ANNALES DE L'I. H. P., SECTION C

IGOR D. CHUESHOV
PIERRE-A. VUILLERMOT

Long-time behavior of solutions to a class of quasilinear parabolic equations with random coefficients

Annales de l'I. H. P., section C, tome 15, nº 2 (1998), p. 191-232
<http://www.numdam.org/item?id=AIHPC_1998__15_2_191_0>

Ann. Inst. Henri Poincaré,

Vol. 15, n° 2, 1998, p. 191-232 *Analyse non linéaire*

Long-time behavior of solutions to a class of quasilinear parabolic equations with random coefficients

by

Igor D. CHUESHOV

Department of Mechanics and Mathematics, Kharkov State University,
4 Svobody Square, Kharkov, 310077 (Ukraine).
E-mail : chueshov@ilt.kharkov.ua

and

Pierre-A. VUILLERMOT

Département de Mathématiques et UMR-CNRS 9973, Université Henri-Poincaré,
Nancy-I, B.P. 239, 54506 Vandoeuvre-lès-Nancy Cedex (France).
E-mail : vuillerm@iecn.u-nancy.fr

ABSTRACT. – In this article we prove new results concerning the long-time behavior of random fields that are almost surely solutions to a class of stochastic parabolic Neumann problems defined on open bounded connected subsets of $\mathbb{R}^N$. Under appropriate ellipticity and regularity hypotheses, we first prove that every such random field stabilizes almost surely in a suitable topology around a spatially homogeneous random process whose statistical properties are entirely determined by those of the given coefficients in the equations. In addition, when the coefficients of the lower-order terms in the equations are stationary random processes, the nature of the equations that we investigate leads us to consider two complementary situations according to whether the average of those processes is zero or not. If their average is different from zero and if the processes are ergodic, we prove that every random field stabilizes almost surely and exponentially rapidly in the uniform topology around a spatially and temporally homogeneous asymptotic state, which depends only on the sign of the average. In this case we can also determine the corresponding Liapunov exponents exactly. In contrast, if the average of the processes is equal to zero we need more structure to identify the asymptotic states properly. The cases where the coefficients of the lower-order terms in the equations are either stationary random processes whose statistics are governed by the central limit theorem, or Gaussian processes that share some of the features of the Ornstein-Uhlenbeck process, are of special interest and we investigate them in detail. In all cases we can also provide estimates for the average time that the random fields spend in small neighborhoods of the asymptotic states. Our methods of proof rest chiefly upon the use of parabolic comparison principles. © Elsevier, Paris

RÉSUMÉ. – Dans cet article nous démontrons de nouveaux résultats concernant le comportement asymptotique en temps de certains champs aléatoires possédant la particularité d'être presque sûrement solutions d'une classe de problèmes de Neumann paraboliques stochastiques définis sur des ouverts bornés connexes de $\mathbb{R}^N$. A l'aide d'hypothèses d'ellipticité et de régularité convenables nous prouvons tout d'abord que ces champs aléatoires se stabilisent presque sûrement, relativement à une topologie appropriée, vers un processus stochastique dont les propriétés statistiques sont entièrement déterminées par celles des coefficients des équations. Nous analysons ensuite le cas où les coefficients des termes d'ordre inférieur des équations sont des processus stationnaires. Ceci nous conduit à considérer deux situations complémentaires suivant que la moyenne de ces processus stationnaires est différente de zéro ou non. Dans le premier cas, si nous supposons en plus que les processus sont ergodiques, nous démontrons que tout champ aléatoire se stabilise presque sûrement et exponentiellement rapidement, relativement à la topologie uniforme, vers un état asymptotique ne dépendant que du signe de la moyenne de ces processus ergodiques ; dans ce cas nous parvenons également à déterminer exactement les exposants de Liapounov correspondants. Dans le second cas, nous avons besoin d'hypothèses légèrement différentes pour pouvoir identifier les états asymptotiques. Les cas où les coefficients des termes d'ordre inférieur des équations sont soit des processus stationnaires satisfaisant aux hypothèses du théorème limite central, soit des processus gaussiens possédant certaines particularités du processus d'Ornstein-Uhlenbeck, présentent un intérêt particulier et nous les analysons en détail. Dans tous les cas nous sommes également en mesure d'estimer les temps moyens de séjour des champs aléatoires dans des voisinages arbitrairement petits des états asymptotiques. Nos méthodes de démonstration reposent essentiellement sur l'existence de principes de comparaison paraboliques. © Elsevier, Paris

---

1. INTRODUCTION AND OUTLINE

Let $(X, \mathcal{F}, \mathbb{P})$ be a complete probability space with σ-algebra $\mathcal{F}$ and probability measure $\mathbb{P}$. In this article we investigate the long-time behavior of real-valued random fields on $(X, \mathcal{F}, \mathbb{P})$ that are $\mathbb{P}$-almost surely classical solutions to quasilinear parabolic Neumann problems of the form

$$\left\{\begin{array}{ll} \partial_t u(x,t,\omega) = \operatorname{div}(k_u(x,t,\omega)\nabla u(x,t,\omega)) & \\ \qquad\qquad + s(t,\omega) g(u(x,t,\omega), \nabla u(x,t,\omega)) & \\ (x,t,\omega) \in \Omega \times \mathbb{R}^+ \times X & \\ u(x,0,\omega) = \varphi(x,\omega) \in (u_0, u_1), & (x,\omega) \in \overline{\Omega} \times X \\ \dfrac{\partial u(x,t,\omega)}{\partial n(u)} = 0, & (x,t,\omega) \in \partial\Omega \times \mathbb{R}^+ \times X \end{array}\right\} \quad (1.1)$$

In relations (1.1) Ω denotes an open bounded connected subset of $\mathbb{R}^N$ with a sufficiently regular boundary $\partial\Omega$, $u_{0,1} \in \mathbb{R}$ with $u_0 < u_1$, φ is a smooth random field such that $x \to \varphi(x,\omega) \in \mathcal{C}^2(\overline{\Omega})$ and $\varphi(x,\omega) \in (u_0, u_1)$ hold $\mathbb{P}$-almost surely, and the third equation in (1.1) stands for the conormal derivative associated with the matrix-valued random field $k_u(x,t,\omega) = k(x,t,u(x,t,\omega),\omega)$. We also assume that φ satisfies the conormal boundary condition. Moreover, the random field k, the random process s and the nonlinearity g satisfy the following hypotheses, respectively :

(K) $(k(x,t,u,.))_{(x,t,u)\in\overline{\Omega}\times\mathbb{R}^+\times[u_0,u_1]}$ is a matrix-valued random field on $(X, \mathcal{F}, \mathbb{P})$ with real-valued entries such that

$$k_{i,j}(.,\omega) = k_{j,i}(.,\omega) \in \mathcal{C}^2(\overline{\Omega} \times \mathbb{R}^+ \times [u_0, u_1])$$

holds $\mathbb{P}$-almost surely for every $i, j \in \{1, \cdots, N\}$. In addition, all partial derivatives of the functions $k_{i,j}$ with respect to (x,t,u) are $\mathbb{P}$-almost surely bounded as functions of (x,t,u,ω). Finally, there exist positive constants $\underline{k}, \overline{k} \in (0, \infty)$ such that the uniform ellipticity condition

$$\underline{k}|q|^2 \leq (k(x,t,u,\omega)q, q)_{\mathbb{R}^N} \leq \overline{k}|q|^2 \quad (1.2)$$

holds $\mathbb{P}$-almost surely for every $(x,t,u,q) \in \overline{\Omega} \times \mathbb{R}^+ \times [u_0, u_1] \times \mathbb{R}^N$. In relation (1.2), $(.,.)_{\mathbb{R}^N}$ stands for the usual Euclidean scalar product in $\mathbb{R}^N$.

(S) $(s(t,.))_{t\in\mathbb{R}}$ is a real-valued random process on $(X, \mathcal{F}, \mathbb{P})$ such that the Hölder continuity of the trajectories $t \to s(t,\omega) \in \mathcal{C}^\mu(\mathbb{R})$ holds $\mathbb{P}$-almost surely for some $\mu \in (0, 1]$.

(G) We have $g \in \mathcal{C}^2([u_0, u_1] \times \mathbb{R}^N)$ with $g(u_0, 0) = g(u_1, 0) = 0$ and $g(u, 0) > 0$ for every $u \in (u_0, u_1)$. Moreover, there exists a constant $c \in (0, \infty)$ such that the inequality

$$(1 + |u|)|g(u, q)| + \left|\frac{\partial g}{\partial u}(u, q)\right| + (1 + |q|)|\nabla_q g(u, q)| \leq c(1 + |q|^2) \quad (1.3)$$

holds for every $(u, q) \in [u_0, u_1] \times \mathbb{R}^N$.

There have been several recent works devoted to the investigation of the long-time behavior of solutions to semilinear and non-random versions of problems of the form (1.1) when s is periodic, almost-periodic or possesses more general recurrence properties ([3], [4], [6], [11]-[13], [16], [17], [24]-[28]). One of the reasons for this is that such problems have played an increasingly important role in the mathematical treatment of many phenomena in various areas of science, ranging from theoretical physics to population dynamics, including the theory of heat diffusion, of nerve pulse propagation and of population genetics ([2]). In this paper, our primary purpose is to investigate the stabilization properties of random fields on $(X, \mathcal{F}, \mathbb{P})$ that are $\mathbb{P}$-almost surely classical solutions to Problem (1.1) when hypotheses (K), (S) and (G) hold. Hypotheses (K) and (S) generalize the models considered thus far in at least two important ways. On the one hand, the structure of the second-order differential operator that appears in the principal part to (1.1) allows one to encode space -and time-dependent random diffusions into the theory. On the other hand, the fact that $(s(t, .))_{t \in \mathbb{R}}$ is a random process makes it possible to consider processes with strong mixing and Markov properties such as Ornstein-Uhlenbeck processes, rather than just nearly deterministic processes such as the almost-periodic ones. With some additional conditions on $(s(t, .))_{t \in \mathbb{R}}$ and g when g depends explicitly on ∇u, we then prove that the solution to (1.1) stabilizes $\mathbb{P}$-almost surely in a suitable topology around a spatially homogeneous random process whose statistical properties are entirely determined by those of the given data. In addition, when the random process $(s(t, .))_{t \in \mathbb{R}}$ is stationary the nature of Problem (1.1) leads us to consider two complementary situations according to whether the average of $(s(t, .))_{t \in \mathbb{R}}$ is zero or not. If the average is different from zero and if the process is ergodic, we prove that the solution to (1.1) stabilizes $\mathbb{P}$-almost surely and exponentially rapidly in the uniform topology around a spatially and temporally homogeneous asymptotic state, which depends only on the sign of the average. In this case we can also determine the corresponding Liapunov exponents exactly. In contrast, if the average of $(s(t, .))_{t \in \mathbb{R}}$ is equal to zero we need a slightly different structure to identify the asymptotic states properly. The cases where $(s(t,.))_{t\in\mathbb{R}}$ is either a random process whose statistics are governed by the central limit theorem or a Gaussian process are of special interest and we analyze them in detail. In all cases we can also provide estimates for the average time that the solution of (1.1) spends in small neighborhoods of the asymptotic states. Our main results are stated precisely and further discussed in Section 2. The corresponding proofs are carried out in Section 3. Our methods of proof there rest upon the use of parabolic maximum principles and upon the existence of exponential dichotomies for a family of random evolution operators associated with the principal part of (1.1). Finally, we devote Section 4 to some concluding remarks and we refer the reader to [7] for a short announcement of the results.

Our work was primarily motivated by the desire to understand the long-time behavior of generalized random fields that are solutions in some sense to semilinear stochastic problems of the form

$$\left.\begin{cases} du(x,t,\omega) = \operatorname{div}(k(x,t)\nabla u(x,t,\omega))dt \\ \qquad\qquad + g(u(x,t,\omega)) \circ dB(t,\omega) \\ (x,t,\omega) \in \Omega \times \mathbb{R}^+ \times X \\ u(x,0,\omega) = \varphi(x,\omega) \in (u_0,u_1), \quad (x,\omega) \in \overline{\Omega} \times X \\ \dfrac{\partial u(x,t,\omega)}{\partial n(u)} = 0, \qquad (x,t,\omega) \in \partial\Omega \times \mathbb{R}^+ \times X \end{cases}\right\} \tag{1.4}$$

In the first equation (1.4), $(B(t,.))_{t\in\mathbb{R}^+}$ stands for the standard one-dimensional Brownian motion starting at the origin and $\circ dB(t,.)$ denotes Stratonovitch's differential. Problems of the form (1.4) define a class of semilinear parabolic problems subjected to homogeneous white noise. Though the theorems of this article do not apply to the solutions of (1.4) directly, it turns out that the analysis of the solutions to (1.4) can be reduced to that developed in the present paper through the combination of a suitable regularization of the Brownian motion with an appropriate limiting procedure. We defer the presentation of the corresponding results to separate publications ([8], [9]).

