ANNALES DE L'I. H. P., SECTION C

YVAN MARTEL

Complete blow up and global behaviour of solutions of $u_t - \Delta u = g(u)$

Annales de l'I. H. P., section C, tome 15, nº 6 (1998), p. 687-723

<http://www.numdam.org/item?id=AIHPC_1998__15_6_687_0>

Ann. Inst. Henri Poincaré,

Vol. 15, n° 6, 1998, p. 687-723 *Analyse non linéaire*

Complete blow up and global behaviour of solutions of $u_t - \triangle u = g(u)$

by

Yvan MARTEL

Laboratoire Analyse Numérique, Université Pierre et Marie Curie,
4, place Jussieu, 75252 Paris Cedex 05.

ABSTRACT. – For $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, we study the global behaviour of solutions of the nonlinear heat equation (1). The domain Ω is smooth and bounded and the nonlinearity g is nonnegative, nondecreasing and convex.

We show in particular that any nondecreasing solution blowing up at the finite time $T_{\max}$ blows up completely in Ω after $T_{\max}$. We apply this result to the description of all possible global behaviours of the solutions of (1) according to the value of λ. We show similar results when we introduce a notion of complete blow up in infinite time. © Elsevier, Paris

RÉSUMÉ. – Pour $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, on étudie le comportement global des solutions de l'équation de la chaleur non-linéaire

$$\begin{cases} u_t - \triangle u = \lambda g(u) \text{ dans } (0,T) \times \Omega, \\ u = 0 \text{ sur } \partial\Omega, \\ u(0) = u_0 \text{ dans } \Omega. \end{cases} \tag{1}$$

Le domaine Ω est borné régulier, et la nonlinéarité g est positive, croissante et convexe.

On montre en particulier que toute solution croissante explosant au temps fini $T_{\max}$ explose totalement dans Ω après $T_{\max}$. On applique ce résultat à la description des comportements globaux possibles des solutions de (1) en fonction de λ. On montre des résultats similaires pour une notion d'explosion totale en temps infini que l'on introduit. © Elsevier, Paris

1. INTRODUCTION

Let $\Omega \subset \mathbf{R}^N$ be a smooth, bounded domain, and let $g : [0, \infty) \to [0, \infty)$ be a C^2 convex, nondecreasing function. For $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, we study the global behaviour of solutions of the nonlinear heat equation

$$\begin{cases} u_t - \Delta u = g(u) \text{ in } (0, T) \times \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u(0) = u_0 \text{ in } \Omega. \end{cases} \tag{1}$$

The possible behaviours of the solutions of (1) depend heavily on which of the following two properties is verified by the nonlinearity g,

$$\textit{There exists } x_0 \geq 0 \textit{ such that } g(x_0) > 0 \textit{ and } \int_{x_0}^{\infty} \frac{ds}{g(s)} < \infty, \tag{2}$$

$$\textit{For all } x_0 \geq 0 \textit{ such that } g(x_0) > 0, \int_{x_0}^{\infty} \frac{ds}{g(s)} = \infty. \tag{3}$$

Indeed it is well known that (2) is a necessary and sufficient condition of existence of blowing up solutions of (1). However, we will see that there exists a parallel between the two cases (2) and (3) in the study of solutions of (1).

Recall that if $u_0 \in L^\infty(\Omega)$ then there exists a unique maximal classical solution u of (1) belonging to $C((0, T_m), L^\infty(\Omega))$. When $T_m < \infty$, we have $\|u(t)\|_{L^\infty} \underset{t \to T_m}{\longrightarrow} \infty$, and we say that u blows up at T_m.

In order to define the notion of **complete** blow up, we consider any sequence (g_n) such that

$$\begin{cases} \text{(i) for all } n > 0,\ g_n \in C([0, \infty), [0, n)), \\ \text{(ii) for all } v \geq 0,\ g_n(v) \uparrow g(v) \text{ as } n \to \infty. \end{cases} \tag{4}$$

It follows that for every $n > 0$, there exists a unique global classical solution of

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \Delta u_n = g_n(u_n) \text{ in } (0, \infty) \times \Omega, \\ u_n = 0 \text{ on } \partial\Omega, \\ u_n(0) = u_0 \text{ in } \Omega. \end{cases} \tag{1_n}$$

It is well known that $u_n \uparrow u$ on $\Omega \times [0, T_m)$ as $n \to \infty$.

Let $\delta(x) = \text{dist}(x, \partial\Omega)$. For $T > 0$, we say that the solution u **blows up completely after** T if

$$\frac{u_n(t,x)}{\delta(x)} \underset{n\to\infty}{\longrightarrow} \infty \text{ uniformly on } [T+\varepsilon, \infty) \times \Omega \tag{5}$$

for every $\varepsilon > 0$. This means in particular that u can not be extended in any sense beyond T. Note that, u_0 being given, the fact for u of blowing up completely after some time T does not depend on the choice of the sequence (g_n) (see Lemma 9).

Our first result shows that every nondecreasing solution of (1) such that $T_m < \infty$ blows up completely after T_m.

THEOREM 1. – *Let $u_0 \in L^\infty(\Omega) \cap W_0^{1,1}(\Omega)$, $u_0 \geq 0$ be such that $\Delta u_0 + g(u_0) \geq 0$. Let u be the unique classical solution of (1) defined on the maximal interval $[0, T_m)$. If $T_m < \infty$ then u blows up completely after T_m.*

The first step of the proof consists in showing that if u does not blow up completely after T_m, then there exists $T_m < T^* \leq \infty$ such that u can be extended by a weak solution U of (1) on (T_m, T^*) (see Definition 1 below for the notion of weak solution). Since u is nondecreasing, there exists $\tau > 0$ such that U verifies the following problem (in the sense of Definition 1)

$$\begin{cases} U_t - \Delta U = g(U) \text{ in } (\tau, T_m + \tau) \times \Omega, \\ U = 0 \text{ on } \partial\Omega, \\ U(\tau) > u_0 \text{ in } \Omega \end{cases}$$

Applying to U a parabolic variant of Theorem 3 of Brezis *et al.* [3] we prove the existence of a bounded solution v of the following problem

$$\begin{cases} v_t - \Delta v = g(v) - \varepsilon \text{ in } (0, T_m) \times \Omega, \\ v = 0 \text{ on } \partial\Omega, \\ v(0) = v_0 > u_0 \text{ in } \Omega, \end{cases}$$

for some $\varepsilon > 0$. Then v allows us to build a super-solution of (1) which is bounded on $(0, T_m) \times \Omega$. By the maximum principle, we obtain a contradiction.

Since we assume $T_m < \infty$ in Theorem 1, we have necessarily (2). For the purposes of the parallel between the cases (2) and (3), let us introduce another notion. We say that the solution u **blows up completely in infinite time** if

$$u \text{ is global and } \quad \frac{u(t,x)}{\delta(x)} \underset{t\to\infty}{\longrightarrow} \infty, \text{ uniformly on } \Omega. \tag{6}$$

If g satisfies (2), no solution has this property (see Lemma 10). On the contrary, when (3) holds, this behaviour happens to have a great interest as we will see below.

As in Brezis *et al* [3], a weak solution of

$$\begin{cases} -\Delta w = g(w) & \text{in} \quad \Omega, \\ w = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{7}$$

is a function $w \geq 0$ almost everywhere, such that

$$w \in L^1(\Omega), \quad g(w)\delta \in L^1(\Omega), \quad -\int_\Omega w\Delta\zeta = \int_\Omega g(w)\zeta, \tag{8}$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$.

The first result related to the notion of complete blow up in infinite time is the following.

THEOREM 2. – *Suppose* (3). *Let* $u_0 \in L^\infty(\Omega) \cap W_0^{1,1}(\Omega)$, $u_0 \geq 0$ *be such that* $\Delta u_0 + g(u_0) \geq 0$. *Let* u *be the global solution of* (1). *Then either* u *blows up completely in infinite time or* $u(t)$ *converges to a weak solution of* (7) *as* $t \to \infty$.

Recall that in [3] Theorem 1, it is shown that when (2) holds, the existence of a global solution of (1) implies the existence of a weak solution of (7) in the sense of (8). Gathering this result and our first two theorems we obtain the following corollary.

COROLLARY 3. – *If there exists a solution of* (1) *which does not blow up completely (neither in finite nor in infinite time) for some* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, *then there exists a weak solution of* (7).

We think that the conclusion of Theorem 1 fails for some $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$. We refer to A. A. Lacey and D. E. Tzanetis [6] and V. A. Galaktionov and J. L. Vazquez [5] for the existence of solutions of (1) which blow up in finite time but continue to exist after T_m (*peaking* solutions) in the case $\Omega = \mathbf{R}^n$. However, this problem seems to be open for Ω bounded.

To deal with these solutions, we will see in Section 3 that if u is such that $T_m < \infty$, but does not blow up completely after $T_m < \infty$ in the sense of

(5), then u can be extended after T_m. Indeed, this extension is obtained as the limit of the sequence (u_n) and continues to satisfy (1) until the complete blow up time (denoted by T^*) in the sense of the following definition.

DEFINITION 1. – *Let u_0 be a nonnegative bounded measure of Ω. A weak solution of* (1) *on* $(0,T)$ *is a function $u \geq 0$ such that for all $0 < S < T$,*

$$u \in L^1((0,S) \times \Omega), \quad \delta g(u) \in L^1((0,S) \times \Omega), \tag{9}$$

and

$$\int_0^S \int_\Omega g(u)\xi = -\int_0^S \int_\Omega u(\xi_t + \triangle\xi) - \int_\Omega u_0\xi(0), \tag{10}$$

for any $\xi \in C^2([0,S] \times \overline{\Omega})$ such that $\xi(S) \equiv 0$ and $\xi = 0$ on $\partial\Omega$. Such a function u also verifies

$$\delta u \in C((0,T), L^1(\Omega)). \tag{11}$$

The notion of weak solutions given by Definition 1 is equivalent to the notion of integral solutions of P. Baras and M. Pierre [2] and P. Baras and L. Cohen [1] (see Section 3).

Uniqueness may fail for weak solutions of (1) in the sense of (10). However, if there exist several weak solutions of (1), then among them there is a minimal one, which is the limit of the nondecreasing sequence (u_n). This unique minimal solution exists on a maximal interval of time $(0,T^*)$, $T^* \leq \infty$.

Therefore, given $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, there exists a unique classical solution on $[0,T_m)$ **and** a minimal weak solution defined on $(0,T^*)$. Since the classical solution is also a weak solution in the sense of Definition 1, we have $0 < T_m \leq T^* \leq \infty$. On the other hand, the two solutions coincide on $(0,T_m)$ and we will denote by u the whole solution on $(0,T^*)$.

If $T^* < \infty$, then (5) holds for all $T \geq T^*$, and the solution u can not be continued in any sense after T^* ; T^* is called the **complete blow up time**. See Lemma 9 for the proof of these results.

We turn now to the behaviour of solutions of the following problem

$$\begin{cases} u_t - \triangle u = \lambda g(u) \text{ in } (0,T) \times \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u(0) = u_0 \text{ in } \Omega. \end{cases} \tag{12_λ}$$

First, we assume $g(0) > 0$ and $g \not\equiv g(0)$ so that there exists $0 < \lambda^* < \infty$ such that the following stationary problem

$$\begin{cases} -\triangle w_\lambda = \lambda g(w_\lambda) \quad \text{in} \quad \Omega, \\ w_\lambda = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{13_λ}$$

admits a minimal classical solution w_λ if $0 \leq \lambda < \lambda^*$ and no solution (even weak) for $\lambda > \lambda^*$. For $\lambda = \lambda^*$, the problem (13_{λ^*}) admits a unique weak solution (sometimes classical), if

$$g'(u) \underset{u \to \infty}{\longrightarrow} \infty. \tag{14}$$

For these results, we refer to [3]. For the uniqueness of w_{λ^*}, see Y. Martel [7]. When $\lim\limits_{u\to\infty} \dfrac{g(u)}{u} = a < \infty$, it is shown in P. Mironescu and V. Radulescu [8] that there exists a solution of (13_{λ^*}) (systematically classical) if and only if

$$\lim_{u \to \infty} (g(u) - au) < 0.$$

We distinguish three cases : $0 \leq \lambda < \lambda^*$, $\lambda = \lambda^*$ and $\lambda > \lambda^*$. First, we present our results in the case (2).

• **Case** $0 \leq \lambda < \lambda^*$**.** For $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, we investigate the behaviour of solutions of the following problem

$$\begin{cases} u_t - \Delta u = \lambda g(u) \text{ in } (0, T^*) \times \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u(0) = \mu u_0 \text{ in } \Omega, \end{cases} \tag{$15_{\lambda,\mu}$}$$

according to the value of μ. In this direction we show the following result.

