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Multibump solutions for a class of Lagrangian systems slowly oscillating at infinity

by

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ABSTRACT. – We prove the existence of infinitely many homoclinic solutions for a class of second order hamiltonian systems of the form $-\ddot{u} + u = \alpha(t)\nabla W(u)$ where W is superquadratic and $\dot{\alpha}(t) \rightarrow 0$, $0 < \liminf \alpha(t) < \limsup \alpha(t)$ as $t \rightarrow +\infty$. In fact we prove that such a kind of systems admit a "multibump" dynamics. © Elsevier, Paris

Key words: Lagrangian systems, homoclinic orbits, multibump solutions, minimax arguments.

RÉSUMÉ. – On montre l'existence d'une infinité de solutions homoclines d'une classe de systèmes hamiltoniens du second ordre de la forme $-\ddot{u} + u = \alpha(t)\nabla W(u)$ où W est superquadratique et $\dot{\alpha}(t) \rightarrow 0$, $0 < \liminf \alpha(t) < \limsup \alpha(t)$ quand $t \rightarrow +\infty$. On montre en particulier que cette famille des systèmes admet une dynamique "multi-bosses". © Elsevier, Paris

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1. INTRODUCTION

In this paper we consider the class of Lagrangian systems

$$-\ddot{u} + u = \alpha(t)\nabla W(u), \quad t \in \mathbf{R},\ u \in \mathbf{R}^N, \tag{L}$$

where we assume

(H_1) $\alpha \in \mathcal{C}^1(\mathbf{R},\mathbf{R})$, $W \in \mathcal{C}^2(\mathbf{R}^N,\mathbf{R})$,

(H_2) there exists $\theta > 2$ such that $0 < \theta W(x) \leq \nabla W(x)x$ for any $x \in \mathbf{R}^N \setminus \{0\}$,

(H_3) $\nabla W(x)x < \nabla^2 W(x)xx$ for any $x \in \mathbf{R}^N \setminus \{0\}$,

(H_4) there exist $\overline{a}$ and $\underline{a} > 0$ such that $\overline{a} \geq \alpha(t) \geq \underline{a}$ for any $t \in \mathbf{R}$,

(H_5) $\underline{\alpha} = \liminf_{t\to+\infty} \alpha(t) < \limsup_{t\to+\infty} \alpha(t) = \overline{\alpha}$ and $\lim_{t\to+\infty} \dot{\alpha}(t) = 0$.

By (H_2) it follows in particular that $\nabla^2 W(0) = 0$ and therefore that the origin in the phase space is a hyperbolic rest point for (L). We look for homoclinic solutions to the origin, i.e. solutions u of (L) such that $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$.

In the recent years, starting with [7], [12] and [23], the homoclinic problem for Hamiltonian systems has been tackled via variational methods by several authors. The variational approach has permitted to study systems with different time dependence of the Hamiltonian. We mention [7], [12], [23], [17], [27], [14], [28], [5], [20], [9], [8], [11], [25], [22] for the periodic and asymptotically periodic case, [6], [29], [13], [24], [21] for the almost periodic and recurrent case.

In these papers different existence and multiplicity results are obtained. Starting from [28], the variational methods have been used to prove shadowing like lemmas and consequently to show the existence of a class of solutions, called multibump solutions, whose presence displays a chaotic dynamics. Such results are always proved assuming some nondegeneracy conditions on the set of "generating" homoclinic solutions which are in general difficult to check. However we quote [5], [8], [11], [25] and [22] where the existence of a multibump dynamics is proved under conditions more general than the classical assumption of transversality between the stable and unstable manifolds to the origin (see e.g. [30]).

In this paper we consider a time behaviour of the Lagrangian different from the ones considered in the papers mentioned above (we refer to [1] for a first study of this kind of systems). This assumption allows us to prove the existence of a multibump dynamics without any others conditions. In fact we prove

THEOREM 1.1. – *If $(H_1) - (H_5)$ hold then (L) admits infinitely many multibump solutions. More precisely there exists $\bar{\delta} > 0$, a sequence of disjoint intervals (Q_j) in $\mathbf{R}^+$ with $|Q_j| \to +\infty$ and an increasing sequence of indices $(\hat{\jmath}_n)$ such that given any increasing sequence of indices (j_n) with $j_i \geq \hat{\jmath}_i$ $(i \in \mathbf{N})$ and $\sigma \in \{0,1\}^{\mathbf{N}}$ there exists $u_{j,\sigma} \in \mathcal{C}^2(\mathbf{R}, \mathbf{R}^N)$ solution of (L) verifying:*

(i) $|u_{j,\sigma}(t)| < \frac{\bar{\delta}}{2}$ *for all* $t \in \mathbf{R} \setminus \cup_{\{i \mid \sigma_i = 1\}} Q_{j_i}$,

(ii) $\|u_{j,\sigma}\|_{L^\infty(Q_{j_i})} \geq \bar{\delta}$ *if* $\sigma_i = 1$.

In addition $u_{j,\sigma}$ is a homoclinic solution of (L) whenever $\sigma_i = 0$ definitively.

Our proof use variational techniques and it is based on a localization procedure related to the time dependence of the Lagrangian. In fact we note that even if the action functional satisfies the geometrical assumptions of the Mountain Pass Theorem, there are simple cases in which there are not Palais Smale convergent sequences at the mountain pass level. However, thanks to the slow oscillations of the Lagrangian at $+\infty$, we can use localized mountain pass classes related to the mountain pass classes of the limit problems at $+\infty$. The use of this localization procedure with a careful analysis of the compactness properties of the action functional give rise to the existence of infinitely many homoclinic solutions. These solutions turn out to be well characterized from the variational point of view and in a certain sense non degenerate. Then to prove theorem 1.1 we can use a product minimax construction somewhat related to the ones used in [28] and [10].

Finally we point out that our construction is possible since the "masses" of the solutions of (L) concentrate in a suitable sense with respect to the slow oscillations of the Lagrangian. In very recent papers, see [2], [15] and [16], it is studied the problem of existence and multiplicity of semiclassical states for nonlinear Schrödinger equations where analogous concentration phenomena occur. In fact with minor changes our proof can be adapted to study also this class of equations.

The current paper is organized as follows. In sections 2 and 3 we state some preliminary results. In section 4 we define the localized minimax classes which we use to prove the existence of infinitely many one-bump homoclinic solutions. The proof of Theorem 1.1 is contained in section 5.

2. VARIATIONAL SETTING AND PRELIMINARY RESULTS

We look for homoclinic solutions of (L) as critical points of the action functional

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}} \alpha(t)W(u)dt,$$

defined on the Sobolev space $X = H^1(\mathbf{R}, \mathbf{R}^N)$ endowed with the scalar product $\langle u, v\rangle = \int_{\mathbf{R}}(\dot{u}\dot{v} + uv)dt$ and the Euclidean norm $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$. In fact it is standard to check that $\varphi \in \mathcal{C}^2(X, \mathbf{R})$ and

$$\varphi'(u)v = \langle u, v\rangle - \int_{\mathbf{R}} \alpha(t)\nabla W(u)v dt, \quad \forall u,\ v \in X$$

so that the critical points of φ are weak and then classical homoclinic solutions of (L) (see e.g. [23]).

In the sequel we will collect some preliminary properties of φ that are standard in almost every paper on homoclinic solutions via variational methods.

First note that the origin in X is a strict local minimum for the functional φ. Indeed by (H_2) there results $\nabla^2 W(0) = 0$ and so, since α is bounded, we can fix $\bar{\delta} > 0$ such that $|\alpha(t)\nabla^2 W(x)| \leq \frac{1}{4}$ for all $t \in \mathbf{R}$ and $x \in \mathbf{R}^N$ with $|x| \leq \bar{\delta}$. In particular this implies that $|\alpha(t)\nabla W(x)| \leq \frac{1}{4}|x|$ and $|\alpha(t)W(x)| \leq \frac{1}{8}|x|^2$ for all $t \in \mathbf{R}$ and $x \in \mathbf{R}^N$ with $|x| \leq \bar{\delta}$. Then we obtain

LEMMA 2.1. – *If* $\|u\|_{L^\infty} \leq \bar{\delta}$ *then* $\varphi(u) \geq \frac{1}{4}\|u\|^2$ *and* $\varphi'(u)u \geq \frac{1}{2}\|u\|^2$.

By the Sobolev Immersion Theorem we can fix $\bar{r} > 0$ such that if I is an interval in $\mathbf{R}$ with $|I| \geq 1$ (where $|I|$ denotes the length of I) then

$$\text{if} \quad \|u\|_I < \bar{r} \quad \text{then} \quad \|u\|_{L^\infty(I)} < \frac{\bar{\delta}}{2}, \tag{2.1}$$

where $\|u\|_I^2 = \int_I(|\dot{u}|^2 + |u|^2)dt$. We denote $r_0 = \frac{\bar{r}}{16}$.

The functional φ does not satisfy the Palais Smale condition. However, thanks to (H_2), we have that

$$\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|^2 \leq \varphi(u) - \frac{1}{\theta}\varphi'(u)u, \quad \forall u \in X. \tag{2.2}$$

Therefore if (u_n) is a Palais Smale (PS for short) sequence for φ at level b, i.e. $\varphi(u_n) \to b$ and $\varphi'(u_n) \to 0$, then (u_n) is bounded in X. Furthermore, by Lemma 2.1, if (u_n) is a PS sequence and $\|u_n\| \leq \bar{r}$ then $u_n \to 0$ in X. By (2.2) this implies:

LEMMA 2.2. – *If* (u_n) *is a* PS *sequence for* φ *at level* b *then either* $b = 0$ *or* $b \geq \bar{\lambda}$, *where* $\bar{\lambda} = (\frac{1}{2} - \frac{1}{\theta})\bar{r}^2$. *Moreover if* $b = 0$ *then* $u_n \to 0$.

We recall that $\varphi' : X \to X$ is weakly continuous. Moreover, setting $\mathcal{K} = \{u \in X \setminus \{0\} \mid \varphi'(u) = 0\}$, arguing as in [14] we obtain:

LEMMA 2.3. – *If (u_n) is a PS sequence for φ at level b then there exists $v \in \mathcal{K} \cup \{0\}$ such that up to a subsequence $u_n \rightharpoonup v$ weakly in X. Moreover $(u_n - v)$ is a PS sequence for φ at level $b - \varphi(v)$.*

By Lemma 2.1, in the spirit of concentration compactness Lemma ([18]) it can be proved that we lose compactness of those PS sequences (u_n) which carry "mass at infinity", in the sense that there exists a sequence (t_n) in $\mathbf{R}$ such that $|t_n| \to \infty$ and $\liminf_{n\to\infty} |u_n(t_n)| \geq \bar{\delta}$.

In order to well describe the behaviour of these PS sequences, and therefore to obtain compactness results, it is useful to introduce the function $T^+ : X \to \mathbf{R} \cup \{-\infty\}$ given by:

$$T^+(u) = \begin{cases} \sup\{t \in \mathbf{R} \mid |u(t)| \geq \bar{\delta}\}, & \text{if } \|u\|_{L^\infty} \geq \bar{\delta}, \\ -\infty\ , & \text{otherwise.} \end{cases}$$

This function is not continuous in X but the following property holds (see e.g. [22]):

LEMMA 2.4. – *If (u_n) is a PS sequence and $(T^+(u_n))$ is bounded in $\mathbf{R}$ then, up to a subsequence, $u_n \rightharpoonup v \in \mathcal{K}$ weakly in X and $T^+(u_n) \to T^+(v)$.*

3. PROBLEMS "AT INFINITY" AND RELATED COMPACTNESS PROPERTIES

In this section we will investigate the lack of compactness of those PS sequences which carry mass at $+\infty$, more precisely PS sequences (u_n) such that $T^+(u_n) \to +\infty$. First we note that by (H_5) such kind of sequences can be characterized in terms of the limit autonomous problems at $+\infty$ associated to (L). More precisely, given $\beta \in [\underline{\alpha}, \overline{\alpha}]$ and considered the functional

$$\varphi_\beta(u) = \tfrac{1}{2}\|u\|^2 - \beta \int_{\mathbf{R}} W(u)dt, \quad \forall u \in X,$$

we have that if (u_n) is a PS sequence with $T^+(u_n) \to +\infty$ then, up to a subsequence, $u_n(\cdot + T^+(u_n)) \rightharpoonup v_\beta$ weakly in X where v_β is a critical point for φ_β, for some $\beta \in [\underline{\alpha}, \overline{\alpha}]$.

We recall some properties of the functionals φ_β.

First note that all the functionals φ_β, as the functional φ, satisfy by (H_2) and (H_4) the geometric assumptions of the Mountain Pass Theorem. Then, setting $\Gamma_\beta = \{\gamma \in \mathcal{C}^0([0,1], X) \mid \gamma(0) = 0,\ \varphi_\beta(\gamma(1)) < 0\}$, we have

$$c_\beta = \inf_{\gamma \in \Gamma_\beta} \sup_{s \in [0,1]} \varphi_\beta(\gamma(s)) > 0.$$

We remark that c_β is a critical level for φ_β (see e.g. [3] and [26]). Moreover, by (H_3), given $v_\beta \in \mathcal{K}_\beta = \{u \in X\setminus\{0\} \mid \varphi'_\beta(u) = 0\}$ and $s_0 \in \mathbf{R}$ such that $\varphi_\beta(s_0 v_\beta) < 0$, if we define $\gamma_\beta(s) = s s_0 v_\beta$ for all $s \in [0,1]$ then we have

LEMMA 3.1. – *For any $v_\beta \in \mathcal{K}_\beta$ there results $\gamma_\beta \in \Gamma_\beta$ and*

(i) $\max_{s\in[0,1]} \varphi_\beta(\gamma_\beta(s)) = \varphi_\beta(v_\beta)$,

(ii) *for any $r > 0$ there exists $h_r > 0$ such that if $\gamma_\beta(s) \in X \setminus B_r(v_\beta)$ then $\varphi_\beta(\gamma_\beta(s)) < \varphi_\beta(v_\beta) - h_r$,*

where $B_r(u) = \{v \in X \mid \|v - u\| < r\}$.