2. STATEMENTS AND DISCUSSION OF THE MAIN THEOREMS

When hypotheses (K), (S) and (G) hold, the standard existence -and regularity theory for parabolic equations implies that there exists a unique random field u_φ which satisfies Problem (1.1) $\mathbb{P}$-almost surely in a classical sense ([20]). It also follows from the classical parabolic maximum principle that $u_\varphi(x,t,\omega) \in (u_0,u_1)$ $\mathbb{P}$-almost surely for every $(x,t) \in \overline{\Omega} \times \mathbb{R}^+$ ([15]). Our primary objective here is to investigate the behavior of u_φ when $t \to \infty$. In this respect, hypotheses (K), (S) and (G) are quite general and the trade-off for this degree of generality is that our first convergence result holds with respect to relatively weak topologies. We illustrate this point first, by showing that u_φ homogeneizes $\mathbb{P}$-almost surely over the region Ω in the $L^p(\Omega)$-topology for any $p \in [1,\infty)$. For this we need the following additional hypothesis for the nonlinearity g.

(QG) There exists a bounded function $c : [u_0,u_1] \to \mathbb{R}^+$ such that the inequality

$$|g(u,q) - g(u,0)| \leq c(u)|q|^2 \tag{2.1}$$

holds for every $u \in [u_0,u_1]$ and every $q \in \mathbb{R}^N$.

We also write $\mathbb{E}$ for the mathematical expectation functional on $(X,\mathcal{F},\mathbb{P})$ and $||.||_p$ for the usual $L^p(\Omega)$-norm. We then have the following.

THEOREM 2.1. – *Assume that hypotheses (K), (S), (G) and (QG) hold. Assume also that there exists $c \in (0,\infty)$ such that the inequality $|s(t,\omega)| \leq c$ holds $\mathbb{P}$-almost surely for every $t \in \mathbb{R}^+$. Then there exists a unique x-independent random process $(\hat{u}(t,.))_{t\in\mathbb{R}}$ on $(X,\mathcal{F},\mathbb{P})$ such that the relation*

$$\lim_{t\to\infty} ||u_\varphi(.,t,\omega) - \hat{u}(t,\omega)||_p = 0 \tag{2.2}$$

holds $\mathbb{P}$-almost surely for every $p \in [1,\infty)$. Moreover, we have

$$\lim_{t\to\infty} \mathbb{E}(||u_\varphi(.,t,.) - \hat{u}(t,.)||_p^r) = 0 \tag{2.3}$$

for every $p, r \in [1,\infty)$.

Remarks.

1. We note that both conditions (1.3) and (2.1) hold trivially when g does not depend on q. In this case, the proof of Theorem 2.1 in Section 3 reveals that the boundedness of $(s(t,.))_{t\in\mathbb{R}}$ is not necessary for relations (2.2) and (2.3) to hold. Thus, in case g does not depend on q, the three conditions (K), (G), (S) alone are sufficient to imply both conclusions of Theorem 2.1.
2. The random process $(\hat{u}(t,.))_{t\in\mathbb{R}}$ of Theorem 2.1 turns out to be an x-independent and $\mathbb{P}$-almost sure solution to Problem (1.1) (compare with the proof of Theorem 2.1 in Section 3). From this, it follows that $(\hat{u}(t..))_{t\in\mathbb{R}}$ is necessarily of the form

$$\hat{u}(t.\omega) = G^{-1}\left\{ \int_0^t d\xi s(\xi.\omega) + G(\hat{\varphi}(\omega)) \right\} \tag{2.4}$$

where $G : (u_0, u_1) \mapsto \mathbb{R}$ stands for any primitive of the function $u \to 1/g(u,0)$, G^{-1} denotes the monotone inverse of G and $\hat{\varphi}$ is the corresponding initial condition. Theorem 2.1 can thus be viewed as an existence statement for the random variable $\hat{\varphi}$ that generates $(\hat{u}(t,.))_{t\in\mathbb{R}}$ through relation (2.4). The fact that the random process $(\hat{u}(t,.))_{t\in\mathbb{R}}$ admits the explicit representation (2.4) will be important below, particularly when $(s(t,.))_{t\in\mathbb{R}}$ has statistical properties governed by the central limit theorem or when it is a Gaussian process.

Now let $H^{1,p}(\Omega)$ be the usual Sobolev space of functions on Ω whose norm we denote by $||.||_{1,p}$. It is natural to ask whether we can replace $||.||_p$ by $||.||_{1,p}$ in Theorem 2.1. Equivalently, we want to know whether we can have $||\nabla u_\varphi(.,t,\omega)||_p \to 0$ $\mathbb{P}$-almost surely as $t \to \infty$. A necessary condition for this is that the relation

$$\lim_{t\to\infty} \int_t^{t+\gamma} d\xi ||\nabla u_\varphi(.,\xi,\omega)||_p^p = 0 \tag{2.5}$$

holds $\mathbb{P}$-almost surely for every $\gamma \in (0,\infty)$, which we prove in Lemma 3.5 of Section 3. Condition (2.5) is, however, not sufficient in general to ensure the homogeneization of u_φ with respect to the strong topology of $H^{1,p}(\Omega)$, unless more is known about the matrix-valued random field k. There is a natural requirement that allows one to dispose of this question readily. Write momentarily $A_{u_\varphi}(t,\omega) = -\text{div}(k_{u_\varphi}(.,t,\omega)\nabla)$ for the family of random linear differential operators that are $\mathbb{P}$-almost surely self-adjoint and positive as operators in $L^2(\Omega)$, when realized on the time - dependent domain $D(A_{u_\varphi}(t,\omega)) = H^{2,2}_{\mathcal{N}(u_\varphi)}(\Omega)$; here we write $H^{2,2}_{\mathcal{N}(u_\varphi)}(\Omega)$ for the vector subspace of $L^2(\Omega)$ that consists of all functions of $H^{2,2}(\Omega)$ which satisfy the conormal boundary condition in (1.1). For every $\tau \in (0,\infty)$ we consider the linear evolution problem

$$\left\{\begin{array}{ll} \partial_t v(.,t,\omega) = \text{div}(k_{u_\varphi}(.,t,\omega)\nabla v(.,t,\omega)), & (t,\omega) \in (\tau,\infty)\times X \\ v(.,\tau,\omega) = \psi(.,\omega), & \omega \in X \end{array}\right\} \tag{2.6}$$

in $L^2(\Omega)$. We then introduce the following hypothesis of unique and global solvability of Problem (2.6), in which $||.||$ stands for the uniform norm of the linear bounded operators on $L^2(\Omega)$.

(LEO) There exists a family of random linear evolution operators $(U(t,\tau,\omega))_{t\geq\tau}$ in $L^2(\Omega)$ associated with Problem (2.6) such that, for every $T \in (0,\infty)$, there exists a constant $c(T) \in (0,\infty)$ such that the estimate

$$||(-\text{div}\,(k_{u_\varphi}(.,t,\omega)\nabla))^{1/2} U(t,\tau,\omega)|| \leq c(T)|t-\tau|^{-1/2} \tag{2.7}$$

holds $\mathbb{P}$-almost surely for every $t \in (\tau, \tau + T]$.

By a family of random linear evolution operators in $L^2(\Omega)$ we mean a family $(U(t, \tau, \omega))_{t \geq \tau}$ of linear bounded operators in $L^2(\Omega)$ which satisfy all the conditions of definition (5.3) in Chapter 5 of [22]. Then there are well-known sufficient conditions that one can impose on the operators $A_{u_\varphi}(t, \omega)$ for estimate (2.7) to hold ([18], [19], [22]). Estimate (2.7) holds, for instance, whenever k_u does not depend on t and u, in which case the family $(U(t, \tau, \omega))_{t \geq \tau}$ reduces to a linear random semigroup. Writing $||.||_\infty$ for the uniform norm of continuous functions on $\overline{\Omega}$, we then obtain the following result.

THEOREM 2.2. – *Assume that all hypotheses of Theorem 2.1 hold. In addition, assume that hypothesis (LEO) holds. Then there exists a unique x-independent random process $(\hat{u}(t, .))_{t \in \mathbb{R}}$ of the form* (2.4) *on $(X, \mathcal{F}, \mathbb{P})$ such that the relation*

$$\lim_{t \to \infty} ||u_\varphi(., t, \omega) - \hat{u}(t, \omega)||_{1,p} = 0 \tag{2.8}$$

holds $\mathbb{P}$-almost surely for every $p \in [1, \infty)$. In particular, we have $\mathbb{P}$-almost surely

$$\lim_{t \to \infty} ||u_\varphi(., t, \omega) - \hat{u}(t, \omega)||_\infty = 0 \tag{2.9}$$

Moreover, we have

$$\lim_{t \to \infty} \mathbb{E}(||u_\varphi(., t, .) - \hat{u}(t, .)||_{1,p}^r) = 0 \tag{2.10}$$

and hence

$$\lim_{t \to \infty} \mathbb{E}(||u_\varphi(., t, .) - \hat{u}(t, .)||_\infty^r) = 0 \tag{2.11}$$

for every $p, r \in [1, \infty)$.

Remark. – It is worth mentioning that with such a degree of generality, the preceding theorems are, to the best of our knowledge, new even in the deterministic case.

We shall now investigate the stabilization properties of u_φ more closely, by keeping the random field k_u quite general while imposing conditions of statistical nature on the random process $(s(t, .))_{t \in \mathbb{R}}$. We describe a first important case in the following hypothesis.

(ES) The process $(s(t, .))_{t \in \mathbb{R}}$ is a stationary, ergodic random process on $(X, \mathcal{F}, \mathbb{P})$ such that

$$\omega \to s(0, \omega) \in L^1(X, \mathbb{P}). \tag{2.12}$$

Recall that such a process can be associated to any periodic or almost-periodic continuous function s in a very natural way ([14]). But the notion of ergodicity also encompasses random processes with exponentially mixing and Markov properties such as Ornstein-Uhlenbeck processes. For this reason, hypothesis (ES) is natural in that it bridges the gap between problems of the form (1.1) where s is periodic or almost-periodic and those where s has strong stochastic properties. Let $<s>$ denote the average of the process $(s(t,.))_{t\in\mathbb{R}}$. Because of hypothesis (ES) and the Birkhoff-Khintchin pointwise ergodic theorem we have

$$<s>= \mathbb{E}(s(t,.)) = \mathbb{E}(s(0,.)) = \lim_{t\longrightarrow\infty} t^{-1}\int_0^t d\xi s(\xi,\omega) \tag{2.13}$$

for every $t \in \mathbb{R}$, where the last equality holds $\mathbb{P}$-almost surely. We begin by investigating the case where $< s > \neq 0$, for which we have the following result.

THEOREM 2.3. – *Assume that hypotheses (K), (S), (G) and (ES) hold. Then the following statements are valid :*

(1) *If* $<s> < 0$ *and if* $g'(u_0,0) > 0$, *the relation*

$$\lim_{t\longrightarrow\infty} t^{-1}\ln(||u_\varphi(.,t,\omega) - u_0||_\infty) =<s> g'(u_0,0) \tag{2.14}$$

holds $\mathbb{P}$*-almost surely.*

(2) *If* $<s> > 0$ *and if* $g'(u_1,0) < 0$, *the relation*

$$\lim_{t\longrightarrow\infty} t^{-1}\ln(||u_\varphi(.,t,\omega) - u_1||_\infty) =<s> g'(u_1,0) \tag{2.15}$$

holds $\mathbb{P}$*-almost surely.*

Remarks.

1. The information provided by relations (2.14) and (2.15) is utmost precise in that it provides both upper and lower exponential decay estimates for $||u_\varphi(.,t,\omega) - u_{0,1}||_\infty$. For instance, for every $\varepsilon \in (0,|<s>|g'(u_0,0))$ there exists $t_\varepsilon(\omega) > 0$ such that the inequalities

$$\exp[(<s> g'(u_0,0) - \varepsilon)t] \leq ||u_\varphi(.,t,\omega) - u_0||_\infty \leq \exp[(<s> g'(u_0,0) + \varepsilon)t] \tag{2.16}$$

hold $\mathbb{P}$-almost surely for every $t \in (t_\varepsilon(\omega),\infty)$. In a completely similar way we have $\mathbb{P}$-almost surely the inequalities

$$\exp[(<s> g'(u_1,0) - \varepsilon)t] \leq ||u_\varphi(.,t,\omega) - u_1||_\infty \leq \exp[(<s> g'(u_1,0) + \varepsilon)t] \tag{2.17}$$

for every $\varepsilon \in (0, <s> |g'(u_1, 0)|)$ and for every $t \in (t_\varepsilon(\omega), \infty)$. Of course, the Liapunov exponents given by relations (2.14) and (2.15) are non-random as a consequence of hypothesis (ES).

2. In the form of relations (2.14) and (2.15), the results of Theorem 2.3 are new also in the deterministic case. In particular, they complete and improve the results of [27] obtained by geometric methods when s is almost-periodic.

3. Obviously, the conclusions of Theorem 2.3 must be consistent with those of Theorems 2.1 and 2.2 when the appropriate hypotheses hold. What this means is that the homogeneous random process $(\hat{u}(t,.))_{t\in\mathbb{R}}$ also converges to u_0 when $<s> < 0$ and $g'(u_0, 0) > 0$, or to u_1 when $<s> > 0$ and $g'(u_1, 0) < 0$. Of course, the statements can be directly verified from the explicit form (2.4) by using the Birkhoff-Khintchin ergodic theorem. The very existence of homogeneous random processes of the form (2.4) that satisfy the above properties is one of the key ingredients in the proof of Theorem 2.3 below.

