to (3.10) we can compute

$$
\begin{aligned}
&\frac{d}{dt}\|f_n - f_m\|_{1,2} \\
&\quad = \int_{\mathbb{R}^3} \operatorname{sgn}\,(f_n - f_m)(Q_n(f_n, f_n) - Q_n(f_m, f_m))(1+|v|^2)\,dv \\
&\qquad + \int_{\mathbb{R}^3} \operatorname{sgn}\,(f_n - f_m)(Q_n(f_m, f_m) - Q_m(f_m, f_m))(1+|v|^2)\,dv \\
&\quad \le C \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (f_n + f_m)\,|f_{n\,1} - f_{m\,1}|\,(|v - v_1| \wedge n)^\beta (1+|v|^2)\,dv_1 dv \\
&\qquad + C \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_m f_{m\,1} 1\!\!1_{\{|v-v_1|>n\}}\,(|v-v_1| \wedge m)^\beta\,(1+|v|^2)\,dv_1 dv \\
&\quad \le C \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (f_n + f_m)\,|f_{n\,1} - f_{m\,1}|\,((|v| \wedge n)^\beta + |v_1|^\beta)(1+|v|^2)\,dv_1 dv \\
&\qquad + C \int\int_{\{|v|>n/2\}\cup\{|v_1|>n/2\}} f_m f_{m\,1} \\
&\qquad\qquad \times ((|v| \wedge m)^\beta + |v_1|^\beta)\,(1+|v|^2)\,dv_1 dv\,.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\frac{d}{dt}\|f_n - f_m\|_{1,2} &\le C\|f_n - f_m\|_{1,2} \int_{\mathbb{R}^3} (f_n + f_m)\,(1 + |v|^2(|v| \wedge n)^\beta)\,dv \\
&+ C\left(\frac{1}{n^{s-2}} + \frac{1}{n^{s-\beta}}\right)\|f_m\|_{1,s} \int_{\mathbb{R}^3} f_m\,(1 + |v|^s(|v| \wedge m)^\beta) dv\,.
\end{aligned}
$$

Setting $Y_{n,m}(t) = \|f_n - f_m\|_{1,2}$ and $h_{n,m}(t) = 1 + \|f_m\|_{1,s} + \int_{\mathbb{R}^3}(f_n + f_m)\,(1 + |v|^s(|v| \wedge m)^\beta)\,dv$ we obtain the differential inequality

$$
\dot{Y}_{n,m} \le C h_{n,m}\left(Y_{n,m} + \frac{1}{n^{s-2}}\right).
$$

But $Y_{n,m}(0) = 0$ and by Lemma 4.2, $h_{n,m}$ is uniformly bounded in $L^1_{loc}[0, +\infty)$, and hence we get (4.2) by Gronwall's lemma. □

Next one would like to know whether solutions can exist also if the initial energy is not bounded. We do not know of any result on existence or non-existence in this case, but the proposition below is a partial result in this direction. The reversed Povzner inequality, Lemma 2.2 ii) is used to prove that if certain moments of order $s < 2$ remain bounded in an interval $[0, T]$, then necessarily the energy is bounded in a closed interval $[0, T_1]$ with $T_1 < T$. The proof actually shows that the energy is non-decreasing.

THEOREM 4.3. – *Let f be a non-negative solution of the Boltzmann equation, and suppose that there are $\varepsilon > 0$ and $T > 0$ such that*

$$f \in L^1([0,T], L^1_{\beta+\varepsilon}) \cap L^\infty([0,T], L^1_\varepsilon).$$

Then, for all $\delta > 0$ and $T_1 < T$,

$$f \in L^1([0,T_1], L^1_{2+\beta-\delta}) \cap L^\infty([0,T], L^1_2),$$

and the energy of f is non-decreasing on $[0,T]$.

Proof. – Let remark that the asumption make on f implies that the collisional term $Q(f,f)$ lies in $L^1([0,T]; L^1(\mathbb{R}^3))$ and thus equation (1.1) make sense in the distributional sense. We carry out this proof in three steps.

