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Weak compactness of wave maps and harmonic maps

by

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ABSTRACT. – We show that a weak limit of a sequence of wave maps in $(1+2)$ dimensions with uniformly bounded energy is again a wave map. Essential ingredients in the proof are Hodge structures related to harmonic maps, $\mathcal{H}^1$ estimates for Jacobians, $\mathcal{H}^1$-BMO duality, a "monotonicity" formula in the hyperbolic context and the concentration compactness method. Application of similar ideas in the elliptic context yields a drastically shortened proof of recent results by Bethuel on Palais-Smale sequences for the harmonic map functional on two dimensional domains and on limits of almost H-surfaces. © Elsevier, Paris

RÉSUMÉ. – La limite faible d'une suite d'applications d'ondes bornée en énergie est de nouveau une application d'ondes (au sens faible). La démonstration utilise de façon essentielle les structures de Hodge associées aux applications harmoniques (ou aux applications d'ondes), des

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estimations dans l'espace $\mathcal{H}^1$ de Hardy pour les Jacobiens (dualité $\mathcal{H}^1$-BMO), une « formule de monotonicité » dans le contexte hyperbolique, et la méthode de concentration-compacité. Dans le cas elliptique, nos idées fournissent une démonstration très simple et naturelle de résultats récents pour la convergence de suites Palais-Smale pour la fonctionnelle liée aux applications harmoniques en dimension deux et pour les surfaces à courbure moyenne prescrite, dues à Bethuel.

1. INTRODUCTION

In this paper we show how the concentration compactness method of P. L. Lions, in combination with other geometrical and analytical estimates, can be used to establish stability results for harmonic maps and wave maps under weak convergence. The idea is that, due to the determinant structure of the equation (in a suitable gauge), one can pass to the limit in the nonlinearity, up to an additional singular term. One can then show that the singular term is supported on a set that is small enough in order not to affect the validity of the (weak) limit equation. A posteriori it turns out that the singular term in fact vanishes. This strategy is closely related to capacity methods (see e.g. Frehse's review [10]). Beginning with the work of Di Perna and Majda similar ideas have also been successfully applied to the two-dimensional Euler equation and related equations (see [1], [7], [20], [32]).

To illustrate this idea we first give drastically shortened proofs of results by Bethuel on the convergence of Palais-Smale sequences for the harmonic map functional on two dimensional domains and on limits of almost H-surfaces. Our main result is that the weak limit of (smooth) wave maps on (1+2) dimensional Minkowski space is a wave map. This was first proved in [12]. The proof given here does not require a detailed analysis of the "concentration set" and some of the finer regularity and interpolation estimates in [12].

In the following N denotes a compact, smooth k-dimensional manifold which we may assume to be isometrically embedded into some $\mathbb{R}^d$. The equivalence of various possible notions of weakly harmonic maps is discussed in the appendix.

THEOREM 1.1 ([2]). – *Let* Ω *be a (bounded) domain in* $\mathbb{R}^2$ *and let* $u^n : \Omega \to N \subset \mathbb{R}^d$ *be a Palais-Smale sequence for the harmonic map functional*, i.e.

$$-\Delta u^n + f^n \perp T_{u^n} N, \tag{1.1}$$

and

$$u^n \text{ bounded in } H^1(\Omega; \mathbb{R}^d), \quad f^n \to 0 \text{ in } H^{-1}(\Omega; \mathbb{R}^d). \tag{1.2}$$

Then, after extraction of a subsequence, $u_n \rightharpoonup u$ *in* $H^1(\Omega; \mathbb{R}^d)$ *and* u *is weakly harmonic,* i.e.

$$-\Delta u \perp T_u N$$

in the weak sense.

THEOREM 1.2 ([2]). – *Let* $H : \mathbb{R}^3 \to \mathbb{R}$ *be Lipschitz and suppose that the maps* $u^n : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ *satisfy*

$$u^n \rightharpoonup u \text{ in } H^1(\Omega; \mathbb{R}^3),$$

$$-\Delta u^n - 2H(u^n)\frac{\partial u^n}{\partial x} \wedge \frac{\partial u^n}{\partial y} \rightharpoonup 0$$

in the sense of distributions. Then

$$-\Delta u = 2H(u)\frac{\partial u}{\partial x} \wedge \frac{\partial u}{\partial y}.$$

Remark. – We do not need to assume $H \in L^\infty(\mathbb{R}^3)$. Therefore the expression $H(u)\frac{\partial u}{\partial x} \wedge \frac{\partial u}{\partial y}$ and the corresponding expression for u^n are in general only defined as distributions (see section 2.2 below and [19], Lemma IV.3).

THEOREM 1.3. – *Let* $u^n : \mathbb{R} \times \mathbb{R}^2 \to N$ *be a sequence of smooth wave maps on* $(1+2)$ *dimensional Minkowski space,* i.e.

$$\Box u^n := \frac{\partial^2 u^n}{\partial t^2} - \Delta u^n \quad \perp T_{u^n} N$$

and suppose that

$$u^n \to u \text{ in } L^2_{\text{loc}}(\mathbb{R}^3),$$

$$\sup_n \sup_{t \in \mathbb{R}} \|Du^n(t, \cdot)\|_{L^2(\mathbb{R}^2)} \leq C,$$

where D *denotes the space-time gradient.*

Then

$$\Box u \perp T_u N,$$

in the weak sense.

Remark. – If $||Du^n(t,\cdot)||_{L^2}$ is finite for some t it is in fact independent of t by the energy identity.

This compactness result is the key ingredient in the recent proof of global existence of weak wave maps for initial data $u_0 := u(0,\cdot), u_1 := \partial_t u(0,\cdot)$ in the energy class $H^1 \times L^2$; see [21] for an approach by a viscous approximation (previously employed for homogeneous targets by Zhou [33] see also [34]) and [23] for finite-difference approximations. In dimensions $m \geq 2$ Shatah [25] had established the existence of weak solutions to the Cauchy problem for wave maps $u : \mathbb{R} \times \mathbb{R}^m \to N$ for such data if the target manifold N is a sphere. The proof, which uses a penalization technique, depends crucially on the symmetry of S^k. This result has been recently generalized by Freire [11] and Zhou [33] to homogeneous spaces as targets. For a slightly different problem that captures the essential difficulties of the Cauchy problem for wave maps Klainerman and Machedon [17], [18] established short time existence, uniqueness and continuous dependence for inital data in $H^{m/2+\epsilon} \times H^{m/2-1+\epsilon}$, for all $\epsilon > 0$, at least for $m = 2, 3$. Their proof uses new Strichartz type estimates that exploit the fact that the wave map equation may be written explicitly as a system of hyperbolic conservation laws with a particular null form structure. This result is further confirmation that the conformally invariant case $m = 2$ is the borderline case. Shatah and Tahvildar-Zadeh [26] proved that, for $m > 2$, solutions may develop singularities in finite time. Uniqueness and (partial) regularity of suitable weak solutions for $m = 2$ remains a challenging open problem. A brief survey of wave maps and further references can be found in [29].

2. THE STATIONARY CASE

2.1. Proof of Theorem 1.1

We first reduce the problem to a periodic setting. It suffices to show that u is harmonic for every square $Q \subset \Omega$. Fix such a square. After translation and scaling we may assume $Q = [-\frac{1}{4}, \frac{1}{4}]^2$. Define maps $v^n \in H^1(T^2)$ by reflection of u^n across the lines $x_1 = \pm\frac{1}{4}$, $x_2 = \pm\frac{1}{4}$ and periodic extension.

Then

$$v^n_{|Q} = u^n \tag{2.1}$$
$$||v^n||^2_{H^1(T^2)} = 4||u^n||^2_{H^1(Q)} \leq C.$$

By a construction of Hélein [16] or of Christoudolou and Tahvildar-Zadeh [5] we may assume that the tangent bundle TN is parallelizable since N can always be realised as a totally geodesic submanifold of a compact manifold with that property. We may thus consider a smooth orthonormal frame $(\overline{e_i})$ of TN, *i.e.* a collection of smooth tangent vectorfields $\overline{e_1}, \dots \overline{e_k}$ such that for every $p \in N$ the fields $\overline{e_1}(p), \dots \overline{e_k}(p)$ form an orthonormal basis of T_pN. For $v \in H^1(T^2, N)$ the fields $e_i = \overline{e_i} \circ v$ form an orthonormal frame of the pull-back bundle $v^{-1}TN$, *i.e.* for every $x \in T^2$ the vectors $\overline{e_1}(v(x)), \dots, \overline{e_k}(v(x))$ are an orthonormal basis of $T_{v(x)}N$. The connection form ω_{ij} of that frame is defined as

$$\omega_{ij} = \langle de_i, e_j \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$. Define 1-forms θ_i by

$$\theta_i = \langle dv, e_i \rangle \quad \text{or} \quad dv = \theta_i e_i.$$

Then

$$\langle -\Delta v, e_i \rangle = \delta\theta_i + \omega_{ij} \cdot \theta_j \tag{2.2}$$

where for two one-forms $\varphi = \varphi_\alpha \, dx^\alpha, \psi = \psi_\alpha \, dx^\alpha$ on T^2 the contraction (with respect to the standard metric of T^2) is given by $\varphi \cdot \psi = \varphi_\alpha \psi_\alpha$ and where $\delta\phi = -(\frac{\partial \phi^1}{\partial x^1} + \frac{\partial \phi^2}{\partial x^2})$. Thus v is harmonic if and only if $\delta\theta_i + \omega_{ij} \cdot \theta_j = 0$.