As a very simple application of Theorem 2.3, we can determine the average time that u_φ spends in a small neighborhood of u_0 and u_1 when $t \to \infty$. Let $T^* \in (0, \infty)$ be given. If $<s> < 0$ and if $g'(u_0, 0) > 0$, define

$$F_{\varepsilon,u_0}(\xi) = \Big\{\omega \in X : \exp[(<s>\, g'(u_0,0) - \varepsilon)\xi] \le ||u_\varphi(.,\xi,\omega) - u_0||_\infty \le \exp[(<s>\, g'(u_0,0) + \varepsilon)\xi]\Big\} \tag{2.18}$$

for every $\xi \in [t, t+T^*]$ and for every $\varepsilon \in (0, |<s>|g'(u_0, 0))$. Similarily, if $<s> > 0$ and if $g'(u_1, 0) < 0$ we define

$$F_{\varepsilon,u_1}(\xi) = \Big\{\omega \in X : \exp[(<s>\, g'(u_1,0) - \varepsilon)\xi] \le ||u_\varphi(.,\xi,\omega) - u_1||_\infty \le \exp[(<s>\, g'(u_1,0) + \varepsilon)\xi]\Big\} \tag{2.19}$$

for every $\xi \in [t, t+T^*]$ and for every $\varepsilon \in (0, <s> |g'(u_1, 0)|)$. Evidently, the sets $F_{\varepsilon,u_0}(\xi)$ and $F_{\varepsilon,u_1}(\xi)$ are $\mathcal{F}$-measurable ; let $\chi_{\varepsilon,u_0}(\xi,.)$ and $\chi_{\varepsilon,u_1}(\xi,.)$ be the corresponding indicator functions. It is clear that the random variables $\omega \to \int_t^{t+T^*} d\xi \chi_{\varepsilon,u_{0,1}}(\xi,\omega)$ measure the fraction of the available time lapse T^* that the random field u_φ spends in the corresponding neighborhood determined by (2.18) or (2.19). We then have the following result.

COROLLARY 2.4. – *The hypotheses are exactly the same as in Theorem 2.3. Then the following statements hold :*

(1) *If* $<s> < 0$ *and if* $g'(u_0, 0) > 0$, *we have*

$$\lim_{t\to\infty} \mathbb{E}\left(\int_t^{t+T^*} d\xi \chi_{\varepsilon,u_0}(\xi,.)\right) = T^* \tag{2.20}$$

for every $T^* \in (0,\infty)$ *and for every* $\varepsilon \in (0, |<s>|g'(u_0,0))$.

(2) *If* $<s> > 0$ *and if* $g'(u_1, 0) < 0$, *we have*

$$\lim_{t\to\infty} \mathbb{E}\left(\int_t^{t+T^*} d\xi \chi_{\varepsilon,u_1}(\xi,.)\right) = T^* \tag{2.21}$$

for every $T^* \in (0,\infty)$ *and for every* $\varepsilon \in (0, <s>\ |g'(u_1,0)|)$.

Remark. – In either case the interpretation of Corollary 2.4 is clear : on the average the random field u_φ spends the entire available time lapse T^* in an arbitrarily small neighborhood of the appropriate asymptotic state when $t \to \infty$. We shall see below that the situation is quite different when $<s>= 0$.

As already noticed, when $<s>= 0$ we need a slightly different structure to identify the asymptotic states properly. The cases where the statistics of $(s(t,.))_{t\in\mathbb{R}}$ are governed by the central limit theorem or by Gaussian distributions are of special interest. We begin with the case of the central limit theorem. We say that the statistics of the random process $(s(t,.))_{t\in\mathbb{R}}$ obey the central limit theorem if the following hypothesis holds :

(CLS) We have $\omega \to s(0,\omega) \in L^2(X,\mathbb{P})$, $(s(t,.))_{t\in\mathbb{R}}$ is stationary and the limit

$$\lim_{t\to\infty} \mathbb{E}\left(\left(t^{-1/2}\int_0^t d\xi(s(\xi,.)- <s>)\right)^2\right) = \sigma^* > 0 \tag{2.22}$$

exists ; in addition we have

$$\lim_{t\to\infty} \mathbb{P}\left\{\omega \in X : t^{-1/2}\int_0^t d\xi(s(\xi,\omega)- <s>) < a^*\right\}$$
$$= (2\pi\sigma^*)^{-1/2}\int_{-\infty}^{a^*} dx \exp\left[\frac{-x^2}{2\sigma^*}\right] \tag{2.23}$$

for every $a^* \in \mathbb{R} \cup \{\pm\infty\}$.

Recall now that $G : (u_0, u_1) \to \mathbb{R}$ stands for any primitive of the function $u \to 1/g(u, 0)$ (compare with Remark 2 following the statement of Theorem 2.1). Our main result concerning the long-time behavior of u_φ is then the following

THEOREM 2.5. – *Assume that hypotheses (K), (S), (G) and (CLS) hold. Assume also that the initial datum φ is not random, that is the function $\omega \longrightarrow \varphi(x, \omega)$ is $\mathbb{P}$-almost surely constant for every $x \in \overline{\Omega}$. Then the following statements are valid :*

(1) *For any function $a : \mathbb{R}^+ \longrightarrow (u_0, u_1)$ such that the limit*

$$a^* = \lim_{t \to \infty} t^{-1/2}(G(a(t)) - \langle s \rangle\, t) \tag{2.24}$$

exists (with $a^* = \pm\infty$ allowed), we have

$$\lim_{t \to \infty} \mathbb{P}\Big\{\omega \in X : u_\varphi(., t, \omega) \leq a(t)\Big\} = (2\pi\sigma^*)^{-1/2} \int_{-\infty}^{a^*} dx \exp\left[\frac{-x^2}{2\sigma^*}\right] \tag{2.25}$$

(2) *For any function $b : \mathbb{R}^+ \longrightarrow (u_0, u_1)$ such that the limit*

$$b^* = \lim_{t \to \infty} t^{-1/2}(G(b(t)) - \langle s \rangle\, t) \tag{2.26}$$

exists (with $b^* = \pm\infty$ allowed), we have

$$\lim_{t \to \infty} \mathbb{P}\{\omega \in X : b(t) \leq u_\varphi(., t, \omega)\} = (2\pi\sigma^*)^{-1/2} \int_{b^*}^{\infty} dx \exp\left[\frac{-x^2}{2\sigma^*}\right] \tag{2.27}$$

We note that both statements of Theorem 2.5 concern the convergence of probabilities, in contrast to all preceding theorems whose convergence statements hold almost surely. While this is in the nature of things because of relation (2.23), the trade-off is that Theorem 2.5 allows for alot of flexibility in the discussion of the asymptotic behavior since the functions a and b are essentially arbitrary. A typical example is the following result, which turns out to be related to Theorem 2.5.

THEOREM 2.6. – *The hypotheses are exactly the same as in Theorem 2.5. Let $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be any continuous function such that $\lim_{t \to \infty} \Phi(t) = \infty$. Then the following statements hold :*

(1) *If $g'(u_0, 0) \neq 0$ and if the limit*

$$a^* = -\lim_{t \to \infty} t^{-1/2}\left(\frac{\Phi(t)}{g'(u_0, 0)} + \langle s \rangle\, t\right) \tag{2.28}$$

exists (with $a^* = \pm\infty$ *allowed), we have*

$$\lim_{t\to\infty} \mathbb{P}\{\omega \in X : ||u_\varphi(.,t,\omega) - u_0||_\infty \leq c\exp[-\Phi(t)]\}$$
$$= (2\pi\sigma^*)^{-1/2} \int_{-\infty}^{a^*} dx \exp\left[\frac{-x^2}{2\sigma^*}\right] \tag{2.29}$$

for every $c \in (0, \infty)$.

(2) *If* $g'(u_1, 0) \neq 0$ *and if the limit*

$$b^* = -\lim_{t\to\infty} t^{-1/2}\left(\frac{\Phi(t)}{g'(u_1,0)} + <s>\, t\right) \tag{2.30}$$

exists (with $b^* = \pm\infty$ *allowed), we have*

$$\lim_{t\to\infty} \mathbb{P}\{\omega \in X : ||u_\varphi(.,t,\omega) - u_1||_\infty \leq c\exp[-\Phi(t)]\}$$
$$= (2\pi\sigma^*)^{-1/2} \int_{b^*}^{\infty} dx \exp\left[\frac{-x^2}{2\sigma^*}\right] \tag{2.31}$$

for every $c \in (0, \infty)$.

Remark. – A glance at relations (2.29) and (2.31) shows that the asymptotic probabilities depend exclusively on the rate function Φ, on some features of the nonlinearity g and on the average of the random process $(s(t,.))_{t\in\mathbb{R}}$. We also stress the fact that explicit expressions such as (2.28) and (2.30) are possible because of the existence of x-independent random processes of the form (2.4), and because of the validity of certain parabolic comparison principles (compare with the methods of proof of Section 3).

In contrast to Theorem 2.3, Theorem 2.6 allows for a detailed analysis of the case $<s>= 0$. In fact, we shall see in Section 3 that suitable choices of Φ lead to the following.

COROLLARY 2.7. – *The hypotheses are exactly the same as in Theorem 2.5. Then the following conclusions hold :*

(1) *If* $<s>\, < 0$ *and if* $g'(u_0, 0) > 0$, *we have*

$$\lim_{t\to\infty} \mathbb{P}\{\omega \in X : c\exp[(<s>\, g'(u_0,0) - \varepsilon)t] \leq ||u_\varphi(.,t,\omega) - u_0||_\infty$$
$$\leq c\exp[(<s>\, g'(u_0,0) + \varepsilon)t]\} = 1 \tag{2.32}$$

for every $c \in (0, \infty)$ *and every* $\varepsilon \in (0, |<s>|g'(u_0, 0))$.

(2) *If* $<s> > 0$ *and if* $g'(u_1, 0) < 0$, *we have*

$$\lim_{t\to\infty} \mathbb{P}\{\omega \in X : c\exp[(<s>\, g'(u_1,0) - \varepsilon)t] \le ||u_\varphi(.,t,\omega) - u_1||_\infty \le c\exp[(<s>\, g'(u_1,0) + \varepsilon)t]\} = 1 \tag{2.33}$$

for every $c \in (0, \infty)$ *and every* $\varepsilon \in (0, <s>\, |g'(u_1, 0)|)$.

(3) *If* $<s>= 0$, $g'(u_0, 0) > 0$, $g'(u_1, 0) < 0$, *and if* $\Psi, \Psi^* : \mathbb{R}^+ \to \mathbb{R}^+$ *are any two continuous functions such that* $\lim_{t\to\infty} \Psi(t) = \lim_{t\to\infty} \Psi^*(t) = \lim_{t\to\infty} \frac{\sqrt{t}}{\Psi(t)} = \infty$, *we have simultaneously*

$$\lim_{t\to\infty} \mathbb{P}\left\{\omega \in X : c\exp[-\sqrt{t}\Psi^*(t)] \le ||u_\varphi(.,t,\omega) - u_0||_\infty \le c\exp\left[-\frac{\sqrt{t}}{\Psi(t)}\right]\right\} = 1/2 \tag{2.34}$$

and

$$\lim_{t\to\infty} \mathbb{P}\left\{\omega \in X : c\exp[-\sqrt{t}\Psi^*(t)] \le ||u_\varphi(.,t,\omega) - u_1||_\infty \le c\exp\left[-\frac{\sqrt{t}}{\Psi(t)}\right]\right\} = 1/2 \tag{2.35}$$

for every $c \in (0, \infty)$.

Of course, with statements (1) and (2) of the preceding corollary, we retrieve a weaker variant of Theorem 2.3, or equivalently of relations (2.16) and (2.17). But the most interesting conclusion is evidently statement (3), whereby the random field u_φ stabilizes about u_0 and u_1 equiprobably. We observe here that the value one-half of the asymptotic probabilities (2.34) and (2.35) cannot be exceeded by virtue of relations (2.28) and (2.30) : while relation (2.28) implies that $a^* \in [-\infty, 0]$ when $<s>= 0$ and $g'(u_0, 0) > 0$, relation (2.30) implies that $b^* \in [0, \infty]$ when $<s>= 0$ and $g'(u_1, 0) < 0$.