Step 1. – Let $\{\Psi_n\}$ be a sequence of concave bounded functions, which converges pointwise to $\Psi(r) = |r|^{1-\gamma}$, with $1-\gamma = \min(\varepsilon, (\varepsilon+\beta)/2)$. Then multiply the Boltzmann equation by $\Psi_n(|v|^2)$ and integrate to find

$$\begin{aligned} \int_{\mathbb{R}^3} f(t_2,v)\Psi_n(|v|^2)\,dv - \int_{t_1}^{t_2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} ff_1\, G_{\Psi_n}(v,v_1)|v-v_1|^\beta\,dv\,dv_1\,d\tau = \\ = \int_{\mathbb{R}^3} f(t_1,v)\Psi_n(|v|^2)\,dv - \int_{t_1}^{t_2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} ff_1\, H_{\Psi_n}(v,v_1)|v-v_1|^\beta\,dv\,dv_1\,d\tau. \end{aligned} \tag{4.5}$$

For almost every $t \in [0,T]$,

$$\int_{\mathbb{R}^3} f(t,v)\Psi_n(|v|^2)\,dv \to \int_{\mathbb{R}^3} f(t,v)|v|^{2(1-\gamma)}\,dv,$$

and in the left hand side, $|G_{\Psi_n}(v,v_1)|$ is bounded by $(\gamma c_1|v||v_1|)^{1-\gamma}$, uniformly in n, and in the right hand side $-H_{\Psi_n}$ increases pointwise to $-H_\gamma \geq c_2\gamma\Big((|v|^2)^{1+\gamma} + (|v_1|^2)^{1+\gamma}\Big)\left(\mathbb{1}_{|v_1|\geq 2|v|} + \mathbb{1}_{|v|\geq 2|v_1|}\right)$; the estimates can be found in Lemma 2.2. Hence, with calculations like in Section 3 of Theorem 1.1′, we find

$$\begin{aligned} Y_{2(1-\gamma)}(t_2) + C\int_{t_1}^{t_2} Y_{1-\gamma}(\tau)Y_{1-\gamma+\beta}(\tau)\,d\tau \\ \geq Y_{2(1-\gamma)}(t_1) + C'\int_{t_1}^{t_2} Y_{2(1-\gamma)+\beta}(\tau)Y_0(\tau)\,d\tau, \end{aligned}$$

for some constants C and C'. Observe that the hypothesis on f implies that the mass is conserved, and therefore one can deduce that for all $T' < T$,

$$f \in L^1([0,T], L^1_{\beta+2\varepsilon}) \cap L^\infty([0,T], L^1_{2\varepsilon}).$$

Step 2. – This step consists in iterating the previous step. After a finite number of steps one has

$$f \in L^1([0,T_1], L^1_{2+\beta-\nu}) \cap L^\infty([0,T_1], L^1_{2-\nu}),$$

and it is possible to take $T_1 < T$, and $\nu > 0$ arbitrarily. Note that since $2+\beta-\nu > 2$, the energy must be bounded for almost every $t \in [0, T_1]$.

Step 3. – Again we return to Step 1, now with $\{\Psi_n\}$ converging to $|\cdot|^{1-\gamma}$, and here the intention is to let $\gamma \to 0$. Let chose t_2 in such a way that $Y_2(t_2) < \infty$, then, using Lemma 2.2 and passing to the limit $n \to +\infty$ in (4.5)

$$\begin{aligned} &Y_{2(1-\gamma)}(t_2) - \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f f_1 G_\gamma \, |v - v_1|^\beta \, dv dv_1 \, d\tau \\ &= Y_{2(1-\gamma)}(t_1) + \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f f_1 \, (-H_\gamma) \, |v - v_1|^\beta \, dv dv_1 \, d\tau \geq Y_{2(1-\gamma)}(t_1). \end{aligned}$$

In the left hand side, the integral is bounded by

$$\gamma c_1 \|f\|_{L^1([0,T_1];L^1_{1+\beta})} \|f\|_{L^\infty([0,T_1];L^1_1)} \,.$$

Hence it is possible to pass to the limit $\gamma \to 0$, and to conclude that

$$\int_{\mathbb{R}^3} f(t_1, v)|v|^2 \, dv \leq \int_{\mathbb{R}^3} f(t_2, v)|v|^2 \, dv$$

for all $t_1 < t_2$. □

5. TIME DISCRETIZATION OF THE BOLTZMANN EQUATION

In this section, some of the estimates from the previous sections are used in order to obtain some results on the convergence of a time discretization of the Boltzmann equation. As a byproduct we find an existence theorem for the continuous equation in the case where only the energy is assumed to be bounded.