We next recall Hélein's idea to choose a good frame (e_i) so that the quadratic term in (2.2) has a determinant structure. If (e_i) is an orthonormal frame of $v^{-1}TN$ so is $(\tilde{e}_i)$, given by

$$\tilde{e_i}(x) = R_{ij}(x) e_j(x), \quad \text{where } R(x) = (R_{ij}(x)) \in \mathrm{SO}(k).$$

A standard lower semicontinuity argument shows that there is a frame $(\widehat{e_i})$ that minimizes $\sum_i \int_{T^2} |\nabla e_i|^2 \, dx$ over all such frames. The necessary condition for the minimizer implies that the corresponding connection form $\widehat{\omega_{ij}}$ satisfies the Coulomb gauge condition

$$\delta \widehat{\omega_{ij}} = 0.$$

To see this, we may assume that the infimum is achieved for $R_{ij} = id$. Recall that $so(k) = T_{id}SO(k) = \{r = (r_{ij}); r_{ij} = -r_{ji}\}$. By minimality, for any smooth map $r: T^2 \to so(k)$ the first variation in direction r vanishes; that is,

$$\begin{aligned}\int_{T^2} \langle \partial_\alpha(\bar{e}_i \circ v), \partial_\alpha\big(r_{ij}(\bar{e}_j \circ v)\big)\rangle\, dx \\ = \int_{T^2} \omega_{ij,\alpha}\partial_\alpha r_{ij}\, dx + \int_{T^2} \langle \partial_\alpha(\bar{e}_i \circ v), \partial_\alpha(\bar{e}_j \circ v)\rangle r_{ij}\, dx \\ = \int_{T^2} \omega_{ij,\alpha}\partial_\alpha r_{ij}\, dx = 0.\end{aligned}$$

Here we also used anti-symmetry of r. Hence $\partial_\alpha \omega^n_{ij,\alpha} = 0$ in the sense of distributions, as claimed.

With a slight change of notation we assume from now on that (e^n_i) and (e_i) are the above minimizers and are thus in Coulomb gauge. In particular by the minimizing property

$$\sum_i \int |\nabla e^n_i|^2 \leq \sum_i \int |\nabla(\overline{e_i} \circ v^n)|^2 \leq C\|Dv^n\|^2_{L^2} \leq C,$$

and, after extraction of a further subsequence, we may assume

$$e^n_i \rightharpoonup e_i \text{ in } H^1(T^2), \quad \omega^n_{ij} \rightharpoonup \omega_{ij} \text{ in } L^2(T^2).$$

Moreover, since $e^n \to e$ in L^2 and $|e^n| = 1$,

$$\theta^n \rightharpoonup \theta \text{ in } L^2(T^2). \tag{2.3}$$

Consider the Hodge decomposition for ω^n_{ij} (resp. ω_{ij})

$$\omega^n_{ij} = da^n_{ij} + \delta b^n_{ij} + H^n_{ij}, \quad \omega_{ij} = da_{ij} + \delta b_{ij} + H_{ij},$$

with the normalization

$$\int_{T^2} a^n_{ij}\, dx = \int_{T^2} a_{ij}\, dx = \int_{T^2} b^n_{ij} = \int_{T^2} b_{ij} = 0.$$

By the Coulomb gauge $\delta\omega^n_{ij} = 0$, whence $\Delta a^n_{ij} = 0$, and the normalization implies

$$a^n_{ij} \equiv a_{ij} \equiv 0.$$

Moreover H^n_{ij} has constant coefficients as a harmonic 1-form on T^2. By L^2-orthogonality of the decomposition

$$\|\delta b^n_{ij}\|^2_{L^2} + \|H^n_{ij}\|^2_{L^2} = \|\omega^n_{ij}\|^2_{L^2} \leq C,$$

and we may thus assume

$$b_{ij}^n \rightharpoonup b_{ij} \text{ in } H^1(T^2), \quad H_{ij}^n \to H_{ij} \text{ as constant forms.}$$

Writing $b_{ij}^n = \beta_{ij}^n \, dx^1 \wedge dx^2$ we see that the crucial term

$$\begin{aligned}\omega_{ij}^n \cdot \theta_j^n &= \delta b_{ij}^n \cdot \theta_j^n + H_{ij}^n \theta_j^n \\ &= -\left\langle \frac{\partial \beta_{ij}^n}{\partial x^1} \frac{\partial v^n}{\partial x^2} - \frac{\partial \beta_{ij}^n}{\partial x^2} \frac{\partial v^n}{\partial x^1}, e_j^n \right\rangle + H_{ij}^n \theta_{ij}^n,\end{aligned}$$

has a determinant structure (up to a harmless term). Thus by a result from concentration compactness ([19], Lemma IV.3)

$$\omega_{ij}^n \cdot \theta_j^n \rightharpoonup \omega_{ij} \cdot \theta_j + \sum_{l \in \mathcal{J}} \nu_l^i \, \delta_{x_l}, \tag{2.4}$$

in the sense of distributions (on T^2), where the set $\mathcal{J}$ is at most countable and $\sum_{l \in \mathcal{J}} |\nu_l| < \infty$. To apply Lions' result in the periodic context it suffices to observe that Wente's inequality ([31], Theorem 3.1)

$$\begin{aligned}\left| \int_{T^2} v^1 \det(Dv^2, Dv^3) \, dx \right| &= \left| \frac{1}{3} \int_{T^2} v \cdot v_{x^1} \wedge v_{x^2} \, dx \right| \\ &\leq C \|Dv^1\|_2 \|Dv^2\|_2 \|Dv^3\|_2\end{aligned}$$

holds in the periodic setting, cf. e.g. Heinz' proof [15] via Fourier series or appendix A of [12].

It now follows from the hypotheses (1.1), (1.2) in conjunction with (2.1) to (2.4) that

$$\delta \theta_i + \omega_{ij} \cdot \theta_j = \sum_{l \in \mathcal{J}} \nu_l^i \, \delta_{x_l} \tag{2.5}$$

in the sense of distributions on Q. The left hand side is in $H^{-1}(Q) + L^1(Q)$ and this space contains no Dirac masses. Hence

$$\delta \theta_i + \omega_{ij} \cdot \theta_j = 0 \quad \text{on } Q \tag{2.6}$$

and thus u is harmonic in Q by (2.2) as claimed.

For a detailed deduction of (2.6) from (2.5) fix $x_l \in Q$, let $B(0,1)$ denote the open unit ball in $\mathbb{R}^2$, consider a test function $\varphi \in C_0^\infty(B(0,1))$ with $\varphi \equiv 1$ on $B(0, 1/2)$, and let $\varphi_k = \varphi(k(x - x_l))$. Then, as $k \to \infty$,

$$\varphi_k \rightharpoonup 0 \quad \text{in } H^1, \quad \varphi_k \to 0 \quad \text{boundedly a.e.}$$

Now test (2.5) with φ_k, let $k \to \infty$ and use the dominated convergence theorem to deduce $0 = \nu_l^i$, whenever $x_l \in Q$. □

2.2. Proof of Theorem 1.2

In this case the nonlinearity already has the determinant structure $f \det(Dg, Dh)$. Hence we deduce as above for a cube $Q \subset \Omega$

$$\Delta u - 2H(u) \cdot \frac{\partial u}{\partial x} \wedge \frac{\partial u}{\partial y} = \sum_{l \in \mathcal{J}} \nu_l \, \delta_{x_l} \quad \text{in } Q, \tag{2.7}$$

where $\mathcal{J}$ is at most countable and $\sum_{l \in \mathcal{J}} |\nu_l| < \infty$.

Recall that for $H \notin L^\infty$ the nonlinear term is defined as a distribution T via $\langle T, \varphi \rangle = \langle \varphi H(u), u_x \wedge u_y \rangle$, see [19], Lemma IV.3. Let $\varphi \in C_0^\infty(B(0,1))$, $\varphi \equiv 1$ on $B(0, 1/2)$ and for a fixed $l \in \mathcal{J}$ set $\varphi_k(x) = \varphi(k(x - x_l))$. Let $u_k(z) = u(x_l + k^{-1}z)$. Test (2.7) with φ_k and rescale to obtain

$$-\int_{B(0,1)} \nabla u_k \, \nabla \varphi \, dx - \left\langle 2H(u_k) \cdot \frac{\partial u_k}{\partial x} \wedge \frac{\partial u_k}{\partial y}, \varphi \right\rangle = \left\langle \sum \nu_l \delta_{x_l}, \varphi_k \right\rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing of distributions and test functions. Since

$$\int_{B(0,1)} |Du_k|^2 \, dx = \int_{B(x_l, \frac{1}{k})} |Du|^2 \, dx \to 0 \quad \text{as } k \to \infty$$

it follows from Wente's inequality that $\langle \varphi H(u_k), (u_k)_x \wedge (u_k)_y \rangle \to 0$ and thus

$$0 = \nu_l.$$

(Strictly speaking we should replace u_k by a periodic function that agrees with u_k outside $\operatorname{supp} \varphi$ to apply Wente's inequality.) The proof is finished. □

3. THE NONSTATIONARY CASE

3.1. Energy inequality

A smooth map $u : \mathbb{R} \times \mathbb{R}^2 \to N \hookrightarrow \mathbb{R}^d$ from $(1+2)$ dimensional Minkowski space into a k-dimensional Riemannian manifold N is called a wave map (or harmonic map with respect to the Lorentzian metric) if

$$\Box u := \left(\frac{\partial^2 u}{\partial t^2} - \Delta u \right) \perp T_u N. \tag{3.1}$$

Here points in Minkowski space are denoted by

$$z = (t, x) = (x^\alpha)_{0 \le \alpha \le 2}\,;\; x = (x^1, x^2),\; t = x^0.$$

The space-time gradient is

$$D = \left(\frac{\partial}{\partial t}, \nabla\right),$$

and $\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2}$ is the spatial Laplacian. For (smooth) wave maps the energy

$$E(u(t)) := \frac{1}{2}\int_{\{t\}\times\mathbb{R}^2} |Du|^2\,dx = \frac{1}{2}\int_{\{t\}\times\mathbb{R}^2} \left[\left|\frac{\partial u}{\partial t}\right|^2 + |\nabla u|^2\right] dx$$

is constant in time (if it is finite for some time t). To state a local energy estimate denote the light cone at a point $z_0 = (t_0, x_0)$ in space-time by

$$K(z_0) = \{z = (t, x) : |x - x_0| \le |t - t_0|\}$$

and its lateral boundary by

$$C(z_0) = \{z \in K(z_0) : |x - x_0| = |t - t_0|\}.$$

For fixed t let

$$D(t) = D(t, z_0) = \{z = (t, x) : z \in K(z_0)\}$$

denote the spatial sections of $K(z_0)$.