The fact that u_φ stabilizes equiprobably around u_0 and u_1 when $<s>= 0$ can be interpreted as an oscillation phenomenon of u_φ between u_0 and u_1. In order to see this we proceed as in the considerations preceding Corollary 2.4. Let $T^* \in (0, \infty)$ be given and let $\Psi, \Psi^* : \mathbb{R}^+ \to \mathbb{R}^+$ be the same functions as in Corollary 2.7. For every $\xi \in [t, t + T^*]$ and every $c \in (0, \infty)$, consider the events

$$F_{u_0}(\xi) = \left\{\omega \in X : c\exp[-\sqrt{\xi}\Psi^*(\xi)] \le ||u_\varphi(.,\xi,\omega) - u_0||_\infty \le c\exp\left[-\frac{\sqrt{\xi}}{\Psi(\xi)}\right]\right\} \tag{2.36}$$

and

$$F_{u_1}(\xi) = \left\{\omega \in X : c\exp[-\sqrt{\xi}\Psi^*(\xi)] \leq ||u_\varphi(.,\xi,\omega) - u_1||_\infty \leq c\exp\left[-\frac{\sqrt{\xi}}{\Psi(\xi)}\right]\right\} \tag{2.37}$$

Let $\chi_{u_0}(\xi,.)$ and $\chi_{u_1}(\xi,.)$ be the corresponding indicator functions. Again, it is clear that the random variables $\omega \to \int_t^{t+T^*} d\xi \chi_{u_{0,1}}(\xi,\omega)$ measure the fraction of the available time lapse T^* that u_φ spends in the corresponding neighborhood of u_0 and u_1 determined by (2.36) and (2.37). The precise result concerning the average times is then the following.

COROLLARY 2.8. – *The hypotheses are exactly the same as in Theorem* 2.5. *Assume that* $<s>= 0$, $g'(u_0,0) > 0$, $g'(u_1,0) < 0$ *and let* $\Psi, \Psi^* : \mathbb{R}^+ \to \mathbb{R}^+$ *be any two continuous functions such that* $\lim_{t\to\infty} \Psi(t) = \lim_{t\to\infty} \Psi^*(t) = \lim_{t\to\infty} \frac{\sqrt{t}}{\Psi(t)} = \infty$. *Then we have simultaneously*

$$\lim_{t\to\infty} \mathbb{E}\left(\int_t^{t+T^*} d\xi \chi_{u_0}(\xi,.)\right) = \frac{T^*}{2} \tag{2.38}$$

and

$$\lim_{t\to\infty} \mathbb{E}\left(\int_t^{t+T^*} d\xi \chi_{u_1}(\xi,.)\right) = \frac{T^*}{2} \tag{2.39}$$

for every $T^* \in (0,\infty)$.

The conclusion is that on the average and for every $T^* \in (0,\infty)$, the random field u_φ spends half of the available time lapse T^* in an arbitrarily small neighborhood of u_0 and the other half in an arbitrarily small neighborhood of u_1, so that an oscillation pattern sets in. Of course, the preceding results do not describe the possible large deviations from the above average behavior.

We conclude our analysis of the central limit theorem case by observing that both Corollaries 2.4 and 2.8 are special cases of a more general result. Our point of departure here is the statement of Theorem 2.6. Let $T^* \in (0,\infty)$ be given and let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be the same function as in that theorem. For every $\xi \in [t, t+T^*]$ and every $c \in (0,\infty)$, define

$$F_{\Phi,u_0}(\xi) = \{\omega \in X : ||u_\varphi(.,\xi,\omega) - u_0||_\infty \leq c\exp[-\Phi(\xi)]\} \tag{2.40}$$

and

$$F_{\Phi,u_1}(\xi) = \{\omega \in X : ||u_\varphi(.,\xi,\omega) - u_1||_\infty \leq c\exp[-\Phi(\xi)]\} \tag{2.41}$$

Again, these two sets are $\mathcal{F}$-measurable and let $\chi_{\Phi,u_0}(\xi,.)$ and $\chi_{\Phi,u_1}(\xi,.)$ be their indicator functions. Clearly, the meaning of the random variables $\omega \to \int_t^{t+T^*} d\xi \chi_{\Phi,u_{0,1}}(\xi,\omega)$ is the same as before and we get the following result.

THEOREM 2.9. – *The hypotheses are exactly the same as in Theorem 2.5. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be any continuous function such that $\lim_{t\to\infty} \Phi(t) = \infty$. Then the following statements hold:*

(1) *If $g'(u_0, 0) \neq 0$ and if the limit*

$$a^* = - \lim_{t\to\infty} t^{-1/2} \left\{ \frac{\Phi(t)}{g'(u_0,0)} + <s>\, t \right\} \tag{2.42}$$

exists (with $a^* = \pm\infty$ allowed), we have

$$\lim_{t\to\infty} \mathbb{E}\left(\int_t^{t+T^*} d\xi \chi_{\Phi,u_0}(\xi,.) \right) = T^*(2\pi\sigma^*)^{-1/2} \int_{-\infty}^{a^*} dx \exp\left[-\frac{x^2}{2\sigma^*} \right] \tag{2.43}$$

for every $T^* \in (0,\infty)$.

(2) *If $g'(u_1, 0) \neq 0$ and if the limit*

$$b^* = - \lim_{t\to\infty} t^{-1/2} \left\{ \frac{\Phi(t)}{g'(u_1,0)} + <s>\, t \right\} \tag{2.44}$$

exists (with $b^* = \pm\infty$ allowed), we have

$$\lim_{t\to\infty} \mathbb{E}\left(\int_t^{t+T^*} d\xi \chi_{\Phi,u_1}(\xi,.) \right) = T^*(2\pi\sigma^*)^{-1/2} \int_{b^*}^{\infty} dx \exp\left[-\frac{x^2}{2\sigma^*} \right] \tag{2.45}$$

for every $T^* \in (0,\infty)$.

Remark. – The conclusions of Theorems 2.5, 2.6, 2.9 and their corollaries hold for an important class of Gaussian processes. Thus, assume that $(s(t,.))_{t\in\mathbb{R}}$ is a stationary Gaussian random process on $(X, \mathcal{F}, \mathbb{P})$ of average $<s>$ and continuous two-point correlation function ρ such that hypothesis (S) holds. Let $\sigma : \mathbb{R} \to \mathbb{R}^+$ be the function defined by

$$\sigma(t) = \int_0^t d\xi \int_0^t d\xi' \rho(\xi - \xi') \tag{2.46}$$

and assume that $\lim_{t\to\infty} t^{-1}\sigma(t) = \sigma^* > 0$. Then it is clear that hypothesis (CLS) holds, for we have

$$t^{-1}\sigma(t) = \mathbb{E}\left(\left(t^{-1/2} \int_0^t d\xi (s(\xi,.) - <s>) \right)^2 \right) \tag{2.47}$$

and

$$\mathbb{P}\left\{\omega \in X : t^{-1/2} \int_0^t d\xi (s(\xi,\omega) - <s>) < a^*\right\}$$
$$= (2\pi\sigma(t))^{-1/2} \int_{-\infty}^{a^* t^{1/2}} dx \exp\left[-\frac{x^2}{2\sigma(t)}\right]$$
$$= (2\pi)^{-1/2} \int_{-\infty}^{a^*(\frac{t}{\sigma(t)})^{1/2}} dx \exp\left[-\frac{x^2}{2}\right]$$
$$\rightarrow (2\pi\sigma^*)^{-1/2} \int_{-\infty}^{a^*} dx \exp\left[-\frac{x^2}{2\sigma^*}\right] \tag{2.48}$$

for every $a^* \in \mathbb{R} \cup \{\pm\infty\}$ as $t \rightarrow \infty$. It is interesting to note here that the class of Gaussian processes just described includes the Ornstein-Uhlenbeck process for which $<s>= 0$ and $\rho(t) = \mathbb{E}(s(t,.)s(0,.)) = \exp[-|t|]$, in which case we get $\sigma(t) = 0(t)$ as $t \rightarrow \infty$. This process is, of course, not only ergodic but also exponentially mixing and Markovian ([10] , [14], [23]). Thus, if $(s(t,.))_{t\in\mathbb{R}}$ is an Ornstein-Uhlenbeck process the oscillation pattern of u_φ between the two stationary states u_0 and u_1 sets in.

The preceding remark makes it natural to ask whether similar results obtain when the statistics of the random process $(s(t,.))_{t\in\mathbb{R}}$ are governed by normal distributions in such a way that the condition $\lim_{t\rightarrow\infty} t^{-1}\sigma(t) = \sigma^* > 0$ does not hold. We shall now see that this is indeed the case, which is hardly a surprise since normal distributions already lurk in relation (2.23). The precise conditions are stated in the following hypothesis.

(NS) The random process $(s(t,.))_{t\in\mathbb{R}}$ is a stationary Gaussian process on $(X, \mathcal{F}, \mathbb{P})$ of average $<s>$ and continuous two-point correlation function ρ such that $\lim_{t\rightarrow\infty} \sigma(t) = \infty$.

The following results also play an important role in our analysis of the homogeneous multiplicative white noise in [9]. We begin with

THEOREM 2.10. – *Assume that hypotheses (K), (S), (G) and (NS) hold. Assume also that the initial datum φ is non-random. Then the following statements are valid :*

(1) *For any function $a : \mathbb{R}^+ \rightarrow (u_0, u_1)$ such that the limit*

$$a^* = \lim_{t\rightarrow\infty} (\sigma(t))^{-1/2}(G(a(t)) - <s>\, t) \tag{2.49}$$

exists (with $a^* = \pm\infty$ allowed), we have

$$\lim_{t\rightarrow\infty} \mathbb{P}\{\omega \in X : u_\varphi(.,t,\omega) \leq a(t)\} = (2\pi)^{-1/2} \int_{-\infty}^{a^*} dx \exp\left[-\frac{x^2}{2}\right] \tag{2.50}$$

(2) *For any function* $b : \mathbb{R}^+ \longrightarrow (u_0, u_1)$ *such that the limit*

$$b^* = \lim_{t \longrightarrow \infty} (\sigma(t))^{-1/2} (G(b(t)) - <s>\, t) \tag{2.51}$$

exists (with $b^* = \pm\infty$ *allowed), we have*

$$\lim_{t \longrightarrow \infty} \mathbb{P}\Big\{\omega \in X : b(t) \leq u_\varphi(.,t,\omega)\Big\} = (2\pi)^{-1/2} \int_{b^*}^{\infty} dx \exp\left[-\frac{x^2}{2}\right] \tag{2.52}$$

Our next result allows us to get the asymptotic probabilities of stabilization for u_φ around the stationary states u_0 and u_1.

THEOREM 2.11. – *The hypotheses are exactly the same as in Theorem 2.10. Let* $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *be any continuous function such that* $\lim_{t \longrightarrow \infty} \Phi(t) = \infty$. *Then the following statements hold :*

(1) *If* $g'(u_0, 0) \neq 0$ *and if the limit*

$$a^* = -\lim_{t \longrightarrow \infty} (\sigma(t))^{-1/2} \left\{ \frac{\Phi(t)}{g'(u_0, 0)} + <s>\, t \right\} \tag{2.53}$$

exists (with $a^* = \pm\infty$ *allowed), we have*

$$\lim_{t \longrightarrow \infty} \mathbb{P}\Big\{\omega \in X : ||u_\varphi(.,t,\omega) - u_0||_\infty \leq c \exp[-\Phi(t)]\Big\} = (2\pi)^{-1/2} \int_{-\infty}^{a^*} dx \exp\left[-\frac{x^2}{2}\right] \tag{2.54}$$

for every $c \in (0, \infty)$.

(2) *If* $g'(u_1, 0) \neq 0$ *and if the limit*

$$b^* = -\lim_{t \longrightarrow \infty} (\sigma(t))^{-1/2} \left\{ \frac{\Phi(t)}{g'(u_1, 0)} + <s>\, t \right\} \tag{2.55}$$

exists (with $b^* = \pm\infty$ *allowed), we have*

$$\lim_{t \longrightarrow \infty} \mathbb{P}\{\omega \in X : ||u_\varphi(.,t,\omega) - u_1||_\infty \leq c \exp[-\Phi(t)]\} = (2\pi)^{-1/2} \int_{b^*}^{\infty} dx \exp\left[-\frac{x^2}{2}\right] \tag{2.56}$$

for every $c \in (0, \infty)$.

Finally, our general result concerning the average times is the following.

THEOREM 2.12. – *The hypotheses are exactly the same as in Theorem 2.10. Let* $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *be any continuous function such that* $\lim_{t\to\infty} \Phi(t) = \infty$, *and let* $\chi_{\Phi,u_0}(\xi,.)$ *and* $\chi_{\Phi,u_1}(\xi,.)$ *be the indicator functions of the sets* $F_{\Phi,u_0}(\xi)$ *and* $F_{\Phi,u_1}(\xi)$ *as defined by relations* (2.40) *and* (2.41). *Then the following statements hold :*

(1) *If* $g'(u_0, 0) \neq 0$ *and if the limit*

$$a^* = -\lim_{t\to\infty} (\sigma(t))^{-1/2} \left\{ \frac{\Phi(t)}{g'(u_0,0)} + <s>\, t \right\} \tag{2.57}$$

exists (with $a^* = \pm\infty$ *allowed), we have*

$$\lim_{t\to\infty} \mathbb{E}\left(\int_t^{t+T^*} d\xi \chi_{\Phi,u_0}(\xi,.) \right) = T^*(2\pi)^{-1/2} \int_{-\infty}^{a^*} dx \exp\left[\frac{-x^2}{2}\right] \tag{2.58}$$

for every $T^* \in (0,\infty)$.