In particular it follows that the critical points of φ_β at the level c_β are mountain pass critical points of φ_β. Moreover we have

LEMMA 3.2. – *For any $\beta \in [\underline{\alpha}, \overline{\alpha}]$ there results $c_\beta = \min_{u\in\mathcal{K}_\beta} \varphi_\beta(u)$.*

As shown in [1] it is easy to see that the function $\beta \to c_\beta$ is strictly monotone. More precisely:

LEMMA 3.3. – *If $\beta_1 < \beta_2$ then $c_{\beta_1} > c_{\beta_2}$.*

In particular we have

$$c_{\overline{\alpha}} = \min_{\beta\in[\underline{\alpha},\overline{\alpha}]} c_\beta = \min_{\beta\in[\underline{\alpha},\overline{\alpha}]} \min_{u\in\mathcal{K}_\beta} \varphi_\beta(u). \tag{3.1}$$

Finally note that the functionals φ_β are invariant under traslations, i.e. $\varphi_\beta(u) = \varphi_\beta(u(\cdot + \tau))$ and $\|\varphi'_\beta(u)\| = \|\varphi'_\beta(u(\cdot + \tau))\|$ for all $u \in X$ and $\tau \in \mathbf{R}$.

Using arguments similar to the ones used in [1] to characterize the asymptotic behaviour of the *PS* sequences (see also [21]), it can be proved the following result:

LEMMA 3.4. – *Let (u_n) be a PS sequence for φ at level b with $T^+(u_n) \to +\infty$. Then there exist $\beta \in [\underline{\alpha}, \overline{\alpha}]$ and $v_\beta \in \mathcal{K}_\beta$ such that, up to a subsequence, there results:*

(i) $\alpha(T^+(u_n)) \to \beta$ *and*

(ii) $u_n(\cdot + T^+(u_n)) \rightharpoonup v_\beta$ *weakly in X.*

Moreover $(u_n - v_\beta(\cdot - T^+(u_n)))$ is a PS sequence for φ at level $b - \varphi_\beta(v_\beta)$.

Using Lemma 3.4 and (3.1) we obtain:

LEMMA 3.5. – *For any $h > 0$ there exists $T > 0$ such that if (u_n) is a PS sequence for φ at level $b > 0$ with $T^+(u_n) \geq T$ for all $n \in \mathbf{N}$ then $b \geq c_{\overline{\alpha}} - h$.*

Proof. – Arguing by contradiction, suppose that there exist $h > 0$ and a PS sequence (u_n) for φ with $T^+(u_n) \to +\infty$ at level b less than $c_{\overline{\alpha}} - h$.

By Lemma 3.4, we have that there exist $\beta \in [\underline{\alpha}, \overline{\alpha}]$ and $v_\beta \in \mathcal{K}_\beta$ such that, up to a subsequence, $u_n(\cdot + T^+(u_n)) \rightharpoonup v_\beta$ and $(u_n - v_\beta(\cdot - T^+(u_n)))$ is a PS sequence for φ at level $b - \varphi_\beta(v_\beta)$. By ((3.1)) we have $\varphi_\beta(v_\beta) \geq c_{\overline{\alpha}}$ and then $b - \varphi_\beta(v_\beta) \leq b - c_{\overline{\alpha}} \leq -h$ in contradiction with Lemma 2.2. ∎

Using the previous results we obtain the following compactness property for φ.

LEMMA 3.6. – *There exist $h_0 > 0$ and $T_0 > 0$ such that for any PS sequence (u_n) for φ at level b strictly less than $c_{\overline{\alpha}} + h_0$ with $T^+(u_n) \geq T_0$ we have:*

(*i*) *if $(T^+(u_n))$ is unbounded then there exist $\beta \in [\underline{\alpha}, \overline{\alpha}]$ and $v_\beta \in \mathcal{K}_\beta$ such that, up to a subsequence, $\alpha(T^+(u_n)) \to \beta$, $u_n(\cdot + T^+(u_n)) \to v_\beta$ strongly in X and $b = \varphi_\beta(v_\beta)$,*

(*ii*) *if $(T^+(u_n))$ is bounded then there exists $v \in \mathcal{K}$ such that, up to a subsequence, $u_n \to v$ strongly in X.*

Proof. – Fix $h_0 \in (0, \frac{\bar{\lambda}}{2})$, where $\bar{\lambda}$ is given in Lemma 2.2. Corresponding to this value h_0 fix $T_0 > 0$ using Lemma 3.5.

To prove (i) suppose that $T^+(u_n) \to +\infty$. Then by Lemma 3.4 we have that there exist $\beta \in [\underline{\alpha}, \overline{\alpha}]$ and $v_\beta \in \mathcal{K}_\beta$ such that, up to a subsequence, $u_n(\cdot + T^+(u_n)) \rightharpoonup v_\beta$. Moreover setting $v_n = u_n - v_\beta(\cdot - T^+(u_n))$ we have that (v_n) is a PS sequence for φ at level $b - \varphi_\beta(v_\beta)$. By (3.1) we have

$$b - \varphi_\beta(v_\beta) \leq b - c_{\overline{\alpha}} < h_0$$

and therefore by the choice of h_0 and Lemma 2.2 we obtain $v_n \to 0$ strongly in X and $b - \varphi_\beta(v_\beta) = 0$, i.e. $u_n(\cdot + T^+(u_n)) \to v_\beta$ and (i) holds.

To prove (ii) suppose that $(T^+(u_n))$ is bounded and $T^+(u_n) \geq T_0$ for all $n \in \mathbf{N}$. Then by Lemmas 2.3 and 2.4 we have that, up to a subsequence, $u_n \rightharpoonup v \in \mathcal{K}$, $T^+(v) \geq T_0$ and $(u_n - v)$ is a PS sequence at level $b - \varphi(v)$. Lemma 3.5 in particular implies that $\varphi(v) \geq c_{\overline{\alpha}} - h_0$. Then, by the choice of h_0, we have

$$b - \varphi(v) \leq b - c_{\overline{\alpha}} + h_0 < 2h_0 < \bar{\lambda}$$

and therefore by Lemma 2.2 we obtain $u_n \to v$ strongly in X. ∎

In particular the following result holds.

LEMMA 3.7. – *There exist $\nu_0 > 0$ and $R_0 > 0$ such that for all $u \in X$ with $\|\varphi'(u)\| < \nu_0$, $T^+(u) \geq T_0$ and $\varphi(u) < c_{\overline{\alpha}} + h_0$ we have*

$$\|u\|_{|t - T^+(u)| > R_0} < r_0.$$

Proof. – Arguing by contradiction suppose that there exists a PS sequence (u_n) in X such that $\varphi(u_n) < c_{\overline{\alpha}} + h_0$, $T^+(u_n) \geq T_0$ and there exists a sequence $(R_n) \subset \mathbf{R}$ such that $R_n \to +\infty$ and

$$\|u_n\|_{|t-T^+(u_n)|>R_n} \geq r_0.$$

This is impossible since Lemma 3.6 implies that, up to a subsequence, $u_n(\cdot + T^+(u_n)) \to v$ in X and then $\|u_n(\cdot + T^+(u_n))\|_{|t|>R_n} \to 0$. ∎

Remark 3.1. – By (3.2) we can fix $M_0 > 0$ such that if $\varphi(u) < c_{\overline{\alpha}} + h_0$ and $\|\varphi'(u)\| < \nu_0$ then $\|u\| < M_0$, where h_0 and ν_0 are given in Lemma 3.6 and Lemma 3.7.

4. EXISTENCE OF INFINITELY MANY ONE BUMP SOLUTIONS

In this section we will prove the existence of infinitely many critical points for φ. Using assumption (H_5) and Lemma 3.7 we will select infinitely many regions in X in which the functional φ is close to $\varphi_{\overline{\alpha}}$ and in which we will look for critical points of φ near critical points of $\varphi_{\overline{\alpha}}$.

First of all we need to state some preliminary properties of the functional φ which are essentially due to (H_5).

Remark 4.1. – Note that by (H_5) we can select a sequence of intervals in which $\alpha(t)$ is close to $\overline{\alpha}$. More precisely, fixed $\varepsilon_0 \in (0, \frac{\overline{\alpha}-\underline{\alpha}}{2})$ and any sequence $\varepsilon_j \to 0$ there exists a sequence (τ_j) in $\mathbf{R}$ such that $\tau_j \to +\infty$ and $\alpha(\tau_j) \to \overline{\alpha}$ as $j \to \infty$. Moreover there exist $(\tau_j^{\pm})$ and $(\sigma_j^{\pm})$ sequences in $\mathbf{R}$ such that for all $j \in \mathbf{N}$ there results:

(i) $\sigma_j^- < \tau_j^- < \tau_j < \tau_j^+ < \sigma_j^+$ and $\tau_j^{\pm} \to +\infty$, $\sigma_j^{\pm} \to +\infty$, $\tau_j^+ - \tau_j^- \to +\infty$, $0 < \sigma_{j+1}^- - \sigma_j^+ \to +\infty$ and $|\sigma_j^{\pm} - \tau_j^{\pm}| \to +\infty$ as $j \to \infty$;

(ii) $\alpha(t) \leq \overline{\alpha} + \varepsilon_j$ for all $t \in [\sigma_j^-, \sigma_j^+]$;

(iii) $\alpha(t) \leq \overline{\alpha} - \varepsilon_0$ for all $t \in [\sigma_j^-, \tau_j^-] \cup [\tau_j^+, \sigma_j^+]$.

In the sequel we will denote $P_j = [\sigma_j^-, \sigma_j^+]$ and $Q_j = [\tau_j^-, \tau_j^+]$.

Moreover, considered T_0 and R_0 given in Lemmas 3.6 and 3.7 respectively, since $\dot{\alpha}(t) \to 0$ as $t \to +\infty$, we have that there exists $j_0 \in \mathbf{N}$ such that for all $j \geq j_0$ we have $\sigma_j^- \geq T_0$ and $\alpha(t) \leq \overline{\alpha} - \frac{\varepsilon_0}{2}$ for all $t \in [\sigma_j^- - R_0, \tau_j^- + R_0] \cup [\tau_j^+ - R_0, \sigma_j^+ + R_0]$.

Given any $h > 0$ and $\nu > 0$, for all $j \in \mathbf{N}$ define

$$\mathcal{A}_j(h, \nu) = \{u \in X \mid \varphi(u) \leq c_{\overline{\alpha}} + h,\ \|\varphi'(u)\| < \nu \text{ and } T^+(u) \in Q_j\}.$$

Then, using Lemma 3.7 we obtain

LEMMA 4.1. – *There exist $\bar{h} \in (0, h_0)$, $\bar{\nu} \in (0, \nu_0)$ and $\bar{\jmath} \geq j_0$ such that if $u \in \mathcal{A}_j(\bar{h}, \bar{\nu})$ for some $j \geq \bar{\jmath}$ then $\|u\|_{\mathbf{R}\setminus Q_j} < r_0$. In particular $\|u\|_{L^\infty(\mathbf{R}\setminus Q_j)} < \frac{\bar{\delta}}{2}$.*

Proof. – Arguing by contradiction, suppose that there exist $h_n \to 0$, $\nu_n \to 0$, $j_n \to \infty$ and $u_n \in \mathcal{A}_{j_n}(h_n, \nu_n)$ such that

$$\|u_n\|_{\mathbf{R}\setminus Q_{j_n}} \geq r_0 \quad \forall n \in \mathbf{N}. \tag{4.1}$$

Then in particular (u_n) is a PS sequence for φ at level less than or equal to $c_{\overline{\alpha}}$ with $T^+(u_n) \to +\infty$. By Lemma 3.7 and (4.1), since $T^+(u_n) \in Q_{j_n}$, we have

$$\inf\{|T^+(u_n) - t| \mid t \in P_{j_n} \setminus Q_{j_n}\} < R_0.$$

Therefore by Remark 4.1, up to a subsequence, we have $\alpha(T^+(u_n)) \to \beta \in [\underline{\alpha}, \overline{\alpha} - \frac{\varepsilon_0}{2}]$ and, by Lemma 3.6 (i), $u_n(\cdot + T^+(u_n)) \to v_\beta \in \mathcal{K}_\beta$. Then, by Lemma 3.3, $\varphi(u_n) \to \varphi_\beta(v_\beta) \geq c_\beta > c_{\overline{\alpha}}$, a contradiction.