LEMMA 3.1. – *For any smooth solution u of* (3.1) *we have*

$$E(u, D(t)) := \frac{1}{2}\int_{D(t)} |Du|^2\,dx \le E(u; D(s)) \tag{3.2}$$

for all $s \le t \le t_0$ (and similarly for $t_0 \le t \le s$).

Proof. – Multiplication of (3.1) by $\frac{\partial u}{\partial t} \in T_uN$ yields

$$0 = \langle \Box u, \frac{\partial u}{\partial t}\rangle = \frac{1}{2}\frac{d}{dt}|Du|^2 - \operatorname{div}\langle \nabla u, \frac{\partial u}{\partial t}\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^k$.

Upon integration over $D(t)$, for $t \leq t_0$, we deduce that

$$\frac{d}{dt}E(u, D(t)) \leq 0$$

and the lemma is proved. □

COROLLARY 3.2. – *Let u be a smooth solution of* (3.1). *Then the "monotonicity formula"*

$$\frac{1}{r}\int_{B(z_0,r)} |Du|^2\, dz \leq \frac{C}{R}\int_{B(z_0,R)} |Du|^2\, dz, \quad \textit{for } r \leq R \tag{3.3}$$

holds.

Proof. – We may consider cylinders $Z(z_0, r) = z_0 + [-r, r] \times B(x_0, r)$ instead of balls, and by scaling and translation we may assume $z_0 = 0$, $r = 1$, $R \geq 4$. Now for $|t| \leq 1$, $|s| \leq R/2$ we have $|t - s| + 1 \leq R$ and thus by Lemma 3.1

$$\int_{\{t\}\times B(0,1)} |Du|^2 \leq \int_{\{s\}\times B(0,R)} |Du|^2.$$

Integration in t over $[-1, 1]$ and in s over $[-R/2, R/2]$ yields the result. □

3.2. Good frames and the determinant structure

Consider a sequence of smooth wave maps with uniformly bounded energy

$$\sup_n \sup_{t\in\mathbb{R}} E(u^n(t)) \leq C.$$

Upon passage to a subsequence we may assume that $u^n \to u$ in L^2_{loc}, and we seek to prove that u satisfies (3.1) in the weak sense.

As in the stationary case we first rewrite (3.1) in a way that makes the underlying determinant structure apparent. First observe that it suffices to verify (3.1) on every cube Q. Fix such a cube. By scaling and translation we may assume $Q = [-1/4, 1/4]^3$. Through reflection at $x^\alpha = \pm 1/4$, $0 \leq \alpha \leq 2$ and periodic extension we can define functions $v^n \in H^1(T^3, N)$ such that

$$v^n = u^n \text{on } Q, \tag{3.4}$$

$$\sup_t \int_{\{t\}\times T^2} |Dv^n|^2\, dx \leq C. \tag{3.5}$$

As in the stationary case we may assume that TN is parallelizable. Let $(\overline{e_1}, \ldots, \overline{e_k})$ be a smooth orthogonal frame of TN and let $e_i^n = \overline{e_i} \circ v^n$

be a frame of the pull-back bundle $(v^n)^{-1}TN$ with connection form $\omega_{ij}^n = \langle de_i^n, e_j^n \rangle$. With $\theta_i^n := \langle dv^n, e_i^n \rangle$ we have

$$\langle \Box v^n, e_i^n \rangle = \langle \delta dv^n, e_i^n \rangle = \delta\theta_i^n + \omega_{ij}^n \cdot \theta_j^n, \tag{3.6}$$

where δ is the codifferential with respect to the Lorenztian metric (which we take with signature $[-,+,+]$) and $\cdot$ denotes the contraction with respect to that metric. Thus

$$\varphi \cdot \psi = -\varphi_0\psi_0 + \varphi_1\psi_1 + \varphi_2\psi_2$$

for one-forms $\varphi = \varphi_\alpha\, dx^\alpha$, $\psi = \psi_\alpha\, dx^\alpha$ and

$$\delta\varphi = \frac{\partial\varphi^0}{\partial x^0} - \frac{\partial\varphi^1}{\partial x^1} - \frac{\partial\varphi^2}{\partial x^2}.$$

To rewrite the quadratic terms in (3.6) we minimize

$$\sum_{i,j} \int_{T^3} |D\widetilde{e}_i|^2$$

over the gauge equivalent frames

$$\widetilde{e}_i(x) = R_{ij}(x)e_j(x), \quad R_{ij}(x) \in \mathrm{SO}(k).$$

The minimizer $\widehat{e_i}^n$ exists by standard lower semicontinuity arguments and satisfies

$$\delta_{\mathrm{eucl}}\hat{\omega}^n = 0$$

(see section 2), where δ_{eucl} is the codifferential with respect to the Euclidean metric on T^3. Moreover, by (3.5), one has

$$\sum_i \|D\hat{e_i}^n\|_{L^2(T^3)} \le \sum_i \|De_i{}^n\|_{L^2(T^3)} \le C,$$

since $\hat{e_i}^n$ is minimizing.

To simplify the notation we write from now on e_i^n instead of $\hat{e_i}^n$. Passing to subsequences we thus may assume

$$e_i^n \rightharpoonup e_i \text{ in } H^1(T^3), \quad e_i^n \to e_i \quad \text{boundedly a.e.} \tag{3.7}$$

$$\omega_{ij}^n \rightharpoonup \omega_{ij} \text{ in } L^2(T^3), \tag{3.8}$$

and thus

$$\theta_i^n \rightharpoonup \theta_i \text{ in } L^2(T^3). \tag{3.9}$$

Moreover

$$\delta_{\text{eucl}}\omega_{ij}^n = 0 \quad \text{for } 1 \leq i, j \leq k. \tag{3.10}$$

Let $*$ be the Hodge$*$ operator with respect to the Lorentzian metric on T^3 and consider the Hodge decomposition of $*\,\theta_j^n$ with respect to the Euclidean metric

$$*\theta_j^n = d\,a_j^n + \delta_{\text{eucl}} b_j^n + c_j^n, \tag{3.11}$$

subject to the normalization

$$\int_{T^3} a_{j,\alpha}^n \, dz = \int_{T^3} b_j^n = 0, \quad 0 \leq \alpha \leq 2, \tag{3.12}$$

where $a_j^n = a_{j,\alpha}^n \, dx^\alpha$ and where the integration is with respect to the Euclidean volume element.

The c_j^n are harmonic two-forms and thus constant forms. More precisely $c_{j,\alpha\beta}^n = \int_{T^3} *\theta_{j,\alpha\beta}^n$ whence

$$c_j^n \to c_j \quad \text{as constant forms.} \tag{3.13}$$

The forms a^n and b^n satisfy

$$-\Delta_z a_j^n = \delta_{\text{eucl}}(*\theta_j^n), \quad -\Delta_z b_j^n = d(*\theta_j^n), \tag{3.14}$$

where $\Delta_z = -(d\delta_{\text{eucl}} + \delta_{\text{eucl}} d)$ is the (three dimensional) Euclidean Laplace operator on forms. In particular

$$a_j^n \to a_j, \quad b_j^n \to b_j \qquad \text{in } L^2(T^3). \tag{3.15}$$

The three-forms b_j^n may be written as

$$b_j^n = \beta_j^n dx^0 \wedge dx^1 \wedge dx^2 = \beta_j^n dz.$$

For a one-form ω and a three-form $b = \beta dz$ ($\beta = *_{\text{eucl}} b \in \Omega^0$) one has:

$$\begin{aligned} \omega \wedge \delta_{\text{eucl}} b &= \omega \wedge \delta_{\text{eucl}}(*_{\text{eucl}}\beta) = -\omega \wedge *_{\text{eucl}} d\beta = -(\omega \cdot_{\text{eucl}} d\beta) dz \\ &= \delta_{\text{eucl}}(\beta\omega) dz - \beta(\delta_{\text{eucl}}\omega) dz \end{aligned}$$

In view of the gauge condition $\delta_{\text{eucl}}\omega_{ij}^n = 0$ we can thus rewrite the nonlinearity as

$$
\begin{aligned}
(3.16)\quad *(\omega_{ij}^n \cdot \theta_j^n) &= \omega_{ij}^n \wedge *\theta_j^n \\
&= \omega_{ij}^n \wedge da_j^n + \omega_{ij}^n \wedge \delta_{\text{eucl}} b_j^n + \omega_{ij}^n \wedge c_j^n \\
&= -d(\omega_{ij}^n \wedge a_j^n) + *_{\text{eucl}}\delta_{\text{eucl}}(\beta_j^n \omega_{ij}^n) + \omega_{ij}^n \wedge c_j + d\omega_{ij}^n \wedge a_j^n .
\end{aligned}
$$

By (3.8), (3.13) and (3.15) the first three terms on the right hand side of (3.16) easily pass to the limit. Now $d\omega_{ij}^n = -de_i^n \wedge de_j^n$ has a determinant structure (here we use the compact notation $d\alpha \wedge d\beta = \sum_{e=1}^d d\alpha^e \wedge d\beta^e$ for two maps $\alpha, \beta : T^3 \to \mathbb{R}^d$). Hence the crucial term becomes

$$
(3.17)\qquad d\omega_{ij}^n \wedge a_j^n = -de_i^n \wedge de_j^n \wedge a_j^n .
$$

In the next two subsections we show how this term can be treated using the $\mathcal{H}^1$-BMO duality ([9]) and the $\mathcal{H}^1$ estimates for determinants ([6]), similar to arguments of Evans [8] and Bethuel [3].