(2) *If* $g'(u_1, 0) \neq 0$ *and if the limit*

$$b^* = -\lim_{t\to\infty} (\sigma(t))^{-1/2} \left\{ \frac{\Phi(t)}{g'(u_1,0)} + <s>\, t \right\} \tag{2.59}$$

exists (with $b^* = \pm\infty$ *allowed), we have*

$$\lim_{t\to\infty} \mathbb{E}\left(\int_t^{t+T^*} d\xi \chi_{\Phi,u_1}(\xi,.) \right) = T^*(2\pi)^{-1/2} \int_{b^*}^{\infty} dx \exp\left[\frac{-x^2}{2}\right] \tag{2.60}$$

for every $T^* \in (0,\infty)$.

The implication of the preceding three theorems is that results similar to Corollaries 2.4, 2.7 and 2.8 hold in the Gaussian case as well. In particular, the oscillation patterns of the random field u_φ also take place in the Gaussian case when $<s>= 0$, $g'(u_0, 0) > 0$, $g'(u_1, 0) < 0$. Evidently, the condition $\sigma(t) \longrightarrow \infty$ implies that the integrated process $(\int_0^t d\xi s(\xi,.))_{t\in\mathbb{R}}$ cannot be $\mathbb{P}$-almost surely bounded in time. Oscillation patterns such as those described above are therefore related to the unboundedness of $t \longrightarrow \int_0^t d\xi s(\xi,.)$ in an essential way.

Remarks.

1. A glance at the proofs given in Section 3 shows that the statements of the last eight theorems and corollaries still hold for random initial conditions φ, provided that there exists a constant c such that the inequalities $u_0 + c \leq \varphi(x, \omega) \leq u_1 - c$ hold $\mathbb{P}$-as. This last condition is, however, somewhat artificial.
2. The random process $(s(t, .))_{t \in \mathbb{R}}$ and all the indicator functions introduced above are continuous in probability. In the first case this property follows from hypothesis (S). In the second case it follows from the continuity of the fonction Φ. As is usual, we may therefore assume that those random processes are jointly measurable in (t, ω) and make no further mention of the matter ([14]). This will justify all our subsequent applications of Fubini's theorem.

The complete proofs of all results will be given in the next section.

3. PROOF OF THE MAIN RESULTS

We begin by outlining briefly our strategy regarding the proof of Theorem 2.1. Our analysis rests upon the introduction of a one-parameter family of auxiliary random fields $v_\alpha : \overline{\Omega} \times \mathbb{R}^+ \times X \rightarrow \mathbb{R}^+$ indexed by a real parameter α, which satisfy $\mathbb{P}$-almost surely a parabolic differential inequality when $|\alpha|$ is sufficiently large. To show what kind of random fields we are looking for we first recall that every x-independent random process $(\hat{u}(t, .))_{t \in \mathbb{R}}$ that solves Problem (1.1) $\mathbb{P}$-almost surely is necessarily of the form (2.4). Since G is strictly monotone, we next observe that for any $\alpha \in \mathbb{R}/\{0\}$, we can rewrite relation (2.4) as

$$\hat{u}(t, \omega) = G^{-1}\left\{\int_0^t d\xi s(\xi, \omega) + \alpha^{-1}\ln(v_\alpha^*(\omega)) + G(\hat{\mu})\right\} \tag{3.1}$$

for some random variable $v_\alpha^* : X \rightarrow \mathbb{R}^+$ and for some $\hat{\mu} \in (u_0, u_1)$. Then, given u_φ that solves Problem (1.1) $\mathbb{P}$-almost surely, we define the $v_\alpha' s$ as the random fields whose relationship to u_φ is formally identical to the relationship between v_α^* and $(\hat{u}(t, .))_{t \in \mathbb{R}}$ in (3.1). This gives

$$u_\varphi(x. t. \omega) = G^{-1}\left\{\int_0^t d\xi s(\xi. \omega) + \alpha^{-1}\ln(v_\alpha(x. t, \omega)) + G(\hat{\mu})\right\} \tag{3.2}$$

or equivalently

$$v_\alpha(x,t,\omega) = \exp\left[\alpha\left\{\int_{\hat{\mu}}^{u_\varphi(x,t,\omega)} \frac{d\xi}{g(\xi,0)} - \int_0^t d\xi s(\xi,\omega)\right\}\right] \tag{3.3}$$

We then proceed by showing that v_α stabilizes $\mathbb{P}$-almost surely around some random variable $v_\alpha^* : X \to \mathbb{R}^+$ in $L^1(\Omega)$ as $t \to \infty$. Of course, this will determine v_α^* uniquely, which in turn will determine the unique random process $(\hat{u}(t,.))_{t\in\mathbb{R}}$ of Theorem 2.1 by means of relation (3.1). From this we shall easily infer both statements of Theorem 2.1.

The derivation of a parabolic differential inequality for v_α requires the control of the dependence of g on ∇u. This is accomplished by using the quadratic growth estimate of hypothesis (QG). The precise result is the following.

LEMMA 3.1. – *Given the random field u_φ, let v_α be the random field given by relation* (3.3) *where $\alpha \in \mathbb{R}/\{0\}$ and $\hat{\mu} \in (u_0, u_1)$. Then for $|\alpha|$ sufficiently large we have $\mathbb{P}$-almost surely*

$$\left\{\begin{aligned} &\partial_t v_\alpha(x,t,\omega) \le \operatorname{div}(k_{u_\varphi}(x,t,\omega)\nabla v_\alpha(x,t,\omega)), && (x,t,\omega)\in\Omega\times\mathbb{R}^+\times X \\ &\frac{\partial v_\alpha(x,t,\omega)}{\partial n(u_\varphi)} = 0, && (x,t,\omega)\in\partial\Omega\times\mathbb{R}^+\times X \end{aligned}\right\} \tag{3.4}$$

Proof. – We first note that

$$\nabla v_\alpha(x,t,\omega) = \alpha \frac{v_\alpha(x,t,\omega)}{g(u_\varphi(x,t,\omega),0)} \nabla u_\varphi(x,t,\omega) \tag{3.5}$$

so that v_α satisfies $\mathbb{P}$-almost surely the homogeneous conormal boundary condition in (3.4) since u_φ does. In order to prove that the differential inequality in (3.4) holds $\mathbb{P}$-almost surely, we calculate each term separately from relation (3.3) by making use of the first equation in (1.1). After regrouping the various contributions we obtain $\mathbb{P}$-almost surely

$$\begin{aligned} &\operatorname{div}\left(k_{u_\varphi}(x,t,\omega)\nabla v_\alpha(x,t,\omega)\right) - \partial_t v_\alpha(x,t,\omega) \\ &= \frac{\alpha v_\alpha(x,t,\omega)}{g^2(u_\varphi(x,t,\omega),0)}\left\{\alpha - \frac{\partial g}{\partial u}(u_\varphi(x,t,\omega),0)\right\} \\ &\quad \times (\nabla u_\varphi(x,t,\omega), k_{u_\varphi}(x,t,\omega)\nabla u_\varphi(x,t,\omega))_{\mathbb{R}^N} \\ &\quad - \alpha\left\{\frac{s(t,\omega)g(u_\varphi(x,t,\omega),\nabla u_\varphi(x,t,\omega))}{g(u_\varphi(x,t,\omega),0)} - s(t,\omega)\right\} v_\alpha(x,t,\omega) \end{aligned} \tag{3.6}$$

Since v_α is positive, we see that the right-hand side of (3.6) is $\mathbb{P}$-almost surely non-negative if, and only if, the inequality

$$\alpha\left\{\alpha - \frac{\partial g}{\partial u}(u_\varphi(x,t,\omega),0)\right\}(\nabla u_\varphi(x,t,\omega), k_{u_\varphi}(x,t,\omega)\nabla u_\varphi(x,t,\omega))_{\mathbb{R}^N}$$
$$\geq \alpha s(t,\omega)g(u_\varphi(x,t,\omega),0)$$
$$\times \{g(u_\varphi(x,t,\omega),\nabla u_\varphi(x,t,\omega)) - g(u_\varphi(x,t,\omega),0)\} \quad (3.7)$$

holds $\mathbb{P}$-almost surely. In order to prove inequality (3.7) for $|\alpha|$ sufficiently large, we now have to distinguish the case $\alpha > 0$ from the case $\alpha < 0$. If $\alpha > 0$, then relation (3.7) holds if, and only if, the inequality

$$\left\{\alpha - \frac{\partial g}{\partial u}(u_\varphi(x,t,\omega),0)\right\}(\nabla u_\varphi(x,t,\omega), k_{u_\varphi}(x,t,\omega)\nabla u_\varphi(x,t,\omega))_{\mathbb{R}^N}$$
$$\geq s(t,\omega)g(u_\varphi(x,t,\omega),0)$$
$$\times \{g(u_\varphi(x,t,\omega),\nabla u_\varphi(x,t,\omega)) - g(u_\varphi(x,t,\omega),0)\} \quad (3.8)$$

holds $\mathbb{P}$-almost surely. In order to prove this last inequality for $\alpha > 0$ sufficiently large, we construct a lower bound for the left-hand side and an upper bound for the right-hand side of (3.8) which still satisfy the above inequality for $\alpha > 0$ large enough. Let $\overline{m} = \max_{u\in[u_0,u_1]} \frac{\partial g}{\partial u}(u,0)$ and choose $\alpha \in \mathbb{R}^+ \cap (\overline{m},\infty)$; on the one hand, by invoking the first inequality in (1.2) we obtain

$$\left\{\alpha - \frac{\partial g}{\partial u}(u_\varphi(x,t,\omega),0)\right\}(\nabla u_\varphi(x,t,\omega), k_{u_\varphi}(x,t,\omega)\nabla u_\varphi(x,t,\omega))_{\mathbb{R}^N}$$
$$\geq (\alpha - \overline{m})\underline{k}|\nabla u_\varphi(x,t,\omega)|^2 \quad (3.9)$$

$\mathbb{P}$-almost surely. On the other hand, owing to the boundedness of $u \to g(u,0)$, that of the random process $(s(t,.))_{t\in\mathbb{R}}$ and by using hypothesis (QG) we get

$$-c|\nabla u_\varphi(x,t,\omega)|^2$$
$$\leq s(t,\omega)g(u_\varphi(x,t,\omega),0)$$
$$\times \{g(u_\varphi(x,t,\omega),\nabla u_\varphi(x,t,\omega)) - g(u_\varphi(x,t,\omega),0)\}$$
$$\leq c|\nabla u_\varphi(x,t,\omega)|^2 \quad (3.10)$$

$\mathbb{P}$-almost surely for some $c \in (0,\infty)$. Inequality (3.9) together with the right-hand side inequality (3.10) then prove relation (3.8) for

$\alpha \in \mathbb{R}^+ \cap [\overline{m} + c\underline{k}^{-1}, \infty)$. Now if $\alpha < 0$, then relation (3.7) holds if, and only if, the inequality

$$\begin{aligned}\left\{\alpha - \frac{\partial g}{\partial u}(u_\varphi(x,t,\omega),0)\right\}&(\nabla u_\varphi(x,t,\omega), k_{u_\varphi}(x,t,\omega)\nabla u_\varphi(x,t,\omega))_{\mathbb{R}^N}\\ &\leq s(t,\omega)g(u_\varphi(x,t,\omega),0)\\ &\quad\times\{g(u_\varphi(x,t,\omega),\nabla u_\varphi(x,t,\omega)) - g(u_\varphi(x,t,\omega),0)\}\end{aligned} \tag{3.11}$$

holds $\mathbb{P}$-almost surely. In order to prove this inequality for $\alpha < 0$ small enough, we construct an upper bound for the left-hand side and a lower bound for the right-hand side of (3.11) which still satisfy the above inequality for $\alpha < 0$ sufficiently small. Let $\underline{m} = \min_{u \in [u_0, u_1]} \frac{\partial g}{\partial u}(u, 0)$ and choose $\alpha \in \mathbb{R}^- \cap (-\infty, \underline{m})$; on the one hand, by invoking the ellipticity condition of the random field k once again we obtain

$$\begin{aligned}\left\{\alpha - \frac{\partial g}{\partial u}(u_\varphi(x,t,\omega),0)\right\}&(\nabla u_\varphi(x,t,\omega), k_{u_\varphi}(x,t,\omega)\nabla u_\varphi(x,t,\omega))_{\mathbb{R}^N}\\ &\leq (\alpha - \underline{m})\underline{k}|\nabla u_\varphi(x,t,\omega)|^2\end{aligned} \tag{3.12}$$

$\mathbb{P}$-almost surely. Inequality (3.12) and the left-hand side inequality (3.10) then prove relation (3.11) for $\alpha \in \mathbb{R}^- \cap (-\infty, \underline{m} - c\underline{k}^{-1}]$. The preceding considerations show that there exists $\alpha_0 > 0$ such that inequality (3.7) holds $\mathbb{P}$-almost surely for every $\alpha \in \mathbb{R}/\{0\}$ with $|\alpha| \geq \alpha_0$. $\square$

Remark. – A glance at relation (3.7) shows that if the nonlinearity g does not depend on ∇u, then the first relation in (3.4) holds if, and only if, the inequality

$$\alpha\left\{\alpha - \frac{\partial g}{\partial u}(u_\varphi(x,t,\omega),0)\right\}(\nabla u_\varphi(x,t,\omega), k_{u_\varphi}(x,t,\omega)\nabla u_\varphi(x,t,\omega))_{\mathbb{R}^N} \geq 0 \tag{3.13}$$

holds $\mathbb{P}$-almost surely. The important point here is that the left-hand side of (3.13) does not depend explicitly on the random process $(s(t,.))_{t\in\mathbb{R}}$, so that inequality (3.13) is true for $|\alpha|$ sufficiently large without any boundedness condition on $(s(t,.))_{t\in\mathbb{R}}$. Thus, in this case the parabolic inequality (3.4) holds for all random processes with $\mathbb{P}$-almost surely Hölder continuous trajectories.