Let $\Delta > 0$ be the step size in a time discretization. We first consider the explicit Euler scheme for the truncated Boltzmann equation, and define the sequence $f^\kappa = f^\kappa_{\Delta,n}$ recursively:

$$\begin{cases} \dfrac{f^{\kappa+1} - f^\kappa}{\Delta} = Q_n(f^\kappa, f^\kappa), \\ f^0(v) = f_0(v) \geq 0. \end{cases} \tag{5.1}$$

First, if we multiply the previous equation with the collision invariants 1, v and $|v|^2$ and integrate, we find

$$\begin{aligned}
&\int_{\mathbb{R}^3} f^{\kappa+1}\, dv = \int_{\mathbb{R}^3} f^{\kappa}\, dv = Y_0\,,\\
&\int_{\mathbb{R}^3} f^{\kappa+1}\, v\, dv = \int_{\mathbb{R}^3} f^{\kappa}\, v\, dv = \int_{\mathbb{R}^3} f_0\, v\, dv\,,\\
&\int_{\mathbb{R}^3} f^{\kappa+1}\, |v|^2\, dv = \int_{\mathbb{R}^3} f^{\kappa}\, |v|^2\, dv = Y_2\,.
\end{aligned} \tag{5.2}$$

Next, we write equation (5.1) as

$$f^{\kappa+1} = f^{\kappa}(1 - \Delta\, L_n(f^{\kappa})) + \Delta\, Q_n^+(f^{\kappa}, f^{\kappa})\,, \tag{5.3}$$

where we have used the notation

$$Q_n(\varphi,\varphi) = Q_n^+(\varphi,\varphi) - Q_n^-(\varphi,\varphi) = Q_n^+(\varphi,\varphi) - \varphi\, L_n(\varphi)\,.$$

Since

$$L_n(f^{\kappa}) = \|b\|_{L^1(S^2)} \int_{\mathbb{R}^3} f_1^{\kappa}(|v-v_1| \wedge n)^{\beta}\, dv_1 \le \|b\|_{L^1(S^2)}\, n^{\beta} \int_{\mathbb{R}^3} f^{\kappa}\, dv\,,$$

we see that if

$$\Delta\, \|b\|_{L^1(S^2)}\, n^{\beta}\, Y_0 \le 1\,, \tag{5.4}$$

the algorithm (5.1) defines a nonnegative sequence $f^{\kappa} \ge 0$, and from which a piecewise constant function $[0,T] \to L_2^1$ can be constructed ($t_k = k\Delta$);

$$f_{\Delta,n}(t,v) = f_{\Delta,n}^{\kappa}(v) \qquad \text{for } t \in [t_{\kappa}, t_{\kappa+1}[\,. \tag{5.5}$$

This function is an approximation of the solution of (1.1).

THEOREM 5.1. – *Let Δ tend to 0 and n to infinity, in such a way that the stability condition (5.4) holds, and assume that $f_0 \in L_2^1(\mathbb{R}^3)$. The family $f_{\Delta,n}$ converges in $L^{\infty}([0,T]; L_2^1(\mathbb{R}^3))$ to the unique solution f of the Boltzmann equation (1.1).*

With a stronger moment condition, it is possible to estimate the rate of convergence:

THEOREM 5.2. – *Let f_0 belong to $L_s^1(\mathbb{R}^3)$, with $s > 2$ and $T > 0$ be fixed. Then, there exist positive constants a_T and C_T such that if Δ tends to 0 and n tends to infinity with $n = \big((-\ln \Delta)/a_T\big)^{1/\beta}$, (this is stronger* *than (5.4)), then the family* $f_\Delta = f_{\Delta,n}$ *converges in* $L^\infty([0,T]; L^1_2(\mathbb{R}^3))$ *to the unique solution* f *of the Boltzmann equation (1.1), and*

$$\sup_{[0,T]} \|f_\Delta - f\|_{1,2} \le \Big(\frac{C_T}{-\ln \Delta}\Big)^{(s-2)/\beta}.$$

Proof of Theorem 5.1. – We perform the proof in two steps.