THEOREM 4. – *Suppose* (2), *let* $\lambda < \lambda^*$, *and let* $u_0 \in L^\infty(\Omega)$ *be such that* $u_0 \geq 0$ *and* $u_0 \not\equiv 0$. *Then there exists* $0 < \mu^* < \infty$ *such that*

(i) $0 \leq \mu < \mu^*$, *the solution* u_μ *of* ($15_{\lambda,\mu}$) *is global bounded and converges to* w_λ *in* $L^\infty(\Omega)$.

(ii) $\mu = \mu^*$, *the solution* u_{μ^*} *of* ($15_{\lambda,\mu}$) *does not blow up completely neither in finite nor in infinite time.*

(iii) $\mu > \mu^*$, *the solution* u_μ *of* ($15_{\lambda,\mu}$) *blows up completely after some time* $T^* < \infty$.

Under certain assumptions, like uniqueness for the stationary problem, we can tell more about the behaviour of u_{μ^*}, but this problem remains in most part open. On the other hand, we do not know whether or not $T_m = T^*$ in (iii).

• **Case** $\lambda = \lambda^*$**.** In view of Corollary 3, it is easy to conclude that if (13_{λ^*}) has no weak solution, then all solutions of the evolution problem (12_{λ^*}) blow up completely after some finite time.

Concerning the special case where the solution of (13_{λ^*}) exists and is classical, we prove the following theorem.

THEOREM 5. – *Let* $\lambda = \lambda^*$. *Suppose* (2), *and that the solution* w_{λ^*} *of* (13_{λ^*}) *is classical. Let* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ *and let* u *be the solution of* (12_{λ^*}). *Then either* u *blows up completely after some time* $T^* < \infty$, *or* u *can be extended for all time by a weak solution of* (1) *in the sense of* (10) *which converges to* w_{λ^*} *in* $L^1(\Omega, \delta(x)dx)$ *as* $t \to \infty$.

Under the same assumptions, for $u_0 \leq w_{\lambda^*}$, $u_0 \not\equiv w_{\lambda^*}$ and u being the corresponding solution of (12_{λ^*}) we prove that $||w_{\lambda^*} - u(t)||_{L^\infty}$ converges to 0 at the rate $\frac{1}{t}$ as $t \to \infty$ (see Proposition 12).

• **Case** $\lambda > \lambda^*$. Here the situation is very simple, and the following result is a corollary of Theorem 1.

COROLLARY 6. – *Suppose* (2) *and* $\lambda > \lambda^*$. *Then for all* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, *the solution* u *of* (12_λ) *blows up completely after some time* $T^* < \infty$.

When (3) holds, we have similar results where "complete blow up in finite time" is to be substituted for "complete blow up after some time $T^* < \infty$". We refer to Section 6 for the statements.

When $g(0) = 0$ and $g'(0) \neq 0$, the critical value λ^* is $\lambda_1/g'(0)$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. When $\lambda < \lambda^*$, Theorem 4 can be stated the same way with $w_\lambda \equiv 0$. For $\lambda > \lambda^*$, the only solution of (13) is the trivial one and all solutions of (12) different from 0 blow up completely (in finite or in infinite time according to which of (2) and (3) is verified).

In this paper, we will use frequently some notions and techniques developed in Brezis *et al.* [3], which deal mainly with the relations between the existence of global solutions of (1) and the existence of weak solutions of (7).

On the other hand, note that Theorem 1 is a generalization of some results of P. Baras and L. Cohen [1] with shorter proof. Recall however that P. Baras and L. Cohen [1] also give a sufficient condition on the nonlinearity to provide complete blow up after T_m without nondecreasing assumption.

Finally, note that a notion of L^1-solutions also appears in W.-M. Ni, P. E. Sacks and J. Tavantzis [9] and A. A. Lacey and D. E. Tzanetis [6] but only for convex Ω. In this framework, Theorem 4 can be viewed as an extension of Theorem 2.5 of A. A. Lacey and D. E. Tzanetis [6]. Similarly, W.-M. Ni, P. E. Şacks and J. Tavantzis [9] are concerned with this kind of results for $g(u) = u^p$. The work of P. Baras and M. Pierre [2] applied to parabolic equations has also a connection with the existence of a critical value μ^* in Theorem 4.

In Section 2, we present the proofs of Theorems 1 and 2. In Section 3, we describe some properties of the weak solutions given by Definition 1. Then, in Sections 4 and 5, we prove Theorems 4 and 5. We state similar results for the case (3) in Section 6. Finally, in Section 7, we give a result on the convergence rate of some solutions of the parabolic problem to the unique solution of the elliptic problem for the case $\lambda = \lambda^*$.

2. PROOFS OF THEOREMS 1 AND 2

We begin with three lemmas. The first one is a parabolic variant of Kato's inequality, and the second one is related to the linear heat semigroup with Dirichlet boundary condition. The third one can be found in [3], we repeat it here for the sake of completeness. We denote by $T(t)$ the linear heat semigroup with Dirichlet boundary condition.

LEMMA 1. – *Let $\Phi \in C^2(\mathbf{R})$ be concave, with Φ' bounded and $\Phi(0) = 0$. Consider $T > 0$, $v_0 \in L^\infty(\Omega)$, and let f, v be such that*

$$v \in L^1((0,T) \times \Omega), \quad f\delta \in L^1((0,T) \times \Omega),$$

and

$$-\int_0^T \int_\Omega v(\xi_t + \triangle\xi) = \int_0^T \int_\Omega f\xi + \int_\Omega v_0\xi(0)$$

for all $\xi \in C^2([0,T] \times \overline{\Omega})$ such that $\xi(T) \equiv 0$ and $\xi = 0$ on $\partial\Omega$. Then

$$-\int_0^T \int_\Omega \Phi(v)(\xi_t + \triangle\xi) \geq \int_0^T \int_\Omega \Phi'(v)f\xi + \int_\Omega \Phi(v_0)\xi(0)$$

for all $\xi \in C^2([0,T] \times \overline{\Omega})$, $\xi \geq 0$ such that $\xi(T) \equiv 0$ and $\xi = 0$ on $\partial\Omega$.

Proof. – The proof is similar to that of Lemma 2 of [3].

LEMMA 2. – *For every $\tau > 0$, there exist $c(\tau)$, $c'(\tau) > 0$ such that for all $\varphi \in L^1_\delta(\Omega)$, $\varphi \geq 0$, one has*

$$c(\tau)\|\varphi\delta\|_{L^1}\delta \leq T(\tau)\varphi \leq c'(\tau)\|\varphi\delta\|_{L^1}\delta \text{ on } \Omega.$$

Proof. – Fix $\tau > 0$. By the $L^p \to W^{2,p} \cap W_0^{1,p}$ smoothing effect of $T(t)$, there exists $c_1(\tau) > 0$ such that

$$\|\delta^{-1}T(\tau/3)\mathbf{1}_\Omega\|_{L^\infty} \leq c_1(\tau). \tag{16}$$

Let φ be such that $\varphi\delta \in L^1(\Omega)$. It follows from (16) that

$$||\delta^{-1}T(\tau)\varphi||_{L^\infty} = ||\delta^{-1}T(\tau/3)[T(2\tau/3)\varphi]||_{L^\infty} \leq c_1(\tau)||T(2\tau/3)\varphi||_{L^\infty}.$$

On the other hand, by the properties of $T(t)$, there exists $c_2(\tau) > 0$ such that

$$||T(2\tau/3)\varphi||_{L^\infty} \leq c_2(\tau)||T(\tau/3)\varphi||_{L^1}.$$

Using (16) again we find

$$\begin{aligned} ||T(\tau/3)\varphi||_{L^1} &= \int_\Omega \varphi T(\tau/3)\mathbf{1}_\Omega \\ &\leq ||\delta\varphi||_{L^1}||\delta^{-1}T(\tau/3)\mathbf{1}_\Omega||_{L^\infty} \leq c_1(\tau)||\delta\varphi||_{L^1}. \end{aligned}$$

We conclude

$$\delta^{-1}T(\tau)\varphi \leq c_2(\tau)c_1^2(\tau)||\delta\varphi||_{L^1}.$$

Turning now to the other inequality, take any ball $\mathcal{B} \subset \Omega$ such that $\overline{\mathcal{B}} \subset \Omega$. There exists $c_3(\tau) > 0$ such that

$$T(\tau/2)\mathbf{1}_\mathcal{B} \geq c_3(\tau)\delta. \tag{17}$$

Observe that there exists $c_4(\tau) > 0$ such that $T(\tau/2)\delta_{x_0} \geq c_4(\tau)\delta$ for all $x_0 \in \mathcal{B}$ (δ_{x_0} is the Dirac distribution supported by x_0). Therefore,

$$T(\tau/2)\varphi(x_0) = \int_\Omega \varphi T(\tau/2)\delta_{x_0} \geq c_4(\tau)\int_\Omega \varphi\delta,$$

for all $x_0 \in \mathcal{B}$, which means

$$T(\tau/2)\varphi \geq c_4(\tau)||\varphi\delta||_{L^1}\mathbf{1}_\mathcal{B}. \tag{18}$$

Finally, by (18) and (17),

$$T(\tau)\varphi \geq c_4(\tau)||\delta\varphi||T(\tau/2)\mathbf{1}_\mathcal{B} \geq c_4(\tau)c_3(\tau)||\delta\varphi||_{L^1}\delta,$$

which completes the proof. □

LEMMA 3. – ([3]) *Assume* (2). *There exist two constants* $K \geq 0$ *and* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$, *there is a function* $\Phi_\varepsilon \in C^2([0,\infty))$, *concave, increasing, with*

$$\Phi_\varepsilon(0) = 0, \tag{19}$$

$$0 < \Phi_\varepsilon(x) \leq x \quad \textit{for} \quad x > 0, \tag{20}$$

$$1 \geq \Phi_\varepsilon'(x) \geq \frac{(g(\Phi_\varepsilon(x)) - \varepsilon K)^+}{g(x)} \quad \textit{for} \quad x \geq 0, \tag{21}$$

$$\Phi_\varepsilon'(x) \underset{\varepsilon\downarrow 0}{\longrightarrow} 1 \quad \textit{uniformly on } [0, M], \textit{ for every } M > 0. \tag{22}$$

Moreover, $\sup_{x\geq 0} \Phi_\varepsilon(x) < \infty$.

Proof of Theorem 1. – We notice first that the existence of a blowing up solution implies necessarily that (2) holds.

Next, note that by Lemma 1.1 of [1] the solution u of (1) is nondecreasing in t on $(0, T_m)$. The sequence (u_n) being defined by (1_n) with $g_n = \min(g, n)$, we set

$$T^* = \sup\,\{T > 0\ ;\ \lim_{n\to\infty} \|u_n(t)\delta\|_{L^1} < \infty \text{ for all } t < T\,\}. \tag{23}$$

Of course, $T_m \leq T^*$.

We now proceed in six steps. Through steps 1 to 4 we show that $T_m = T^*$. In step 5 we prove that (5) holds with $T = T_m$. Finally, in step 6, we show that T^* and the property (5) do not depend on the choice of the sequence (g_n) satisfying (4).