In order to prove that v_α stabilizes around some positive random variable $v_\alpha^* : X \to \mathbb{R}^+$ in $L^1(\Omega)$, we need a few more preparatory results. The first one is an easy consequence of Lemma 3.1.

LEMMA 3.2. – *There exist two random variables* $v_\alpha^\pm : X \to \mathbb{R}^+$ *such that the inequalities*

$$0 < v_\alpha^-(\omega) \leq v_\alpha(.,t,\omega) \leq v_\alpha^+(\omega) < \infty \tag{3.14}$$

hold $\mathbb{P}$*-almost surely for* $\alpha > 0$ *sufficiently large and for every* $t \in \mathbb{R}^+$.

Proof. – Let $\alpha_0 > 0$ be the positive constant in the proof of Lemma 3.1. For $|\alpha| \geq \alpha_0$, define the random variable $v_\alpha^+(\omega) = \sup_{x\in\overline{\Omega}} v_\alpha(x,0,\omega)$; then for every $(x,t) \in \overline{\Omega} \times \mathbb{R}^+$ we have $v_\alpha(x,t,\omega) \leq v_\alpha^+(\omega) < \infty$ $\mathbb{P}$-almost surely by the parabolic maximum principle applied to Problem (3.4). If $\alpha \geq \alpha_0$, this implies that $v_{-\alpha}(x,t,\omega) = (v_\alpha(x,t,\omega))^{-1} \leq v_{-\alpha}^+(\omega)$ $\mathbb{P}$-almost surely because of relation (3.3), which means that there exists a random variable $v_\alpha^- : X \to \mathbb{R}^+$ such that the inequalities $0 < v_\alpha^-(\omega) \leq v_\alpha(x,t,\omega)$ hold $\mathbb{P}$-almost surely for every $(x,t) \in \overline{\Omega} \times \mathbb{R}^+$; in fact, it is sufficient to choose $v_\alpha^- = (v_{-\alpha}^+)^{-1}$. □

The next result is also critical to our proof of convergence. It involves yet another auxiliary random field for which we can prove a property of exponential dichotomy in $L^2(\Omega)$ by means of a simple version of the Poincaré-Wirtinger inequality. We write I for the identity operator in $L^2(\Omega)$, $||.||_2$ for the usual L^2-norm and Q for the orthogonal projection operator onto the constant functions, that is $Qf = |\Omega|^{-1}\int_\Omega dx f(x)$ for every $f \in L^2(\Omega)$.

LEMMA 3.3. – *Given the random field* u_φ, *let* v_α *be the random field given by relation* (3.3). *Let* $\tau > 0$ *and let* v *be the random field that solves* $\mathbb{P}$*-almost surely the linear initial-boundary value problem*

$$\left\{\begin{aligned} &\partial_t v(x,t,\omega) = \operatorname{div}(k_{u_\varphi}(x,t,\omega)\nabla v(x,t,\omega)), && (x,t,\omega) \in \Omega \times (\tau,\infty) \times X \\ &v(x,\tau,\omega) = v_\alpha(x,\tau,\omega), && (x,\omega) \in \overline{\Omega} \times X \\ &\frac{\partial v(x,t,\omega)}{\partial n(u_\varphi)} = 0, && (x,t,\omega) \in \partial\Omega \times (\tau,\infty) \times X \end{aligned}\right\} \tag{3.15}$$

Then the following statements hold :

(1) *We have* $v_\alpha(.,t,\omega) \leq v(.,t,\omega)$ $\mathbb{P}$*-almost surely for* $|\alpha|$ *sufficiently large and for every* $t \in [\tau,\infty)$.

(2) *The equality* $Qv(.,t,\omega) = Qv_\alpha(.,\tau,\omega)$ *holds* $\mathbb{P}$*-almost surely for every* $\alpha \in \mathbb{R}/\{0\}$ *and for every* $t \in [\tau,\infty)$.

(3) *Let* $\underline{k}$ *be the ellipticity constant in relation* (1.2) *and let* λ_1 *be the largest negative eigenvalue of the* $L^2(\Omega)$*-realization of Laplace's*

operator on $H^{2,2}_{\mathcal{N}}$. Then the inequality

$$||(I-Q)v(.,t,\omega)||_2 \leq \exp[-\underline{k}|\lambda_1|(t-\tau)]||(I-Q)v_\alpha(.,\tau,\omega)||_2 \tag{3.16}$$

holds $\mathbb{P}$-almost surely for every $\alpha \in \mathbb{R}/\{0\}$ and for every $t \in [\tau,\infty)$.

Proof. – Statement (1) is an immediate consequence of relations (3.4), (3.15) and of the parabolic maximum principle applied to the difference $v_\alpha(.,t,\omega) - v(.,t,\omega)$. As for statement (2), the second relation in (3.15) implies that $Qv(.,\tau,\omega) = Qv_\alpha(.,\tau,\omega)$ so that it is sufficient to prove the relation $Qv(.,t,\omega) = Qv(.,\tau,\omega)$ $\mathbb{P}$-almost surely for every $t \in [\tau,\infty)$. But this relation is clearly satisfied since from the first and third equations in (3.15) and owing to the definition of Q we get

$$\frac{d}{dt}Qv(.,t,\omega) = |\Omega|^{-1}\int_\Omega dx\,\mathrm{div}\,(k_{u_\varphi}(x,t,\omega)\nabla v(x,t,\omega)) = 0 \tag{3.17}$$

$\mathbb{P}$-almost surely by invoking Gauss' divergence theorem. We now prove statement (3). We first notice that the projected random field $(I-Q)v(.,t,\omega)$ satisfies the same linear initial-boundary value problem as v does since Q commutes with the differential operator in (3.15). This means that we have $\mathbb{P}$-almost surely

$$\left\{\begin{array}{c} \partial_t(I-Q)v(x,t,\omega) = \mathrm{div}\,(k_{u_\varphi}(x,t,\omega)\nabla(I-Q)v(x,t,\omega)), \\ (x,t,\omega) \in \Omega\times(\tau,\infty)\times X \\ (I-Q)v(x,t,\omega) = (I-Q)v_\alpha(x,\tau,\omega), \\ (x,\omega)\in\overline{\Omega}\times X \\ \dfrac{\partial(I-Q)v(x,t,\omega)}{\partial n(u_\varphi)} = 0, \\ (x,t,\omega)\in\partial\Omega\times(\tau,\infty)\times X \end{array}\right\} \tag{3.18}$$

By using successively relations (3.18), integration by parts and the first inequality in (1.2) we then get $\mathbb{P}$-almost surely

$$\begin{aligned} &\frac{d}{dt}||(I-Q)v(.,t,\omega)||_2^2 \\ &\quad = 2\int_\Omega dx(I-Q)v(x,t,\omega)\,\mathrm{div}\,(k_{u_\varphi}(x,t,\omega)\nabla(I-Q)v(x,t,\omega)) \\ &\quad = -2\int_\Omega dx(\nabla(I-Q)v(x,t,\omega), k_{u_\varphi}(x,t,\omega)\nabla(I-Q)v(x,t,\omega))_{\mathbb{R}^N} \\ &\quad \leq -2\underline{k}||\nabla(I-Q)v(.,t,\omega))||_2^2 \end{aligned} \tag{3.19}$$

for every $t \in (\tau, \infty)$. Now since the operator $I - Q$ amounts to subtracting off the spatial average of the function, an $L^2(\Omega)$-version of the Poincaré-Wirtinger inequality gives

$$||(I-Q)v(.,t,\omega)||_2 \leq |\lambda_1|^{-1/2}||\nabla(I-Q)v(.,t,\omega)||_2 \tag{3.20}$$

$\mathbb{P}$-almost surely for every $t \in (\tau, \infty)$ (see for instance [5] and [21] for a discussion of the general Poincaré-Wirtinger inequality). The substitution of relation (3.20) into relation (3.19) then leads to

$$\frac{d}{dt}||(I-Q)v(.,t,\omega)||_2^2 + 2\underline{k}|\lambda_1|\,||(I-Q)v(.,t,\omega)||_2^2 \leq 0 \tag{3.21}$$

for every $t \in (\tau, \infty)$, which immediately implies relation (3.16) because of the initial condition in (3.15). □

The preceding results now allow us to give the following.

Proof of Theorem 2.1. – According to the general strategy outlined above we first prove that for $\alpha > 0$ sufficiently large, there exists a random variable $v_\alpha^* : X \to \mathbb{R}^+$ such that $||v_\alpha(.,t,\omega) - v_\alpha^*(\omega)||_1 \to 0$ $\mathbb{P}$-almost surely as $t \to \infty$. We first note that the operator Q is positivity preserving. Then the application of Q on both sides of the differential inequality (3.4) along with the boundary condition in (3.4) imply that $\frac{d}{dt}Qv_\alpha(.,t,\omega) \leq 0$ $\mathbb{P}$-almost surely. Consequently, the function $t \to Qv_\alpha(.,t,\omega)$ is monotone decreasing on $(0,\infty)$ and we define the random variable v_α^* by $v_\alpha^*(\omega) = \inf_{t\in\mathbb{R}_0^+} Qv_\alpha(.,t,\omega) = \lim_{t\to\infty} Qv_\alpha(.,t,\omega)$. We then have $\mathbb{P}$-almost surely the estimates

$$\begin{aligned}
&||v_\alpha(.,t,\omega) - v_\alpha^*(\omega)||_1 \\
&\leq \int_\Omega dx|v_\alpha(x,t,\omega) - Qv_\alpha(.,t,\omega)| + |\Omega||Qv_\alpha(.,t,\omega) - v_\alpha^*(\omega)| \\
&= \int_\Omega dx(v_\alpha(x,t,\omega) - Qv_\alpha(.,t,\omega))^+ \\
&- \int_\Omega dx(v_\alpha(x,t,\omega) - Qv_\alpha(.,t,\omega))^- + |\Omega||Qv_\alpha(.,t,\omega) - v_\alpha^*(\omega)| \\
&= 2\int_\Omega dx(v_\alpha(x,t,\omega) - Qv_\alpha(.,t,\omega))^+ + |\Omega||Qv_\alpha(.,t,\omega) - v_\alpha^*(\omega)| \qquad (3.22)
\end{aligned}$$

where $(v_\alpha(.,t,\omega) - Qv_\alpha(.,t,\omega))^\pm$ denotes the positive and the negative part of $v_\alpha(.,t,\omega) - Qv_\alpha(.,t,\omega)$, respectively. The last equality in (3.22) follows from the fact that

$$\begin{aligned}\int_\Omega dx(v_\alpha(x,t,\omega) - Qv_\alpha(.,t,\omega)) &= |\Omega|Qv_\alpha(.,t,\omega) - |\Omega|Qv_\alpha(.,t,\omega)\\ = 0 = &\int_\Omega dx(v_\alpha(x,t,\omega) - Qv_\alpha(.,t,\omega))^+\\ &+ \int_\Omega dx(v_\alpha(x,t,\omega) - Qv_\alpha(.,t,\omega))^-\end{aligned}$$