In particular, by (2.1), we obtain that $\|u\|_{L^\infty(\mathbf{R}\setminus Q_j)} < \frac{\bar{\delta}}{2}$. ∎

From now on we will denote $\mathcal{A}_j = \mathcal{A}_j(\bar{h}, \bar{\nu})$. Note that it is not restrictive to assume $\bar{\nu} < \frac{\bar{r}}{2}$. Then, setting $B_{\bar{r}}(\mathcal{A}_j) = \{u \in X \mid \inf_{v\in\mathcal{A}_j} \|u - v\| < \bar{r}\}$, we have

LEMMA 4.2. – *If $u \in B_{\bar{r}}(\mathcal{A}_j) \setminus \mathcal{A}_j$ for some $j \geq \bar{\jmath}$ and $\varphi(u) \leq c_{\overline{\alpha}} + \bar{h}$ then*

$$\|\varphi'(u)\| \geq \bar{\nu}, \quad \textit{and} \quad \|u\|_{L^\infty(\mathbf{R}\setminus Q_j)} < \bar{\delta}.$$

Proof. – By Lemma 4.1 if $v \in \mathcal{A}_j$ for some $j \geq \bar{\jmath}$ then $|v(t)| < \frac{\bar{\delta}}{2}$ for all $t \notin Q_j$. By the choice of $\bar{r}$, this implies that if $u \in B_{\bar{r}}(\mathcal{A}_j)$ then $|u(t)| < \bar{\delta}$ for all $t \notin Q_j$. In particular it follows that either $T^+(u) \in Q_j$ or $T^+(u) = -\infty$. In the first case if $u \notin \mathcal{A}_j$, we get that $\|\varphi'(u)\| \geq \bar{\nu}$, by definition of $\mathcal{A}_j$. In the second case we have $\|u\|_\infty < \bar{\delta}$ and then, by Lemma 2.1, we obtain $\|\varphi'(u)\| \geq \frac{1}{2}\|u\|$. This prove the lemma since if $u \in B_{\bar{r}}(\mathcal{A}_j)$ then $\|u\| \geq \bar{r} > 2\bar{\nu}$. Indeed if $v \in \mathcal{A}_j$ then $T^+(v) \in Q_j$ which implies $\|v\|_\infty \geq \bar{\delta}$. Then by (2.1) we get $\inf_{\mathcal{A}_j} \|v\| \geq 2\bar{r}$ from which $\inf_{B_{\bar{r}}(\mathcal{A}_j)} \|u\| \geq \bar{r}$. ∎

Now we introduce a sequence of mountain pass classes for φ "located" in $\mathcal{A}_j$. First we fix some notation.

Let $\gamma_{\overline{\alpha}}$ be the mountain pass path for $\varphi_{\overline{\alpha}}$ corresponding (as in Lemma 3.1) to some fixed critical point $v_{\overline{\alpha}} \in \mathcal{K}_{\overline{\alpha}}$ with $T^+(v_\beta) = 0$ and $\varphi_\beta(v_\beta) = c_\beta$.

In the sequel we will denote by γ_j the path given by $\gamma_j(s) = \gamma_{\overline{\alpha}}(s)(\cdot - \tau_j)$ for all $s \in [0,1]$, where (τ_j) is the sequence given in Remark 4.1.

Remark 4.2. – Let $M > 2M_0$ (M_0 given in Remark 3.1) be such that $M \geq 2\|\gamma_{\overline{\alpha}}(s)\|$ for all $s \in [0,1]$. Since W is locally Lipschitz continuous, we can fix $K_M > 0$ such that $W(x) \leq K_M |x|^2$ for all $|x| \leq M$.

We define a sequence (Γ_j) of local mountain pass classes for φ and the corresponding sequence of mountain pass levels (c_j) by setting

$$\Gamma_j = \{\gamma \in \mathcal{C}^0([0,1], X) \mid \gamma(0) = 0,\ \varphi(\gamma(1)) < \frac{1}{2}\varphi_{\overline{\alpha}}(\gamma_{\overline{\alpha}}(1)),$$

$$|\gamma(s)(t)| \leq \overline{\delta}\ \forall t \notin Q_j \text{ and } \|\gamma(s)\| \leq M\ \forall s \in [0,1]\}$$

and

$$c_j = \inf_{\gamma \in \Gamma_j} \sup_{s \in [0,1]} \varphi(\gamma(s))$$

for all $j \in \mathbf{N}$, where $\overline{\delta}$ is given by Lemma 2.1 and M by Remark 4.2.

By construction we obtain that the sequence (c_j) converges to the mountain pass level $c_{\overline{\alpha}}$ for $\varphi_{\overline{\alpha}}$.

LEMMA 4.3. – *There results* $\lim_{j\to\infty} c_j = c_{\overline{\alpha}}$ *and in particular*

$$\lim_{j\to\infty} \max_{s\in[0,1]} |\varphi(\gamma_j(s)) - \varphi_{\overline{\alpha}}(\gamma_j(s))| = 0.$$

Proof. – Let $h > 0$ be fixed.
By (H_2) there exists $\delta_h \in (0, \overline{\delta})$ such that if $\|u\| \leq M$ then

$$\int_{\{t\in\mathbf{R} \mid |u(t)|\leq\delta_h\}} W(u)dt \leq \frac{h}{4a} \tag{4.2}$$

where $a = \sup_{\mathbf{R}} \alpha(t)$.

Moreover, since $\gamma_{\overline{\alpha}}([0,1])$ is compact in X, there exists $R_h > 0$ such that

$$\sup_{|t|\geq R_h} |\gamma_{\overline{\alpha}}(s)(t)| \leq \delta_h, \quad \forall s \in [0,1]. \tag{4.3}$$

By Remark 4.1 there exists $j_1 = j_1(h) \in \mathbf{N}$ such that for all $j \geq j_1$ we have $[\tau_j - R_h, \tau_j + R_h] \subset Q_j$ and

$$\sup_{t\in[\tau_j - R_h, \tau_j + R_h]} |\alpha(t) - \overline{\alpha}| < \frac{h}{2K_M M^2}. \tag{4.4}$$

Therefore for all $j \geq j_1$ and $s \in [0,1]$, using (4.2), (4.3), (4.4) and Remark 4.2 we obtain

$$|\varphi(\gamma_j(s)) - \varphi_{\overline{\alpha}}(\gamma_j(s))| \leq \int_{|t-\tau_j|>R_h} |\overline{\alpha} - \alpha(t)| W(\gamma_j(s))dt + \\ + \int_{|t-\tau_j|\leq R_h} |\overline{\alpha} - \alpha(t)| W(\gamma_j(s))dt \leq h.$$

Then in particular $\varphi(\gamma_j(1)) \leq \varphi_{\overline{\alpha}}(\gamma_j(1)) + h < \frac{1}{2}\varphi_{\overline{\alpha}}(\gamma_{\overline{\alpha}}(1))$ if h is small enough. Hence by definition of γ_j and (4.3), we have $\gamma_j \in \Gamma_j$ for all $j \geq j_1$ and then

$$c_j \leq \varphi(\gamma_j(s)) \leq \varphi_{\overline{\alpha}}(\gamma_j(s)) + h, \quad \forall s \in [0,1].$$

By definitions of γ_j this proves that $c_j \leq c_{\overline{\alpha}} + h$ for all $j \geq j_1$.

Now to prove that definitively $c_j \geq c_{\overline{\alpha}} - h$ we introduce the following minimum problem. Fixed any $\tau \in \mathbf{R}$ and $x \in \mathbf{R}^N$ such that $|x| \leq \bar{\delta}$, we set $\mathbf{R}_\tau^+ = [\tau, +\infty)$ and $\mathbf{R}_\tau^- = (-\infty, \tau]$. Define

$$\varphi_{\tau^\pm}(u) = \frac{1}{2}\|u\|^2_{H^1(\mathbf{R}_\tau^\pm)} - \int_{\mathbf{R}_\tau^\pm} \alpha(t) W(u(t))dt$$

and

$$\mathcal{U}_{\tau^\pm,x} = \{u \in H^1(\mathbf{R}_\tau^\pm) \mid u(\tau) = x, \; \|u\|_{L^\infty(\mathbf{R}_\tau^\pm)} \leq \bar{\delta}\}.$$

The minimum problem

$$\min\{\varphi_{\tau^\pm}(u) \mid u \in \mathcal{U}_{\tau^\pm,x}\}$$

admits a unique solution $u_{\tau^\pm,x}$ for any $\tau \in \mathbf{R}$ and $|x| \leq \bar{\delta}$. Indeed, by the choice of $\bar{\delta}$, we have that $\varphi_{\tau^\pm}$ is strictly convex on the convex set $\mathcal{U}_{\tau^\pm,x}$. Note that $u_{\tau^\pm,x}$ is the unique solution of (L) on $\mathbf{R}_\tau^\pm$ which verifies the conditions $u_{\tau^\pm,x}(\tau) = x$ and $\|u_{\tau^\pm,x}\|_{L^\infty(\mathbf{R}_\tau^\pm)} \leq \bar{\delta}$. Then, by the maximum principle, we infer that for any $\tau \in \mathbf{R}$ and $|x| \leq \bar{\delta}$ there results

$$|u_{\tau^\pm,x}(t)| \leq \bar{\delta} e^{-\frac{|t-\tau|}{4}}, \quad \forall t \in \mathbf{R}_\tau^\pm.$$

It follows that there exists $r_h > 0$ such that for any $\tau \in \mathbf{R}$ and $|x| \leq \bar{\delta}$ we have

$$|u_{\tau^\pm,x}(t)| \leq \delta_h, \quad \forall t \in \mathbf{R}_\tau^\pm \text{ with } |t-\tau| \geq r_h \tag{4.5}$$

where δ_h is given in (4.2).

Given any $\gamma \in \Gamma_j$ we denote $x^{\pm}(s) = \gamma(s)(\tau_j^{\pm})$ and $u^{\pm}(s)(\cdot) = u_{\tau^{\pm},x^{\pm}(s)}$ for all $s \in [0,1]$. Therefore it is well defined and continuous the path $\tilde{\gamma} : [0,1] \to X$ given by

$$\tilde{\gamma}(s)(t) = \begin{cases} u^-(s)(t), & \text{if } t \le \tau_j^-, \\ \gamma(s)(t), & \text{if } \tau_j^- \le t \le \tau_j^+, \\ u^+(s)(t), & \text{if } \tau_j^+ \le t \end{cases} \quad \forall s \in [0,1].$$

By construction $\varphi(\gamma(s)) \ge \varphi(\tilde{\gamma}(s))$ for any $s \in [0,1]$. Moreover, by (4.5), $|\tilde{\gamma}(s)(t)| \le \delta_h$ for all $t \in \mathbf{R}$ with $t \le \tau_j^- - r_h$ or $t \ge \tau_j^+ + r_h$. Then, since $|\tau_j^{\pm} - \sigma_j^{\pm}| \to \infty$ as $j \to \infty$, we have that there exists $j_2 = j_2(h) \ge j_1$ such that $[\tau_j^- - r_h, \tau_j^+ + r_h] \subset P_j$ for all $j \ge j_2$. Therefore we have

$$|\tilde{\gamma}(s)(t)| \le \delta_h, \quad \forall t \notin P_j,$$

for all $\gamma \in \Gamma_j$ with $j \ge j_2$. Then by (4.2) and the choice of P_j we obtain

$$\begin{aligned} \varphi(\tilde{\gamma}(s)) &= \varphi_{\overline{\alpha}}(\tilde{\gamma}(s)) - \int_{\mathbf{R}} (\alpha(t) - \overline{\alpha}) W(\tilde{\gamma}(s)) dt \\ &\ge \varphi_{\overline{\alpha}}(\tilde{\gamma}(s)) - \varepsilon_j \int_{P_j} W(\tilde{\gamma}(s)) dt - \sup_{\mathbf{R}} |\alpha(t) - \overline{\alpha}| \int_{\mathbf{R} \setminus P_j} W(\tilde{\gamma}(s)) dt \\ &\ge \varphi_{\overline{\alpha}}(\tilde{\gamma}(s)) - \varepsilon_j K_M M^2 - \frac{h}{2} \end{aligned}$$

from which we conclude, since $\varepsilon_j \to 0$, that there exists $j_3 = j_3(h) \ge j_2$ such that

$$\varphi(\gamma(s)) \ge \varphi(\tilde{\gamma}(s)) \ge \varphi_{\overline{\alpha}}(\tilde{\gamma}(s)) - h$$

for all $s \in [0,1]$ and $\gamma \in \Gamma_j$ with $j \ge j_3$. In particular

$$\varphi_{\overline{\alpha}}(\tilde{\gamma}(1)) \le \varphi(\gamma(1)) + h < \frac{1}{2} \varphi_{\overline{\alpha}}(\gamma_{\overline{\alpha}}(1)) + h.$$

Therefore if h is small enough we have $\tilde{\gamma} \in \Gamma_{\overline{\alpha}}$ and then

$$\max_{s \in [0,1]} \varphi(\gamma(s)) \ge \max_{s \in [0,1]} \varphi_{\overline{\alpha}}(\tilde{\gamma}(s)) - h \ge c_{\overline{\alpha}} - h$$

for all $\gamma \in \Gamma_j$ with $j \ge j_3$. Then $c_j \ge c_{\overline{\alpha}} - h$ for all $j \ge j_3$ and the proof is complete. ∎

Remark 4.3. – Note that by the choice of M and Remark 3.1 we have $\mathcal{A}_j \subset \{u \in X \mid \|u\| \le \frac{M}{2}\}$. Therefore we can assume $\overline{r}$ so small that there results $B_{\bar{r}}(\mathcal{A}_j) \subset \{u \in X \mid \|u\| \le M\}$ for all $j \in \mathbf{N}$. Moreover since $\varphi_{\overline{\alpha}}(\gamma_{\overline{\alpha}}(1)) < 0$, we can also assume that $\gamma_j(1) \notin B_{\bar{r}}(\mathcal{A}_j)$ for any $j \ge \bar{\jmath}$.