Step 1. We suppose for the sake of contradiction that there exists $\tau > 0$ such that $T_m + \tau < T^*$. Take also $\tau < T_m$ for later convenience. By the definition of T^*, there exists $C_1 > 0$ such that

$$\|u_n(\tau)\delta\|_{L^1} \leq C_1 \quad \text{and} \quad \|u_n(T_m+\tau)\delta\|_{L^1} \leq C_1. \tag{24}$$

Let $\xi \in C^2([\tau, T_m+\tau] \times \overline{\Omega})$ be such that $\xi = 0$ on $\partial\Omega$. Multiplying (1_n) by ξ and integrating on $(\tau, T_m + \tau)$ we obtain

$$\int_\tau^{T_m+\tau}\int_\Omega g_n(u_n)\xi = -\int_\tau^{T_m+\tau}\int_\Omega u_n(\xi_t + \Delta\xi) + \int_\Omega u_n(T_m+\tau)\xi(T_m+\tau) - \int_\Omega u_n(\tau)\xi(\tau). \tag{25}$$

Taking $\xi(t) = T(T_m + \tau - t)\delta$ in (25) we find

$$\int_\tau^{T_m+\tau}\int_\Omega g_n(u_n)(T(T_m+\tau-t)\delta) = \int_\Omega u_n(T_m+\tau)\,\delta - \int_\Omega u_n(\tau)(T(T_m)\delta). \tag{26}$$

Since there exists a constant $C_2 > 0$ such that $T(s)\delta > C_2\delta$ for every $0 < s < T_m$, it follows from (24) and (26) that

$$\int_\tau^{T_m+\tau}\int_\Omega g_n(u_n)\ \delta \leq C_1/C_2. \tag{27}$$

Let χ be the solution of

$$\begin{cases} -\dfrac{\partial \chi}{\partial t} - \Delta \chi = 1 \text{ in } (0, T_m + \tau) \times \Omega, \\ \chi = 0 \text{ on } \partial\Omega, \\ \chi(T_m + \tau) = 0 \text{ in } \Omega. \end{cases}$$

Using (25) again we obtain

$$\int_\tau^{T_m+\tau} \int_\Omega u_n = \int_\tau^{T_m+\tau} \int_\Omega g_n(u_n)\chi + \int_\Omega u_n(\tau)\chi(\tau).$$

There exists $C_3 > 0$ such that $\chi(s) \leq C_3\delta$ for $\tau < s < T_m + \tau$, so that by (24) and (27)

$$\int_\tau^{T_m+\tau} \int_\Omega u_n \leq C_1 C_3 / C_2 + C_1. \tag{28}$$

By (28) and the monotone convergence theorem, there exists $U \in L^1((\tau, T_m+\tau)\times\Omega)$ such that $(u_n)_{n\in\mathbf{N}}$ converges to U in $L^1((\tau, T_m+\tau)\times\Omega)$ and almost everywhere on $(\tau, T_m + \tau) \times \Omega$. By (27) we have in addition $g(U)\delta \in L^1((\tau, T_m + \tau) \times \Omega)$ and $(g_n(u_n)\delta)_{n\in\mathbf{N}}$ converges to $g(U)\delta$ in $L^1((\tau, T_m + \tau) \times \Omega)$. Of course, $U = u$ on $(\tau, T_m) \times \Omega$.

On the other hand by letting $n \to \infty$ in (25) it follows that

$$\int_\tau^{T_m+\tau} \int_\Omega g(U)\xi = -\int_\tau^{T_m+\tau} \int_\Omega U(\xi_t + \Delta\xi) - \int_\Omega u(\tau)\xi(\tau) \tag{29}$$

for all $\xi \in C^2([\tau, T_m + \tau] \times \overline{\Omega})$ such that $\xi(T_m + \tau) \equiv 0$ and $\xi = 0$ on $\partial\Omega$.

Step 2. Let $u_1 = u(\tau)$. Let ε_0, Φ_ε and K be as in Lemma 3. Take $U_\varepsilon(t) = \Phi_\varepsilon(U(t + \tau))$ for $t \in (0, T_m) \times \Omega$. By Lemma 3, we have

$$U_\varepsilon \in L^\infty((0, T_m) \times \Omega). \tag{30}$$

On the other hand, by (27), (28) and (29) we may apply Lemma 1 to $U(. + \tau)$. We obtain

$$-\int_0^{T_m} \int_\Omega U_\varepsilon(\xi_t + \Delta\xi) \geq \int_0^{T_m} \int_\Omega (g(U_\varepsilon) - K\varepsilon)^+\xi + \int_\Omega \Phi_\varepsilon(u_1)\xi(0) \tag{31}$$

for all $\xi \in C^2([0, T_m] \times \overline{\Omega})$, $\xi \geq 0$ such that $\xi(T_m) \equiv 0$ and $\xi = 0$ on $\partial\Omega$. By a standard iteration argument and (31), it follows that the solution v of

$$\begin{cases} v_t - \Delta v = (g(v) - K\varepsilon)^+ \text{ in } (0, T) \times \Omega, \\ v = 0 \text{ on } \partial\Omega, \\ v(0) = \Phi_\varepsilon(u_1) \text{ in } \Omega, \end{cases}$$

satisfies $0 \leq v \leq U_\varepsilon$ on $(0, T_m) \times \Omega$ and then $v \in L^\infty((0, T_m) \times \Omega)$ by (30).

Step 3. Lemma 7 of [3] proves that there exists $0 < \varepsilon_1 < \varepsilon_0$ such that for $0 < \varepsilon \leq \varepsilon_1$, the solution Z of

$$\begin{cases} Z_t - \triangle Z = -K\varepsilon \text{ in } (0, \infty) \times \Omega, \\ Z = 0 \text{ on } \partial\Omega, \\ Z(0) = c_0\delta/2 \text{ in } \Omega, \end{cases}$$

satisfies $Z \geq 0$ on $[0, T_m] \times \Omega$.

Step 4. There exists $c_0 > 0$ such that $u_1 \geq u_0 + c_0\delta$. Otherwise u is a stationary solution and $T_m = \infty$. Using (22), since $u_1 \in L^\infty(\Omega)$, there exists $0 < \varepsilon_2 < \varepsilon_1$ such that if $0 < \varepsilon \leq \varepsilon_2$ we have

$$\Phi_\varepsilon(u_1) \geq u_1 - c_0\delta/2 \geq u_0 + c_0\delta/2.$$

(See also [3], proof of Theorem 2, step 4.)

Take $0 < \varepsilon \leq \varepsilon_2$, then $z(t) = v(t) - Z(t)$ verifies

$$\begin{cases} z_t - \triangle z \geq g(v) \geq g(z) \text{ in } (0, T_m) \times \Omega, \\ z = 0 \text{ on } \partial\Omega, \\ z(0) = \Phi_\varepsilon(u_1) - c_0\delta/2 \geq u_0 \text{ in } \Omega. \end{cases}$$

By the maximum principle we have $u \leq z$ on $[0, T_m)$ which is absurd by $z \in L^\infty((0, T_m) \times \Omega)$. We conclude $T_m = T^*$.

Step 5. Since $u_0 \in L^\infty(\Omega)$, for n large enough, one has $g_n(u_0) = g(u_0)$ and u_n is nondecreasing in time. On the other hand, by (23), and $T_m = T^*$, we have

$$\lim_{t \to T_m} \|u_n(T_m + \varepsilon/2)\delta\|_{L^1} = \infty,$$

for every $\varepsilon > 0$. From $u_n(T_n + \varepsilon) \geq T(\varepsilon/2)u_n(T_n + \varepsilon/2)$ and Lemma 2, it follows that

$$\frac{u_n(T_m + \varepsilon)}{\delta} \underset{n \to \infty}{\longrightarrow} \infty \quad \text{uniformly on } \Omega. \tag{32}$$

Finally, we obtain (5) by (32) and u_n nondecreasing in time.

Step 6. Take another sequence $(\widetilde{g}_n)$ satisfying (4). Consider $\widetilde{u}_n$, the corresponding nondecreasing sequence of approximate solutions and $\widetilde{T}^*$ being defined as in (23). Since $\widetilde{g}_n \leq g_n$, we have $\widetilde{u}_n \leq u_n$ and then $\widetilde{T}^* \geq T^*$.

On the other hand, assume that $\widetilde{T}^* > T^*$. As in step 1, there exists a weak solution $\widetilde{U}$ of (1) on $(0, \widetilde{T}^*)$ with $\|\widetilde{U}\delta\|_{L^1} < \infty$ for all $t < \widetilde{T}^*$. By a standard iteration argument, we have $u_n \leq \widetilde{U}$ almost everywhere on $(0, \widetilde{T}^*) \times \Omega$, for all $n > 0$. This contradicts the definition of T^*. Finally, (5) for the sequence $(\widetilde{u}_n)$ is established as in step 5 (see also the proof of Lemma 9). □

In the proof of Theorem 2, we will distinguish two cases according to whether or not g satisfies the following condition

$$\textit{There exists } x_0 \textit{ such that } g(x) \geq \lambda_1 x \textit{ for all } x \geq x_0 \ ; \ g \not\equiv \lambda_1 x. \tag{33}$$

We establish two lemmas related to (33).

LEMMA 4. – *Suppose* (3) *and* (33). *Let* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ *and let* u *be the global solution of* (1). *Then either* u *blows up completely in infinite time or* $\|u(t)\delta\|_{L^1} \leq C_g$, *for all* $t > 0$, *where* $C_g > 0$ *does not depend on* u_0.

Proof. – We first prove that there exists $C_g \geq 0$ such that either $\|u(t)\delta\|_{L^1} < C_g$, or $\|u(t)\delta\|_{L^1}$ is nondecreasing for t large and converges to ∞ as $t \uparrow \infty$.

Since (3) holds, the solution u is global and we can multiply (1) by φ_1 (the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$), integrate on Ω, and apply Jensen's inequality. We find

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 \geq g\Big(\int_\Omega u(t)\varphi_1\Big) - \lambda_1 \int_\Omega u(t)\varphi_1. \tag{34}$$

Since g satisfy (33) and is convex, nondecreasing, either

(i) $g(x) \geq \lambda_1 x$, for all $x \geq 0$, $g(x) \not\equiv \lambda_1 x$

or

(ii) there exist C_g and $\Lambda > \lambda_1$ such that $g(x) \geq \Lambda x$, for all $x \geq C_g$.

In case (i), all nonnegative solutions of (1) are such that $\|u(t)\delta\|_{L^1}$ converges to ∞ as $t \uparrow \infty$. Indeed, suppose the contrary for the solution v of (1) with $v(0) \equiv 0$. Since v is nondecreasing, it converges to a weak solution w of (7) (see [3], proof of Theorem 1). But taking φ_1 as test function for w leads to

$$\lambda_1 \int_\Omega w\varphi_1 = \int_\Omega g(w)\varphi_1 > \lambda_1 \int_\Omega w\varphi_1,$$

and we obtain the desired contradiction. On the other hand, it is clear from (34) that $t \to \int_\Omega u(t)\varphi_1$ is nondecreasing.

In case (ii), if there exists t_0 such that $\int_\Omega u(t_0)\varphi_1 \geq C_g$, then (34) implies that $t \longrightarrow \int_\Omega u(t)\varphi_1$ is nondecreasing for $t \geq t_0$ and

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 \geq (\Lambda - \lambda_1)\int_\Omega u(t)\varphi_1, \quad \text{for all } t \geq t_0.$$

It is now clear that in this case $\|u(t)\delta\|_{L^1}$ converges to ∞.

To complete the proof of Lemma 4, it suffices to show that if u does not blow up completely in infinite time, then there exists a sequence $s_n \uparrow \infty$ such that $\|u(s_n)\delta\|_{L^1}$ is bounded uniformly in n.

To see that, let us take a constant C_1, a sequence (t_n, x_n), $t_n \uparrow \infty$ and $x_n \in \Omega$ such that

$$\frac{u(t_n, x_n)}{\delta(x_n)} \leq C_1. \tag{35}$$

Applying Lemma 2 with $\tau = 1$, and then (35), it follows that

$$c(1)\|u(t_n - 1)\delta\|_{L^1}\delta(x_n) \leq (T(1)u(t_n - 1))(x_n) \leq u(t_n, x_n) \leq C_1\delta(x_n). \tag{36}$$

Setting $s_n = t_n - 1$ in (36), we obtain $\|u(s_n)\delta\|_{L^1} \leq C$. Hence the result. □

LEMMA 5. – *Suppose that* (33) *does not hold then all solutions of* (1) *with* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ *are bounded.*

Proof. – Let $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, and let $\tau > 0$. Since $u(\tau) \in C^1(\overline{\Omega})$, there exists $N_0 > 0$ such that $u(\tau) \leq N_0\varphi_1$.

If $g \equiv \lambda_1 x$ then $N_0\varphi_1 \geq u(t)$, for all $t \geq 0$ and u is bounded.