3.3. BMO and $\mathcal{H}^1$ estimates

We first derive an estimate for a^n in the space BMO of functions of bounded mean oscillation. Consider the seminorm and the norm

$$
\begin{aligned}
[f]_{2,\lambda}^2 &:= \sup_{0<r<1} \sup_{y \in T^3} \frac{1}{r^\lambda} \int_{B(y,r)} |f - f_{y,r}|^2 \, dz, \\
\|f\|_{L^{2,\lambda}}^2 &:= \sup_{0<r<1} \sup_{y \in T^3} \frac{1}{r^\lambda} \int_{B(y,r)} |f|^2 \, dz,
\end{aligned}
$$

where $f_{y,r} := \fint_{B(y,r)} f$ denotes the average over $B(y, r)$. The Campanato and Morrey spaces are given by

$$
\begin{aligned}
\mathcal{L}^{2,\lambda}(T^3) &:= \{f \in L^2(T^3) : [f]_{2,\lambda} < \infty\}, \\
L^{2,\lambda}(T^3) &:= \{f \in L^2(T^3) : \|f\|_{L^{2,\lambda}} < \infty\},
\end{aligned}
$$

with norms $\|f\|_{L^2} + [f]_{2,\lambda}$ and $\|f\|_{L^{2,\lambda}}$ respectively. One has ([13], Prop. 3.2, Cor. 4.2) $L^{2,\lambda} \simeq \mathcal{L}^{2,\lambda}$ for $0 \le \lambda < 3$ and $\mathcal{L}^{2,3} \simeq$ BMO, with equivalent norms. We will use the seminorm

$$
[f]_{\text{BMO}(T^3)}^2 := \sup_{0<r<1} \sup_{y \in T^3} \fint_{B(y,r)} |f - f_{y,r}|^2 \, dz
$$

and the norm

$$
\|f\|_{\text{BMO}(T^3)} = [f]_{\text{BMO}(T^3)} + \left| \fint_{T^3} f \, dz \right| .
$$

Equivalent norms are ([13], Cor. 4.1, Cor. 4.2)

$$[f]_{\mathrm{BMO}(T^3)} + \|f\|_{L^p}, \quad \text{for } 1 \le p < \infty.$$

For an open set $U \subset T^3$ define the local BMO seminorm by

$$[f]^2_{\mathrm{BMO}(U)} = \sup\left\{ \fint_{B(y,r)} |f - f_{y,r}|^2 \, dz : B(y,r) \subset U \right\}.$$

Campanato made the fundamental observation that a good elliptic regularity theory can be developed based on the spaces $\mathcal{L}^{2,\lambda}$. In particular, if $F \in \mathcal{L}^{2,1}(T^3)(\simeq L^{2,1}(T^3))$, if $A(D) = a_\alpha(\partial/\partial x^\alpha)$ is a first order operator with constant coefficients and if U is the (weak) solution of

$$-\Delta U = A(D)F \text{ in } T^3, \quad \int_{T^3} U = 0,$$

then by estimate (10.2) in [4] (see also [13], Theorem 3.3) and Poincaré's inequality

$$\|U\|_{\mathrm{BMO}} \le C\|DU\|_{\mathcal{L}^{2,1}} \le C\|F\|_{\mathcal{L}^{2,1}} \le C\|F\|_{L^{2,1}}. \tag{3.18}$$

In view of (3.5) one has $\|\theta_j^n\|_{L^{2,1}} \le \|Dv^n\|_{L^{2,1}} \le C$ and hence by (3.12), (3.14) and (3.18)

$$\|a^n\|_{\mathrm{BMO}} \le C. \tag{3.19}$$

The Hardy space $\mathcal{H}^1(T^3) \subset L^1(T^3)$ can be defined as follows. For $f \in L^1(T^3)$ and $\varphi \in C_0^\infty(B(0,1))$ with $\int \varphi = 1$ consider the regularized maximal function

$$\mathcal{M}f(x) := \sup_{0<r<1} \left| \int_{T^3} r^{-3} \varphi\left(\frac{x-y}{r}\right) f(y) \, dy \right|,$$

and let

$$\mathcal{H}^1(T^3) := \{ f \in L^1(T^3) : \int_{T^3} f = 0, \quad \|\mathcal{M}f\|_{L^1(T^3)} < \infty \},$$
$$\|f\|_{\mathcal{H}^1} := \|\mathcal{M}f\|_{L^1(T^3)}.$$

Different φ lead to the same space with equivalent norms. See [9], [24] or [28] for further information on $\mathcal{H}^1$ and [14] for local versions of $\mathcal{H}^1$.

Our argument makes crucial use of the $\mathcal{H}^1$-BMO duality due to Fefferman and Stein and of the $\mathcal{H}^1$ estimates for Jacobians of Coifman, Lions, Meyer and Semmes.

THEOREM 3.3. – ([9]) *The space* $\mathrm{BMO}(T^3)/\ \{constant\ functions\}$ *is the dual of* $\mathcal{H}^1(T^3)$. *On the other hand* $\mathcal{H}^1(T^3)$ *is the dual of* $\mathrm{VMO}(T^3)/\ \{constants\}$, *where* $\mathrm{VMO}(T^3)$ *is the closure of smooth functions in* $\mathrm{BMO}(T^3)$.

THEOREM 3.4. – ([6]) *Let* $g, h \in H^1(T^3)$ *and let* M *be a a minor of order two. Then* $M(Dg, Dh) \in \mathcal{H}^1(T^3)$ *and*

$$\|M(Dg, Dh)\|_{\mathcal{H}^1(T^3)} \leq C\|Dg\|_{L^2(T^3)}\|Dh\|_{L^2(T^3)}. \tag{3.20}$$

Moreover, if $g^n \rightharpoonup g$, $h^n \rightharpoonup h$ *in* $H^1(T^3)$ *then*

$$M(Dg^n, Dh^n) \overset{*}{\rightharpoonup} M(Dg, Dh) \text{ in } \mathcal{H}^1(T^3).$$

Remark. – These results are usually stated for the whole space situation. The proofs can, however, be easily localized and adapted to the periodic case. The crucial estimate $|\int_{T^3} fM(Dg, Dh)| \leq C\|f\|_{\mathrm{BMO}}\|Dg\|_2\|Dh\|_2$ can also be easily deduced from the whole space result by a suitable cut-off (see Appendix A of [12]).

3.4. Concentration compactness

The defect in the weak continuity of the nonlinear term $de_i^n \wedge de_j^n \wedge a^n$ can now be precisely characterized.

THEOREM 3.5. – *Suppose that*

$$g^n \rightharpoonup g, \quad h^n \rightharpoonup h \quad in \quad H^1(T^3), \quad \gamma^n \rightharpoonup \gamma \quad in \quad L^2(T^3),$$

$$\|\gamma^n\|_{L^{2,1}(T^3)} \leq C.$$

Let $A(D) = a_\alpha \frac{\partial}{\partial x^\alpha}$ *be a first order differential operator on* T^3 *with constant coefficients, let* f^n *be the weak solution of*

$$-\Delta_z f^n = A(D)\gamma^n, \quad \int_{T^3} f^n = 0, \tag{3.21}$$

and let f *be defined in the same way by* γ.

Then, for any minor M *of order two one has, after passage to a subsequence,*

$$f^n M(Dg^n, Dh^n) \rightharpoonup fM(Dg, Dh) + \nu \tag{3.22}$$

in the sense of distributions, where ν *is a (signed) Radon measure and*

$$\operatorname{supp} \nu \subset \{z \in T^3 : \limsup_{R\to 0} \limsup_{n\to\infty} \|\chi_{B(z,R)}\gamma^n\|_{L^{2,1}} > 0\}. \tag{3.23}$$

Remark. – By elliptic regularity (see (3.18))

$$\|f^n\|_{BMO} \leq C\|\gamma^n\|_{L^{2,1}}. \tag{3.24}$$

The product $f^n M(Dg^n, Dh^n)$ can then be defined as a distribution; see the proof of the theorem for further details.

Combining Theorem 3.5 with the construction of optimal frames in section 3.2 we obtain the following result. For future reference we observe that the result does not only hold for solutions to (3.1); in fact uniform bounds in the Morrey space $L^{2,1}$ suffice.

THEOREM 3.6. – *Let N be a parallelizable compact k-dimensional Riemannian manifold (isometrically embedded into $\mathbb{R}^d$) and suppose that the maps $v^n : T^3 \to N$ satisfy*

$$v^n \rightharpoonup v \quad in\ H^1(T^3;\mathbb{R}^d), \quad \|Dv^n\|_{L^{2,1}} \leq C.$$

Then there exist orthonormal frames $\{e_1^n, \ldots, e_k^n\}$ of the pull-back bundles $(v^n)^{-1}TN$ and an orthonormal frame $\{e_1, \ldots, e_k\}$ of $v^{-1}TN$ such that $e_i^n \rightharpoonup e_i$ in $H^1(T^3;\mathbb{R}^d)$. Moreover, with the usual notation $\theta_j^n = \langle dv^n, e_j^n\rangle$, $\omega_{ij}^n = \langle de_i^n, e_j\rangle$, $\theta_j = \langle dv, e_j^n\rangle$ and $\omega_{ij} = \langle de_i, e_j\rangle$, one has

$$\omega_{ij}^n \cdot \theta_j^n \rightharpoonup \omega_{ij} \cdot \theta_j + \nu_i$$

as distributions, where ν_i is a Radon measure and

$$\operatorname{supp} \nu_i \subset \{z \in T^3 : \limsup_{R\to 0} \limsup_{n\to\infty} \|\chi_{B(z,R)}\theta^n\|_{L^{2,1}} > 0\}.$$

Theorem 3.5 above is adapted to the special structure of our problem (see (3.14), (3.16) and (3.17)). More generally the following result holds.

THEOREM 3.7. – *Suppose that*

$$f^n \to f \quad in\ L^2(T^m), \quad \|f^n\|_{\mathrm{BMO}} \leq C, \quad g^n \overset{*}{\rightharpoonup} g \quad in\ \mathcal{H}^1(T^m).$$

Then

$$f^n g^n \rightharpoonup fg + \nu$$

in the sense of distributions and ν is a Radon measure that satisfies

$$\operatorname{supp} \nu \subset \{x \in T^m : \limsup_{R\to 0} \limsup_{n\to\infty} [f^n]_{\mathrm{BMO}(B(x,R))} > 0\}. \tag{3.25}$$

The proof of Theorem 3.5 and a sketch of the proof of Theorem 3.7 are given at the end of this section. We first proceed with the proof of our main result, Theorem 1.3.