Now, using deformation arguments, by Lemma 4.2 and Lemma 4.3 we can prove the existence of infinitely many one bump solutions of (L).

THEOREM 4.1. – *There exists $\hat{\jmath} \in \mathbf{N}$ such that $\mathcal{K} \cap \mathcal{A}_j \ne \emptyset$ for all $j \ge \hat{\jmath}$.*

Proof. – For any $j \ge \bar{\jmath}$, let $\eta_j : [0,1] \times X \to X$ be the flow associated to the Cauchy problem

$$\begin{cases} \frac{d}{dt}\eta_j(t,u) = -\psi(\eta_j(t,u))\frac{\inf(1,\|\varphi'(\eta_j(t,u))\|)}{\|\varphi'(\eta_j(t,u))\|}\varphi'(\eta_j(t,u)), \\ \eta_j(0,u) = u, \quad \forall u \in X, \end{cases}$$

where $\psi : X \to [0,1]$ is a locally Lipschitz continuous function such that $\psi(u) = 1$ for all $u \in B_{\frac{\bar{r}}{2}}(\mathcal{A}_j)$ and $\psi(u) = 0$ for all $u \in X \setminus B_{\bar{r}}(\mathcal{A}_j)$. It is standard to check that φ decreases along the flow lines and moreover that $X \setminus B_{\bar{r}}(\mathcal{A}_j)$ is invariant under η_j. By Lemma 4.2 and Remark 4.3 this implies in particular that the class Γ_j is invariant under the flow η_j, i.e. for all $\gamma \in \Gamma_j$ and for all $t > 0$ we have $\eta_j(t, \gamma(\cdot)) \in \Gamma_j$.
Furthermore by Lemma 4.3 if $u \in B_{\frac{\bar{r}}{4}}(\mathcal{A}_j)$ and there exists $t > 0$ such that $\eta_j(t,u) \notin B_{\frac{\bar{r}}{2}}(\mathcal{A}_j)$ then

$$\varphi(\eta_j(t,u)) \le \varphi(u) - \frac{\bar{r}\bar{\nu}}{4}. \tag{4.6}$$

By Lemma 4.3 and Lemma 3.1 for any $h \in (0, \frac{1}{2}\Delta_{\bar{r}})$, where $\Delta_{\bar{r}} = \min(\frac{\bar{r}\bar{\nu}}{4}, h_{\frac{\bar{r}}{4}})$ and $h_{\frac{\bar{r}}{4}}$ is given in Lemma 3.1 (ii), there exists $\hat{\jmath} \ge \bar{\jmath}$ such that for all $j \ge \hat{\jmath}$ we have $\gamma_j \in \Gamma_j$ and moreover:

(i) if $\gamma_j(s) \notin B_{\frac{\bar{r}}{4}}(\mathcal{A}_j)$ then $\varphi(\gamma_j(s)) < c_j - h$,
(ii) $\max_{s \in [0,1]} \varphi(\gamma_j(s)) \le c_j + h$.

We claim that for all $j \ge \hat{\jmath}$ there exists $s_j \in [0,1]$ such that $\gamma_j(s_j) \in B_{\frac{\bar{r}}{4}}(\mathcal{A}_j) \cap \{\varphi > c_j - h\}$ and for all $t \ge 0$ there results:

(a) $\varphi(\eta_j(t, \gamma_j(s_j))) > c_j - h$,
(b) $\eta_j(t, \gamma_j(s_j)) \in B_{\frac{\bar{r}}{2}}(\mathcal{A}_j)$.

From the claim we derive that for all $j \ge \hat{\jmath}$ there exists a PS sequence (u_n^j) for φ in $B_{\frac{\bar{r}}{2}}(\mathcal{A}_j)$. In particular, since $(u_n^j) \subset B_{\frac{\bar{r}}{2}}(\mathcal{A}_j)$ by Lemma 4.2 we have that $u_n^j \in \mathcal{A}_j$, and then $T^+(u_n^j) \in Q_j$, definitively. Therefore by Lemma 3.6 (ii) we have that (u_n^j) is precompact in X and then we obtain a critical point for φ in $\mathcal{A}_j$ for all $j \ge \hat{\jmath}$.

To prove the claim, first we note that (b) plainly follows from (a). Indeed if $\eta_j(t, \gamma_j(s_j)) \notin B_{\frac{\bar{r}}{2}}(\mathcal{A}_j)$ for some $t > 0$ then by (4.6) and (ii) we obtain that $\varphi(\eta_j(t, \gamma_j(s_j))) \le \varphi(\gamma_j(s_j)) - \Delta_{\bar{r}} \le c_j + h - \Delta_{\bar{r}} \le c_j - h$ which is impossible by (a).

To prove (a) we argue by contradiction assuming that for all $s \in [0,1]$ for which $\gamma_j(s) \in B_{\frac{\bar{r}}{4}}(\mathcal{A}_j) \cap \{\varphi > c_j - h\}$ there exists $t > 0$ such that

$$\varphi(\eta_j(t, \gamma_j(s))) \le c_j - h. \tag{4.7}$$

Then for any $s \in [0,1]$ set $T(s) = \inf\{t \ge 0 \ : \ \varphi(\eta_j(t, \gamma_j(s))) \le c_j - h\}$. By (i) and (4.7) we obtain that $T : [0,1] \to \mathbf{R}^+$ is well defined and continuous. Therefore setting $\hat{\gamma}_j(s) = \eta_j(T(s), \gamma_j(s))$ for all $s \in [0,1]$ we obtain $\hat{\gamma}_j \in \Gamma_j$ and then a contradiction, since by construction there results $\varphi(\hat{\gamma}_j(s)) \le c_j - h$ for all $s \in [0,1]$. This complete the proof. ∎

5. MULTIBUMP SOLUTIONS

In the previous section we proved the existence of infinitely many one bump solutions of (L). In fact, by Theorem 4.1, for any $j \ge \hat{\jmath}$ there is a homoclinic solution of (L) which has L^∞-norm greater than $\bar{\delta}$ only in the time interval Q_j. In other words such trajectory leaves and returns in the $\bar{\delta}$ neighbourhood of the origin in the configuration space only in the time interval Q_j.

In this last section we look for k-bump homoclinic solutions of (L). More precisely we show that there exists a sequence of indices $(\hat{\jmath}_n)$ such that if $j_1 < ... < j_k \in \mathbf{N}$ verify $j_i \ge \hat{\jmath}_i$, $i = 1, \dots, k$ then there is a homoclinic trajectory of (L) which leaves and returns in the $\bar{\delta}$-neighbourhood of the origin in the configuration space only in the time interval Q_{j_i}, $i = 1, \dots, k$. Considering the C^1_{loc}-closure of the set of k-bump solutions we obtain a multibump dynamics proving Theorem 1.1 stated in the introduction.

First of all we introduce some notation.

Fixed $k \in \mathbf{N}$ and k indices $j_1 < \dots < j_k$ we denote

$$\begin{aligned} I_1 &= \left(-\infty, \frac{\sigma^+_{j_1} + \sigma^-_{j_2}}{2}\right), \\ I_i &= \left(\frac{\sigma^+_{j_{i-1}} + \sigma^-_{j_i}}{2}, \frac{\sigma^+_{j_i} + \sigma^-_{j_{i+1}}}{2}\right), \qquad i = 2, \dots, k-1, \\ I_k &= \left(\frac{\sigma^+_{j_{k-1}} + \sigma^-_{j_k}}{2}, +\infty\right), \end{aligned}$$

where the sequences $(\sigma^\pm_j)$ are given in Remark 4.1.

Then the family of intervals $\{I_i, i = 1, \ldots, k\}$ is a partition of $\mathbf{R}$. Moreover each interval P_{j_i} is strictly contained in the interval I_i. Let M_i be the complement of the interval P_{j_i} in I_i.

We also define the "truncated" functionals $\varphi_i : X \to \mathbf{R}$ by setting

$$\varphi_i(u) = \tfrac{1}{2}\|u\|_{I_i}^2 - \int_{I_i} \alpha(t) W(u(t))\, dt, \quad \forall u \in X,\ i = 1, \ldots, k.$$

Note that $\varphi(u) = \sum_{i=1}^k \varphi_i(u)$ and $\varphi_i \in C^1(X, \mathbf{R})$ with $\varphi_i'(u)v = \langle u, v\rangle_{I_i} - \int_{I_i} \alpha(t) W(u(t))\, dt$ for all $u, v \in X$.

Finally, given $r > 0$ and $J = (j_1, \ldots, j_k)$ with $\bar{\jmath} < j_1 < \ldots < j_k$ we consider the set

$$\mathcal{B}_r(J) = \{u \in X \mid \inf_{v \in \mathcal{A}_{j_i}} \|u - v\|_{I_i} < r,\ i = 1, \ldots, k\}.$$

By Lemma 4.1 if $v \in \mathcal{A}_j$ for some $j \geq \bar{\jmath}$ then $\|v\|_{L^\infty(\mathbf{R} \setminus Q_{j_i})} < \frac{\bar{\delta}}{2}$. Therefore if $r \in (0, \bar{r}]$ and $v \in \mathcal{B}_r(J)$ then $\|v\|_{L^\infty(I_i \setminus Q_{j_i})} < \bar{\delta}$. In other words the functions in $\mathcal{B}_r(J)$ can be outside the $\bar{\delta}$-neighbourhood of the origin only in the intervals Q_{j_i}. Therefore we will look for k-bump solutions of (L) in these sets. To this end we investigate some compactness properties of φ in $\mathcal{B}_r(J)$.

Note that the action of the functional φ on $\mathcal{B}_r(J)$ separates on the actions of the functionals φ_i and, roughly speaking, that each functional φ_i acts on $\mathcal{B}_r(J)$ as the functional φ acts on $B_r(A_{j_i})$. Then, starting from the compactness properties of φ on $B_{\bar{r}}(A_j)$ proved in the previous sections, see Lemmas 2.1, 3.5 and 4.2, we can obtain analogous properties of φ on $\mathcal{B}_r(J)$.

Let we fix $\bar{\mu} = \frac{1}{8}\min\{\frac{r_0^2}{16}, \frac{\bar{\nu}}{8}\}$, $\tilde{h} = \frac{1}{8}\min\{h_{r_0}, \bar{h}, \bar{\mu} r_0\}$ (where h_{r_0} is given by Lemma 3.1 (ii) with $r = \frac{r_0}{2}$ and $\beta = \overline{\alpha}$) and a decreasing sequence (h_i) such that $0 < \sum_{i=1}^\infty h_i < \tilde{h}$. We set also $r_1 = \frac{\bar{r}}{2}$, $r_2 = \frac{2\bar{r}}{3}$ and $r_3 = \frac{3\bar{r}}{4}$. Defining $\mathcal{E}_k = \{u \in X\,;\, \|u\|_{M_l}^2 \leq \frac{h_l}{8},\ l = 1, \ldots, k\}$ and $\Phi_k = \cap_{i=1}^k \{\varphi_i \geq c_{\overline{\alpha}} - h_i\} \cap \{\varphi \leq k c_{\overline{\alpha}} + \tilde{h}\}$ we have

LEMMA 5.1. – *There exists an increasing sequence of indices $(\tilde{\jmath}_i) \in \mathbf{N}$ such that given $k \in \mathbf{N}$ and $J = (j_1, \ldots, j_k)$ with $j_1 < \ldots < j_k$ and $j_i \geq \tilde{\jmath}_i$ $(i = 1, \ldots, k)$, then if $\mathcal{E}_k \cap \Phi_k \cap \mathcal{B}_{\bar{r}}(J) \cap \mathcal{K} = \emptyset$ there exists a locally Lipschitz continuous vector field $\mathcal{F} : X \to X$ which verifies the following properties:*

($\mathcal{F}1$) $\|\mathcal{F}(u)\|_{I_i} \leq 1$ $(i = 1, \ldots, k)$, $\varphi'(u)\mathcal{F}(u) \geq 0$ *for any* $u \in X$ *and* $\mathcal{F}(u) = 0$ *for any* $u \in X \setminus \mathcal{B}_{r_3}(J)$;

($\mathcal{F}2$) *if* $u \in \mathcal{B}_{r_2}(J)$, $r_1 \le \inf_{v\in\mathcal{A}_{j_i}} \|u-v\|_{I_i}$ *and* $\varphi_i(u) \le c_{\overline{\alpha}} + 2\tilde{h}$ *then* $\varphi_i'(u)\mathcal{F}(u) \ge \bar{\mu}$;

($\mathcal{F}3$) *if* $u \in \mathcal{B}_{r_3}(J)$ *and* $\varphi_i(u) \le c_{\overline{\alpha}} - h_i$ *then* $\varphi_i'(u)\mathcal{F}(u) \ge 0$;

($\mathcal{F}4$) *if* $u \in \mathcal{B}_{r_3}(J) \setminus \mathcal{E}_k$ *then* $\langle u, F(u)\rangle_{M_i} \ge 0$ *for any* $i \in \{1,\dots,k\}$;

($\mathcal{F}5$) *there exists* $\mu_J > 0$ *such that* $\varphi'(u)\mathcal{F}(u) \ge \nu_J$ *for any* $u \in B_{r_2}(J) \cap \{\varphi < kc_{\overline{\alpha}} + \tilde{h}\}$.