Step 1. – We start by proving that the sequence $f_{\Delta,n}$ lies in a weakly compact set of $L^1([0,T];\, L^1_2(\mathbb{R}^3))$. To this end we multiply the equation (5.1) by $|v|$ and we integrate to find

$$Y_1^{\kappa+1} - Y_1^\kappa = \Delta \int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3} f^\kappa f_1^\kappa (|v-v_1| \wedge n)^\beta K_{1/2}\, dv dv_1.$$

Then we use the reverse Povzner inequality ii) of lemma 2.2 which, thanks to Young inequality, writes $K_{1/2} \ge c_1'(|v|+|v_1|) - c_2'|v|^{1/2}|v_1|^{1/2}$ and the elementary inequality $|z+y| \wedge n \le |z| \wedge n + |y| \wedge n$, and we obtain

$$Y_1^{\kappa+1} - Y_1^\kappa + \Delta\, C_2 \int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3} f^\kappa f_1^\kappa (|v| \wedge n)^\beta \big(|v|^{1/2}|v_1|^{1/2} + |v_1|\big)\, dv dv_1 \ge$$
$$\ge \Delta\, C_1 \int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3} f^\kappa f_1^\kappa |v| (|v| \wedge n)^\beta \, dv dv_1.$$

Proceding like in lemma 4.2 and using conservation of mass and energy (5.2) one gets

$$Y_1^{\kappa+1} - Y_1^\kappa + \Delta\, C_2'\, Y_0\, \|f_0\|_{1,2} \ge \Delta\, \frac{C_1}{2}\, Y_0 \int_{\mathbb{R}^3} f^\kappa |v| (|v| \wedge n)^\beta \, dv,$$

which in turn implies

$$\sup_n \int_0^t \int_{\mathbb{R}^3} f_{n,\Delta}(t,v)\, |v| (|v| \wedge n)^\beta \, dv \le C\,(1+t). \tag{5.6}$$

Next, we can choose a function Ψ with $\Psi(r)/r \to +\infty$ when $r \to +\infty$ and $\int_{\mathbb{R}^3} f_0(v)\, \Psi(|v|^2)\, dv < +\infty$, and we proceed just like in Step 1 Theorem 1.1′; using (5.6) we get

$$\sup_{t \ge 0} \int_{\mathbb{R}^3} f_{\Delta,n}(t,v)\, \Psi(|v|^2)\, dv \le C\,. \tag{5.7}$$

Moreover,

$$f^{\kappa+1} = f^\kappa + \Delta\big(Q_n^+(f^\kappa, f^\kappa) - Q_n^-(f^\kappa, f^\kappa)\big) \le f^\kappa + \Delta Q^+(f^\kappa, f^\kappa),$$

from what it follows by iteration

$$f_{\Delta,n}(t,v) \le f_0(v) + \int_0^t Q^+\big(f_{\Delta,n}(\tau,v), f_{\Delta,n}(\tau,v)\big)\,d\tau,$$

and therefore we get

$$\begin{aligned} f_{\Delta,n}(t,v) \le\ & f_0(v) + t\ Q^+(f_0,f_0)(v) \\ & + \int_0^t \int_0^\tau \Big(Q^+\big(f_0, Q^+(f_{\Delta,n}, f_{\Delta,n})(\tau_1)\big) \\ & + Q^+\big(Q^+(f_{\Delta,n}, f_{\Delta,n})(\tau_1), f_{\Delta,n}(\tau)\big)\Big)\, d\tau_1 d\tau. \end{aligned}$$

Lemma 2.1 implies that for all $\varepsilon > 0$ and $T > 0$ there exists η such that if A is a Borel set with measure $\mu(A) \le \eta$, then for all $t \le T$,

$$\int_A f_{\Delta,n}(t,v)\,dv \le \varepsilon\,. \tag{5.8}$$

The Dunford-Pettis lemma together with (5.7) and (5.8) imply that the sequence $f_{\Delta,n}$ is weakly compact in $L^1([0,T]; L^1_2(\mathbb{R}^3))$.

Step 2. – We wish to pass to the limit in equation (5.1). We next note that for all test functions $\psi \in L^\infty(\mathbb{R}^3)$,

$$\frac{d}{dt}\int_{\mathbb{R}^3} f_{\Delta,n}(t,v)\,\psi(v)\,dv = \sum_\kappa \Big(\Delta \int_{\mathbb{R}^3} Q_n(f^\kappa, f^\kappa)(v)\,\psi(v)\,dv\Big)\,\delta_{t=t_\kappa},$$