Otherwise, since g is convex there exist $x_1 > 0$ and $c > 0$ such that

$$g(x) \leq \lambda_1 x - c, \text{ for all } x > x_1. \tag{37}$$

For $N > 0$, let η_N^1, η_N^2 be solutions of

$$\begin{cases} -\Delta\eta_N^1 = g(x_1)\mathbf{1}_{\{N\varphi_1 < x_1\}} & \text{in} \quad \Omega, \\ \eta_N^1 = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

$$\begin{cases} -\Delta\eta_N^2 = c\mathbf{1}_{\{N\varphi_1 > x_1\}} & \text{in} \quad \Omega, \\ \eta_N^2 = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{38}$$

Note that for every $1 \le p < \infty$,

$$\|\mathbf{1}_{\{N\varphi_1 < x_1\}}\|_{L^p} \underset{N\to\infty}{\longrightarrow} 0,$$

$$\|\mathbf{1}_{\{N\varphi_1 > x_1\}}\|_{L^p} \underset{N\to\infty}{\longrightarrow} |\Omega|^{1/p},$$

so that by the properties of non homogeneous heat equation, there exists $N_1 > 0$ such that for every $N > N_1$, we have

$$\eta_N^1 \le \eta_N^2 \quad \text{on } \Omega. \tag{39}$$

In addition, from g nondecreasing and (37) it follows that

$$(\lambda_1 N\varphi_1 - c)\mathbf{1}_{\{N\varphi_1 > x_1\}} + g(x_1)\mathbf{1}_{\{N\varphi_1 < x_1\}} \ge g(N\varphi_1). \tag{40}$$

Setting $\psi = N\varphi_1 - \eta_N^2 + \eta_N^1$ for $N > N_1$, by (40) and (39) we obtain

$$\begin{cases} -\Delta\psi = \lambda_1 N\varphi_1 - c\mathbf{1}_{\{N\varphi_1 > x_1\}} + g(x_0)\mathbf{1}_{\{N\varphi_1 < x_1\}} \\ \qquad\quad \ge g(N\varphi_1) \ge g(\psi) \quad \text{in} \quad \Omega, \\ \psi = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

In view of (38), there exists $C > 0$ such that $\eta_N^2 \le C\delta$ for all $N > 0$. Hence, there exists $N_2 > 0$ such that $\psi \ge N\varphi_1 - C\delta \ge N_0\varphi_1 \ge u(\tau)$ for $N > N_2$. For $N > \max(N_1, N_2)$, ψ is a super-solution of the problem (1) with $u_0 = u(\tau)$, and then by the maximum principle $u(t) \le \psi$, for all $t > \tau$. Hence u is bounded. □

Proof of Theorem 2. – When (33) does not hold, the solution u is bounded by Lemma 5 and so it converges to a classical solution of (7).

Otherwise, when (33) is verified, if we assume that u does not blow up completely in infinite time, by Lemma 4 we have $\|u(t)\delta\|_{L^1} \le C_g$, for all $t > 0$. Since u is nondecreasing we can now apply the argument of [3], proof of Theorem 1 to conclude that u converges to a weak solution of (7). □

3. WEAK SOLUTIONS

In this section, we prove what we claimed in the introduction about weak solutions. First, Lemmas 6 and 7 give some properties of weak solutions of the linear non homogeneous heat equation. In particular, these two lemmas prove the equivalence between weak solution in the sense of Definition 1 and the notion of integral solution of P. Baras and M. Pierre [2], P. Baras and L. Cohen [1]. Then, Lemma 8 proves property (11) in Definition 1.

Finally, Lemma 9 proves the existence of a unique minimal weak solution to (1), gives a characterization of the complete blow up time T^* and shows that the classical solution of (1) and the minimal weak solution coincide on $(0, T_m)$.

We define $N(f)(t,x) = \int_0^t \int_\Omega G(t-s,x,y)f(s,y)dsdy$ where G is the Green function of the heat equation with Dirichlet boundary condition. Let $L^1_\delta(\Omega) = L^1(\Omega, \delta(x)dx)$ and $L^1_\delta(I \times \Omega) = L^1(I, L^1_\delta(\Omega))$.

LEMMA 6. – *Let $T^* > 0$ and let $f \in L^1_\delta((0,T) \times \Omega)$ for all $0 < T < T^*$. Then there exists a unique function v, $v \in L^1((0,T)\times\Omega)$ for all $0 < T < T^*$ which is a weak solution of*

$$\begin{cases} v_t - \triangle v = f \text{ in } (0,T^*) \times \Omega, \\ v = 0 \text{ on } \partial\Omega, \\ v(0) = 0 \text{ in } \Omega, \end{cases}$$

in the following sense

$$\int_0^T \int_\Omega f\xi = -\int_0^T \int_\Omega v(\xi_t + \triangle\xi) \tag{41}$$

for all $\xi \in C^2([0,T] \times \overline{\Omega})$ such that $\xi(T) \equiv 0$ and $\xi = 0$ on $\partial\Omega$ and for all $0 < T < T^*$. Moreover if $f \geq 0$ a.e. in $(0,T^*) \times \Omega$ then $v \geq 0$ a.e. in $(0,T^*) \times \Omega$ and v is given by

$$v = N(f).$$

Proof. – First we prove the uniqueness. Let v_1 and v_2 be two solutions of (41) and $v = v_1 - v_2$. Then for all $0 < T < T^*$

$$\int_0^T \int_\Omega v(\xi_t + \triangle\xi) = 0$$

for all $\xi \in C^2([0,T] \times \overline{\Omega})$ such that $\xi(T) \equiv 0$ and $\xi = 0$ on $\partial\Omega$. Given $\varphi \in \mathcal{D}((0,T) \times \Omega)$, let ξ_φ be the solution of

$$\begin{cases} -\dfrac{\partial\xi_\varphi}{\partial t} - \triangle\xi_\varphi = \varphi \text{ in } (0,T) \times \Omega, \\ \xi_\varphi = 0 \text{ on } \partial\Omega, \\ \xi_\varphi(T) = 0 \text{ in } \Omega. \end{cases}$$

It follows that

$$\int_0^T \int_\Omega v\varphi = 0$$

for all $0 < T < T^*$, $\varphi \in \mathcal{D}((0,T)\times\Omega)$, we deduce $v = 0$ a.e. on $(0,T^*)\times\Omega$.

For the existence, we may assume that $f \geq 0$ (the equation is linear and so we can write $f = f_+ - f_-$). For all $k \leq 0$, we set $f_k(t, x) = \min(f(t, x), k)$, so that $f_k \to f$ in $L^1_\delta((0, T) \times \Omega)$ for all $0 < T < T^*$. Let

$$v_k = N(f_k). \tag{42}$$

and fix $0 < T < T^*$. The sequence (v_k) is monotone nondecreasing and is bounded in $L^1((0, T) \times \Omega)$. Indeed

$$\int_0^T \int_\Omega v_k = \int_0^T \int_\Omega f_k \xi_1 \leq C \int_0^T \int_\Omega f\delta,$$

where ξ_1 is defined by

$$\begin{cases} -\dfrac{\partial \xi_1}{\partial t} - \Delta \xi_1 = 1 \text{ in } (0, T) \times \Omega, \\ \xi_1 = 0 \text{ on } \partial\Omega, \\ \xi_1(T) = 0 \text{ in } \Omega. \end{cases}$$

We define v as the limit of the sequence (v_k). Then $v \in L^1((0, T) \times \Omega)$ for all $0 < T < T^*$ and $v \in \mathcal{D}'((0, T^*) \times \Omega)$. Passing to the limit in

$$\int_0^T \int_\Omega f_k \xi = -\int_0^T \int_\Omega v_k(\xi_t + \Delta\xi)$$

for all $\xi \in C^2([0, T] \times \overline{\Omega})$ such that $\xi(T) = 0$ and $\xi = 0$ on $\partial\Omega$ and in expression (42) we complete the proof. □

LEMMA 7. – *Let $T^* > 0$, $f \in L_+((0, T^*) \times \Omega)$ and let*

$$v = N(f) \in L^1_{loc}((0, T^*) \times \Omega).$$

Then for all $0 < T < T^*$, v and f satisfy

$$\begin{aligned} &v \in L^1((0, T) \times \Omega), \\ &f \in L^1_\delta((0, T) \times \Omega), \end{aligned} \tag{43}$$

and

$$\int_0^T \int_\Omega f\xi = -\int_0^T \int_\Omega v(\xi_t + \Delta\xi),$$

for all $\xi \in C^2([0, T] \times \overline{\Omega})$ such that $\xi(T) \equiv 0$ and $\xi = 0$ on $\partial\Omega$.

Proof. – In view of Lemma 6, it suffices to show (43). Let $0 < T < T' < T^*$ and let $\psi \in \mathcal{D}(\Omega)$, $\psi \geq 0$, $\psi \not\equiv 0$. Let $\varphi(t,x) = \psi(x)$, for $(t,x) \in [T'/2, T'] \times \Omega$ and $\varphi(t,x) = 0$, for $(t,x) \in [0, T'/2) \times \Omega$. We consider

$$\begin{cases} -\dfrac{\partial \xi_\varphi}{\partial t} - \triangle \xi_\varphi = \varphi \text{ in } (0,T') \times \Omega, \\ \xi_\varphi = 0 \text{ on } \partial\Omega, \\ \xi_\varphi(T') = 0 \text{ in } \Omega, \end{cases}$$

Using the properties of the heat semigroup with Dirichlet boundary condition, there exists $\varepsilon > 0$ such that

$$\xi_\varphi(x,t) \geq \varepsilon\delta(x), \text{ for } (t,x) \in (0,T) \times \Omega.$$

For $k \geq 0$, let $f_k(x) = \min(f(x), k)$ and let $v_k = N(f_k)$. Multiply $\dfrac{\partial v_k}{\partial t} - \triangle v_k = f_k$ by ξ_φ, and integrate in space and time. It follows that

$$\varepsilon \int_0^T \int_\Omega f_k \delta \leq \int_0^{T'} \int_\Omega f_k \xi_0 = \int_0^{T'} \int_\Omega v_k \varphi \leq \int_{T'/2}^{T'} \int_\Omega v\varphi < \infty.$$

Passing to the limit as $k \to \infty$ we obtain (43). □

The function v given by Lemma 7 also enjoys another property.

LEMMA 8. – *Let* $T^* > 0$ *and let* v, f *be such that*

$$v \in L^1((0,T) \times \Omega), \quad f \in L^1_\delta((0,T) \times \Omega)$$

and satisfying (41). *Then* $v \in C([0,T^*), L^1(\Omega, \delta(x)dx))$.

Proof. – Let f_k, v_k defined as in Lemmas 6 and 7. Let $0 < T < T^*$ and consider the solution χ of

$$\begin{cases} -\chi_t - \triangle\chi = 0 \text{ in } (0,T) \times \Omega, \\ \chi = 0 \text{ on } \partial\Omega, \\ \chi(T) = \delta \text{ in } \Omega, \end{cases}$$

There exist two constants $C > 0$, $C' > 0$ such that

$$C'\delta(x) \leq \chi(t,x) \leq C\delta(x), \text{ for } (t,x) \in (0,T) \times \Omega.$$

Therefore, for every $0 < t < T$,

$$\begin{aligned} \int_\Omega (v_k(t) - v_j(t))\delta &\leq \frac{1}{C'} \int_\Omega (v_k(t) - v_j(t))\chi(t) \\ &= \frac{1}{C'} \int_0^t \int_\Omega (f_k - f_j)\chi \leq \frac{C}{C'} \int_0^T \int_\Omega (f_k - f_j)\delta. \end{aligned}$$

By uniform Cauchy convergence, we conclude

$$v \in C([0,T), L^1(\Omega, \delta(x)dx)).$$ □

Returning now to the nonlinear problem (1), let u_0 be a nonnegative bounded measure of Ω. Consider U verifying

$$U(t,x) = \int_\Omega G(t,x,y)u_0(y)dy + \int_0^t \int_\Omega G(t-s,x,y)g(U(s,y))dyds,$$

on $(0,T^*) \times \Omega$, *i.e.* U is an integral solution of (1) in the sense of Baras-Pierre. We have necessarily $U \in L^1_{loc}((0,T^*) \times \Omega)$. Indeed, thank to properties of the Green function, we have $g(U) \in L^1_{loc}((0,T^*) \times \Omega)$. If there exist $C > 0$ and $v_0 \geq 0$ such that $g(v) > Cv$ for every $v > v_0$, it is clear that $U \in L^1_{loc}((0,T^*) \times \Omega)$. Otherwise, g is constant, and then U is classical.

Applying Lemma 7 to $v(t,x) = \int_0^t \int_\Omega G(t-s,x,y)g(U(s,y))dyds$, we obtain that U is a weak solution in the sense of Definition 1. By Lemma 8, we obtain (11).

Conversely, if $u \geq 0$ is a weak solution of (1), then according to Lemma 6 applied to $u - T(.)u_0 = N(f)$, we obtain that u is an integral solution.

REMARK 1. – From Lemma 8 it follows that a weak solution of (1) also satisfies

$$\int_0^T \int_\Omega g(u)\xi = \int_\Omega u(T)\xi(T) - \int_\Omega u_0\xi(0) - \int_0^T \int_\Omega u(\xi_t + \triangle\xi),$$

for all $\xi \in C^2([0,T] \times \overline{\Omega})$ such that $\xi = 0$ on $\partial\Omega$, for all $0 < T < T^*$.

This property shows in particular that our definition of weak solution is equivalent to the notion of L^1-solution of [6], [9], given only for Ω convex.