This kind of result is classical in the multibump construction (see [28]). The proof is based on the use of a suitable cutoff procedure, it is quite technical and we postpone it to the Appendix.

We set $\tilde{J}_k = \{(j_1,\dots,j_k);\ j_1 < j_2 < \dots < j_k,\ j_i \ge \tilde{j}_i\}$. As a consequence of Lemma 5.1 we get that if $J \in \tilde{J}_k$ and $\mathcal{B}_{\bar{r}}(J) \cap \mathcal{K} = \emptyset$ then the set $\mathcal{B}_{r_1}(J) \cap \{\varphi \le kc_{\overline{\alpha}} + \tilde{h}\}$ can be continuously deformed in the set $\bigcup_{i=i}^k \{\varphi_i \le c_{\overline{\alpha}} - h_i\}$. In fact we have

LEMMA 5.2. – *Given* $k \in \mathbf{N}$ *and* $J \in \tilde{J}_k$, *if* $\mathcal{E}_k \cap \Phi_k \cap \mathcal{B}_{\bar{r}}(J) \cap \mathcal{K} = \emptyset$ *then there exists* $\eta \in C(X,X)$ *such that*

i) $\eta|_{X\setminus\mathcal{B}_{r_3}(J)} \equiv I$;

ii) $\eta(\mathcal{E}_k) \subset \mathcal{E}_k$;

iii) $\eta(\{\varphi_i \le c_{\overline{\alpha}} - h_i\}) \subset \{\varphi_i \le c_{\overline{\alpha}} - h_i\}$;

iv) *if* $u \in \mathcal{B}_{r_1}(J) \cap \{\varphi < kc_{\overline{\alpha}} + \tilde{h}\}$ *then* $\eta(u) \in \bigcup_{i=1}^k \{\varphi_i \le c_{\overline{\alpha}} - h_i\}$.

Proof. – Let us consider the Cauchy problem

$$\begin{cases} \dfrac{d\eta}{ds} = -\mathcal{F}(\eta) \\ \eta(0,u) = u \end{cases} \tag{5.1}$$

where $\mathcal{F}$ is the bounded locally Lipschitz continuous vector field given by Lemma 5.1. For any $u \in X$ there exists a unique solution $\eta(\cdot,u) \in C(\mathbf{R}^+, X)$ of (5.1), depending continuously on $u \in X$.

By ($\mathcal{F}1$), since $\mathcal{F}(u) = 0$ for any $u \in X \setminus \mathcal{B}_{r_3}(J)$, we obtain that

$$\eta(s,u) = u \quad \forall u \in X \setminus \mathcal{B}_{r_3}(J),\ \forall s > 0. \tag{5.2}$$

By ($\mathcal{F}4$), if $\eta(s,u) \in X \setminus \mathcal{E}_k$ then

$$\frac{d}{ds}\|\eta(s,u)\|_{M_i}^2 = -\langle \eta(s,u), \mathcal{F}(\eta(s,u))\rangle_{M_i} \le 0.$$

Therefore the set $\mathcal{E}_k$ is positively invariant w.r.t. the flow η, i.e.

$$\eta(s,\mathcal{E}_k) \subset \mathcal{E}_k, \quad \forall s > 0. \tag{5.3}$$

By $(\mathcal{F}3)$, $\frac{d}{ds}\varphi_i(\eta(s,u)) = -\varphi_i'(\eta(s,u))\mathcal{F}(\eta(s,u) \leq 0$ if $\varphi_i(\eta(s,u)) \leq c_{\overline{\alpha}} - h_i$. Hence also for the sets $\{\varphi_i \leq c_{\overline{\alpha}} - h_i\}$ we have

$$\eta(s, \{\varphi_i \leq c_{\overline{\alpha}} - h_i\}) \subset \{\varphi_i \leq c_{\overline{\alpha}} - h_i\}, \quad \forall s > 0. \tag{5.4}$$

Finally note that since φ sends bounded sets into bounded sets, by $(\mathcal{F}5)$ there exists $\mathcal{T} > 0$ such that

$$\forall u \in \mathcal{B}_{r_1}(J)\ \exists s_u \in (0, \mathcal{T}) \quad \text{such that} \quad \eta(s_u, u) \in X \setminus \mathcal{B}_{r_2}(J). \tag{5.5}$$

By (5.5) for all $u \in \mathcal{B}_{r_1}(J) \cap \{\varphi \leq kc_{\overline{\alpha}} + \tilde{h}\}$ there is an index $i_u \in \{1, \ldots, k\}$ and an interval $[s_1, s_2] \subset (0, \mathcal{T})$ such that $\inf_{v \in \mathcal{A}_{j_{i_u}}} \|\eta(s_1, u) - v\|_{I_{i_u}} = r_1$, $\inf_{v \in \mathcal{A}_{j_{i_u}}} \|\eta(s_2, u) - v\|_{I_{i_u}} = r_2$ and $r_1 \leq \inf_{v \in \mathcal{A}_{j_{i_u}}} \|\eta(s, u) - v\|_{I_{i_u}} \leq r_2$ for any $s \in (s_1, s_2)$. In particular, by $(\mathcal{F}1)$ we obtain

$$r_2 - r_1 \leq \|\eta(s_2, u) - \eta(s_1, u)\|_{I_{i_u}} \leq \int_{s_1}^{s_2} \|\mathcal{F}(\eta(s, u))\|_{I_{i_u}} ds \leq s_2 - s_1. \tag{5.6}$$

Now, let $u \in \mathcal{B}_{r_1}(J) \cap \{\varphi \leq kc_{\overline{\alpha}} + \tilde{h}\}$. We claim that there exists $i \in \{1, \ldots, k\}$ such that $\varphi_i(\eta(s, u)) < c_{\overline{\alpha}} - h_i$ for some $s \in [0, s_2]$ and therefore

$$\varphi_i(\eta(\mathcal{T}, u)) \leq c_{\overline{\alpha}} - h_i. \tag{5.7}$$

Indeed if not we have $\inf_{i=1,\ldots,k} \varphi_i(\eta(s, u)) \geq c_{\overline{\alpha}} - h_i$ for any $s \in [0, s_2]$. Then since, by $(\mathcal{F}1)$, $\varphi(\eta(s, u)) \leq kc_{\overline{\alpha}} + \tilde{h}$ we obtain that $\sup_{i=1,\ldots,k} \varphi_i(\eta(s, u)) \leq c_{\overline{\alpha}} + 2\tilde{h}$ for any $s \in [0, s_2]$ (recall that $\sum_{i=1}^{\infty} h_i < \tilde{h}$). Then, by $(\mathcal{F}2)$ and (5.6), we get

$$\begin{aligned} 3\tilde{h} \geq c_{\overline{\alpha}} + 2\tilde{h} - c_{\overline{\alpha}} + h_{i_u} &\geq \varphi_{i_u}(\eta(s_1, u)) - \varphi_{i_u}(\eta(s_2, u)) \\ &= \int_{s_1}^{s_2} \varphi_{i_u}'(\eta(s, u))\mathcal{F}(\eta(s, u)) ds \\ &\geq \bar{\mu}(s_2 - s_1) \geq \bar{\mu}(r_2 - r_1) \end{aligned}$$

in contradiction with the choice of $\tilde{h}$ (recall that $\tilde{h} \leq \bar{\mu}\frac{(r_2 - r_1)}{4}$).

With abuse of notation we set $\eta(\cdot) \equiv \eta(\mathcal{T}, \cdot)$ and the lemma follows by (5.2), (5.3), (5.4) and (5.7). ∎

Now we are able to prove the existence of k-bump solutions applying the Séré's product minimax.

THEOREM 5.1. – *There exists an increasing sequence of indices $(\hat{j}_i) \subset \mathbf{N}$ such that if $k \in \mathbf{N}$ and $J = (j_1, \ldots, j_k)$ verifies $j_1 < \ldots < j_k$ and $j_i \geq \hat{j}_i$ $(i = 1, \ldots, k)$ then $\mathcal{E}_k \cap \Phi_k \cap \mathcal{B}_{\bar{r}}(J) \cap \mathcal{K} \neq \emptyset$.*

Proof. – For all $j \in \mathbf{N}$ consider the cutoff function $\chi_j \in C(\mathbf{R}, [0,1])$ defined by $\chi_j(t) = \min\{1, dist(t, \mathbf{R} \setminus Q_j)\}$ and the paths $\tilde{\gamma}_j(s) = \chi_j \gamma_j(s)$, $s \in [0,1]$, where the paths γ_j are given in section 4.

It is immediate to recognize that $\sup_{s \in (0,1)} \|\tilde{\gamma}_j(s) - \gamma_j(s)\| \to 0$ as $j \to \infty$. Therefore since φ is uniformly continuous on the bounded sets, by Remark 4.3 and Lemmas 3.1 and 4.3, we can fix an increasing sequence of indices $(\hat{j}_i) \subset \mathbf{N}$, $\hat{j}_i \geq \tilde{j}_i$ ($i \in \mathbf{N}$), such that:

(γ_1) $c_j \geq c_{\overline{\alpha}} - \frac{h_i}{4}$ for every $j \geq \hat{j}_i$;
(γ_2) $\tilde{\gamma}_j \in \Gamma_j$ and $\tilde{\gamma}_j(1) \notin B_{\bar{r}}(\mathcal{A}_j)$ for every $j \geq \hat{j}_1$;
(γ_3) if $j \geq \hat{j}_1$ and $\tilde{\gamma}_j(s) \in X \setminus B_{r_0}(\mathcal{A}_j)$ then $\varphi(\tilde{\gamma}_j(s)) \leq c_{\overline{\alpha}} - \frac{h_{r_0}}{2}$;
(γ_4) $\max_{s \in [0,1]} \varphi(\tilde{\gamma}_j(s)) \leq c_{\overline{\alpha}} + h_i \ \forall j \geq \hat{j}_i$.

Let $k \in \mathbf{N}$ and $J = (j_1, \ldots, j_k)$ be such that $j_1 < \ldots < j_k$ and $j_i \geq \hat{j}_i$ for all $i = 1, \ldots, k$. We define the surface $G \in C([0,1]^k, X)$ by setting $G(\theta) = \sum_{i=1}^k \tilde{\gamma}_{j_i}(\theta_i)$. We have

(G_1) $\max_{\theta \in [0,1]^k} \varphi(G(\theta)) \leq kc_{\overline{\alpha}} + \tilde{h}$;
(G_2) if $G(\theta) \in X \setminus \mathcal{B}_{r_1}(J)$ then there exists $i_\theta \in \{1, \ldots, k\}$ for which $\varphi_{i_\theta}(G(\theta)) < c_{\overline{\alpha}} - h_{i_\theta}$;
(G_3) $G(\theta) \in \mathcal{E}_k$ for every $\theta \in [0,1]^k$.

Indeed (G_1) plainly follows by (γ_4) since $\sum_{i=1}^k h_i < \tilde{h}$. Moreover we obtain ($G_2$) by ($\gamma_3$) simply noting that if $G(\theta) \in X \setminus \mathcal{B}_{r_1}(J)$ then there is $i_\theta \in \{1, \ldots, k\}$ such that $r_0 < r_1 < \inf_{v \in \mathcal{A}_{j_{i_\theta}}} \|G(\theta) - v\|_{I_{i_\theta}} \leq \inf_{v \in \mathcal{A}_{j_{i_\theta}}} \|\tilde{\gamma}_{j_{i_\theta}}(\theta_{i_\theta}) - v\|$. Finally since $supp\ G(\theta) \subset \cup_{i=1}^k Q_{j_i}$ we obtain (G_3).

Now, arguing by contradiction assume that $\mathcal{E}_k \cap \Phi_k \cap \mathcal{B}_{\bar{r}}(J) \cap \mathcal{K} = \emptyset$. Then we can consider the surface $\tilde{G}(\cdot) = \eta(G(\cdot))$ where η is given by Lemma 5.2. By Lemma 5.2 and (G_1)-(G_3) we obtain

($\tilde{G}_1$) if $G(\theta) \in X \setminus \mathcal{B}_{r_3}(J)$ then $\tilde{G}(\theta) = G(\theta)$ and in particular $\tilde{G}|_{\partial[0,1]^k} = G|_{\partial[0,1]^k}$;
($\tilde{G}_2$) $\forall \theta \in [0,1]^k$ there exists $i_\theta \in \{1, ..., k\}$ such that $\varphi_{i_\theta}(\tilde{G}(\theta)) < c_{\overline{\alpha}} - h_{i_\theta}$;
($\tilde{G}_3$) $\tilde{G}(\theta) \in \mathcal{E}_k$ for every $\theta \in [0,1]^k$.

Indeed ($\tilde{G}_1$) plainly follows by Lemma 5.2-(i) since by (γ_2) $G(\partial[0,1]^k) \subset X \setminus \mathcal{B}_{r_3}(J)$. Also ($\tilde{G}_3$) is an immediate consequence of Lemma 5.2-(ii) and (G_3).

To prove ($\tilde{G}_2$) we consider the following alternative: $G(\theta) \in X \setminus \mathcal{B}_{r_1}(J)$ or $G(\theta) \in \mathcal{B}_{r_1}(J)$.