is a bounded measure on $[0,T]$, and therefore $\int_{\mathbb{R}^3} f_{\Delta,n}(t,v)\,\psi(v)\,dv$ is bounded in $BV([0,T])$, for all T; hence it converges almost everywhere. Furthermore, (5.7) implies that the same holds for every measurable function ψ, such that $|\psi(v)| \le C\,(1+|v|^2)$ almost everwhere. The collision operator is essentially a convolution operator, in which the kernel B is not growing faster than $(1+|v|^2)$, and therefore on can pass to the limit with $Q_n(f_{\Delta,n}, f_{\Delta,n})$, and prove that the limit f is a solution to the Boltzmann equation (1.1) for the initial data f_0. Moreover, the bound (5.7) implies that f belongs to $C([0,T], L^1_2(\mathbb{R}^3))$, that f conserves mass, momentum and energy, that (5.7) holds for f, and from (5.6) that $\|f(t,.)\|_{1,1+\beta} \in L^1_{loc}([0,+\infty))$. By Theorem 3.1 this solution is the unique solution of the Boltzmann equation (1.1) and therefore the full sequence $f_{\Delta,n}$ converges to f weakly in $L^1([0,T]; L^1_2(\mathbb{R}^3))$. In order to prove the strong convergence we write

$$\begin{aligned} f^{\kappa+1} &= \Delta Q_n^+(f^\kappa, f^\kappa) + \big(1 - \Delta L_n(f^\kappa)\big) f^\kappa \\ &= \big(1 - \Delta L_n(f^\kappa)\big) \dots \big(1 - \Delta L_n(f_0)\big)\, f_0 \\ &\quad + \sum_{j=0}^{\kappa} \Delta Q_n^+(f^j, f^j)\,\big(1 - \Delta L_n(f^\kappa)\big) \dots \big(1 - \Delta L_n(f^j)\big). \end{aligned} \tag{5.9}$$

The sequence $L_n(f_{n,\Delta})$ satisfies

$$0 \leq L_n(f_{n,\Delta}) \leq C\,(1+|v|^\beta) \quad \text{and} \quad L_n(f_{n,\Delta}) \longrightarrow L(f) \quad \text{for a.e. } t, v\,,$$

so that the function $E_{n,\Delta}(t,v) = \prod_{j=0}^{\kappa}(1-\Delta\, L_n(f^j))$ if $t \in [t_\kappa, t_{\kappa+1}[$ satisfies $0 \leq E_{n,\Delta} \leq 1$ and

$$\begin{aligned} E_{n,\Delta} &= \exp\Big\{-\sum_{j=0}^{\kappa}\Delta\, L_n(f^j) + \sum_{j=0}^{\kappa}\big(\ln(1-\Delta\, L_n(f^j)) + \Delta\, L_n(f^j)\big)\Big\} \\ &= \exp\Big\{-\int_0^{t_\kappa} L_n(f_{n,\Delta})\, d\tau + \mathcal{O}(\Delta\,(1+|v|)^\beta)\Big\} \\ &\longrightarrow \exp\Big(-\int_0^t L(f)\, d\tau\Big) \quad \text{for a.e. } t, v\,. \end{aligned}$$

The regularity property of the Q^+ term, see [10], shows that $Q_n^+(f_{n,\Delta}, f_{n,\Delta})$ converges strongly to $Q^+(f,f)$. Therefore the right hand side of the equation (5.9) converges almost everywhere to

$$f(t) = f_0\, e^{-\int_0^t L(f)\, ds} + \int_0^t Q^+(f,f)(s)\, e^{-\int_s^t L(f)\, d\sigma}\, ds.$$

We have obtained the strong convergence of $f_{\Delta,n}$ to f in $L^1([0,T]\times\mathbb{R}^3)$ and we obtain the strong convergence in $L^\infty([0,T];L^1(\mathbb{R}^3))$ using estimate (3.10). $\square$

Remark 1. – In particular, we have proved that the existence part of Theorem 1.1 holds in the general case $f_0 \in L^1_2(\mathbb{R}^3)$ and $\beta \in]0,2]$. In fact, estimates (5.6) and (5.7) still hold when $\beta > 2$. Furthermore, in Remark 2 following Theorem 1.1′ we proved that the bounds ii) and iii) of Theorem 1.1 can also be generalized to the case $\beta > 2$. Therefore, we are able to prove that for all $\beta > 2$ and $f_0 \in L^1_2(\mathbb{R}^3)$ there exists a distributional solution $f \in C([0,+\infty); L^1_2(\mathbb{R}^3)) \cap L^1_{loc}(0,+\infty; L^1_{\beta+1}(\mathbb{R}^3))$ of equation (1.1) conserving mass and energy, which furthermore satisfies properties i), ii), iii) of Theorem 1.1 . (One has to observe that the above *a priori* bounds on f imply that $Q(f,f) \in L^1_{loc}([0,+\infty); L^1(\mathbb{R}^3))$.)