3.5. Proof of Theorem 1.3

Since u^n is a wave map we have in the notation of Section 3.2 (see (1.4) and (1.6))

$$\delta\theta_i^n + \omega_{ij}^n \cdot \theta_j^n = 0 \quad \text{in } Q.$$

We have to show the same identity for the weak limits θ_i, ω_{ij}. In view of (3.13), (3.15)-(3.17) as well as (3.14), (3.18) and Theorem 3.5 (applied with $\gamma^n = *\theta_n^j$ and $f^n = a_j^n$)

$$\omega_{ij}^n \cdot \theta_j^n \rightharpoonup \omega_{ij} \cdot \theta_j + \nu$$

and thus

$$\delta\theta_i - \omega_{ij} \cdot \theta_j = \nu \quad \text{on } Q \tag{3.26}$$

in the sense of distributions, where ν is a Radon measure with

$$\operatorname{supp}\nu \subset S := \{z \in Q : \limsup_{R\to 0} \limsup_{n\to\infty} \sum_j \|\chi_{B(z,R)}\theta_j^n\|_{L^{2,1}} > 0\}.$$

We have $\sum_j |\theta_j^n|^2 \le |Du^n|^2$ and we may assume (after passage to a subsequence)

$$|Du^n|^2 \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(Q).$$

Now for $\rho \ge R$ trivially

$$\frac{1}{\rho}\int_{B(z_0,\rho)} \chi_{B(z,R)}|Du^n|^2\,dz' \le \frac{1}{R}\int_{B(z,R)} |Du^n|^2\,dz',$$

while for $\rho \le R$ the expression on the left hand side vanishes unless $z_0 \in B(z,2R)$. In the latter case the monotonicity formula in Corollary 3.2 yields (for $B(z,3R) \subset Q$)

$$\begin{aligned}\frac{1}{\rho}\int_{B(z_0,\rho)} \chi_{B(z,R)}|Du^n|^2\,dz' &\le \frac{C}{R}\int_{B(z_0,R)} |Du^n|^2\,dz' \\ &\le \frac{C}{3R}\int_{B(z,3R)} |Du^n|^2\,dz'.\end{aligned}$$

It follows that

$$S \subset \{z : \lim_{r\to 0}\frac{1}{r}\mu(\overline{B(z,r)}) > 0\}.$$

By a standard covering argument $S \cap Q$ is a countable union of sets of finite one-dimensional Hausdorff measure and hence has vanishing $H^{1,2}$-capacity in T^3. On the other hand the left hand side of (3.26) is in $H^{-1} + L^1$.

Thus $\nu_{|Q} = 0$ and the proof is finished. In fact we may choose compact sets $K_k \subset S$ such that $\nu(S \backslash K_k) \to 0$ and smooth functions φ_k satisfying $0 \le \varphi_k \le 1$, $\varphi_k = 0$ on K_k, $\varphi_k \to 1$ in H^1 and almost everywhere. Upon testing (3.26) with $\varphi\varphi_k$, for $\varphi \in C_0^\infty$, we can use the dominated convergence theorem to pass to the limit $k \to \infty$ to obtain $\langle \delta\theta_i, \varphi \rangle - (\omega_{ij} \cdot \theta_j)\varphi = 0$. □

3.6. Proof of the first concentration theorem

We will use the following fact about BMO.

PROPOSITION 3.8. – *Suppose that* $f \in \mathrm{BMO}(T^3)$ *and that* φ *is Lipschitz. Then* $\varphi f \in \mathrm{BMO}(T^3)$ *and*

$$(3.27)\quad [\varphi f]_{\mathrm{BMO}} \le \sqrt{2} \sup_{y \in T^3} \sup_{0<r<1} \left\{ \sup_{B(y,r)} |\varphi| \left(\fint_{B(y,r)} |f - f_{y,r}|^2 \, dz \right)^{1/2} + \left(\fint_{B(y,r)} |\varphi - \varphi_{y,r}|^2 \, dz \right)^{1/2} |f_{y,r}| \right\}.$$

In particular

$$\|\varphi f\|_{\mathrm{BMO}} \le C \sup_{T^3} (|\varphi| + |D\varphi|) \|f\|_{\mathrm{BMO}}.$$

Proof. – Let $g = f - f_{y,r}$. Then

$$\varphi f - (\varphi f)_{y,r} = (\varphi g - (\varphi g)_{y,r}) + (\varphi - \varphi_{y,r}) f_{y,r}.$$

Since for any L^2 function h

$$\int_{B(y,r)} |h - h_{y,r}|^2 \le \int_{B(y,r)} |h|^2,$$

the first estimate follows easily. To prove the second estimate note that

$$(3.28)\quad \begin{aligned} |f_{y,r/2} - f_{y,r}| &= \left| \fint_{B(y,r/2)} (f - f_{y,r}) \, dz \right| \\ &\le \left\{ \fint_{B(y,r/2)} |f - f_{y,r}|^2 \, dz \right\}^{1/2} \\ &\le 2^{3/2} \left\{ \fint_{B(y,r)} |f - f_{y,r}|^2 \, dz \right\}^{1/2} \end{aligned}$$

which implies that

$$|f_{y,r}| \le C(1 + |\log r|) \|f\|_{\mathrm{BMO}}\,.$$

Since

$$\int_{B(y,r)} |\varphi - \varphi_{y,r}|^2 \, dz \leq Cr \sup |D\varphi|$$

the estimate for $[\varphi f]_{\text{BMO}}$ follows from (3.27) while the estimate for $|\int \varphi f \, dz| \leq \sup |\varphi| \|f\|_{L^2} \leq C \sup |\varphi| \|f\|_{\text{BMO}}$ is obvious. □

Proof of Theorem 3.5. – The proof follows the usual line of reasoning in the theory of concentration compactness.

Step 1. – (Definition and distributional convergence of $f^n M(Dg^n, Dh^n)$).

For $f \in$BMO(T^3) and $\beta \in \mathcal{H}^1(T^3)$ the product is defined as a distribution T via

$$\langle T, \varphi \rangle = \langle f\varphi, \beta \rangle_{\text{BMO},\mathcal{H}^1} \cdot$$

By Proposition 3.8 and $\mathcal{H}^1$-BMO duality

$$|\langle T, \varphi \rangle| \leq \|f\|_{\text{BMO}} \; \|\beta\|_{\mathcal{H}^1} \; \|\varphi\|_{C^1}. \tag{3.29}$$

Now by (3.24) the sequence (f^n) is bounded in BMO while $M(Dg^n, Dh^n)$ is bounded in $\mathcal{H}^1$ by Theorem 3.4. Thus the product (in the above sense) $f^n M(Dg^n, Dh^n)$ is bounded in the dual of C^1 whence (for a subsequence)

$$f^n M(Dg^n, Dh^n) \rightharpoonup \tilde{T}$$

in distributions. The goal is to characterize the defect

$$\nu = \tilde{T} - fM(Dg, Dh).$$

Note that by (3.29) and Theorem 3.4 the product $f^n M(Dg^n, Dh^n)$ defines a continuous map from BMO$\times(H^1)^2$ into the dual of C^1. By density we may thus assume

$$g^n, h^n, g, h \in C^\infty \tag{3.30}$$

and work with the pointwise, rather than the distributional, definition of the product.

Step 2 (Reduction to $g = h = 0$). – Let $\tilde{g}^n = g^n - g, \tilde{h}^n = h^n - h$. Then

$$\begin{aligned} f^n M(Dg^n, Dh^n) = {} & f^n M(Dg, Dh) + f^n M(D\tilde{g}^n, D\tilde{h}^n) \\ & + f^n M(D\tilde{g}^n, Dh) + f^n M(Dg, D\tilde{h}^n). \end{aligned} \tag{3.31}$$

Since f^n is bounded in BMO we may assume

$$f^n \overset{*}{\rightharpoonup} f \quad \mathrm{BMO}(T^3). \tag{3.32}$$

In view of (3.21) and (3.18) Df^n is bounded in L^2 and

$$f^n \to f \quad \text{in } L^2(T^3) \qquad \text{(and hence in all } L^p \text{ by (3.32)).} \tag{3.33}$$

Therefore the first term on the right hand side of (3.31) converges to $fM(Dg, Dh)$ in distributions, while the last two converge to zero by (3.30). To determine ν we may thus assume from now on

$$g = h = 0. \tag{3.34}$$

Step 3. – (ν is a Radon measure).

Let $\varphi \in C^1(T^3)$. By the Sobolev embedding theorem and (3.34) one has $g^n \to 0$ in $L^p(T^3)$ for $p < 6$. We may further assume (for a subsequence)

$$|Dg^n|^2 + |Dh^n|^2 \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(T^3).$$

Thus by (3.33) and Theorem 3.4

$$\begin{aligned} |\langle \nu, \varphi \rangle| &\le \limsup_{n\to\infty} \left| \int_{T^3} \varphi f^n M(Dg^n, Dh^n)\, dz \right| \\ &= \limsup_{n\to\infty} \left| \int_{T^3} f^n M(D(\varphi g^n), Dh^n)\, dz \right| \\ &\le \limsup_{n\to\infty} \|f^n\|_{\mathrm{BMO}} \|D(\varphi g^n)\|_{L^2} \|Dh^n\|_{L^2} \\ &\le C \limsup_{n\to\infty} \|\varphi Dg^n\|_{L^2} \\ &\le C(\langle \mu, |\varphi|^2 \rangle)^{1/2} \le C\|\mu\|_{\mathcal{M}}^{1/2} \sup |\varphi|. \end{aligned}$$

It follows that ν is a Radon measure and ν is absolutely continuous with respect to μ.

Step 4. – (support of ν).