In the first case by (G_2) there exists i_θ such that $\varphi_{i_\theta}(G(\theta)) < c_{\overline{\alpha}} - h_{i_\theta}$ and, by Lemma 5.2-(iii), we obtain $\varphi_{i_\theta}(\tilde{G}(\theta)) < c_{\overline{\alpha}} - h_{i_\theta}$. In the second case by (G_1) we have that $G(\theta) \in \mathcal{B}_{r_1}(J) \cap \{\varphi < kc_{\overline{\alpha}} + \tilde{h}\}$ and therefore, by Lemma 5.2-(iv), also in this case there exists i_θ such that $\varphi_{i_\theta}(\tilde{G}(\theta)) < c_{\overline{\alpha}} - h_{i_\theta}$. Then $(\tilde{G}_2)$ holds.

Thanks to $(\tilde{G}_3)$ we can select on $[0,1]^k$ a path ξ joining two opposite faces $\{\theta_i = 0\}$ and $\{\theta_i = 1\}$ along which the function $\varphi_i \circ \bar{G}$ takes values less than $c_{\overline{\alpha}} - \frac{3h_i}{4}$ for some $i \in \{1, \dots, k\}$. Precisely:

$(\tilde{G}_4)$ there exists $\iota \in \{1, \dots, k\}$ and $\xi \in C([0,1],[0,1]^k)$ such that $\xi(0) \in \{\theta_\iota = 0\}$, $\xi(1) \in \{\theta_\iota = 1\}$ and $\varphi_\iota(\tilde{G}(\theta)) < c_{\overline{\alpha}} - \frac{3h_\iota}{4}$, for any $\theta \in \operatorname{range}\xi$

Indeed, assuming the contrary, the set $D_i = \{\theta \in [0,1]^k \; : \; \varphi_i(\bar{G}(\theta)) \geq c_{\overline{\alpha}} - \frac{3h_i}{4}\}$ for any $i \in \{1, \dots, k\}$ separates in $[0,1]^k$ the faces $F_i^0 = \{\theta_i = 0\}$ and $F_i^1 = \{\theta_i = 1\}$. For any $i \in \{1, \dots, k\}$ let C_i be the component of $[0,1]^k \setminus D_i$ which contains the face F_i^1 and let us define the functions $f_i : [0,1]^k \to \mathbf{R}$ as follows:

$$f_i(\theta) = \begin{cases} \operatorname{dist}(\theta, D_i) & \text{if } \theta \in [0,1]^k \setminus C_i \\ -\operatorname{dist}(\theta, D_i) & \text{if } \theta \in C_i. \end{cases}$$

Then, $f_i \in C([0,1]^k, \mathbf{R})$, $f_i|_{F_i^0} \geq 0$, $f_i|_{F_i^1} \leq 0$ and $f_i(\theta) = 0$ if and only if $\theta \in D_i$. Using the Miranda fixed point Theorem (see [19]), we get that there exists $\theta \in [0,1]^k$ such that $f_i(\theta) = 0$ for all $i \in \{1, \dots, k\}$, hence $\bigcap_i D_i \neq \emptyset$, which is in contradiction with the property $(\tilde{G}_2)$.

Using $(\tilde{G}_4)$ we define the cutoff function $\chi \in C(\mathbf{R}, [0,1])$ by setting $\chi(t) = \min\{1, dist(t, \mathbf{R} \setminus I_\iota)\}$, and we consider the path $\gamma \in C([0,1], X)$ given by $\gamma(s) = \chi\tilde{G}(\xi(s))$. We claim that $\gamma \in \Gamma_{j_\iota}$.

Indeed since $supp\,\tilde{\gamma}_{j_\iota}(s) \subset Q_{j_\iota}$ and $Q_{j_\iota} \subset \{t\,;\, \chi(t) = 1\}$, we have

$$\gamma(0) = 0 \quad \text{and} \quad \gamma(1) = \tilde{\gamma}_{j_\iota}(1).$$

In particular $\varphi(\gamma(1)) = \varphi(\tilde{\gamma}_{j_\iota}(1)) < \frac{1}{2}\varphi_{\overline{\alpha}}(\gamma_{\overline{\alpha}}(1))$.

Moreover if $\gamma(s) \in B_{\bar{r}}(\mathcal{A}_{j_\iota})$ then $\|\gamma(s)\| \leq M$ and, by Lemma 4.2, if $t \notin Q_{j_\iota}$ then $|\gamma(s)(t)| < \bar{\delta}$.

Otherwise, if $\gamma(s) \notin B_{\bar{r}}(\mathcal{A}_{j_\iota})$, by Lemma 4.1, we have $\bar{r} \leq \inf_{v \in \mathcal{A}_{j_\iota}} \|\gamma(s) - v\| \leq \inf_{v \in \mathcal{A}_{j_\iota}} \|\gamma(s) - v\|_{I_\iota} + r_0 = \inf_{v \in \mathcal{A}_{j_\iota}} \|\chi\tilde{G}(\xi(s)) - v\|_{I_\iota} + r_0$. Therefore since $\tilde{G}(\theta) \in \mathcal{E}_k$ for any $\theta \in [0,1]^k$ we obtain $\inf_{v \in \mathcal{A}_{j_\iota}} \|\tilde{G}(\xi(s)) - v\|_{I_\iota} \geq \bar{r} - r_0 - \|(1-\chi)\tilde{G}(\xi(s))\|_{I_\iota} \geq \bar{r} - 2\|\tilde{G}(\xi(s)\|_{M_\iota} \geq \bar{r} - (\frac{h_\iota}{8})^{\frac{1}{2}} > r_3$.

Then we conclude that $\tilde{G}(\xi(s)) \notin \mathcal{B}_{r_3}(J)$ and, by $(\tilde{G}_1)$, that $\gamma(s) = \tilde{\gamma}_{j_\iota}(s)$. Therefore, since $\tilde{\gamma}_{j_\iota} \in \Gamma_{j_\iota}$ we have also in this case $\|\gamma(s)\| \leq M$ and if $t \notin Q_{j_\iota}$ then $|\gamma(s)(t)| < \bar{\delta}$.

Then $\gamma \in \Gamma_{j_i}$ and if we show that $\varphi(\gamma(s)) < c_{\overline{\alpha}} - \frac{h_\iota}{4}$, using (γ_1), we obtain a contradiction.

By the choice of $\bar{\delta}$ and by $(\tilde{G}_3)$ we have

$$\begin{aligned}\varphi(\gamma(s)) &= \varphi_\iota(\gamma(s)) = \varphi_\iota(\tilde{G}(\xi(s))) + \varphi_\iota(\gamma(s)) - \varphi_\iota(\tilde{G}(\xi(s))) \\ &\le c_{\overline{\alpha}} - \tfrac{3h_\iota}{4} + \tfrac{1}{2}\|\chi\tilde{G}(\xi(s))\|^2_{M_\iota} + \int_{M_\iota} \alpha(t)W(\tilde{G}(\xi(s)))dt \\ &\le c_{\overline{\alpha}} - \tfrac{3h_\iota}{4} + \tfrac{h_\iota}{4} + \tfrac{h_\iota}{64} < c_{\overline{\alpha}} - \tfrac{h_\iota}{4}\end{aligned}$$

and the theorem follows. ∎

As a consequence of Theorem 5.1 we have

COROLLARY 5.1. – *For every $k \in \mathbf{N}$ and $J = (j_1, \dots, j_k)$ with $j_1 < \dots < j_k$ and $j_i \ge \hat{j}_i$ $(i = 1, \dots, k)$ there exists $u \in \mathcal{C}^2(\mathbf{R}, \mathbf{R}^N)$ solution of (L) such that*

$$\|u\|_{L^\infty(Q_{j_i})} > \bar{\delta} \ \ \forall i = 1, \dots, k \quad \textit{and} \quad \|u\|_{L^\infty(\mathbf{R}\setminus\cup_{i=1}^k Q_{j_i})} < \frac{\bar{\delta}}{2}.$$

Proof. – By Theorem 5.1 there exists $u \in \mathcal{E}_k \cap \Phi_k \cap \mathcal{B}_{\bar{r}}(J) \cap \mathcal{K}$. Then for all $i = 1, ..., k$ consider the cutoff function $\chi_i(t) = dist\,(t, \mathbf{R} \setminus I_i)$.

Now, since $u \in \mathcal{B}_{\bar{r}}(J)$ if $\|u\|_{L^\infty(Q_{j_i})} \le \bar{\delta}$ for some $i \in \{1, \dots, k\}$ then $\|u\|_{L^\infty(I_i)} \le \bar{\delta}$ too and therefore $\|\chi_i u\|_{L^\infty} \le \bar{\delta}$. By the choice of $\bar{\delta}$ we obtain $\varphi'(\chi_i u)\chi_i u \ge \frac{1}{2}\|\chi_i u\|^2$. Since $u \in \mathcal{B}_{\bar{r}}(J)$ we have $\|u\|_{I_i} \ge \bar{r}$ and, since $u \in \mathcal{E}_k$, we obtain $\|\chi_i u\|_{I_i} \ge \|u\|_{I_i} - \|(1-\chi_i)u\|_{I_i} \ge \bar{r} - \frac{h_i}{4} \ge \frac{\bar{r}}{2}$. This implies that $\varphi'(\chi_i u)\chi_i u \ge \frac{\bar{r}^2}{8}$. Then we have $|\varphi'(\chi_i u)\chi_i u - \varphi'(u)\chi_i u| \le |\langle(1-\chi_i)u, \chi_i u\rangle_{M_i}| + |\int_{M_i} \alpha(t)(\nabla W(\chi_i u) - \nabla W(u))\chi_i u\, dt| \le 5\|u\|^2_{M_i} < h_i$ and we conclude that $\varphi'(u)\chi_i u \ge \frac{\bar{r}^2}{16}$ in contradiction with $u \in \mathcal{K}$.

Moreover arguing as above it is also easy to prove that since $u \in \Phi_k \cap \mathcal{K}$ we have $\|\varphi'(\chi_i u)\| < \bar{\nu}$ and $\varphi(\chi_i u) < c_{\overline{\alpha}} + \bar{h}$ for all $i = 1, ..., k$. Then, since we have already proved that $T^+(\chi_i u) \in Q_{j_i}$, we obtain $\chi_i u \in \mathcal{A}_{j_i}$. Then, by Lemma 5.1 we obtain that $\|\chi_i u\|_{L^\infty(\mathbf{R}\setminus Q_{j_i})} < \frac{\bar{\delta}}{2}$ $(i = 1, \dots, k)$. This complete the proof. ∎

Considering the C^1_{loc} closure of the set of k-bump solution, using the Ascoli Arzelá theorem, by Theorem 5.1 and Corollary 5.1 we obtain Theorem 1.1 stated in the introduction.

6. APPENDIX

In this section we prove Lemma 5.1.

First of all we recall two properties which we will use in the sequel (see Lemmas 4.2 and 3.7 respectively):

(A) if $u \in B_{\bar{r}}(\mathcal{A}_j) \setminus \mathcal{A}_j$ and $\varphi(u) \le c_{\overline{\alpha}} + \bar{h}$ then $\|\varphi'(u)\| \ge \bar{\nu}$ (*Annuli property*),

(S) for every $h > 0$ there exists $j_h \in \mathbf{N}$ and $\nu_h > 0$ such that if $\varphi(u) \le c_{\overline{\alpha}} - h$ and $u \in B_{\bar{r}}(\mathcal{A}_j)$ for some $j \ge j_h$ then $\|\varphi'(u)\| \ge \nu_h$ (*Slices property*)

Using the *slices property* fixed a non increasing sequence of positive numbers (ν_i) we obtain an increasing sequence of indices $(j_i(h))$ such that:

(S_i) if $u \in B_{\bar{r}}(\mathcal{A}_j)$ for some $j \ge j_i(h)$ and $\varphi(u) \le c_{\overline{\alpha}} - \frac{1}{2}h_i$ then $\|\varphi'(u)\| \ge \nu_i$.

Now note that if $u \in \mathcal{B}_r(J)$ then $\|u\|_{I_i} \le \sup_{v \in \mathcal{A}_{l_i}} \|v\| + r \le M$. In other words the mass of the functions in $\mathcal{B}_r(J)$ in each interval I_i is bounded independently of the number k. Then we obtain

LEMMA 6.1. – *Given any sequence of positive real numbers (ξ_i) there exists a monotone increasing sequence of indices $(j_i(\xi))$ for which if $k \in \mathbf{N}$ and $j_1 < j_2 < \ldots < j_k \in \mathbf{N}$ verify $j_i \ge j_i(\xi)$ $(i = 1, \ldots, k)$ then for any $u \in \mathcal{B}_r(J)$ there exist two intervals $N^-_{u,i} \subset (\sigma^-_{j_i}, \tau^-_{j_i})$ and $N^+_{u,i} \subset (\tau^+_{j_i}, \sigma^+_{j_i})$ such that*

$$|N^{\pm}_{u,i}| = 1, \quad \textit{and} \quad \|u\|_{N^-_{u,i} \cup N^+_{u,i}} \le \xi_i, \quad \forall i \in \{1, \ldots, k\}.$$

Proof. – We recall that $L_j = \min\{|\sigma^-_j - \tau^-_j|, |\sigma^+_j - \tau^+_j|\} \to \infty$ as $j \to \infty$. Then we can fix an increasing sequence of indices $(j_i(\xi))$ such that $\frac{M^2}{[L_j]} \le \xi_i^2$ for every $j \ge j_i(\xi)$ (where $[x]$ denotes the entire part of x).