LEMMA 9. – *Let $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ and let (u_n) be the sequence given by (1_n). There exists a unique minimal weak solution U of (1) in the sense of Definition 1, defined on a maximal interval $(0,T^*)$. This solution U coincides with the classical solution of (1) on $(0,T_m)$. Moreover T^* satisfies*

$$T^* = \sup\{T > 0\ ;\ \lim_{n\to\infty} \|u_n(t)\delta\|_{L^1} < \infty \quad \text{for} \quad 0 < t < T\},$$

and does not depend on the choice of the sequence (g_n) satisfying (4). If $T^* < \infty$ then (5) is verified for all $T \geq T^*$.

Proof. – Consider the sequence (u_n) defined by (1_n) with $g_n = \min(g,n)$. As in the proof of Theorem 1, neither T^* nor (5) depend on this choice. We proceed in three steps.

Step 1. Define

$$S^* = \sup\{S > 0 \; ; \; \lim_{n\to\infty} \|u_n(t)\delta\|_{L^1} < \infty \quad \text{for} \quad 0 < t < S\}.$$

Reasoning as in proof of Theorem 1, step 1, the definition of S^* implies that there exists a weak solution U of (1) on $(0, S^*)$ obtained as the limit of the sequence (u_n).

Take V a weak solution of (1) defined on $(0, S^*(V))$. By a standard iteration argument and Lemma 6, we have $V \geq u_n$ almost everywhere on $(0, T^*(V)) \times \Omega$, for every $n > 0$. It follows that $V \geq U$ almost everywhere on $(0, \min(S^*, S^*(V))) \times \Omega$.

On the other hand, the classical solution u is a weak solution of (1) in the sense of Definition 1 and then $u \geq U$ on $(0, \min(T_m, S^*))$. By uniqueness of the classical solution, we have $S^* \geq T_m$ and $u = U$ on $(0, T_m) \times \Omega$.

Step 2. Suppose $S^* < \infty$. By the definition of S^*, for every $\varepsilon > 0$ there exists $t \in (S^*, S^* + \frac{\varepsilon}{2})$ such that $\|u_n(t)\delta\|_{L^1} \underset{n\to\infty}{\longrightarrow} \infty$.

Fix $\varepsilon > 0$, it follows from by Lemma 2 that

$$\frac{u_n(S^* + \varepsilon)}{\delta} \underset{n\to\infty}{\longrightarrow} \infty \text{ uniformly on } \Omega. \tag{44}$$

Since g is convex and $T_m \leq S^* < \infty$, it follows that there exist x_1 and $c > 0$ such that for all $x > x_1 - 1$,

$$g(x) \geq (\lambda_1 + 2c)x. \tag{45}$$

For $N > 0$, consider η_N^1 and η_N^2 such that

$$\begin{cases} -\Delta\eta_N^1 = \lambda_1 N\varphi_1 \mathbf{1}_{\{N\varphi_1 < x_1\}} \leq \lambda_1 x_1 \mathbf{1}_{\{N\varphi_1 < x_1\}} & \text{in} \quad \Omega, \\ \eta_N^1 = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

$$\begin{cases} -\Delta\eta_N^2 = cN\varphi_1 \mathbf{1}_{\{N\varphi_1 > x_1\}} & \text{in} \quad \Omega, \\ \eta_N^2 = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

By the properties of the Laplace equation, there exist c_1, c_2, $c_3 > 0$ such that for all $N > 1$

$$\eta_N^1 \leq c_1\delta, \quad Nc_2\delta \leq \eta_N^2 \leq Nc_3\delta. \tag{46}$$

Hence there exists $N_1 > 0$ such that for every $N > N_1$, we have

$$\eta_N^1 \leq \eta_N^2 \quad \text{on} \quad \Omega. \tag{47}$$

Note that by (45),

$$g(N\varphi_1) \geq (\lambda_1 + c)N\varphi_1 \mathbf{1}_{\{N\varphi_1 > x_1\}}.$$

Setting $\psi_N = N\varphi_1 - \eta_N^1 + \eta_N^2$, for $N > N_1$ we obtain by (47),

$$\begin{cases} -\triangle\psi_N = (\lambda_1 + c)N\varphi_1\mathbf{1}_{\{N\varphi_1 > x_1\}} & \text{in} \quad \Omega, \\ \psi_N = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

By (44) and (46), there exists K_N such that $u_{K_N}(T^*(u_0)+\varepsilon) \geq N\varphi_1+\eta_N^2 \geq \psi_N$. On the other hand, by (45) and by possibly choosing larger K_N, we may assume that for all $x_1 < x < \|\psi_N\|_{L^\infty}$,

$$g_{K_N}(x) \geq (\lambda_1 + c)x.$$

It follows that

$$(\lambda_1 + c)N\varphi_1\mathbf{1}_{\{N\varphi_1 > x_1\}} \leq g_{K_N}(N\varphi_1) \leq g_{N_K}(\psi_N), \quad \text{in} \quad \Omega.$$

Therefore, for $N > N_1$, the function ψ_N is a sub-solution of the problem (1_{K_N}) after time $T^*(u_0) + \varepsilon$. We conclude that $u_{K_N}(t) \geq \psi_N$, for all $t > T^*(u_0) + \varepsilon$, which proves (5) for every $T > S^*$.

Step 3. Since T^* is the maximal time of existence of the weak solution U, we have $S^* \leq T^*$. By step 2, it is impossible to obtain a weak solution of (1) after S^* and then $S^* = T^*$. □

4. PROOF OF THEOREM 4

We begin with two lemmas. The first one is well-known and we give it for the sake of completeness. The second one is a convexity result : for $\lambda < \lambda^*$ and $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ such that the solution of (12_λ) does not blow up completely in finite time, all $v_0 \in L^\infty(\Omega)$, $0 \leq v_0 \leq u_0$, $v_0 \not\equiv u_0$ lead to global bounded solutions of (12_λ).

LEMMA 10. – *Suppose* (2), *let* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, *and assume that the solution* u *of* (1) *is global. Then* $\|u(t)\delta\|_{L^1} \leq C_g$, *for all* $t > 0$, *where* $C_g > 0$ *does not depend on* u_0.

Proof. – Since in [3], proof of Theorem 1, there exists $C_g > 0$ such that if $\|u(t_0)\delta\|_{L^1} \geq C_g$ then for every $t > t_0$ we have

$$\frac{d}{dt}\int_\Omega u(t)\varphi_1 \geq \frac{1}{2}g\Big(\int_\Omega u(t)\varphi_1\Big). \tag{48}$$

Since we assume u global, (48) and (2) lead to a contradiction and then $\|u(t_0)\delta\|_{L^1} < C_g$ for all $t > 0$. (Observe that this argument does not require u to be nondecreasing.) □

LEMMA 11. – *Suppose* (2), *let* $\lambda < \lambda^*$, *let* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ *and let* u *be the solution of* (12_λ). *Assume that* u *does not blow up completely in finite time. Then for every* $v_0 \in L^\infty(\Omega)$, $0 \leq v_0 \leq u_0$, $v_0 \not\equiv u_0$, *the solution* v *of* (12_λ) *with* $v(0) = v_0$ *is global bounded.*

Proof. – Let u be the minimal weak solution of (1) and suppose that $T^*(u) = \infty$. Set $w_\varepsilon = \Phi_\varepsilon(u)$, where Φ_ε is the function defined in Lemma 3. As in the proof of Theorem 1, we can make use of Lemma 1 to show that w_ε is a super-solution of the following problem

$$\begin{cases} w_t - \triangle w = \lambda(g(w) - K\varepsilon)^+ \text{ in } (0,\infty) \times \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w(0) = \Phi_\varepsilon(u_0) \text{ in } \Omega, \end{cases} \tag{49}$$

Since Φ_ε is bounded, we have $w_\varepsilon \in L^\infty((0,\infty) \times \Omega)$, and thus the solution w of (49) is global bounded.

Take now v_0 as in the statement of the lemma. Fix $0 < \tau < T_m(u_0)$, there exists $c_0 > 0$ such that $u(\tau) - v(\tau) \geq T(\tau)(u_0 - v_0) \geq c_0\delta$. Taking $u(\tau)$ and $v(\tau)$ instead of u_0 and v_0 and reasoning as in the proof of Theorem 1, Step 4, there exist $0 < \eta < 1$ and $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$,

$$\eta\Phi_\varepsilon(u_0) \geq u_0 - c_0\delta \geq v_0.$$

Such η and ε_1 being fixed, consider the solution z of

$$\begin{cases} z_t - \triangle z = \lambda\Big(g(z) + \dfrac{K\eta}{1-\eta}\varepsilon\Big) \text{ in } (0,\infty) \times \Omega, \\ z = 0 \text{ on } \partial\Omega, \\ z(0) = 0 \text{ in } \Omega. \end{cases}$$

Since $\lambda < \lambda^*$, there exists $0 < \varepsilon_2 \leq \varepsilon_1$ such that for $0 < \varepsilon \leq \varepsilon_2$, z is global bounded.

We now set $0 < \varepsilon < \varepsilon_2$ and $Z = \eta w + (1-\eta)z$. Since g is convex, Z satisfies

$$\begin{cases} Z_t - \triangle Z = \lambda(\eta g(w) + (1-\eta)g(z)) \geq \lambda g(Z), \text{ in } (0,\infty) \times \Omega, \\ Z = 0 \text{ on } \partial\Omega, \\ Z(0) = \eta\Phi_\varepsilon(u_0) \geq v_0 \text{ in } \Omega. \end{cases}$$

Since Z is global bounded we deduce v global bounded by the maximum principle. □

We are now in position to prove Theorem 4.

Proof of Theorem 4. – Let u_0 be as in the statement of the theorem and let u be the solution of $(15_{\lambda,\mu})$. We proceed in four steps.

Step 1. We prove that there exists $\mu_1 > 0$ such that for every $0 \leq \mu \leq \mu_1$ the solution u of $(15_{\lambda,\mu})$ is global bounded.

Since $u_0 \in L^\infty(\Omega)$ and $u \in C((0,T_m) \times \Omega)$ with $T_m > 0$, there exist $\mu_2 > 0$ and $\tau_1 > 0$ such that for $0 < \mu < \mu_2$ and $0 \leq t \leq \tau_1$, $u(t) \leq 1$. On the other hand, there exists $0 < \tau_2 \leq \tau_1$ such that

$$\lambda \int_0^{\tau_2} T(\tau_2 - s)g(u)ds \leq \lambda \int_0^{\tau_2} T(\tau_2 - s)g(1)ds \leq w_\lambda/2, \tag{50}$$

where w_λ is the minimal classical solution of (13_λ). Such $\tau_2 > 0$ being fixed, $T(\tau_2)u_0 \in C^1(\overline{\Omega})$ and there exists $0 < \mu_1 \leq \mu_2$ such that

$$\mu_1 T(\tau_2)u_0 \leq w_\lambda/2. \tag{51}$$

For $0 \leq \mu \leq \mu_1$, by (50), (51), we have

$$u(\tau_2) = \mu T(\tau_2)u_0 + \lambda \int_0^{\tau_2} T(\tau_2 - s)g(u)ds \leq w_\lambda,$$

and then $u(t) \leq w_\lambda$ for all $t \geq \tau$ by the maximum principle.

Step 2. Set $\mu^* = \sup\{\mu > 0;$ the solution u of $(15_{\lambda,\mu})$ is global bounded$\}$. We have $\mu^* < \infty$, indeed by Lemma 10, for $\int_\Omega \mu u_0 \varphi_1$ large enough, the solution u of $(15_{\lambda,\mu})$ blows up in finite time.

We show that for every $\mu < \mu^*$, the solution u of $(15_{\lambda,\mu})$ converges to w_λ in $L^\infty(\Omega)$. First, since $\lambda_1(-\Delta - \lambda g'(w_\lambda)) > 0$, if there exists a sequence (t_n), $t_n \uparrow \infty$, such that $\|u(t_n) - w_\lambda\|_{L^\infty} \to 0$, then the whole sequence $u(t)$ converges to w_λ.

Suppose by contradiction that there exists no subsequence (t_n), $t_n \uparrow \infty$ such that $u(t_n)$ converges to w_λ. We can assume $u_0 > w_\lambda$. Indeed since u is bounded, the ω-limit set of u contains a solution w of (13_λ), with $w \geq w_\lambda + c_0\delta$. Therefore there exists a subsequence t_n such that $u(t_n)$ converges to w in $C^1(\overline{\Omega})$, and for n large enough we have $u(t_n) > w_\lambda$. On the other hand, by possibly taking $u(\tau)$ instead of u_0, we may suppose that $u_0 \in C^1(\overline{\Omega})$.