Let $\varphi \in C^2(T^3)$. As in Step 3

$$|\langle \nu, \varphi^3 \rangle| \le \limsup_{n\to\infty} \left| \int_{T^3} \varphi f^n M(D(\varphi g^n), D(\varphi h^n))\, dz \right|.$$

To proceed we replace φf^n by a term whose BMO norm is more easily estimated. Let $\tilde{f}^n$ be the weak solution of

$$-\Delta_z \tilde{f}^n = A(D)(\varphi \gamma^n), \qquad \int_{T^3} \tilde{f}^n = 0.$$

Then $\Delta_z(\varphi f^n - \tilde{f}^n)$ is bounded in $L^{2,1}$ and hence by elliptic regularity (apply (3.18) to $U = (\partial/\partial z^\alpha)(\varphi f^n - \tilde{f}^n)$) the term $D(\varphi f^n - \tilde{f}^n)$ is bounded in BMO $\subset L^q$ for all $q < \infty$. It follows that $\varphi f^n - \tilde{f}^n$ is compact in $C^{0,\alpha}$. On the other hand $M(D(\varphi g^n), D(\varphi h^n)) \overset{*}{\rightharpoonup} 0$ in $\mathcal{H}^1$ (since $g = h = 0$) and in particular weak$*$ in $\mathcal{M}(T^3) = (C^0(T^3))^*$.

Hence

$$\begin{aligned}|\langle \nu, \varphi^3 \rangle| &\leq \limsup_{n\to\infty} |\int \tilde{f}^n M(D(\varphi g^n), D(\varphi h^n))\, dz| \\ &\leq C \limsup_{n\to\infty} \|\tilde{f}^n\|_{\mathrm{BMO}} \|\varphi D g^n\|_{L^2} \|\varphi D h^n\|_{L^2} \\ &\leq C \limsup_{n\to\infty} \|\varphi\gamma^n\|_{L^{2,1}} \langle \mu, \varphi^2 \rangle.\end{aligned}$$

Now consider $\varphi_k \in C^1(T^3)$ with $\varphi_k \nearrow \chi_{B(y,r)}$. Then

$$\nu(B(y,R)) \leq C \limsup_{n\to\infty} \|\chi_{B(y,r)}\gamma^n\|_{L^{2,1}}\, \mu(B(y,R)).$$

Let S be the set on the right hand side of (3.23). The estimate above implies that for $y \notin S$ the Radon-Nikodyn derivative

$$\frac{d\nu}{d\mu}(y) = \lim_{R\to 0} \frac{\nu(B(y,R))}{\mu(B(y,R))}$$

vanishes. Since ν is absolutely continuous with respect to μ

$$\nu = \frac{d\nu}{d\mu}\mu.$$

Hence $\operatorname{supp} \nu \subset S$ and the theorem is proved. □

3.7. Proof of the second concentration theorem

We sketch the proof of Theorem 3.7 that directly employs the $\mathcal{H}^1$-BMO duality. Although this result is not needed in the convergence proof for wave maps we have included it for its conceptual simplicity. We also indicate how Theorem 3.5 can be deduced from Theorem 3.7.

The proof of Theorem 3.6 is parallel to that of Theorem 3.5 and makes use of the following two lemmas that will be proved at the end of this section.

LEMMA 3.9. – *Let $g^n \in \mathcal{H}^1(T^m)$ and let $\mathcal{M}g^n$ be the corresponding regularized maximal functions. Suppose*

$$g^n \overset{*}{\rightharpoonup} 0 \textit{ in } \mathcal{H}^1(T^m),$$
$$\mathcal{M}g^n \overset{*}{\rightharpoonup} \mu \textit{ in } \mathcal{M}(T^m).$$

Then for $\varphi \in C^1(T^m)$ *and* $c^n = \int_{T^m} \varphi g^n$ *the functions* $g^n - c^n$ *belong* $\mathcal{H}^1(T^m)$ *and*

$$\limsup_{n\to\infty} \|\varphi g^n\|_{\mathcal{H}^1(T^m)} \le C\langle \mu, |\varphi|\rangle.$$

LEMMA 3.10. – *Let* $f^n \in BMO(T^m)$, $y \in T^m$ *and suppose that* $\|f^n\|_{\mathrm{BMO}} \le C$ *and*

$$\lim_{R\to 0} \limsup_{n\to\infty} [f^n]_{\mathrm{BMO}(B(y,R))} = 0.$$

Then, for each $R > 0$ *sufficiently small, there exists* $f_R^n \in \mathrm{BMO}(T^m)$ *such that*

$$f_R^n = f^n \quad on\ B(y,R)$$

$$\lim_{R\to 0} \limsup_{n\to\infty} \|f_R^n\|_{\mathrm{BMO}(T^m)} = 0.$$

Proof of Theorem 3.6. – The $\mathcal{H}^1$-BMO duality implies that

$$f^n g \rightharpoonup fg$$

as distributions and we may assume that $g = 0$. As in the proof Theorem 3.5, Step 1, we may assume that $f^n g^n$ has a distributional limit ν and that $g^n \in C^\infty$.

Since g^n is bounded in $\mathcal{H}^1$, the functions $\mathcal{M}g^n$ are bounded in L^1 and we may assume $\mathcal{M}g^n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(T^m)$. For $\varphi \in C^1(T^m)$ let $c^n = \int_{T^m} \varphi g^n$. Then Lemma 3.9 yields

$$\begin{aligned}\langle \nu, \varphi\rangle &\le \limsup_{n\to\infty} \left|\int_{T^m} f^n \varphi g^n\right| \\ &\le \limsup_{n\to\infty} \|f^n\|_{BMO} \|\varphi g^n\|_{\mathcal{H}^1} \le C\langle \mu, |\varphi|\rangle.\end{aligned}$$

Hence ν is a Radon measure and is absolutely continuous with respect to μ. Let

$$S := \{x \in T^m : \limsup_{R\to 0} \limsup_{n\to\infty} [f^n]_{\mathrm{BMO}(B(x,R))} > 0\}.$$

For $y \in T^m \setminus S$ consider f_R^n as in Lemma 3.10. Then for all $\varphi \in C_0^1(B(y,R))$

$$|\langle \nu, \varphi\rangle| \le C \limsup_{n\to\infty} \|f_R^n\|_{\mathrm{BMO}} \langle \mu, |\varphi|\rangle$$

and, after passage to characteristic functions, we deduce

$$\frac{d\nu}{d\mu}(y) = \lim_{R\to 0} \frac{\nu(B(y,R))}{\mu(B(y,R))} = 0, \text{ for } y \in T^m \backslash S.$$

Hence $\operatorname{supp} \nu \subset S$ and the theorem is proved. □

Deduction of Theorem 3.5 *from Theorem* 3.7. – Apply the following lemma to f^n and γ^n with $\rho = \sqrt{r}$. □

LEMMA 3.11. – *Let $A(D) = a^\alpha \frac{\partial}{\partial x^\alpha}$ be a first order differential operator on T^3 with constant coefficients. Let $\gamma \in L^{2,1}(T^3)$ and let $f \in H^1(T^3)$ be the weak solution of*

$$-\Delta f = A(D)\gamma, \quad \int_{T^3} f = 0.$$

Then, for $r \le \rho/2 < 1/4$, $y \in T^3$

$$[f]_{\mathrm{BMO}(B(y,r))} \le C\left\{\frac{r}{\rho}\|f\|_{\mathrm{BMO}(B(y,\rho))} + \|\gamma\chi_{B(y,\rho)}\|_{L^{2,1}}\right\}.$$

Proof. – Let $f = f^1 + f^2$ on $B(y,\rho)$ where

$$-\Delta f^1 = 0, \quad f^1 = f \text{ on } \partial B(y,\rho),$$
$$-\Delta f^2 = A(D)\gamma, \quad f^2 = 0 \text{ on } \partial B(y,\rho).$$

By [4], Theorema 16.I, and Poincaré's inequality

$$[f^2]_{\mathrm{BMO}(B(y,\rho))} \le C\|Df^2\|_{L^{2,1}(B(y,\rho))} \le C\|\chi_{B(y,\rho)}\gamma\|_{L^{2,1}}.$$

Standard estimates for harmonic functions and scaling yield

$$\sup_{B(y,\rho/2)} |Df^1| \le \frac{C}{\rho}\left(\fint_{B(y,\rho)} |f^1 - f^1_{y,\rho}|^2\right)^{1/2}$$

and thus

$$\begin{aligned} \mathrm{osc}_{B(y,r)} f^1 &\le r \sup_{B(y,r)} |Df^1| \le C\frac{r}{\rho}\left(\fint_{B(y,\rho)} |f^1 - f^1_{y,\rho}|^2\right)^{1/2} \\ &\le C\frac{r}{\rho}[f^1]_{\mathrm{BMO}(B(y,\rho))} \\ &\le \frac{r}{\rho}[f]_{\mathrm{BMO}(B(y,\rho))} + C[f^2]_{\mathrm{BMO}(B(y,\rho))}. \end{aligned}$$

The lemma follows by combining the estimates for f^1 and f^2. □

Proof of Lemma 3.9. – Fix $\varphi \in C^1(T^m)$. For the rest of the proof we suppress possible dependence of constants on φ. Let $\rho \in C_0^\infty(B(0,1))$ with $\int \rho\, dz = 1$ and let $\rho_\varepsilon(z) = \varepsilon^{-m}\rho(\frac{z}{\varepsilon})$. Then

$$\begin{aligned}\mathcal{M}(\varphi g^n)(z) := \sup_{0<\varepsilon<1} \Big| \int_{T^m} \rho_\varepsilon(z-y)\varphi(y)g^n(y)\,dy\Big| \\ \le \sup_{0<\varepsilon<1} |\varphi(z)| \Big| \int_{T^m} \rho_\varepsilon(z-y)g^n(y)\,dy\Big| \\ + \sup_{0<\varepsilon<1} \Big| \int_{T^m} \rho_\varepsilon(z-y)(\varphi(y)-\varphi(z))g^n(y)\,dy\Big|.\end{aligned}$$