Let $k \in \mathbf{N}$ and $j_i \in \mathbf{N}$ with $j_i \ge j_i(\xi)$ for all $i = 1, \ldots, k$. If $u \in \mathcal{B}_r(J)$ we have

$$\begin{aligned} M^2 \ge \|u\|^2_{I_i} &\ge \sum_{l=1}^{[L_i]} [\|u\|^2_{(\tau^-_{j_i} - l, \tau^-_{j_i} - l + 1)} + \|u\|^2_{(\tau^+_{j_i} + l - 1, \tau^+_{j_i} + l)}] \\ &\ge [L_i] \min_{l = 1, \ldots, [L_i]} [\|u\|^2_{(\tau^-_{j_i} - l, \tau^-_{j_i} - l + 1)} + \|u\|^2_{(\tau^+_{j_i} + l - 1, \tau^+_{j_i} + l)}] \quad i = 1, \ldots, k \end{aligned}$$

and the lemma follows by the choice of $j_i(\xi)$. ∎

We fix a decreasing sequence $(\xi_i) \subset (0,1)$ with $\xi_i \le \frac{1}{8}\min\{r_0, \bar{\nu}, \nu_i, h^{\frac{1}{2}}_{i+1}\}$ for any $i \in \mathbf{N}$. We will denote $J_k(\xi) = \{(j_1, \ldots, j_k);\ j_1 < j_2 < \ldots < j_k,\ j_i \ge \max\{j_i(h), j_i(\xi)\}\}$ where $j_i(h)$ is given in (S_i) and $j_i(\xi)$ in Lemma 6.1.

By Lemma 6.1 if $J = (j_1, \ldots, j_k) \in J_k(\xi)$ and $u \in \mathcal{B}_r(J)$ then each interval P_{j_i} contains two subintervals, one on the right and one on the left of Q_{j_i}, over which the norm of u is controlled by ξ_i. We will use this property to produce a suitable cutoff procedure controlling the errors via the sequence (ξ_i).

Then, given $J \in J_k(\xi)$ and $u \in \mathcal{B}_{\bar{r}}(J)$ we define the cutoff functions by

$$\beta_{u,i}(t) = \begin{cases} 0 & \text{if } t \le \inf N^-_{u,i} \\ t - \inf N^-_{u,i} & \text{if } t \in N^-_{u,i} \\ 1 & \text{if } \sup N^-_{u,i} \le t \le \inf N^+_{u,i} \\ \sup N^+_{u,i} - t & \text{if } t \in N^+_{u,i} \\ 0 & \text{if } t \ge \sup N^+_{u,i} \end{cases} \quad i = 1, \dots, k.$$

We define also the "complement" functions by

$$\bar{\beta}_{u,0}(t) = \begin{cases} 1 - \beta_{u,1}(t) & \text{if } t \le \tau_{j_1} \\ 0 & \text{otherwise} \end{cases}$$
$$\bar{\beta}_{u,i}(t) = \begin{cases} 1 - \beta_{u,i}(t) - \beta_{u,i+1}(t) & \text{if } \tau_{j_i} \le t \le \tau_{j_{i+1}} \\ 0 & \text{otherwise} \end{cases} \quad i = 1, \dots, k-1$$
$$\bar{\beta}_{u,k}(t) = \begin{cases} 1 - \beta_{u,k}(t) & \text{if } t \ge \tau_{j_k} \\ 0 & \text{otherwise.} \end{cases}$$

Setting $\beta_u = \sum_{i=1}^k \beta_{u,i}$ and $\bar{\beta}_u = \sum_{C=0}^k \bar{\beta}_{u,i}$ we have that $\beta_u(t) + \bar{\beta}_u(t) = 1$ for any $t \in \mathbf{R}$.
We denote $B_{u,l} = \{t \in \mathbf{R};\, \bar{\beta}_{u,l}(t) \neq 0\}$, $B_u = \cup_{l=0}^k B_{u,l}$, $A_{u,l} = \{t \in \mathbf{R};\, \bar{\beta}_{u,l}(t) = 1\}$, $A_u = \cup_{l=0}^k A_{u,l}$.
Note that if β is anyone of the above defined cutoff functions then $|\dot{\beta}(t)| \le 1$, a.e. on $\mathbf{R}$. Then if A is a measurable subset of $\mathbf{R}$, a direct computation shows that $\|\beta v\|_A^2 \le 3\|v\|_A^2$ for any $v \in X$.

We will use these cutoff functions to study for every $u \in B_{\bar{r}}(J)$ the different contributions to $\varphi'(u)$ due to the behaviour of $u(t)$ on each interval I_i. In fact, as one argues from the following lemma, if $\|\varphi'(\beta_{u,i}u)\|$ is sufficiently large with respect to ξ_i, then we get informations on both $\varphi'(u)$ and $\varphi_i'(u)$.

LEMMA 6.2. – *If $J \in J_k(\xi)$, $u \in \mathcal{B}_{\bar{r}}(J)$ then* $\forall i \in \{1, \dots, k\}$

$$\sup_{\|V\|=1} |\varphi'(\beta_{u,i}u)V - \varphi'(u)\beta_{u,i}V| = \sup_{\|V\|=1} |\varphi'(\beta_{u,i}u)V - \varphi_i'(u)\beta_{u,i}V| \le 2\xi_i.$$

Proof. – Note that if $u \in \mathcal{B}_{\bar{r}}(J)$ we have $\|u\|_{L^\infty(\mathbf{R}\setminus\cup_{i=1}^k Q_{j_i})} \le \bar{\delta}$ for all $i = 1, \cdots, k$ and in particular $\|u\|_{L^\infty(N_{u,i})} \le \bar{\delta}$ where $N_{u,i} = N^-_{u,i} \cup N^+_{u,i}$. Therefore by the choice of $\bar{\delta}$ for every $V \in X$ with $\|V\| = 1$ we obtain

$$\begin{aligned} &|\varphi'(\beta_{u,i}u)V - \varphi'(u)\beta_{u,i}V| \\ \le &|\langle \beta_{u,i}u, V\rangle - \langle u, \beta_{u,i}V\rangle| + |\int_{N_{u,i}} \alpha(t)(\nabla W(\beta_{u,i}u) - \nabla W(u)\beta_{u,i})V\,dt \\ \le &|\int_{N_{u,i}} \dot{\beta}_{u,i}(u\dot{V} - \dot{u}V)dt| + \tfrac{1}{2}\int_{N_{u,i}} |u||V|dt \le 2\|u\|_{N_{u,i}} \le 2\xi_i. \qquad \blacksquare \end{aligned}$$

Now note that if $u \in \mathcal{B}_r(J)$ we have $\|u\|_{L^\infty(\mathbf{R}\setminus\cup_{i=1}^k Q_{j_i})} \leq \bar{\delta}$ and therefore $\|\bar{\beta}_{u,i}u\|_{L^\infty} \leq \bar{\delta}$ too. With the agreement that $\xi_0 = \xi_1$, we define for $l \in \{0,\dots,k\}$

$$\sigma_{u,l} = \begin{cases} 1 & \text{if } \|u\|^2_{A_{u,l}} \geq 2\xi_l^2 \\ \dfrac{1}{2}\left(\dfrac{\xi_k^2}{\sum_{i=0}^k \xi_i^2}\right) & \text{otherwise} \end{cases}$$

and

$$\mathcal{W}_u = \sum_{l=0}^k \sigma_{u,l}\bar{\beta}_{u,l}u.$$

Then, by the choice of $\bar{\delta}$, if the mass of the function u on M_i (which is always contained in $I_i \cap A_u$) is sufficiently large w.r.t. ξ_i, then $\mathcal{W}_u$ is an increasing direction both for φ and φ_i. In fact we have

LEMMA 6.3. – *For every $J \in J(\xi)$ and $u \in \mathcal{B}_{\bar{r}}(J)$, we have*

$$\varphi'(u)\mathcal{W}_u \geq \tfrac{1}{2}\sum_{l=0}^k \sigma_{u,l}(\|u\|^2_{A_{u,l}} - 2\xi_l^2),\ \varphi_i'(u)\mathcal{W}_u \geq \tfrac{1}{2}\sum_{l=0}^k \sigma_{u,l}(\|u\|^2_{I_i\cap A_{u,l}} - 2\xi_l^2)$$

for all $i \in \{1,\dots,k\}$.

Proof. – By the choice of $\bar{\delta}$ we have

$$\begin{aligned}
\varphi'(u)\mathcal{W}_u &= \sum_{l=0}^k \sigma_{u,l}(\langle u, \bar{\beta}_{u,l}u\rangle - \int \nabla W(u)\bar{\beta}_{u,l}u\, dt) \\
&\geq \sum_{l=0}^k \sigma_{u,l}(\langle u, \bar{\beta}_{u,l}u\rangle - \frac{1}{4}\int \bar{\beta}_{u,l}u^2 dt) \\
&\geq \sum_{l=0}^k \sigma_{u,l}(\tfrac{3}{4}\|u\|^2_{A_{u,l}} + \int_{B_{u,l}\setminus A_{u,l}} [\bar{\beta}_{u,l}(\dot{u}^2 + \tfrac{3}{4}u^2) + \dot{\bar{\beta}}_{u,l}u\dot{u}]\, dt) \\
&\geq \sum_{l=0}^k \sigma_{u,l}(\tfrac{3}{4}\|u\|^2_{A_{u,l}} - \int_{B_{u,l}\setminus A_{u,l}} |\dot{\bar{\beta}}_{u,l}|u\dot{u}\, dt) \\
&\geq \sum_{l=0}^k \sigma_{u,l}(\tfrac{3}{4}\|u\|^2_{A_{u,l}} - \tfrac{1}{2}\|u\|^2_{B_{u,l}\setminus A_{u,l}}) \geq \tfrac{1}{2}\sum_{l=0}^k \sigma_{u,l}(\|u\|^2_{A_{u,l}} - 2\xi_l^2).
\end{aligned}$$

The computation for φ_i' is analogous. ∎

Remark 6.1. – Note that by construction we have

$$\min\{\varphi'(u)\mathcal{W}_u, \varphi_i'(u)\mathcal{W}_u\} \geq - \sum_{\{l;\|u\|^2_{A_{u,l}}<2\xi_l^2\}} \sigma_{u,l}\xi_l^2 \geq -\tfrac{1}{2}\xi_k^2 \tag{6.1}$$

and

$$\langle u, \mathcal{W}_u\rangle_{M_i} \geq \tfrac{1}{2}\Big(\frac{\xi_k^2}{\sum_{l=0}^k \xi_l^2}\Big)\|u\|^2_{M_i} \tag{6.2}$$

for all $i = 1, ..., k$.

Now we are able to prove Lemma 5.1 with the sequence of indices $\tilde{j}_i = \max\{j_i(h), j_i(\xi)\}$.

Proof of Lemma 5.1. – We will show that if $k \in \mathbf{N}$ and $J = (j_1, \ldots, j_k)$ verifies $j_1 < \ldots < j_k$ and $j_i \geq \tilde{j}_i$ $(i = 1, \ldots, k)$ and if $\mathcal{E}_k \cap \Phi_k \cap \mathcal{B}_{\bar{r}}(J) \cap \mathcal{K} = \emptyset$ then for any $u \in \mathcal{B}_{r_3}(J)$ there exists $F_u \in X$ with $\|F_u\|_{I_i} \leq 1$ which verifies the listed properties ($\mathcal{F}1$)-($\mathcal{F}5$). Then the existence of a locally Lipschitz vector field will follow with a classical pseudo-gradient construction.

Given $u \in B_{r_3}(J)$ we set

$$\mathcal{I}_1(u) = \{i \in \{1, \ldots, k\};\, r_1 \leq \inf_{v \in \mathcal{A}_{j_i}} \|u - v\|_{I_i},\, \varphi_i(u) \leq c_{\overline{\alpha}} + 2\tilde{h}\},$$

$$\mathcal{I}_2(u) = \{i \in \{1, \ldots, k\}\,;\, \varphi_i(u) \leq c_{\overline{\alpha}} - h_i\}.$$

For $i \in \mathcal{I}_1(u)$, we have either $\|u\|_{I_i \cap A_u} \geq r_0$ or $\|u\|_{I_i \cap A_u} < r_0$.

In the first case we have that $\max\{\|u\|^2_{A_{u,i-1}\cap I_i}, \|u\|^2_{A_{u,i}\cap I_i}\} \geq \frac{1}{2}r_0^2$. Therefore, since $\xi_1 \leq \frac{r_0}{3}$ (and so every ξ_i), by Lemma 6.3, we get

$$\begin{aligned} &\varphi'(u)\mathcal{W}_u \geq \\ &\geq \frac{1}{2}(\max\{\|u\|^2_{A_{u,i-1}\cap I_i}, \|u\|^2_{A_{u,i}\cap I_i}\} - 2\xi_{i-1}^2) - \frac{1}{2} \sum_{\{l\;;\;\|u\|^2_{A_{u,l}}<2\xi_l^2\}} \sigma_{u,l}\xi_l^2 \\ &\geq \frac{1}{2}\left(\frac{1}{2}r_0^2 - \frac{1}{4}r_0^2\right) - \frac{1}{2}\xi_k^2 \geq \frac{1}{16}r_0^2. \end{aligned}$$

The same computation shows also that $\varphi_i'(u)\mathcal{W}_u \geq \frac{1}{16}r_0^2$ and we conclude

$$\min\{\varphi'(u)\mathcal{W}_u, \varphi_i'(u)\mathcal{W}_u\} \geq \frac{1}{16}r_0^2. \tag{6.3}$$

In this case we set $F_{u,i} = 0$.