Set $z(t) = u(t) - w_\lambda > 0$, there exists $C_1 > 0$ such that $\|z(t)\|_{L^\infty} \geq C_1$ for all $t > 0$. By Lemma 2, since $G = \|u\|_{L^\infty((0,\infty)\times\Omega)} < \infty$, we find

$$\|z(t)\delta\|_{L^1} \geq ce^{g'(0)}(T(1)z(t-1))(x) \geq ce^{(g'(0)-G)}z(t,x) \text{ for every } x \in \Omega.$$

and then

$$\|z(t)\delta\|_{L^1} \geq ce^{(g'(0)-G)}C_1 = C_2 > 0. \tag{52}$$

Let φ^T be the solution of

$$\begin{cases} -\varphi_t^T - \Delta\varphi^T = g'(u)\varphi^T & \text{in} \quad (0,\infty)\times\Omega, \\ \varphi^T = 0 \quad \text{on} \quad \partial\Omega, \\ \varphi^T(T) = \delta \quad \text{in} \quad \Omega. \end{cases}$$

We have

$$\frac{d}{dt}\int_\Omega \varphi^T(u(t)-w_\lambda) = \int_\Omega (g(u)-g(w_\lambda)-g'(u)(u-w_\lambda))\varphi^T \leq 0.$$

It follows from (52) that

$$\int_\Omega \varphi^T(0)(u_0 - w_\lambda) \geq \int_\Omega \varphi^T(T)z(T) \geq C_2. \tag{53}$$

Let now ψ be the solution of

$$\begin{cases} \psi_t - \Delta\psi = g'(u)\psi & \text{in} \quad (0,\infty)\times\Omega, \\ \psi = 0 \quad \text{on} \quad \partial\Omega, \\ \psi(0) = \delta \quad \text{in} \quad \Omega. \end{cases}$$

Easy calculations lead to $\int_\Omega \psi(T)\delta = \int_\Omega \varphi^T(0)\delta$. By (53) and $u_0 \in C^1(\overline{\Omega})$, we conclude $\int_\Omega \psi(T)\delta \geq C$, for every $T > 0$, where $C > 0$ does not depend on T.

Let $\mu < \mu' < \mu^*$ and let v be the solution of (15) corresponding to $\mu' u_0$. Using the convexity of g, we have

$$\begin{cases} (v-u)_t - \Delta(v-u) \geq g'(u)(v-u) & \text{in} \quad (0,\infty)\times\Omega, \\ v - u = 0 \quad \text{on} \quad \partial\Omega, \\ v(0) - u(0) = (\mu' - \mu)u_0 \geq \kappa\delta \quad \text{in} \quad \Omega, \end{cases}$$

where $\kappa > 0$. By using ψ, there exists $C_3 > 0$ such that

$$\int_\Omega (v(t)-u(t))\delta \geq C_3 \quad \text{for every} \quad t > 0. \tag{54}$$

Since $\mu' < \mu^*$, v is bounded and then $v(t)$ converges up to a sub-sequence to a classical solution W of (13_λ). By (54) and $v > u$, we have $w_\lambda < w < W$, but the existence of a triple of ordered solutions of (13) is impossible (see H. Fujita [4]). The contradiction shows that $u(t)$ converges to w_λ as $t \to \infty$.

Step 3. Consider a nondecreasing sequence $\mu_n \uparrow \mu^*$, $\mu_n < \mu^*$. For each $n > 0$, the solution u_{μ_n} is global. By Lemma 10, we have $\|u_{\mu_n}\|_{L^1_\delta} \le C_g$. By using the technique of [3] proof of Theorem 1, we obtain

$$\|g(u_{\mu_n})\|_{L^1_\delta((T,T+1)\times\Omega)} \le C \quad \text{and} \quad \|u_{\mu_n}\|_{L^1((T,T+1)\times\Omega)} \le C \qquad (55)$$

where C depends neither on n nor on T. We define u_{μ^*} as the limit of the nondecreasing sequence (u_{μ_n}), by (55) and taking the limit in all terms of (10), u_{μ^*} is a weak solution of (1) and $T^*(u_{\mu^*}) = \infty$.

Step 4. We prove (iii) by contradiction. Suppose that there exists $\mu^{**} > \mu^*$ such that $T^*(\mu^{**}u_0) = \infty$, then for $\mu^* < \mu' < \mu^{**}$, the solution of $(15_{\lambda,\mu'})$ is global bounded by Lemma 11, which contradicts the definition of μ^*. □

REMARK 2. – Following the argument of [6], Lemma 2.1, we know that u_{μ^*} does not converge to w_{λ^*} in $L^\infty(\Omega)$ even up to a subsequence. Therefore, when there exists only one classical solution of the stationary problem, the solution u_{μ^*} is not bounded. However, the exact behaviour of u_{μ^*} remains mainly unknown.

Theorem 4 has a corollary concerning the instability of the weak solutions of (13_λ) different from w_λ.

COROLLARY 7. – *Let $\lambda < \lambda^*$. Let w_λ be the minimal classical solution of (13_λ) and suppose that there exists another weak solution $\overline{w}_\lambda$ of (13_λ). Let $u_0 \in L^\infty(\Omega)$ be such that $u_0 \le \overline{w}_\lambda$, $0 \le u_0 \not\equiv \overline{w}_\lambda$. Then the solution u of (12_λ) is global bounded and converges to w_λ as $t \to \infty$.*

5. PROOF OF THEOREM 5

Instead of Theorem 5, we prove the following result which clearly implies Theorem 5.

PROPOSITION 8. – *Suppose (2), $\lambda = \lambda^*$ and $w_{\lambda^*} \in L^\infty(\Omega)$. Let $u_0 \in L^\infty(\Omega)$ be such that $u_0 \ge 0$ and $u_0 \not\equiv 0$. Then there exist $\mu^* > 0$ and $\mu^{**} > 0$ such that*

(i) *$0 \le \mu < \mu^*$, the solution u_μ of $(15_{\lambda^*,\mu})$ is global bounded and converges to w_{λ^*} as $t \to \infty$.*

(ii) *$\mu^* \le \mu < \mu^{**}$, the solution u_μ of $(15_{\lambda^*,\mu})$ is global and converges to w_{λ^*} in $L^1_\delta(\Omega)$ as $t \to \infty$.*

(iii) $\mu = \mu^{**}$, *the minimal weak solution of* $(15_{\lambda^*,\mu^{**}})$ *converges to* w_{λ^*} *in* $L^1_\delta(\Omega)$ *as* $t \to \infty$.

(iv) $\mu > \mu^{**}$, *the solution* u_μ *of* $(15_{\lambda^*,\mu})$ *blows up completely in finite time.*

REMARK 3. – We do not know whether or not $\mu^* = \mu^{**}$. If this is the case then (ii) does not occur. For $\mu^* \leq \mu \leq \mu^{**}$, we do not know if the solution u is bounded.

To prove Proposition 8 we will use two lemmas. The first one is a weaker form of Lemma 11 which holds in the case $\lambda = \lambda^*$. The second one proves a instability property of the minimal solution of (13_{λ^*}).

For $\lambda = \lambda^*$, Lemma 11 can fail. However, we can prove the following weaker result without any restriction on λ.

LEMMA 12. – *Suppose* (2), *let* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, *and assume that the solution* u *of* (1) *does not blow up completely in finite time. Then for every* $v_0 \in L^\infty(\Omega)$, $0 \leq v_0 \leq u_0$, $v_0 \not\equiv u_0$, *the solution* v *of* (1) *corresponding to* v_0 *is global.*

Proof. – The proof is just an adaptation (slightly improved) of the proof of Theorem 2 of [3].

As in the proof of Lemma 11, the idea is to use a function Φ_ε, which is bounded, increasing and concave, in order to obtain a super-solution of a "perturbation" of (1). Here, we take Φ_ε depending also on t. For $g(0) \neq 0$ and $\varepsilon < g(0)$, the function Φ_ε would be

$$\Phi_\varepsilon(t,u) = \widetilde{h}_t^{-1}(h(u)),$$

where

$$h(u) = \int_0^u \frac{ds}{g(s)}, \quad \widetilde{h}_t(u) = \int_0^u \frac{ds}{g(s) - \varepsilon e^{-2\lambda_1 t}}.$$

If $g(0) = 0$, it is possible to take another function in the spirit of Lemma 3. (See the proof of Lemma 6 of [3].)

Let u be a weak solution of (1) such that $T^*(u) = \infty$, and let $w_\varepsilon(t,x) = \Phi_\varepsilon(t, u(t,x))$. Easy calculations and Lemma 1 show that w_ε is a super-solution of the following problem

$$\begin{cases} w_t - \Delta w = g(w) - \varepsilon e^{-2\lambda_1 t} \text{ in } (0,\infty) \times \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w(0) = \Phi_\varepsilon(0, u_0) \text{ in } \Omega, \end{cases} \tag{56}$$

Since Φ_ε is bounded, we have $w_\varepsilon \in L^\infty_{loc}((0,\infty), L^\infty(\Omega))$, so that the unique classical solution w of (56) is global.

Take now v_0 as in the statement of the lemma. Fix $0 < \tau < T_m(u_0)$. There exists $c_0 > 0$ such that $u(\tau) - v(\tau) \geq T(\tau)(u_0 - v_0) \geq c_0\delta$. Taking $u(\tau)$ and $v(\tau)$ instead of u_0 and v_0 and reasoning as in the proof of Theorem 1 step 4, there exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon < \varepsilon_1$

$$v_0 \leq \Phi_\varepsilon(0, u_0) - c_0\delta/2.$$

Denote by $c > 0$ a constant verifying

$$c\chi \leq \varphi_1, \tag{57}$$

where $\varphi_1 \geq 0$ is the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$, and χ satisfies

$$\begin{cases} -\Delta\chi = 1 \text{ in } \Omega, \\ \chi = 0 \text{ on } \partial\Omega. \end{cases}$$

Set $k(t) = (2\frac{\varphi_1}{c} - \chi)e^{-2\lambda_1 t}$, k is a sub-solution of the problem

$$\begin{cases} Z_t - \Delta Z = -e^{-2\lambda_1 t} \text{ in } (0,\infty) \times \Omega, \\ Z = 0 \text{ on } \partial\Omega, \\ Z(0) = 2\frac{\varphi_1}{c} \text{ in } \Omega. \end{cases} \tag{58}$$

By (57), we have $k \geq 0$ on $(0,\infty) \times \Omega$, so that the solution Z of (58) is also nonnegative on $(0,\infty) \times \Omega$.

Now, we set $z = w - \varepsilon Z$, for ε small enough, we have $\varepsilon Z \leq c_0\delta/2$ and z is a super-solution of the problem verified by v. Since z is global, v is also global by the maximum principle. □

When the solution w_{λ^*} is classical, it is unstable from above. Indeed, we prove the following lemma.

LEMMA 13. – *Suppose* (2), $\lambda = \lambda^*$ *and* $w_{\lambda^*} \in L^\infty(\Omega)$. *Let* $u_0 \in L^\infty(\Omega)$, $u_0 \geq w_{\lambda^*}$ *and* $u_0 \not\equiv w_{\lambda^*}$. *Then the solution* u *of* (12_λ) *blows up completely in finite time.*

Proof. – By contradiction suppose $T^*(u) = +\infty$. In that case we may assume that u is global, by taking $\widetilde{u}_0 \in L^\infty(\Omega)$, $w_{\lambda^*} \leq \widetilde{u}_0 \leq u_0$ and $\widetilde{u}_0 \not\equiv w_{\lambda^*}$, $\widetilde{u}_0 \not\equiv u_0$ instead of u_0 and applying Lemma 12. As usual we may assume without loss of generality that there exists $c_0 > 0$ such that $u_0 - w_{\lambda^*} \geq c_0\delta$. We proceed in three steps.

Step 1. There exists $A \subset w_{\lambda^*}(\Omega)$, $|A| \neq 0$ such that

$$g''(v) > 0, \text{ for all } v \in A.$$

Indeed, otherwise we have $g(w_{\lambda^*}) \equiv g(0) + g'(0)w_{\lambda^*}$ and everything happens for w_{λ^*} as if g were linear. Since $g(0) > 0$, g verifies the monotone case of P. Mironescu and V. Radulescu [8] and the existence of w_{λ^*} contradicts the definition of λ^*.