Now by definition of the random variable v_α^*, the second term of (3.22) converges to zero $\mathbb{P}$-almost surely as $t \to \infty$. It remains to show that the first term of (3.22) converges to zero as well. For every $t \in [\tau, \infty)$, define the set $\Omega_t^+(\omega)$ of those $x \in \Omega$ such that the inequality $v_\alpha(x,t,\omega) - Qv_\alpha(.,t,\omega) \geq 0$ holds. Using successively the first two statements of Lemma 3.3, Schwarz inequality and the third statement of Lemma 3.3, we obtain $\mathbb{P}$-almost surely

$$\begin{aligned}&\int_\Omega dx(v_\alpha(x,t,\omega) - Qv_\alpha(.,t,\omega))^+ = \int_{\Omega_t^+(\omega)} dx(v_\alpha(x,t,\omega) - Qv_\alpha(.,t,\omega))\\ &\leq \int_\Omega dx|(I-Q)v(x,t,\omega)| + |\Omega_t^+(\omega)||Qv_\alpha(.,t,\omega) - Qv_\alpha(.,\tau,\omega)|\\ &\leq |\Omega|^{1/2} ||(I-Q)v(.,t,\omega)||_2 + |\Omega||Qv_\alpha(.,t,\omega) - Qv_\alpha(.,\tau,\omega)|\\ &\leq |\Omega|^{1/2} \exp[-\underline{k}|\lambda_1|(t-\tau)] \, ||(I-Q)v_\alpha(.,\tau,\omega)||_2\\ &\quad + |\Omega| \, |Qv_\alpha(.,t,\omega) - Qv_\alpha(.,\tau,\omega)|\\ &\leq |\Omega|^{1/2} \exp[-\underline{k}|\lambda_1|(t-\tau)] \, ||v_\alpha(.,\tau,\omega)||_2\\ &\quad + |\Omega| \, |Qv_\alpha(.,t,\omega) - Qv_\alpha(.,\tau,\omega)| \qquad (3.23)\end{aligned}$$

for every $t \in [\tau, \infty)$. Since $\tau > 0$ is arbitrary in the first place, we can choose $\tau = t/2$ and invoke the upper bound of Lemma 3.2 along with the fact that $t \to Qv_\alpha(.,t,\omega)$ is monotone decreasing. From relation (3.23) we get $\mathbb{P}$-almost surely

$$\begin{aligned}&\int_\Omega dx(v_\alpha(x,t,\omega) - Qv_\alpha(.,t,\omega))^+\\ &\quad \leq |\Omega| \left(\exp\left[-\underline{k}|\lambda_1|\frac{t}{2}\right] v_\alpha^+(\omega) + \left(Qv_\alpha\left(.,\frac{t}{2},\omega\right) - v_\alpha^* \right) \right) \qquad (3.24)\end{aligned}$$

which implies the desired result as $t \to \infty$. Now from relations (3.14) of Lemma 3.2 we infer that the inequalities

$$v_\alpha^-(\omega) \leq \inf_{(x,t) \in \overline{\Omega} \times \mathbb{R}^+} v_\alpha(x,t,\omega) \qquad (3.25)$$

and

$$v_\alpha^-(\omega) \leq \inf_{(x,t)\in\overline{\Omega}\times\mathbb{R}^+} Qv_\alpha(.,t,\omega) = v_\alpha^*(\omega) \tag{3.26}$$

hold $\mathbb{P}$-almost surely. From relations (3.1), (3.2), (3.25), (3.26), the fact that G^{-1} has a uniformly bounded derivative on $\mathbb{R}$, we then conclude that

$$\begin{aligned}
&||u_\varphi(.,t,\omega) - \hat{u}(t,\omega)||_1 = \int_\Omega dx|u_\varphi(x,t,\omega) - \hat{u}(t,\omega)| \\
&\leq 0(1)\int_\Omega dx|\ln(v_\alpha(x,t,\omega)) - \ln(v_\alpha^*(\omega))| \\
&\leq 0(1)\max((\inf v_\alpha(x,t,\omega))^{-1},(v_\alpha^*(\omega))^{-1})\int_\Omega dx|v_\alpha(x,t,\omega) - v_\alpha^*(\omega)| \\
&\leq 0(1)\max((v_\alpha^-(\omega))^{-1},(v_\alpha^*(\omega))^{-1})||v_\alpha(.,t,\omega) - v_\alpha^*(\omega)||_1 \\
&= 0(1)(v_\alpha^-(\omega))^{-1}||v_\alpha(.,t,\omega) - v_\alpha^*(\omega)||_1 \to 0
\end{aligned}$$

$\mathbb{P}$-almost surely as $t \to \infty$, so that $u_\varphi(.,t,\omega) - \hat{u}(t,\omega) \to 0$ $\mathbb{P}$-almost surely strongly in $L^1(\Omega)$. Since $u_\varphi(.,t,\omega) \in (u_0, u_1)$, $\hat{u}(t,\omega) \in (u_0, u_1)$ $\mathbb{P}$-almost surely for any $t \in \mathbb{R}^+$, we have

$$||u_\varphi(.,t,\omega) - \hat{u}(t,\omega)||_\infty = \sup_{x\in\overline{\Omega}}|u_\varphi(x,t,\omega) - \hat{u}(t,\omega)| \leq c$$

$\mathbb{P}$-almost surely for some $c \in (0,\infty)$, so that relation (2.2) holds for every $p \in [1,\infty)$. Relation (2.3) then follows immediately from dominated convergence. □

Remark. – A glance at all the preceding proofs shows that the boundedness of the random process $(s(t,.))_{t\in\mathbb{R}}$ is required only in Lemma 3.1. This observation and the remark following the proof of Lemma 3.1 then lead to the conclusion of Remark (1) following the statement of Theorem 2.1.

We now turn to the proof of Theorem 2.2, for which we need additional preparatory results. We begin by stating the existence of some uniform bounds that pertain to the random field u_φ.

LEMMA 3.4. – *There exists a constant* $c \in (0,\infty)$ *such that the two estimates*

$$\sup_{(x,t)\in\overline{\Omega}\times\mathbb{R}^+} |\nabla u_\varphi(x,t,\omega)| \leq c \tag{3.27}$$

$$\sup_{(x,t)\in\overline{\Omega}\times\mathbb{R}^+} |g(u_\varphi(x,t,\omega),\nabla u_\varphi(x,t,\omega))| \leq c \tag{3.28}$$

hold $\mathbb{P}$*-almost surely.*

Proof. – Since we have $u_\varphi(x,t,\omega) \in (u_0, u_1)$ $\mathbb{P}$-almost surely for every $(x,t) \in \overline{\Omega} \times \mathbb{R}^+$, there exists $\widetilde{c} \in (0,\infty)$ such that $\sup_{(x,t)\in\overline{\Omega}\times\mathbb{R}^+} |u_\varphi(x,t,\omega)| \le \widetilde{c}$ holds $\mathbb{P}$-almost surely. The fact that inequality (3.27) holds then follows from the standard *a priori* estimates for quasilinear parabolic equations [20]. The second inequality (3.28) is an immediate consequence of (1.3) and (3.27). □

Our next preparatory result provides a proof of condition (2.5).

LEMMA 3.5. – *For every $\gamma \in (0,\infty)$ and every $p \in [1,\infty)$ we have $\mathbb{P}$-almost surely*

$$\lim_{t\to\infty} \int_t^{t+\gamma} d\xi ||\nabla u_\varphi(.,\xi,\omega)||_p^p = 0 \tag{3.29}$$

Proof. – We first notice that the projected random field $(I-Q)u_\varphi(.,t,\omega)$ satisfies $\mathbb{P}$-almost surely the initial-boundary value problem

$$\left\{\begin{array}{ll} \partial_t(I-Q)u_\varphi(x,t,\omega) = \operatorname{div}(k_{u_\varphi}(x,t,\omega)\nabla(I-Q)u_\varphi(x,t,\omega)) & \\ +s(t,\omega)(I-Q)g(u_\varphi(x,t,\omega),\nabla u_\varphi(x,t,\omega)), & (x,t,\omega)\in\Omega\times\mathbb{R}^+\times X \\ (I-Q)u_\varphi(x,0,\omega) = (I-Q)\varphi(x,\omega), & (x,\omega)\in\overline{\Omega}\times X \\ \dfrac{\partial(I-Q)u_\varphi(x,t,\omega)}{\partial n(u_\varphi)} = 0, & (x,t,\omega)\in\partial\Omega\times\mathbb{R}^+\times X \end{array}\right\} \tag{3.30}$$

By using successively relations (3.30), integration by parts, the first inequality in (1.2), the boundedness of $(s(t,.))_{t\in\mathbb{R}}$ along with inequality (3.28), we obtain $\mathbb{P}$-almost surely

$$\begin{aligned} &\frac{d}{d\xi}||(I-Q)u_\varphi(.,\xi,\omega)||_2^2 \\ &= 2\int_\Omega dx(I-Q)u_\varphi(x,\xi,\omega)\operatorname{div}(k_{u_\varphi}(x,\xi,\omega)\nabla(I-Q)u_\varphi(x,\xi,\omega)) \\ &+ 2s(\xi,\omega)\int_\Omega dx(I-Q)u_\varphi(x,\xi,\omega)(I-Q)g(u_\varphi(x,\xi,\omega),\nabla u_\varphi(x,\xi,\omega)) \\ &= -2\int_\Omega dx(\nabla(I-Q)u_\varphi(x,\xi,\omega),k_{u_\varphi}(x,t,\omega)\nabla(I-Q)u_\varphi(x,\xi,\omega))_{\mathbb{R}^N} \\ &+ 2s(\xi,\omega)\int_\Omega dx(I-Q)u_\varphi(x,\xi,\omega)(I-Q)g(u_\varphi(x,\xi,\omega),\nabla u_\varphi(x,\xi,\omega)) \\ &\le -2\underline{k}||\nabla(I-Q)u_\varphi(.,\xi,\omega)||_2^2 + c||(I-Q)u_\varphi(.,\xi,\omega)||_2 \end{aligned} \tag{3.31}$$

for some $c \in (0, \infty)$. For $\gamma, t \in (0, \infty)$ we now integrate inequality (3.31) over the interval $[t, t+\gamma]$; we then get $\mathbb{P}$-almost surely the estimate

$$\begin{aligned} &||(I-Q)u_{\varphi}(.,t+\gamma,\omega)||_2^2 - ||(I-Q)u_{\varphi}(.,t,\omega)||_2^2 \\ &+ 2\underline{k}\int_t^{t+\gamma} d\xi ||\nabla(I-Q)u_{\varphi}(.,\xi,\omega)||_2^2 \\ &\leq c\int_t^{t+\gamma} d\xi ||(I-Q)u_{\varphi}(.,\xi,\omega)||_2 \qquad (3.32) \end{aligned}$$

which in turn leads to the inequality

$$\begin{aligned} \int_t^{t+\gamma} d\xi ||\nabla(I-Q)u_{\varphi}(.,\xi,\omega)||_2^2 &\leq (2\underline{k})^{-1}||(I-Q)u_{\varphi}(.,t,\omega)||_2^2 \\ &+ (2\underline{k})^{-1}c\int_t^{t+\gamma} d\xi ||(I-Q)u_{\varphi}(.,\xi,\omega)||_2 \qquad (3.33) \end{aligned}$$

Owing now to the fact that the random process (2.4) is x-independent and that the operator $I - Q$ is an orthogonal projector in $L^2(\Omega)$, we have $\mathbb{P}$-almost surely

$$\begin{aligned} ||(I-Q)u_{\varphi}(.,t,\omega)||_2 &= ||(I-Q)(u_{\varphi}(.,t,\omega) - \hat{u}(t,\omega))||_2 \\ &\leq ||u_{\varphi}(.,t,\omega) - \hat{u}(t,\omega)||_2 \to 0 \qquad (3.34) \end{aligned}$$

as $t \to \infty$, by virtue of Theorem 2.1 for $p = 2$. Schwarz inequality and relations (3.33), (3.34) then imply that the estimates

$$\begin{aligned} 0 &\leq \limsup_{t\to\infty} \int_t^{t+\gamma} d\xi ||\nabla u_{\varphi}(.,\xi,\omega)||_2 \\ &\leq \sqrt{\gamma}\limsup_{t\to\infty}\left\{\int_t^{t+\gamma} d\xi ||\nabla u_{\varphi}(.,\xi,\omega)||_2^2\right\}^{1/2} \\ &= \sqrt{\gamma}\lim_{t\to\infty}\left\{\int_t^{t+\gamma} d\xi ||\nabla(I-Q)u_{\varphi}(.,\xi,\omega)||_2^2\right\}^{1/2} = 0 \qquad (3.35) \end{aligned}$$

hold $\mathbb{P}$-almost surely for every $\gamma \in (0, \infty)$. This and the *a priori* estimate (3.27) now imply relation (2.5) or (3.29). □

The passage from relation (2.5) to the first statement of Theorem 2.2 relies on hypothesis (LEO) in an essential way. We first show that the existence of the family of random linear evolution operators $(U(t, \tau, \omega))_{t\geq\tau}$ allows us to get the following integral representation for the projected random field $(I - Q)u_{\varphi}(., t, \omega)$.