Remark 2. – We would like to emphasize that the proof can be notably simplified when $0 < \beta < 2$. In this case, with only the conservation of mass and energy at hand, but without estimates (5.6) and (5.7), we can pass to the limit in equation (5.1) and get a solution f of the Boltzmann equation (1.1).

At this stage we have lost the conservation of energy, however by Fatou's lemma, we have $\int_{\mathbb{R}^3} f(t,v)|v|^2\,dv \le \lim_{\Delta\to 0,\, n\to\infty} \int_{\mathbb{R}^3} f_{\Delta,n}(t,v)|v|^2\,dv \le \int_{\mathbb{R}^3} f(0,v)|v|^2\,dv$. Then f satisfies the conditions of Theorem 4.3, and therefore also $\int_{\mathbb{R}^3} f(t,v)\,dv \ge \int_{\mathbb{R}^3} f_0(v)\,dv$. We conclude that f conserves mass and energy and it is the unique solution of Theorem 1.1′.

Proof of Theorem 5.2. – Here, the result essentially follows from estimate (4.2). Let $e^\kappa = f_n(t_\kappa) - f^\kappa$, where f_n is the solution of the Boltzmann equation (4.1) and $f^\kappa = f^\kappa_{\Delta,n}$ the solution of the Euler scheme (5.1):

$$f^{\kappa+1} = f^\kappa + \Delta\, Q_n(f^\kappa, f^\kappa).$$

The exact solution is continuous in t, and can be computed at $t = t_k$:

$$f_n(t_{\kappa+1}) = f_n(t_\kappa) + \Delta\, Q_n(f_n(t_\kappa), f_n(t_\kappa)) + \mathcal{A}_\kappa\,,$$

where A_k is given by

$$\mathcal{A}_\kappa = \int_{t_\kappa}^{t_{\kappa+1}} \big(Q_n(f_n(\tau), f_n(\tau)) - Q_n(f_n(t_\kappa), f_n(t_\kappa))\big)\,d\tau\,. \tag{5.10}$$

Similarly

$$e^{\kappa+1} = e^\kappa + \Delta\,\big(Q_n(f_n(t_\kappa), f_n(t_\kappa)) - Q_n(f^\kappa, f^\kappa)\big) + \mathcal{A}_\kappa\,.$$

In (5.10),

$$\|Q_n(f_n(t_\kappa), f_n(t_\kappa)) - Q_n(f^\kappa, f^\kappa)\|_{1,2} \le C_1\, n^\beta\, \|f\|_{1,2}\, e^\kappa\,,$$

and taking the calculation one step further gives

$$\begin{aligned}\mathcal{A}_\kappa = \int_{t_\kappa}^{t_{\kappa+1}} \int_{t_\kappa}^{\tau} \Big(&Q_n(Q_n(f_n(\tau_1), f_n(\tau_1)), f_n(\tau)) \\ &+ Q_n\big(f_n(\tau_1), Q_n(f_n(\tau_1), f_n(\tau))\big)\Big)\,d\tau_1 d\tau\,.\end{aligned}$$

Hence

$$\|\mathcal{A}_\kappa\|_{1,2} \le \Delta^2\, n^{2\beta}\, C_2 \sup_{[t_\kappa, t_{\kappa+1}]} \|f_n\|^3_{1,2}\,,$$

and then

$$\|e^{\kappa+1}\|_{1,2} \le (1 + C_1\, n^\beta\, \Delta)\|e^\kappa\|_{1,2} + C_2'\, \Delta^2\, n^{2\beta}\,,$$

which implies

$$\|e^\kappa\|_{1,2} \leq \Delta \exp(C_1\, n^\beta\, \kappa\, \Delta)\, \frac{C_2'}{C_1}\, n^\beta\,.$$