Let $K_\varepsilon(z,y) = \rho_\varepsilon(z-y)(\varphi(y)-\varphi(z))$ and for $\delta > 0$ define

$$R_1^n(z) = \sup_{0<\varepsilon<\delta} \Big| \int_{T^m} K_\varepsilon(z,y)g^n(y)\,dy\Big|,$$
$$R_2^n(z) = \sup_{\delta<\varepsilon<1} \Big| \int_{T^m} K_\varepsilon(z,y)g^n(y)\,dy\Big|.$$

Then

$$\mathcal{M}(\varphi g^n) \le |\varphi|\mathcal{M}(g^n) + R_1^n + R_2^n.$$

Since

$$\sup_{\varepsilon\in(\delta,1)} \left\{ |K_\varepsilon(z,y)| + \left|\frac{\partial K_\varepsilon}{\partial y}(z,y)\right| \right\} \le C(\delta)$$

the set $\{K_\varepsilon(z,\cdot) : z \in T^m,\ \varepsilon \in (\delta,1)\} \subset C^0(T^m)$ is precompact. On the other hand by assumption $(g^n) \overset{*}{\rightharpoonup} 0$ in $\mathcal{H}^1$ and hence in $\mathcal{M}(T^m)$. It follows that $R_2^n(z) \to 0$ for all $z \in T^m$. Moreover $|R_2^n(z)| \le C(\delta)$ and therefore

$$\|R_2^n\|_{L^1} \to 0.$$

To estimate R_1^n note that for $\varepsilon < \delta$

$$\begin{aligned}|K_\varepsilon(z,y)| &\le \sup|D\varphi|\ |\rho_\varepsilon(z-y)|\ |z-y| \\ &\le C\sup|D\varphi| \begin{cases} |z-y|^{-(m-1)}, & |z-y|<\delta, \\ 0, & |z-y| \ge \delta. \end{cases}\end{aligned}$$

Denote the right hand side by $f_\delta(z-y)$. Then

$$\|R_1^n\|_{L^1} \le \|f_\delta * |g^n|\,\|_{L^1} \le C\|f_\delta\|_{L^1} \le C\delta.$$

Finally

$$\begin{aligned}&\limsup_{n\to\infty} \|\mathcal{M}(\varphi g^n)\|_{L^1} \\ &\le \limsup_{n\to\infty} \int_{T^m} |\varphi|\mathcal{M}(g^n)\,dz + C\delta = \langle \mu, |\varphi|\rangle + C\delta.\end{aligned}$$

The lemma follows since $c^n = \int \varphi g^n \to 0$, and $\delta > 0$ was arbitrary. $\square$

Proof of Lemma 3.10. – Let $y \in T^m$, $f \in$ BMO and

$$\omega(r) := [f]_{\mathrm{BMO}(B(y,r))}.$$

It follows from (3.28) (with the factor $2^{3/2}$ replaced by $2^{m/2}$) and the obvious estimate $|f_{y,s}| \leq C\|f\|_{L^2(T^m)} \leq C\|f\|_{\mathrm{BMO}(T^m)}$ for $1/4 \leq s \leq 1/2$ that

$$|f_{y,R}| \leq C \int_R^1 \frac{\omega(r)}{r}\, dr + C\|f\|_{\mathrm{BMO}}. \tag{3.35}$$

To define the extension f_R it is useful to split f into a constant and a function with zero mean on $B(y, 4R)$. Let

$$\begin{aligned} g &= f - f_{y,4R}, \\ \psi_R(z) &= \psi\left(\frac{z-y}{4R}\right), \quad \psi \in C_0^\infty(B(0,1/2)), \quad \psi_{|B(0,1/4)} \equiv 1, \\ \eta_R(z) &= \begin{cases} 1, & |z-y| \leq R \\ (\log(1/4R))^{-1}\log(1/4|y-z|), & R \leq |z-y| \leq \frac{1}{4} \\ 0, & |z-y| \geq \frac{1}{4}, \end{cases} \\ f_R &= \psi_R\, g + \eta_R f_{y,4R}. \end{aligned}$$

Consider the rescaled function

$$\tilde{g}(z) = g(y + 4Rz).$$

Then

$$\|\tilde{g}\|_{\mathrm{BMO}(B(0,1))} = \|g\|_{\mathrm{BMO}(B(y,4R))} \leq \omega(4R)$$

since $\int_{B(y,4R)} g = 0$. Thus by the analogue of Proposition 3.8 for balls in $\mathbb{R}^m$

$$\|\psi_R\, g\|_{\mathrm{BMO}(B(y,4R))} = \|\psi\, \tilde{g}\|_{\mathrm{BMO}(B(0,1))} \leq C\|\tilde{g}\|_{\mathrm{BMO}(B(0,1))} \leq C\omega(4R).$$

As $\psi_R g$ vanishes outside $B(y, 2R)$ one easily deduces (for $R < 1/8$)

$$\|\psi_R\, g\|_{\mathrm{BMO}(T^m)} \leq C\omega(4R). \tag{3.36}$$

Moreover $\ln|x|^{-1} \in \mathrm{BMO}(\mathbb{R}^m)$ and thus

$$\|\eta_R\|_{\mathrm{BMO}(T^m)} \leq C(\log(1/4R))^{-1}.$$

In combination with (3.35) and (3.36) we deduce

$$\|f_R\|_{\mathrm{BMO}(T^m)} \leq C\omega(4R) + C(\log 1/4R)^{-1} \int_{4R}^{1} \frac{\omega(r)}{r}\, dr + C(\log(1/4R))^{-1}\|f\|_{\mathrm{BMO}} \tag{3.37}$$

Now let f^n be as in the Lemma and define ω^n by f^n. Let $\Omega^n(R) = \|f_R^n\|_{\mathrm{BMO}(T^m)}$ and denote by $\overline{\omega}$ and $\overline{\Omega}$ the limites superiores of these quantities as $n \to \infty$. Then by the dominated convergence theorem and (3.37)

$$\overline{\Omega}(R) \leq C\overline{\omega}(4R) + C(\log(1/4R))^{-1} \int_{4R}^{1} \frac{\overline{\omega}(r)}{r}\, dr + C(\log(1/4R))^{-1}.$$

By assumption $\lim_{R\to 0} \overline{\omega}(R) = 0$. Hence $\lim_{R\to 0} \overline{\Omega}(R) = 0$, and the lemma is proved. □

APPENDIX

Weakly harmonic maps

Let N be a smooth compact Riemannian manifold of dimension k, isometrically embedded into $\mathbb{R}^d$. A smooth map $u : \Omega \subset \mathbb{R}^m \to N$ is harmonic if

$$-\Delta u \perp T_u N \tag{A.1}$$

or, equivalently,

$$-\Delta u = A(u)(\partial^\alpha u, \partial_\alpha u), \tag{A.2}$$

where A denotes the second fundamental form of N and where $\partial^\alpha = \partial_\alpha$ denotes the partial derivative with respect to x^α.

If N is parallelizable and $(e_1, \ldots, e_k)$ is an orthonormal frame of the pull-back bundle $u^{-1}TN$ (*i.e.*, $e_1(x), \ldots, e_k(x)$ is a basis of $T_{u(x)}N$), if $\theta_i := \langle du, e_i \rangle$ and $\omega_{ij} := \langle de_i, e_j \rangle$, then the above are equivalent to

$$\delta\theta_i + \omega_{ij} \cdot \theta_j = 0. \tag{A.3}$$

The lemma below states that the corresponding weak formulations are also equivalent. While this is well-known to experts in the field we are not aware of a standard reference and thus include a proof for the convenience of the reader. Recall that

$$H^1(\Omega, N) := \{u \in H^1(\Omega, \mathbb{R}^d) \,:\, u(x) \in N \text{ a.e.}\}.$$

If u is in $H^1(\Omega; N)$ then the weak derivative $Du \in L^2(\Omega; \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^d))$ satisfies

$$\text{Range}\, Du(x) \subset T_{u(x)}N \text{ a.e.}$$

We use the compact notation $\langle du; d\psi\rangle = du^k \cdot d\psi^k = \partial^\alpha u^k \partial_\alpha \psi^k = \langle du, e_i\rangle \cdot \langle d\psi, e_i\rangle$ and $A(u)(du; du) = A(u)(\partial^\alpha u, \partial_\alpha u) = A(u)(e_i, e_j)\langle du, e_i\rangle \cdot \langle du, e_j\rangle$. Let

$$X_{\tan} := \{\phi \in H_0^1(\Omega, \mathbb{R}^d) \,:\, \phi(x) \in T_{u(x)}N \text{ a.e.}\}.$$

LEMMA A.1. – *Suppose that N is parallelizable and that $u \in H^1(\Omega; N)$, $e_i \in H^1(\Omega; \mathbb{R}^d)$, $1 \leq i \leq k$, is a frame of the pull-back bundle $u^{-1}TN$. Then the following five statements are equivalent.*

(i) *Equation* (A.3) *holds in the sense of distributions,* i.e.,

$$\int_\Omega \theta_i \cdot d\eta + \omega_{ij} \cdot \theta_j \, \eta \, dx = 0 \quad \forall \eta \in C_0^\infty(\Omega);$$

(ii)

$$\int_\Omega \theta_i \cdot d\eta + \omega_{ij} \cdot \theta_j \, \eta \, dx = 0 \quad \forall \eta \in (H_0^1 \cap L^\infty)(\Omega);$$

(iii)

$$\int_\Omega \langle du; d\psi\rangle \, dx = \int_\Omega \langle A(u)(du; du), \psi\rangle \, dx \quad \forall \psi \in (H_0^1 \cap L^\infty)(\Omega; \mathbb{R}^d);$$

(iv)

$$\int_\Omega \langle du; d\phi\rangle = 0 \quad \forall \phi \in X_{\tan} \cap L^\infty(\Omega; \mathbb{R}^d);$$

(v)

$$\int_\Omega \langle du; d\phi\rangle = 0 \quad \forall \phi \in X_{\tan}.$$

Remarks. – 1. The result applies equally for wave maps if we take $\partial^0 = -\partial_0$, $\partial^i = \partial_i$ and use contraction with respect to the Lorentzian metric. In fact the statement and proof of the lemma are independent of the metric on Ω.