Therefore, there exist $\eta > 0$, $0 < K_1 < K_2 \leq \|w_{\lambda^*}\|_{L^\infty}$ such that

$$g''(u) \geq \eta, \text{ for all } u \in [K_1, K_2]. \tag{59}$$

Step 2. Since g is convex, $w = u - w_{\lambda^*}$ verifies

$$\begin{cases} w_t - \Delta w \geq \lambda^* g'(w_{\lambda^*})w \text{ in } (0, T^*(u_0)) \times \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w(0) = u_0 - w_{\lambda^*} \geq c_0\delta \text{ in } \Omega, \end{cases} \tag{60}$$

On the other hand, we denote by Ψ_1 the first eigenfunction of $(-\Delta - \lambda^* g'(w_{\lambda^*}))$ in $H_0^1(\Omega)$, the corresponding eigenvalue being 0. The function Ψ_1 being chosen such that $\|\Psi_1\|_{L^1} = 1$, there exists C such that

$$\Psi_1 \leq C\delta. \tag{61}$$

Note that by (60) and (61), w is a super-solution of the problem verified by $C_1\Psi_1$ for some $C_1 > 0$. By the maximum principle, we conclude that

$$w(t) \geq C_1\Psi_1 \quad \text{in} \quad (0, \infty) \times \Omega. \tag{62}$$

Step 3. We set $f(t) = \int_\Omega u(t)\Psi_1 dx$. Then

$$f'(t) = \int_\Omega [g(u(t)) - g'(w_{\lambda^*})u(t)]\Psi_1 dx.$$

Since

$$\int_\Omega [g(w_{\lambda^*}) - g'(w_{\lambda^*})w_{\lambda^*}]\Psi_1 dx = 0$$

it follows that

$$\begin{aligned} f'(t) &= \int_\Omega [g(u(t)) - g(w_{\lambda^*}) - g'(w_{\lambda^*})w(t)]\Psi_1 dx \\ &= \int_\Omega \Big(\int_{w_{\lambda^*}}^{u(t)} \Big(\int_{w_{\lambda^*}}^{s} g''(\sigma)d\sigma\Big) ds\Big)\Psi_1 dx \geq 0. \end{aligned}$$

The idea is to use (59) to show that $f'(t)$ does not converge to 0. There exists a subset $\widetilde{\Omega}$ of Ω, such that $|\widetilde{\Omega}| \neq 0$ and

$$w_{\lambda^*}(x) \geq K_1, \text{ for all } x \in \widetilde{\Omega}.$$

Since $\widetilde{\Omega}$ is compact in Ω, there exists a constant $C_2 > 0$ such that

$$\Psi_1(x) \geq C_2, \text{ for all } x \in \widetilde{\Omega}. \tag{63}$$

Next, there exists a subset $\widetilde{\widetilde{\Omega}}$ of $\widetilde{\Omega}$ such that $|\widetilde{\widetilde{\Omega}}| \neq 0$ and

$$K_1 \leq w_{\lambda^*}(x) \leq \min\left(K_1 + \frac{C_1 C_2}{2}, \frac{K_2 + K_1}{2}\right), \text{ for all } x \in \widetilde{\widetilde{\Omega}}.$$

By (62) and (63), we obtain

$$K_1 + C_1 C_2 \leq u(t,x), \text{ in } [0,\infty) \times \widetilde{\widetilde{\Omega}}. \tag{64}$$

Hence by (59) and (64) we obtain

$$f'(t) \geq C_2 \eta \int_{\widetilde{\widetilde{\Omega}}} \left(\int_{\min(K_1 + \frac{C_1 C_2}{2}, \frac{K_2 + K_1}{2})}^{\min(K_1 + C_1 C_2, K_2)} \left(\int_{\min(K_1 + \frac{C_1 C_2}{2}, \frac{K_2 + K_1}{2})}^{s} d\sigma \right) ds \right) dx \geq C > 0.$$

We conclude that f is not bounded. By Lemma 10, we obtain a contradiction. □

REMARK 4. – This result proves that for $w_{\lambda^*} \in L^\infty(\Omega)$, there does not exist a second solution of (13_{λ^*}) even in the weak sense.

Proof of Proposition 8. – We define $\mu^* > 0$ as in the proof of Theorem 4. Similarly, for μ large enough the solution u of $(15_{\lambda^*,\mu})$ blows up completely in finite time, so that we can define

$$\mu^{**} = \inf\{\mu > 0; \text{ the solution } u \text{ of } (15_{\lambda^*,\mu}) \text{ blows up completely in finite time}\}.$$

Of course $\mu^{**} \geq \mu^*$. For $\mu < \mu^{**}$, the solutions are global by Lemma 12. Then using Lemma 10 and reasoning as in the proof of Theorem 4, the weak solution u^{**} corresponding to μ^{**} is obtained as the limit of the solutions (u_μ) for $\mu \uparrow \mu^{**}$. Now we show that u^{**} converges s to w_{λ^*} in $L^1_\delta(\Omega)$ as $t \to \infty$. The result will follow for $\mu^* \leq \mu < \mu^{**}$.

Let v be the global classical solution of (12_{λ^*}), with $v(0) = 0$. We have

$$\|w_{\lambda^*} - v(t)\|_{L^\infty} \underset{t\to\infty}{\longrightarrow} 0,$$

so that for every $n > 0$ there exists $s_n > 0$ such that for any $t > s_n$

$$w_\lambda - v(t) \leq 1/n \text{ on } \Omega. \tag{65}$$

Suppose for the sake of contradiction that $||w_{\lambda^*} - u^{**}(t)||_{L^1_\delta}$ does not converge to 0 as $t \to \infty$. There exist $C > 0$ and a sequence (t_n) such that $t_n > s_n$ and

$$||w_{\lambda^*} - u^{**}(t_n)||_{L^1_\delta} > C. \tag{66}$$

Since $u^{**}(t) \geq v(t)$ almost everywhere on $(0, \infty) \times \Omega$, by (65) we find

$$||(u^{**}(t_n) - w_{\lambda^*})^-||_{L^\infty} < 1/n. \tag{67}$$

Set $w(t) = u^{**}(t) - w_{\lambda^*}$, since g is convex, w satisfies (in the weak sense)

$$\begin{cases} w_t - \Delta w \geq g'(w_{\lambda^*})w \text{ in } (0, \infty) \times \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w(0) = u_0 - w_{\lambda^*} \text{ in } \Omega, \end{cases}$$

By (67) and (66), we have $w^-(t_n) \leq 1/n$ and $||w^+(t_n)||_{L^1_\delta} \geq \frac{C}{2}$ for n large enough.

With $c = ||g'(w_{\lambda^*})||_{L^\infty((0,\infty)\times\Omega)} < \infty$, it follows that (in the weak sense)

$$\begin{cases} w_t - \Delta w \geq -cw^- \text{ in } (t_n, \infty) \times \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w(t_n) \geq w^+(t_n) - 1/n \text{ in } \Omega, \end{cases}$$

Fix $\tau > 0$. We claim that there exists $n > 0$ such that $w(\tau + t_n) > 0$ (a.e.). Indeed, consider z_1 the solution of

$$\begin{cases} \dfrac{\partial z_1}{\partial t} - \Delta z_1 = cz_1 \text{ in } (t_n, \infty) \times \Omega, \\ z_1 = 0 \text{ on } \partial\Omega, \\ z_1(0) = 1/n \text{ in } \Omega, \end{cases}$$

It is clear that $z_1(\tau) \leq c_1\delta/n$ where $c_1 > 0$ does not depend on n.

On the other hand, for $z_2(t) = T(t)w^+(t_n)$, it follows from Lemma 2 that $z_2(\tau) \geq c_2\delta$ where c_2 does not depend on n. Therefore for n large enough $z_2(\tau) - z_1(\tau) > 0$. Since $w(t_n + .)$ is a super-solution of the problem verified by $z_2 - z_1$, we obtain the claim.

Since $w(t_n + \tau) > 0$, there exists $u_1 \in L^\infty(\Omega)$ such that $w_{\lambda^*} < u_1 \leq u^{**}(t_n + \tau)$. By Lemma 13, the solution of (1) corresponding to u_1 blows up completely in finite time, and we obtain a contradiction. □

6. THE CASE $\int^{\infty} \frac{ds}{g(s)} = \infty$ (3)

We can adapt Theorems 4, 5 and Corollary 6 to the case (3). We obtain the following results.

THEOREM 9. – *Suppose* (3). *Let* $\lambda < \lambda^*$, *and* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ *and* $u_0 \not\equiv 0$. *Then there exists* $0 < \mu^* \leq \infty$ *such that*

(i) $0 \leq \mu < \mu^*$, *the global solution* u_μ *of* $(15_{\lambda,\mu})$ *is bounded and converges to* w_λ *in* $L^\infty(\Omega)$ *as* $t \to \infty$.

(ii) $\mu = \mu^*$, *the global solution* u_{μ^*} *of* $(15_{\lambda,\mu^*})$ *does not blow up completely in infinite time.*

(iii) $\mu > \mu^*$, *the global solution* u_μ *of* $(15_{\lambda,\mu^*})$ *blows up completely in infinite time.*

Futhermore, $\mu^* < \infty$ *if and only if* λg *satisfies* (33).

THEOREM 10. – *Let* $\lambda = \lambda^*$. *Suppose* (3), *and that there exists a classical solution* w_{λ^*} *of* (13_{λ^*}). *Let* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ *and let* u *be the corresponding global solution of* (12_{λ^*}). *Then either* u *blows up completely in infinite time, or* $u(t)$ *converges to* w_{λ^*} *in* $L^1(\Omega, \delta(x)dx)$ *as* $t \to \infty$.

COROLLARY 11. – *Suppose* (3) *and let* $\lambda > \lambda^*$. *Then for all* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, *the global solution* u *of* (12_λ) *blows up completely in infinite time.*

The proof of Theorem 5 can be adapted with obvious modification to show Theorem 10. Note that in the case (3), Lemma 12 is useless. On the other hand Corollary 11 is a direct consequence of Corollary 3.

Finally, to prove Theorem 9, we need only an equivalent of Lemma 11 which holds for (3). This is the object of the next lemma.

LEMMA 14. – *Suppose* (3). *Let* $\lambda < \lambda^*$, *let* $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, *and let* u *be the global solution of* (12_λ). *Assume that* u *does not blow up completely in infinite time. Then for every* $v_0 \in L^\infty(\Omega)$, $0 \leq v_0 \leq u_0$, $v_0 \not\equiv u_0$, *the solution* v *of* (12_λ) *with* $v(0) = v_0$ *is bounded.*

Proof. – In view of the proof of Lemma 11 it suffices to show that for every v_0 chosen as in the statement of the lemma, there exist $K > 0$, $\eta < 1$ and $\varepsilon_0 > 0$, such that for $0 < \varepsilon < \varepsilon_0$ the solution w of

$$\begin{cases} w_t - \Delta w = \lambda g(w) - K\varepsilon \text{ in } (0, \infty) \times \Omega, \\ w = 0 \text{ on } \partial\Omega, \\ w(0) = \dfrac{v_0}{\eta} \text{ in } \Omega. \end{cases} \tag{68}$$

is bounded.

We will make use of the following continuous embeddings

$$W^{1,p}((0,1),L^p(\Omega))\cap L^p((0,1),W^{2,p}(\Omega))\to L^r((0,1)\times\Omega), \tag{69}$$

for all $p\leq\dfrac{N+2}{2}$ and all $r<\dfrac{p}{1-\frac{2p}{N+2}}$,

$$W^{1,p}((0,1),L^p(\Omega))\cap L^p((0,1),W^{2,p}(\Omega))\to C([0,1]\times\overline{\Omega}), \tag{70}$$

for all $p>\frac{N+2}{2}$ and

$$W^{1,p}((0,1),L^p(\Omega))\cap L^p((0,1),W^{2,p}(\Omega))\to C([0,1],C^1(\overline{\Omega})), \tag{71}$$

for all $p>N+2$.