Therefore the result follows by a combination of

$$\sup_{[0,T]} \|f_{n,\Delta} - f_n\|_{1,2} \leq \Delta\, \exp(n^\beta\, a_T/2)$$

and estimate (4.2). □

We now consider the implicit Euler scheme,

$$\begin{cases} \dfrac{f^{\kappa+1} - f^\kappa}{\Delta} = Q_n(f^{\kappa+1}, f^{\kappa+1}) \\ f^0(v) = f_0(v) \geq 0\,, \end{cases} \tag{5.11}$$

and as before we set $f_{\Delta,n}(t,v) = f^\kappa$ for $t \in [t_\kappa, t_{\kappa+1}[$. In order to see that the scheme is well defined, we introduce the map T from $L^1_+(\mathbb{R}^3)$ into itself, defined by

$$h = T\, g$$

and

$$h + \lambda\Delta\, h = \Delta\, Q_n^+(g,g) + \Delta(\lambda - L_n g)g + f^\kappa\,,$$

where $\lambda = \|b\|_{L^1(S^2)}\, Y_0\, n^\beta$. It is easy to see that if

$$3\lambda\Delta < 1\,, \tag{5.12}$$

then T has a unique fix point, which is the solution of (5.11), and which will be denoted $f^{\kappa+1}$. We note that (5.12) is a stability condition similar to (5.4).

In the following theorem, the main interest is the convergence rate of the implicit Euler scheme. We get a better rate of convergence by requiring an extra moment condition. However to prove only convergence, one can proceed as in Theorem 5.1 with initial data $f_0 \in L^1_2$.

THEOREM 5.3. – *Let Δ tend to 0 and n to infinity in such a way that (5.12) is satisfied, and assume that $f_0 \in L^1_s(\mathbb{R}^3)$, with $s \geq 2 + 2\beta$. Then the family $f_{\Delta,n}$ converges in $L^\infty([0,T]; L^1_2(\mathbb{R}^3))$ to the unique solution f of the Boltzmann equation (1.1), and*

$$\sup_{[0,T]} \|f_{\Delta,n} - f\|_{1,2} \leq C_T \left(\Delta + \frac{1}{n^{s-2}}\right). \tag{5.13}$$

Proof. – We start with *a priori* bounds. In the same way as for the explicit scheme, we see that the mass and the energy are conserved. An adaptation of Lemma 4.1 shows that there exists a constant C such that for all n, Δ and all time $t \geq 0$

$$\|f_{\Delta,n}(t,.)\|_{1,s} \leq C\,. \tag{5.14}$$

Then, we set $e^{\kappa} = f(t_\kappa) - f^{\kappa}$, where f is the solution of the Boltzmann equation (1.1) and let $f^{\kappa} = f^{\kappa}_{\Delta,n}$ the solution of the Euler scheme (5.11). We can write

$$f^{\kappa+1} = f^{\kappa} + \Delta\, Q_n(f^{\kappa+1}, f^{\kappa+1})$$

and

$$f(t_{\kappa+1}) = f(t_\kappa) + \Delta\, Q_n(f(t_{\kappa+1}), f(t_{\kappa+1})) + \mathcal{A}_\kappa + \mathcal{B}_\kappa$$

where

$$\mathcal{A}_\kappa = \Delta\left(Q(f(t_{\kappa+1}), f(t_{\kappa+1})) - Q_n(f(t_{\kappa+1}), f(t_{\kappa+1})\right)$$

and

$$\mathcal{B}_\kappa = \int_{t_\kappa}^{t_{\kappa+1}} \left(Q(f(\tau), f(\tau)) - Q(f(t_{\kappa+1}), f(t_{\kappa+1}))\right) d\tau\,.$$

Keeping the notation from the proof of Theorem 5.2, we have

$$e^{\kappa+1} = e^{\kappa} + \Delta\left(Q_n(f(t_{\kappa+1}), f(t_{\kappa+1})) - Q_n(f^{\kappa+1}, f^{\kappa+1})\right) + \mathcal{A}_\kappa + \mathcal{B}_\kappa\,.$$

Then, multiplying by $\operatorname{sgn}\,(f(t_{\kappa+1}) - f^{\kappa+1})$ and integrating we get

$$\begin{aligned}\|e^{\kappa+1}\|_{1,2} &\leq \|e^{\kappa}\|_{1,2} + C\,\|f\|_{1,2+\beta}\,\Delta\,\|e^{\kappa+1}\|_{1,2}\\ &+ \Delta\,\frac{C}{n^{s-2}}\,\|f\|_{1,s+\beta}\,\|f\|_{1,s} + \Delta^2\, C\,\|f\|_{1,2+\beta}\,\|f\|^2_{1,2+2\beta}\,.\end{aligned}$$

Gronwall lemma together with (5.14) give the error control (5.13). □

APPENDIX

In this Appendix we construct a function Φ that satisfies all properties needed in Sections 2 and 3. The existence of such a function is probably a classical result, but we have not found any general reference for such a construction, and in any case, the result seems not to be very well known. However, similar constructions have previously been used in kinetic theory [12].