2. Since (iii) and (iv) are independent of the frame (e_i) it follows that (i) or (ii) hold for all (H^1-) frames provided that they hold for one such frame.

Proof. – The implications (ii) $\Longrightarrow$ (i) and (v) $\Longrightarrow$ (iv) are obvious. We show (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (ii) and (iv) $\Longrightarrow$ (v). Recall first that, by the product rule, $H^1 \cap L^\infty$ is an algebra under pointwise multiplication.

(i) $\Longrightarrow$ (ii): Let $\eta \in H_0^1 \cap L^\infty$. By density there exists $\eta_k \in C_0^\infty$ that converge to η in H^1 and almost everywhere. We may assume in addition that the η_k are uniformly bounded. Indeed if $M = \|\eta\|_\infty$, let Ψ be a function in $C_0^\infty(-3M, 3M)$, which is the identity on $(-2M, 2M)$. Then $\tilde{\eta}_k = \Psi \circ \eta_k$ has the desired properties. Now apply (i) with $\tilde{\eta}_k$, let $k \to \infty$ and use the dominated convergence theorem to derive (ii).

(ii) $\Longrightarrow$ (iii): We have

$$\langle du; d\psi \rangle = \langle du, e_i \rangle \cdot \langle d\psi, e_i \rangle = \theta_i \cdot d\langle \psi, e_i \rangle - \theta_j \cdot \langle \psi, de_j \rangle.$$

Now $\langle de_j, e_i \rangle = \omega_{ji} = -\omega_{ij}$ and by the definition of the second fundamental form $(\partial_\alpha e_j)^{\text{nor}} = -A(u)(e_j, \partial_\alpha u)$ (to verify this formula for the H^1 vectorfields e_j one can consider a smooth frame $(\bar{e}_1, \ldots, \bar{e}_k)$ of N and expand $e_j = \langle e_j, \bar{e}_l \circ u \rangle \bar{e}_l \circ u$). Hence

$$-\langle \psi, de_j \rangle = \omega_{ij} \langle \psi, e_i \rangle + \langle A(u)(e_j, e_l), \psi \rangle \langle du, e_l \rangle$$

and

$$\langle du; d\psi \rangle = \theta_i \cdot d\langle \psi, e_i \rangle + \omega_{ij} \cdot \theta_j \, \langle \psi, e_i \rangle + \langle A(u)(du; du), \psi \rangle$$

The integral over the first two terms on the right hand side vanishes by (ii) and thus (iii) holds.

(iii) $\Longrightarrow$ (iv): This is obvious since the second fundamental form takes its values in the normal bundle.

(iv) $\Longrightarrow$ (ii): Let $\phi = \eta e_i$. We have

$$\langle du; d\phi \rangle = \langle du, e_j \rangle \cdot \langle d\phi, e_j \rangle = \theta_j \cdot (\delta_{ij} d\eta + \eta \omega_{ij}),$$

and hence (iv) implies (ii).

(iv) $\Longrightarrow$ (v): Let $a : [0, \infty) \to [0, \infty)$ be a bounded and smooth function that agrees with the identity on $[0, 1]$, and for $z \in \mathbb{R}^d$ let $\Psi_R(z) = Ra(|z|/R)z/|z|$. Suppose $\phi \in X_{\text{tan}}$. Then $\phi_k := \Psi_k \circ \phi \in X_{\text{tan}} \cap L^\infty$ and $\phi_k \to \phi$ almost everywhere and thus in H^1 by the chain rule. The implication follows.

The proof is finished. □

REFERENCES

[1] F. BETHUEL and J.-M. GHIADAGLIA, Weak limits of solutions to the steady incompressible two-dimensional Euler equations in a bounded domain, *Asympt. Anal.*, Vol. **8**, 1994, pp. 277-291.

[2] F. BETHUEL, Weak convergence of Palais-Smale sequences for some critical functionals, *Calc. Var.*, Vol. **1**, 1993, pp. 267-310.

[3] F. BETHUEL, On the singular set of stationary harmonic maps, *manusc. math.*, Vol. **78**, 1993, pp. 417-443.

[4] S. CAMPANATO, Sistemi ellittiche del secondo ordine e spazi $\mathcal{L}^{2,\lambda}$, *Ann. Mat. Pura Appl.*, Vol. **69**, 1965, pp. 321-380.

[5] D. CHRISTODOULOU and A.S. TAHVILDAR-ZADEH, On the regularity of spherically symmetric wave maps, *Comm. Pure Appl. Math.*, Vol. **46**, 1993, pp. 1041-1091.

[6] R. COIFMAN, P.-L. LIONS, Y. MEYER and S. SEMMES, Compensated compactness and Hardy spaces, *J. Math. Pures Appl.*, Vol. **72**, 1993, pp. 247-286.

[7] R. DIPERNA and A. MAJDA, Reduced Hausdorff dimension and concentration cancellation for two-dimensional incompressible flows, *J. Amer. Math. Soc.*, Vol. **1**, 1988, pp. 59-95.

[8] L. C. EVANS, Partial regularity for stationary harmonic maps into spheres, *Arch. Rat. Mech. Anal.*, Vol. **116**, 1991, 101-113.

[9] C. FEFFERMAN and E.M. STEIN, H^p spaces of several variables, *Acta Math.*, Vol. **129**, 1972, pp. 137-193.

[10] J. FREHSE, Capacity methods in the theory of partial differential equations, *Jahresb. Dt. Math. Ver.*, Vol. **84**, 1982, pp. 1-44.

[11] A. FREIRE, Global weak solutions of the wave map system to compact homogeneous spaces, *manuscripta math.*, Vol. **91**, 1996, pp. 525-533.

[12] A. FREIRE, S. MÜLLER and M. STRUWE, Weak convergence of wave maps from (2+1) dimensional Minkowski space to Riemannian manifolds, *Invent. Math.*, Vol. **130**, 1997, pp. 589-617.

[13] M. GIAQUINTA, Introduction to regularity theory for nonlinear elliptic systems, *Lectures in Mathematics, Birkhäuser*, 1993.

[14] D. GOLDBERG, A local version of the Hardy space, *Duke Math. J.*, Vol. **46**, 1979, pp. 27-42.

[15] E. HEINZ, Elementare Bemerkung zur isoperimetrischen Ungleichung im $\mathbb{R}^3$, *Math. Z.*, Vol. **132**, 1973, pp. 319-322.

[16] F. HÉLEIN, Régularité des applications faiblement harmoniques entre une surface et une variété Riemannienne, *C. R. Acad. Sci. Paris Ser. I Math.*, Vol. **312**, 1991, pp. 591-596.

[17] S. KLAINERMAN and M. MACHEDON, Space-time estimates for null forms and the local existence theorem, *Comm. Pure Appl. Math.*, Vol. **46**, 1993, pp. 1221-1268.

[18] S. KLAINERMAN and M. MACHEDON, Smoothing estimates for null forms and applications, *Int. Math. Res. Not.*, Vol. **4**, 1994, pp. 383-389.

[19] P.-L. LIONS, The concentration compactness principle in the calculus of variations, the limit case, Part 2 , *Rev. Mat. Iberoam.*, Vol. **1/2** , 1985, pp. 45-121.

[20] E. J. NUSSENZWEIG LOPES, An estimate on the Hausdorff dimension of a concentration set for the incompressible 2-D Euler equations, *Indiana Univ. Math. J.*, Vol. **43**, 1994, pp. 521-534.

[21] S. MÜLLER and M. STRUWE, Global existence of wave maps in 1+2 dimensions for finite energy data, *Top. Methods Nonlinear Analysis*, Vol. **7**, 1996, pp. 245-259.

[22] S. MÜLLER, M. STRUWE and V. ŠVERÁK, Harmonic maps on planar lattices, *Ann. SNS Pisa*, to appear.

[23] S. MÜLLER and M. STRUWE, Spatially discrete wave maps on (1+2)-dimensional space-time, *Top. Methods Nonlinear Analysis*, to appear.

[24] S. SEMMES, A primer on Hardy spaces, and some remarks on a theorem of Evans and Müller, *Comm. P.D.E.*, Vol. **19**, 1994, 277-319.

[25] J. SHATAH, Weak solutions and development of singularities in the $SU(2)$ σ-model, *Comm. Pure Appl. Math.*, Vol. **41**, 1988, pp. 459-469.

[26] J. SHATAH and A.S. TAHVILDAR-ZADEH, On the Cauchy problem for equivariant wave maps, *Comm. Pure Appl. Math.*, Vol. **45**, 1992, pp. 947-971.

[27] E. STEIN, Singular integrals and differentiability properties of functions, *Princeton Univ. Press,* 1970.

[28] E. STEIN, Harmonic analysis, *Princeton Univ. Press*, 1993.

[29] M. STRUWE, Wave maps, in: *Nonlinear partial differential equations in geometry and physics-Barrett Lectures* 1995 (Eds. G. Baker et al.), Birkhäuser, 1997, pp. 113-153.

[30] L. TARTAR, Remarks on oscillations and Stokes's equation, in: Macroscopic modelling of turbulent flows (Nice, 1984), Lect. Notes Phys., Vol. **230**, *Springer*, 1985, pp. 24-31.

[31] H. C. WENTE, An existence theorem for surfaces of constant mean curvature, *J. Math. Anal. Appl.*, Vol. **26**, 1969, pp. 318-344.

[32] Y. ZHENG, Regularity of solutions to a two dimensional modified Dirac-Klein-Gordon system of equations, *Comm. Math. Phys.*, Vol. **151**, 1993, pp. 67-87.

[33] YI ZHOU, Global weak solutions for the $2 + 1$ dimensional wave maps with small energy, preprint, 1995.

[34] YI ZHOU, Global weak solutions for $1 + 2$ dimensional wave maps into homogeneous spaces, *Ann. IHP – Analyse non linéaire*, to appear.

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