In the second case, i.e. $i \in \mathcal{I}_1(u)$ and $\|u\|_{I_i\cap A_u} < r_0$, we claim that $\beta_{u,i}u \in B_{\bar r}(\mathcal{A}_{j_i}) \setminus \mathcal{A}_{j_i}$ and $\varphi(u) \le c_{\overline{\alpha}} + \bar h$.

Indeed we obtain easily that $\beta_{u,i}u \in X\setminus\mathcal{A}_{j_i}$ since by Lemma 6.1 we have

$$\begin{aligned}\inf_{v\in\mathcal{A}_{j_i}} \|\beta_{u,i}u - v\| &\ge \inf_{v\in\mathcal{A}_{j_i}} \|\beta_{u,i}u - v\|_{I_i}\\ &\ge \inf_{v\in\mathcal{A}_{j_i}} \|u - v\|_{I_i} - \|u-\beta_{u,i}u\|_{I_i}\\ &\ge r_1 - (\|u\|^2_{A_u\cap I_i} + 4\xi_i^2)^{\frac12} \ge r_1 - (r_0^2 + \tfrac12 r_0^2)^{\frac12} > 0.\end{aligned}$$

On the other hand we recall that by Lemma 4.1, since $Q_{j_i}\subset I_i$, we have that $\sup_{\mathcal{A}_{j_i}} \|v\|_{\mathbf{R}\setminus I_i} \le r_0$. Therefore

$$\begin{aligned}\inf_{v\in\mathcal{A}_{j_i}} \|\beta_{u,i}u - v\| &\le \inf_{v\in\mathcal{A}_{j_i}} \|\beta_{u,i}u - v\|_{I_i} + r_0\\ &\le \inf_{v\in\mathcal{A}_{j_i}} \|u - v\|_{I_i} + \|\beta_{u,i}u - u\|_{I_i} + r_0\\ &\le r_3 + (r_0^2 + \frac12 r_0^2)^{\frac12} + r_0 < \bar r\end{aligned}$$

To prove our claim we have to show that $\varphi(\beta_{u,i}u) \le c_{\overline{\alpha}} + \bar h$.

To this end we observe that since $\|u\|_{L^\infty(I_i\setminus Q_{j_i})} < \bar\delta$, we have $\frac12\|u\|^2_{B_u\cap I_i} - \int_{B_u\cap I_i}\alpha(t)W(u)dt \ge 0$. Then since $\varphi_i(u) \le c_{\overline{\alpha}} + 2\tilde h$ and $\xi_i^2 < \frac{\tilde h}{3}$, we obtain

$$\begin{aligned}\varphi(\beta_{u,i}u) = \varphi_i(\beta_{u,i}u) &= \frac12\|u\|^2_{I_i\setminus B_u} + \frac12\|\beta_{u,i}u\|^2_{N_{u,i}} +\\ &\quad - \int_{I_i\setminus B_u}\alpha(t)W(u)dt - \int_{N_{u,i}}\alpha(t)W(u)dt\\ &\le \varphi_i(u) + 3\xi_i^2 \le c_{\overline{\alpha}} + 3\tilde h < c_{\overline{\alpha}} + \bar h.\end{aligned}$$

Then by the *annuli property* there exists $V_{u,i}\in X$, $\|V_{u,i}\| = 1$, such that $\varphi'(\beta_{u,i}u)V_{u,i} \ge \frac{\bar\nu}{2}$. By Lemma 6.2, since $\xi_i^2 < \xi_1 \le \frac{\bar\nu}{8}$, we obtain

$$\min\{\varphi(u)\beta_{u,i}V_{u,i}, \varphi_i'(u)\beta_{u,i}V_{u,i}\} \ge \tfrac{\bar\nu}{4}.$$

Then if $i\in\mathcal{I}_1(u)$ and $\|u\|_{I_i\cap A_u} < r_0$ we set $F_{u,i} = \beta_{u,i}V_{u,i}$. Since $\xi_k^2 \le \frac{\bar\nu}{8}$, by (6.1) we have

$$\min\{\varphi'(u)(F_{u,i} + \mathcal{W}_u), \varphi_i'(u)(F_{u,i}+\mathcal{W}_u)\} \ge \tfrac{\bar\nu}{8}. \tag{6.4}$$

Defining

$$F_u^{(1)} = \begin{cases}\sum_{i\in\mathcal{I}_1(u)} F_{u,i} + \mathcal{W}_u & \text{if } \mathcal{I}_1(u)\ne\emptyset\\ 0 & \text{otherwise,}\end{cases}$$

by (6.3) and (6.4), recalling that $\bar{\mu} = \frac{1}{8}\min\{\frac{r_0^2}{16}, \frac{\bar{\nu}}{8}\}$, we finally obtain that if $\mathcal{I}_1(u) \neq \emptyset$ then

$$\begin{cases} \varphi'(u)F_u^{(1)} \geq 8\bar{\mu} & \\ \varphi_i'(u)F_u^{(1)} \geq 8\bar{\mu} & \text{if } i \in \mathcal{I}_1(u) \\ \langle u, F_u^{(1)}\rangle_{M_i} = \langle u, \mathcal{W}_u\rangle_{M_i} \geq \frac{1}{2}\left(\frac{\xi_k^2}{\sum_{l=0}^k \xi_l^2}\right)\|u\|_{M_i}^2. & i = 1, \ldots, k. \end{cases} \tag{6.5}$$

Now we consider the case $i \in \mathcal{I}_2(u)$.

Considered $\lambda_i^2 = \min\{\frac{h_i}{2}, r_0^2\}$, we have either $\|u\|_{A_u \cap I_i} \geq \lambda_i$ or $\|u\|_{A_u \cap I_i} < \lambda_i$.

In the first case we set $\tilde{F}_{u,i} = 0$ and we observe that, replacing r_0 with λ_i, the same estimative with which we obtained (6.3), give now

$$\min\{\varphi'(u)\mathcal{W}_u, \varphi_i'(u)\mathcal{W}_u\} \geq \frac{1}{16}\lambda_i^2. \tag{6.6}$$

In the second case we claim that $\beta_{u,i}u \in B_{\bar{r}}(A_{j_i})$ and $\varphi(\beta_{u,i}u) \leq c_{\overline{\alpha}} - \frac{h_i}{2}$. Indeed, since $\lambda_i \leq r_0$ we have already prove that $\beta_{u,i}u \in B_{\bar{r}}(A_{j_i})$. Moreover, since $\lambda_i^2 \leq \frac{h_i}{2}$, $\xi_i^2 \leq \frac{h_i}{16}$ and $\|u\|_{L^\infty(I_i \cap B_u} \leq \bar{\delta}$, we have

$$\begin{aligned} |\varphi_i(u) - \varphi(\beta_{u,i}u)| &= \frac{1}{2}(\|u\|_{I_i}^2 - \|\beta_{u,i}u\|_{I_i}^2) - \int_{I_i} \alpha(t)(W(u) - W(\beta_{u,i}u))dt \\ &\leq \frac{1}{2}(\|u\|_{I_i \cap A_u}^2 + 4\xi_i^2) - \frac{1}{4}\int_{N_{u,i}} |u|^2\, dt \\ &\leq \frac{1}{2}\|u\|_{I_i \cap A_u}^2 + 3\xi_i^2 < \frac{1}{2}h_i \end{aligned}$$

and since $\varphi_i(u) \leq c_{\overline{\alpha}} - h_i$ the claim is proved.

By (S_i) there exists $\tilde{V}_{u,i} \in X$, $\|\tilde{V}_{u,i}\| = 1$, such that $\varphi'(\beta_{u,i}u)\tilde{V}_{u,i} \geq \frac{\nu_i}{2}$. By Lemma 6.2, since $\xi_i^2 < \xi_i \leq \frac{\nu_i}{8}$ we have

$$\min\{\varphi(u)\beta_{u,i}\tilde{V}_{u,i}, \varphi_i'(u)\beta_{u,i}\tilde{V}_{u,i}\} \geq \tfrac{\nu_i}{4}.$$

Then if $i \in \mathcal{I}_2(u)$ and $\|u\|_{I_i \cap A_u} < \lambda_i$ we set $\tilde{F}_{u,i} = \beta_{u,i}\tilde{V}_{u,i}$. Therefore, since $\xi_k^2 \leq \xi_i \leq \frac{\nu_i}{8}$, by (6.1) we have

$$\min\{\varphi'(u)(\tilde{F}_{u,i} + \mathcal{W}_u), \varphi_i'(u)(\tilde{F}_{u,i} + \mathcal{W}_u)\} \geq \tfrac{\nu_i}{8}. \tag{6.7}$$

We define

$$F_u^{(2)} = \begin{cases} \sum_{i \in \mathcal{I}_2(u)} \tilde{F}_{u,i} + \mathcal{W}_u & \text{if } \mathcal{I}_2(u) \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

and by (6.6), (6.7) and (6.2), we have that if $\mathcal{I}_2(u) \neq \emptyset$ then

$$\begin{cases} \varphi'(u)F_u^{(2)} \geq \min\left\{\dfrac{1}{16}\lambda_k^2, \tfrac{1}{8}\nu_k\right\} & \\ \varphi_i'(u)F_u^{(1)} \geq \min\left\{\dfrac{1}{16}\lambda_k^2, \tfrac{1}{8}\nu_k\right\} & \text{if } i \in \mathcal{I}_2(u) \\ \langle u, F_u^{(2)}\rangle_{M_i} = \langle u, \mathcal{W}_u\rangle_{M_i} \geq \dfrac{1}{2}\left(\dfrac{\xi_k^2}{\sum_{l=0}^k \xi_l^2}\right)\|u\|_{M_i}^2. & i = 1, \dots, k. \end{cases} \tag{6.8}$$

Finally we consider the case $\mathcal{I}_1(u) = \mathcal{I}_2(u) = \emptyset$. Also in this case we distinguish between the two following alternative cases:

$$\max_{1 \leq i \leq k}(\|u\|_{M_i}^2 - 8\xi_{i-1}^2) \geq 0 \quad \text{or} \quad \max_{1 \leq i \leq k}(\|u\|_{M_i}^2 - 8\xi_{i-1}^2) < 0.$$

In the first case there exists $i \in \{1, \dots, k\}$ for which $\|u\|_{M_i}^2 \geq 8\xi_{i-1}^2$ and by Lemma 6.3 we have

$$\varphi'(u)\mathcal{W}_u \geq \frac{1}{2}(\max\{\|u\|_{A_{u,i-1}\cap M_i}^2, \|u\|_{A_{u,i}\cap M_i}^2\} - 2\xi_{i-1}^2) - \frac{1}{2}\xi_k^2 \geq \frac{1}{2}\xi_k^2. \tag{6.9}$$

In the second case if $u \in \{\varphi < kc_{\overline{\alpha}} + \tilde{h}\}$, since $\frac{h_i}{8} \geq 8\xi_{i-1}^2$ $(1 \leq i \leq k)$, we have that $u \in \mathcal{E}_k \cap \Phi_k \cap \mathcal{B}_{r_3}(J)$. Since $\mathcal{E}_k \cap \Phi_k \cap \mathcal{B}_{\bar{r}}(J) \cap \mathcal{K} = \emptyset$ and since in $\mathcal{B}_{\bar{r}}(J)$ the Palais Smale sequences are precompact, there exists $\tilde{\nu}_J > 0$ such that for any $u \in \mathcal{E}_k \cap \Phi_k \cap \mathcal{B}_{r_3}(J)$ there exists $V_u \in X$, $\|V_u\| = 1$ and such that $\varphi'(u)V_u \geq \tilde{\nu}_J$.
Setting $\nu_J = \frac{1}{8}\min\{\tilde{\nu}_J, \frac{1}{2}\xi_k^2\}$ and

$$F_u^{(3)} = \begin{cases} \mathcal{W}_u & \text{if } \mathcal{I}_1(u) = \mathcal{I}_2(u) = \emptyset \text{ and } \max_{1 \leq i \leq k}(\|u\|_{M_i}^2 - 8\xi_{i-1}^2) \geq 0, \\ V_u & \text{if } \mathcal{I}_1(u) = \mathcal{I}_2(u) = \emptyset \text{ and if } \max_{1 \leq i \leq k}(\|u\|_{M_i}^2 - 8\xi_{i-1}^2) < 0, \\ 0 & \text{otherwise,} \end{cases}$$

we have that if $\mathcal{I}_1(u) = \mathcal{I}_2(u) = \emptyset$ then

$$\varphi'(u)F_u^{(3)} \geq 8\nu_J \tag{6.10}$$

and moreover, if $u \in \mathcal{B}_{r_3}(J) \setminus \mathcal{E}_k$, by (6.2) we have

$$\langle u, F_u^{(3)}\rangle_{M_i} = \langle u, \mathcal{W}_u\rangle_{M_i} \geq \frac{1}{2}\left(\frac{\xi_k^2}{\sum_{i=0}^k \xi_i^2}\right)\|u\|_{M_i}^2 \quad i = 1, \dots, k. \tag{6.11}$$

We define

$$F_u = \frac{1}{6}F_u^{(1)} + \frac{1}{6}F_u^{(2)} + \frac{1}{6}F_u^{(3)}$$

obtaining our results by (6.5), (6.8), (6.10) and (6.11). We note that it is not restrictive to assume $\|F_u\|_{I_i} \leq 1$ choosing $\bar{r}$ smaller if necessary. ∎

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