PROPOSITION A1. – *Let $f = f(v)$ be a function such that $(1+|v|^2)f(v) \in L^1(\mathbb{R}^3)$. There exists a concave function $\Phi(r)$, depending on f, such that $\Phi(r) \to \infty$ as $r \to \infty$, $r\Phi(r)$ is convex, and such that for all $\varepsilon > 0$ and $\alpha \in]0,1[$, $\big(\Phi(r) - \Phi(\alpha r)\big)r^\varepsilon \to \infty$ as $r \to \infty$, and such that $(1+|v|^2)\Phi(|v|^2)f(v) \in L^1(\mathbb{R}^3)$.*

Proof. – There is no loss of generality in assuming that f is positive, and that $\int_{\mathbb{R}^3}(1+|v|^2)f(v)\,dv = 1$. Take $0 = r_0 < r_1 < ... < r_n...$ such that

$$\int_{r_{j-1}<|v|^2\le r_j} (1+|v|^2)f(v)\,dx = 2^{-j}\,.$$

Since the theorem is trivial if f is compactly supported, it is only necessary to consider the case where $r_j \to \infty$ as $j \to \infty$. Let Φ_1 be linear in each interval $[r_j, r_{j+1}[$ and such that $\Phi_1(r_j) = j$. Clearly $\Phi_1(r)$ is increasing to infinity with r, and

$$\int_{\mathbb{R}^3}(1+|v|^2)f(v)\Phi_1(|v|^2)\,dx \le \sum_{j=1}^{\infty}(j+2)2^{-j} < \infty\,.$$

Clearly Φ_1 is concave if $r_{j+1} - r_j$ is an increasing sequence, and one can always assume that this holds. For if that were not the case, one could replace the sequence r_j by a new sequence $\tilde r_j$ given by $\tilde r_{j+1} = \max\big(r_{j+1}, \tilde r_j + (\tilde r_j - \tilde r_{j-1})\big)$, and take $\tilde\Phi_1$ as above, but defined by the sequence $\tilde r_j$. Then $\tilde\Phi_1$ is also increasing to infinity (because $\tilde r_j$ is finite for each finite j), concave and pointwise bounded by Φ_1. Finally we take

$$\Phi(r) = \frac{1}{r}\int_0^r \frac{1}{y}\int_0^y \Big(\Phi_1(z) + 1 - (\log(e+z))^{-1}\Big)\,dz\,dy$$

in order to make $r\Phi(r)$ convex. Since Φ_1 is concave and increasing to infinity, the same holds for Φ. And

$$\Phi(\alpha r) = \frac{1}{r}\int_0^r \frac{1}{y}\int_0^y \Big(\Phi_1(\alpha z) + 1 - (\log(e+\alpha z))^{-1}\Big)\,dz\,dy\,,$$

and therefore

$$\begin{aligned}\Phi(r) - \Phi(\alpha r) &\ge \frac{1}{r}\int_0^r \frac{1}{y}\int_0^y \Big((\log(e+\alpha z))^{-1} - (\log(e+z))^{-1}\Big)\,dz\,dy \\ &\ge C(\log(e+r))^{-2}\,.\end{aligned}$$

Remark A.1. – Note that $1 - (\log(e+r))^{-1}$ was added to Φ_1 only in order to obtain the lower bound for $\Phi(r) - \Phi(\alpha r)$ without having to be too careful with in the construction of the sequence r_j. Clearly $\Phi(r) - \Phi(\alpha r)$ decays more slowly than any power of r.

Remark A.2. – Let Ψ be the function constructed above, and let

$$\Psi_n(r) = \begin{cases} \Psi(r) & \text{if } r \leq n, \\ r\Psi'(n) + \Psi(n) - n\Psi'(n), & \text{otherwise} \end{cases}$$

i.e., Ψ_n is not growing faster than linearly, and it is converging pointwise monotonically to Ψ as $n \to \infty$. The point to note is that $\Psi_{n+1} - \Psi_n$ is still a convex function. Similarly it is possible to construct a sequence of bounded, concave functions Φ_n, which converge pointwise to Φ, and such that $\Phi_{n+1} - \Phi_n$ are concave. For example, if Φ is twice differentiable, then a possible choice is

$$\Phi_n(r) = r\big(\Phi'(0) - \Phi'(n)\big) + \int_0^r (r-s)\Phi''(s)\mathbb{1}_{s\leq n}(s)\, ds.$$

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