For later convenience we also recall that if

$$\begin{cases}\eta_t-\triangle\eta=f \text{ in } (0,1)\times\Omega,\\ \eta=0 \text{ on } \partial\Omega,\\ \eta(0)=0 \text{ in } \Omega.\end{cases}$$

and $f\in L^p((0,1)\times\Omega)$ for some $1<p<\infty$, then

$$\|\eta_t\|_{L^p((0,1)\times\Omega)}+\|\triangle\eta\|_{L^p((0,1)\times\Omega)}\leq C_p\|f\|_{L^p((0,1)\times\Omega)}. \tag{72}$$

Returning now to our problem, let us define the following functions

$$\Phi_{j,\varepsilon}(u)=h_{j+1}^{-1}(h_j(u)),\quad h_j(u)=\int_0^u\frac{ds}{\lambda g(s)-j\varepsilon},\quad \text{for}\quad \mathrm{j}\geq 0.$$

Set $u_1=\Phi_{0,\varepsilon}(u)$ and observe that by concavity of h_1,

$$h_1(u)-h_1(u_1)\leq(u-u_1)h_1'(u_1)=\frac{u-u_1}{\lambda g(u_1)-\varepsilon}. \tag{73}$$

From $h_0(u)=h_1(u_1)$ and (73), it follows that

$$g(u_1)=\frac{\varepsilon}{\lambda}+\frac{u-u_1}{\lambda(h_1(u)-h_0(u))}\leq\frac{C}{\varepsilon}(u+1).$$

Since u does not blow up completely in infinite time, by Lemma 4 there exists $C>0$ such that

$$\|u(t)\varphi_1\|_{L^1}\leq C,\ \text{for all } t>0. \tag{74}$$

Multiplying (1) by φ_1 and integrating on $(T, T+1)$ we find

$$\int_T^{T+1}\int_\Omega g(u)\varphi_1 \le (1+\lambda_1)C, \quad \int_T^{T+1}\int_\Omega u \le (2+\lambda_1)C. \tag{75}$$

We now take v_1 the global solution of

$$\begin{cases} \dfrac{\partial v_1}{\partial t} - \triangle v_1 = \lambda g(v_1) - \varepsilon \text{ in } (0,\infty)\times\Omega, \\ v_1 = 0 \text{ on } \partial\Omega, \\ v_1(0) = \Phi_{0,\varepsilon}(u_0) \text{ in } \Omega. \end{cases} \tag{76}$$

By Lemma 1, u_1 is a super-solution for problem (76) and so $v_1 \le u_1$. It follows from (74) and (75) that

$$\int_\Omega v_1(T)\delta \le C, \quad \int_T^{T+1}\int_\Omega v_1 \le C, \quad \int_T^{T+1}\int_\Omega g(v_1) \le C, \tag{77}$$

where $C > 0$ does not depend on T. For $p > N+2$, define η_1^T by

$$\begin{cases} -\dfrac{\partial \eta_1^T}{\partial t} - \triangle \eta_1^T = v_1^{1/p}(T+.) \text{ in } (0,1)\times\Omega, \\ \eta_1^T = 0 \text{ on } \partial\Omega, \\ \eta_1^T(1) = 0 \text{ in } \Omega. \end{cases}$$

By (77), (72) and (71), we obtain $\eta_1^T \in C([0,1], C^1(\overline{\Omega}))$ and

$$\|\eta_1^T\|_{L^\infty((0,1),W^{1,\infty}(\Omega))} \le C, \tag{78}$$

where C does not depend on T. Multiply (76) by η_1^T and integrate on $(T, T+1)$, it follows that

$$\int_T^{T+1}\int_\Omega g(v_1)\eta_1^T = \int_T^{T+1}\int_\Omega v_1^{1+1/p} - \int v_1(T)\eta_1^T(0).$$

By (77) and (78), we conclude

$$\int_T^{T+1}\int_\Omega v_1^{1+1/p} \le C,$$

where C does not depend on T.

Iterating this argument on v_1, we obtain

$$\int_T^{T+1}\int_\Omega v_1^\gamma \leq C,$$

for all $\gamma < \frac{N+2}{N+1}$.

Iterating this argument j times (and using (69)) one proves that the solution v_j of

$$\begin{cases} \dfrac{\partial v_j}{\partial t} - \Delta v_j = \lambda g(v_j) - j\varepsilon \text{ in } (0,\infty)\times\Omega, \\ v_j = 0 \text{ on } \partial\Omega, \\ v_j(0) = \Phi_{j,\varepsilon}(\ldots(\Phi_{0,\varepsilon}(u_0))) \text{ in } \Omega. \end{cases}$$

satisfies

$$\int_T^{T+1}\int_\Omega v_j^\gamma \leq C, \tag{79}$$

for every $\gamma < \frac{N+2}{N-(j-1)}$. Taking $j = N$ in (79) and considering v_{N+1} we obtain

$$\int_T^{T+1}\int_\Omega g(v_{N+1})^\gamma \leq C,$$

for all $\gamma < N+2$, so that we can apply (72) with $\frac{N+2}{2} < p < N+2$, and then (70) directly to v_{N+1}.

We have proved $\|v_{N+1}\|_{L^\infty((T,T+1)\times\Omega)} \leq C$, where C does not depend on T, *i.e.* v_{N+1} is uniformly bounded. Finally, observe that for every $c_0 > 0$, there exists $\varepsilon_0 > 0$ and $\eta < 1$ such that $\eta\ \Phi_{N,\varepsilon}(\ldots\Phi_{0,\varepsilon}(u_0)) \geq v_0$. The function v_{N+1} is a super-solution for problem (68) with $K = N+1$ and so w is bounded. □

7. CONVERGENCE RATE

We give a last result concerning the convergence of some solutions of (1) to the unique solution of the stationary problem.

PROPOSITION 12. – *Let* $\lambda = \lambda^*$, $w_{\lambda^*} \in L^\infty(\Omega)$, *and* $u_0 \in L^\infty(\Omega)$, $0 \leq u_0 \leq w_{\lambda^*}$, $u_0 \not\equiv w_{\lambda^*}$. *Let* u *be the global classical solution of*

$$\begin{cases} u_t - \Delta u = g(u) \text{ in } (0,\infty)\times\Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u(0) = u_0 \text{ in } \Omega, \end{cases}$$

There exist C, $C' > 0$ *such that*

$$\frac{C}{t+1} \le \|w_{\lambda^*} - u(t)\|_{L^\infty} \le \frac{C'}{t+1},\ \textit{for all } t > 0. \tag{80}$$

Proof. – Fix $\tau > 0$, for every u_0 as in the statement of the proposition, we have $u(\tau) \le w_{\lambda^*} - c_0\delta$ for some $c_0 > 0$. Let v be the solution of (1) corresponding to $v_0 \equiv 0$, then $v(t)$ converges to w_{λ^*} in $C^1(\overline{\Omega})$ as $t \uparrow \infty$. It follows that there exists t such that $u(\tau) \le v(t)$. Therefore, it suffices to show (80) for v.

First we prove an estimate for $\|(w_{\lambda^*} - v(t))\delta\|_{L^1}$. As in Lemma 13, there exist $\eta > 0$ and $0 < K_1 < K_2 \le \|w_{\lambda^*}\|_{L^\infty}$ such that

$$g''(u) \ge \eta,\ \text{for all } u \in [K_1, K_2]. \tag{81}$$

We denote by Ψ_1 the first eigenfunction of $(-\triangle - \lambda^* g'(w_{\lambda^*}))$.

We set $w(t) = w_{\lambda^*} - v(t)$. w is nonnegative, nonincreasing and converges to 0 in $L^1(\Omega, \delta(x)dx)$ as $t \uparrow \infty$. In what follows we will take t large enough to have

$$c'(1)\|\Psi_1\|_{L^\infty} e^{g'(\|w_{\lambda^*}\|_{L^\infty})}\|w(t-1)\Psi_1\|_{L^1} \le \frac{K_2 - K_1}{2}, \tag{82}$$

where $c'(1)$ is defined in Lemma 2 (in which we substituted Ψ_1 for δ).

Define $f(t) = \int_\Omega w(t)\Psi_1 dx$. Then as in the proof of Lemma 13,

$$\begin{aligned} f'(t) &= \int_\Omega [g(v(t)) - g(w_{\lambda^*}) + g'(w_{\lambda^*})w(t)]\Psi_1 dx \\ &= -\int_\Omega \Big(\int_{v(t)}^{w_{\lambda^*}} \Big(\int_{v(t)}^{s} g''(\sigma)d\sigma\Big) ds\Big)\Psi_1 dx \le 0. \end{aligned}$$

There exists a compact subset $\widetilde{\Omega}$ of Ω, such that

$$\frac{K_1 + K_2}{2} \le w_{\lambda^*}(x) \le K_2,\ \text{for all } x \in \widetilde{\Omega}.$$

Since $\widetilde{\Omega}$ is compact in Ω, there exists a constant $C_2 > 0$ such that

$$\Psi_1(x) \ge C_2,\ \text{for all } x \in \widetilde{\Omega}.$$

On the other hand, by Lemma 2 (in which we substituted Ψ_1 for δ), we have

$$\begin{aligned} e^{g'(0)}c(1)\|w(t-1)\Psi_1\|_{L^1}\Psi_1 &\le w_{\lambda^*} - v(t) \\ &\le e^{g'(\|w_{\lambda^*}\|_{L^\infty})}c'(1)\|w(t-1)\Psi_1\|_{L^1}\Psi_1. \end{aligned} \tag{83}$$

By (82) and (83), we obtain

$$v(t) \geq w_{\lambda^*} - e^{g'(\|w_{\lambda^*}\|_{L^\infty})} c'(1) \|w(t-1)\Psi_1\|_{L^1} \Psi_1 \geq K_1 \text{ on } \widetilde{\Omega}.$$

We conclude by (81) that

$$\begin{aligned} f'(t) &\leq -C_2 \eta \int_{\widetilde{\Omega}} \int_{v(t)}^{w_{\lambda^*}} \Big(\int_{v(t)}^{s} d\sigma \Big) ds dx \\ &\leq -\frac{C_2 \eta}{2} \int_{\widetilde{\Omega}} (w_{\lambda^*} - v(t))^2 dx \\ &\leq -C_3 \, \|w(t-1)\Psi_1\|_{L^1}^2 \\ &\leq -C_3 \, \|w(t)\Psi_1\|_{L^1}^2 = -C_3 \, f^2(t). \end{aligned}$$

Therefore, we get

$$f(t) \leq \frac{C}{t+1}, \tag{84}$$

for a certain constant C, for every $t > 0$.

On the other hand we have

$$g''(\sigma) \leq G, \text{ for all } \sigma \in w_{\lambda^*}(\Omega),$$

and by (83)

$$\begin{aligned} f'(t) &\geq -\frac{G}{2} \int_{\Omega} w^2(t) \Psi_1 dx \\ &\geq -C_4 \|w(t-1)\Psi_1\|_{L^1}^2. \end{aligned}$$

Using (84) we find

$$f'(t) \geq -C_4 f^2(t-1) \geq -\frac{C_5}{t^2},$$

which provides the result for $\|w(t)\Psi_1\|_{L^1}$. Using (83) again we find

$$\frac{\delta C}{1+t} \leq w_{\lambda^*} - v(t) \leq \frac{\delta C'}{1+t},$$

which proves (80). □

REMARK 5. – For $\lambda < \lambda^*$, and $u_0 \leq w_\lambda$, we can prove by the same technique that $\|w_\lambda - v(t)\|_{L^\infty}$ converges to 0 like $e^{-\lambda_1(\lambda)t}$ as $t \to \infty$, where $\lambda_1(\lambda)$ is the first eigenvalue of $(-\Delta - \lambda g'(w_\lambda))$.

ACKNOWLEDGMENTS

The author would like to thank Prof. H. Brezis and Prof. T. Cazenave for useful suggestions concerning this work.

REFERENCES

[1] P. Baras and L. Cohen, Complete blow after $T_{\max}$ for the solution of a semilinear heat equation, *J. Funct. Anal.*, Vol. **71**, 1987, pp. 142-174.
[2] P. Baras and M. Pierre, Critère d'existence de solutions positives pour des équations semi-linéaires non monotones, *Ann. Inst. Henri Poincaré*, Vol. **2**, 1985, pp. 185-212.
[3] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, Blow up for $u_t - \triangle u = g(u)$ revisited, *Adv. Diff. Eq.*, Vol. **1**, 1996, pp. 73-90.
[4] H. Fujita, On the nonlinear equations $\triangle u + e^u = 0$ and $\frac{\partial u}{\partial t} = \triangle u + e^u$, *Bull. Amer. Math. Soc.*, Vol. **75**, 1969, pp. 132-135.
[5] V. A. Galaktionov and J. L. Vazquez, Continuation of blow up solutions of nonlinear heat equations in several space dimensions, to appear.
[6] A. A. Lacey and D. E. Tzanetis, Global, unbounded solutions to a parabolic equation, *J. Diff. Eq.*, Vol. **101**, 1993, pp. 80-102.
[7] Y. Martel, Uniqueness of weak extremal solutions of nonlinear elliptic problems, to appear in *Houston Journal of Math.*, 1997.
[8] P. Mironescu and V. D. Radulescu, The study of a bifurcation problem associated to an asymptotically linear function, *Nonlinear Analysis TMA*, Vol. **26**, 1996, pp. 857-875.
[9] W.-M. Ni, P. E. Sacks and J. Tavantzis, On the asymptotic behavior of solutions of certain quasilinear parabolic equations, *J. Diff. Eq.*, Vol. **54**, 1984, pp. 97-120.

(Manuscript received July 20, 1996.)