LEMMA 3.6. – *For every* $\tau \in (0, \infty)$ *and every* $t \in [\tau, \infty)$ *we have* $\mathbb{P}$*-almost surely*

$$(I - Q)u_{\varphi}(., t, \omega) = U(t, \tau, \omega)(I - Q)u_{\varphi}(., \tau, \omega) + \int_{\tau}^{t} d\xi U(t, \xi, \omega)s(\xi, \omega)(I - Q)\Big(g(u_{\varphi}(., \xi, \omega), \nabla u_{\varphi}(., \xi, \omega)) - g(Qu_{\varphi}(., \xi, \omega), 0)\Big) \tag{3.36}$$

Proof. – Relation (3.36) is an immediate consequence of the variation of constants formula, together with the observation that the relations $(I - Q)U(t, \tau, \omega) = U(t, \tau, \omega)(I - Q)$ and $(I - Q)g(Qu_{\varphi}(., t, \omega), 0) = 0$ both hold $\mathbb{P}$-almost surely. □

We can now give the following

Proof of Theorem 2.2. – We begin by proving that $||\nabla u_{\varphi}(., t, \omega)||_p \to 0$ $\mathbb{P}$-almost surely as $t \to \infty$ for every $p \in [1, \infty)$. Since the *a priori* estimate (3.27) holds, it is sufficient to prove the result for $p = 1$. Our first objective is to prove that

$$\lim_{t \to \infty} ||(-\operatorname{div}(k_{u_{\varphi}}(., t, \omega)\nabla))^{1/2}(I - Q)u_{\varphi}(., t, \omega)||_2 = 0 \tag{3.37}$$

$\mathbb{P}$-almost surely. By using successively Lemma 3.6, hypothesis (LEO), the boundedness of the random process $(s(t, .))_{t \in \mathbb{R}}$ and a limited Taylor expansion for g we obtain $\mathbb{P}$-almost surely the estimates

$$||(-\operatorname{div}(k_{u_{\varphi}}(., t, \omega)\nabla))^{1/2}(I - Q)u_{\varphi}(., t, \omega)||_2 \leq c(T)\Big\{(t - \tau)^{-1/2}||(I - Q)u_{\varphi}(., \tau, \omega)||_2 + \int_{\tau}^{t} d\xi (t - \xi)^{-1/2}(||(I - Q)u_{\varphi}(., \xi, \omega)||_2 + ||\nabla u_{\varphi}(., \xi, \omega)||_2)\Big\} \tag{3.38}$$

for every $\tau, T \in (0, \infty)$, every $t \in (\tau, \tau + T]$ and some $c(T) \in (0, \infty)$. Now let $\beta \in (1, 2)$ and let $\beta^* \in (2, \infty)$ be the dual exponent. Clearly, the function $\xi \to (t - \xi)^{-\frac{\beta}{2}}$ is integrable on (τ, t) so that we can invoke Hölder's inequality to handle the second term in the last term of relation (3.38). We obtain $\mathbb{P}$-almost surely the inequality

$$\int_{\tau}^{t} d\xi (t - \xi)^{-1/2}(||(I - Q)u_{\varphi}(., \xi, \omega)||_2 + ||\nabla u_{\varphi}(., \xi, \omega)||_2) \leq c(\beta, T)\Bigg\{\left(\int_{\tau}^{t} d\xi ||(I - Q)u_{\varphi}(., \xi, \omega)||_2^{\beta^*}\right)^{1/\beta^*} + \left(\int_{\tau}^{t} d\xi ||\nabla u_{\varphi}(., \xi, \omega)||_2^{\beta^*}\right)^{1/\beta^*}\Bigg\} \tag{3.39}$$

for some $c(\beta, T) \in (0, \infty)$. The substitution of relation (3.39) into relation (3.38) then leads $\mathbb{P}$-almost surely to the estimate

$$\begin{aligned} &\|(-\operatorname{div}(k_{u_\varphi}(.,t,\omega)\nabla))^{1/2}(I-Q)u_\varphi(.,t,\omega)\|_2 \\ &\leq c(\beta,T)\left\{ (t-\tau)^{-1/2}\|(I-Q)u_\varphi(.,\tau,\omega)\|_2 \right. \\ &\quad + \left(\int_\tau^t d\xi \|(I-Q)u_\varphi(.,\xi,\omega)\|_2^{\beta^*}\right)^{1/\beta^*} \\ &\quad \left. + \left(\int_\tau^t d\xi \|\nabla u_\varphi(.,\xi,\omega)\|_2^{\beta^*}\right)^{1/\beta^*} \right\} \end{aligned} \tag{3.40}$$

for some $c(\beta, T) \in (0, \infty)$ and for every $\tau, T \in (0, \infty)$, $t \in (\tau, \tau + T]$. Since we eventually want to investigate relation (3.40) for t sufficiently large and since $\tau > 0$ is *a priori* arbitrary, we now consider estimate (3.40) for $\tau \in [t-2, t-1]$. Then estimate (3.40) implies $\mathbb{P}$-almost surely the inequality

$$\begin{aligned} &\|(-\operatorname{div}(k_{u_\varphi}(.,t,\omega)\nabla))^{1/2}(I-Q)u_\varphi(.,t,\omega)\|_2 \\ &\leq c(\beta,T)\left\{ \|(I-Q)u_\varphi(.,\tau,\omega)\|_2 \right. \\ &\quad + \left(\int_{t-2}^t d\xi \|(I-Q)u_\varphi(.,\xi,\omega)\|_2^{\beta^*}\right)^{1/\beta^*} \\ &\quad \left. + \left(\int_{t-2}^t d\xi \|\nabla u_\varphi(.,\xi,\omega)\|_2^{\beta^*}\right)^{1/\beta^*} \right\} \end{aligned} \tag{3.41}$$

for t large enough. Considering now inequality (3.41) as a function of τ and integrating both sides with respect to τ over the interval $[t-2, t-1]$, we obtain $\mathbb{P}$-almost surely the estimate

$$\begin{aligned} &\|(-\operatorname{div}(k_{u_\varphi}(.,t,\omega)\nabla))^{1/2}(I-Q)u_\varphi(.,t,\omega)\|_2 \\ &\leq c(\beta,T)\left\{ \int_{t-2}^{t-1} d\xi \|(I-Q)u_\varphi(.,\xi,\omega)\|_2 \right. \\ &\quad + \left(\int_{t-2}^t d\xi \|(I-Q)u_\varphi(.,\xi,\omega)\|_2^{\beta^*}\right)^{1/\beta^*} \\ &\quad \left. + \left(\int_{t-2}^t d\xi \|\nabla u_\varphi(.,\xi,\omega)\|_2^{\beta^*}\right)^{1/\beta^*} \right\} \end{aligned} \tag{3.42}$$

for t sufficiently large. Now by virtue of estimate (3.34), the first two terms of relation (3.42) go to zero $\mathbb{P}$-almost surely as $t \to \infty$, while the same conclusion holds for the third term as a consequence of Lemma 3.5 so that relation (3.37) holds. We now combine relation (3.37) with the ellipticity estimate of relation (1.2) and integration by parts to get $\mathbb{P}$-almost surely

$$\begin{aligned} ||\nabla(I-Q)u_\varphi(.,t,\omega)||_1^2 &\leq |\Omega|\,||\nabla(I-Q)u_\varphi(.,t,\omega)||_2^2 \\ &\leq ||(-\operatorname{div}(k_{u_\varphi}(.,t,\omega)\nabla))^{1/2}(I-Q)u_\varphi(.,t,\omega)||_2^2 \to 0 \end{aligned} \tag{3.43}$$

as $t \to \infty$. Equivalently, $||\nabla u_\varphi(.,t,\omega)||_1 \to 0$ and hence $||\nabla u_\varphi(.,t,\omega)||_p \to 0$ $\mathbb{P}$-almost surely for every $p \in [1,\infty)$ as $t \to \infty$ because of the *a priori* estimate (3.27). This and Theorem 2.1 now imply that relation (2.8) holds. As in the proof of Theorem 2.1, relation (2.10) then follows from dominated convergence while (2.9) and (2.11) follow from the existence of the continuous embedding $H^{1,p}(\Omega) \to C(\overline{\Omega})$ for p sufficiently large ([1]). □

The proofs of the remaining theorems of Section 2 rely on yet another version of the parabolic maximum principle. We begin with the proof of Theorem 2.3, for which we need two preparatory results. In the first one we prove structural inequalities for the nonlinearity g.

LEMMA 3.7. – *Assume that $g'(u_0,0) \neq 0$. Then for every constant $c \in (0, u_1 - u_0)$, there exist real constants $c_{1,2} \in \mathbb{R}$ such that the inequalities*

$$c_1 + G(y) \leq (g'(u_0,0))^{-1}\ln(y-u_0) \leq c_2 + G(y) \tag{3.44}$$

hold for every $y \in (u_0, u_1 - c)$. Similarly, assume that $g'(u_1,0) \neq 0$. Then for every constant $c \in (0, u_1 - u_0)$, there exist real constants $c_{3,4} \in \mathbb{R}$ such that the inequalities

$$c_3 + G(y) \leq (g'(u_1,0))^{-1}\ln(u_1-y) \leq c_4 + G(y) \tag{3.45}$$

hold for every $y \in (u_0 + c, u_1)$.

Proof. – Define the function $h_0 : (u_0, u_1) \to \mathbb{R}$ by

$$h_0(\xi) = (g(\xi,0))^{-1} - (g'(u_0,0)(\xi-u_0))^{-1} \tag{3.46}$$

It follows from the first part of hypothesis (G) and from the appropriate Taylor expansion around u_0 that h_0 can be continued to $[u_0, u_1)$ and is bounded on $[u_0, u_1 - c]$ for every $c \in (0, u_1 - u_0)$. Furthermore, from the definition of G and relation (3.46) we get

$$G(y) = \int_{\hat{\mu}}^{y} d\xi h_0(\xi) + (g'(u_0,0))^{-1}\ln(y-u_0) + 0(1) \tag{3.47}$$

for every fixed $\hat{\mu} \in (u_0, u_1)$ and every $y \in (u_0, u_1 - c)$. Now we infer from the boundedness of h_0 that $\int_{\hat{\mu}}^{y} d\xi |h_0(\xi)| = 0(1)$ for such $y's$. This remark together with relation (3.47) imply relation (3.44). We can prove inequalities (3.45) in a similar way by introducing the function

$$h_1(\xi) = (g(\xi, 0))^{-1} - (g'(u_1, 0)(\xi - u_1))^{-1}$$

for $\xi \in (u_0, u_1)$. □

In the following lemma, we establish a comparison between the random field u_φ and certain random processes of the form (2.4).

LEMMA 3.8. – *Given the random field u_φ, there exist two random processes $(\hat{u}_-(t, .))_{t\in\mathbb{R}}$ and $(\hat{u}_+(t, .))_{t\in\mathbb{R}}$ of the form* (2.4) *such that the inequalities*

$$\hat{u}_-(t, \omega) \leq u_\varphi(., t, \omega) \leq \hat{u}_+(t, \omega) \tag{3.48}$$

hold $\mathbb{P}$-almost surely for every $t \in \mathbb{R}^+$.

Proof. – Since $\overline{\Omega}$ is compact and since $\varphi(., \omega)$ is continuous $\mathbb{P}$-as., there exists a random variable c such that the inequalities $u_0 + c(\omega) \leq \varphi(x, \omega) \leq u_1 - c(\omega)$ hold for every $x \in \overline{\Omega}$. Now let $(\hat{u}_-(t, .))_{t\in\mathbb{R}}$ be the random process of the form (2.4) generated by the initial condition $u_0 + c(\omega)$; in a similar way, let $(\hat{u}_+(t, .))_{t\in\mathbb{R}}$ be the random process of the form (2.4) generated by the initial condition $u_1 - c(\omega)$. Since the two random processes $(\hat{u}_\pm(t, .))_{t\in\mathbb{R}}$ and the random field u_φ satisfy the same parabolic boundary-value problem, inequalities (3.48) follow from the parabolic maximum principle ([15]). □

Our proof of Theorem 2.3 now follows from Lemmata 3.7, 3.8 and the Birkhoff-Khintchin pointwise ergodic theorem.

Proof of Theorem 2.3. – We begin by proving statement (1). Since $<s> < 0$ implies that $\int_0^t d\xi s(\xi, \omega) \to -\infty$ $\mathbb{P}$-almost surely when $t \to \infty$, we may assume that $\hat{u}_\pm(t, \omega) \leq u_1 - c$ for some fixed $c \in (0, u_1 - u_0)$ and t sufficiently large. By using successively the second inequality in (3.44), the second inequality in (3.48) and relation (2.4) we then get

$$\begin{aligned}
&(g'(u_0, 0))^{-1}\ln(||u_\varphi(., t, \omega) - u_0||_\infty) \\
&\quad \leq (g'(u_0, 0))^{-1}\ln(\hat{u}_+(t, \omega) - u_0) \\
&\quad \leq c_2 + G(\hat{u}_+(t, \omega)) = c_2 + \int_0^t d\xi s(\xi, \omega) + G(u_1 - c(\omega))
\end{aligned}$$

or

$$t^{-1}\ln(||u_\varphi(., t, \omega) - u_0||_\infty) \leq g'(u_0, 0)t^{-1}\left(c_2 + \int_0^t d\xi s(\xi, \omega) + G(u_1 - c(\omega))\right) \tag{3.49}$$

$\mathbb{P}$-almost surely. Inequality (3.49) together with the Birkhoff-Khintchin pointwise ergodic theorem now imply that the estimate

$$\limsup_{t\to\infty} t^{-1}\ln(||u_\varphi(.,t,\omega)-u_0||_\infty) \leq <s>\, g'(u_0,0) \tag{3.50}$$

holds $\mathbb{P}$-almost surely. Now, by using successively the first inequality in (3.44), the first inequality in (3.48) and relation (2.4), we obtain the sequence of estimates

$$(g'(u_0,0))^{-1}\ln(||u_\varphi(.,t,\omega)-u_0||_\infty) \geq (g'(u_0,0))^{-1}\ln(\hat{u}_-(t,\omega)-u_0)$$
$$\geq c_1 + G(\hat{u}_-(t,\omega)) = c_1 + \int_0^t d\xi s(\xi,\omega) + G(u_0+c(\omega))$$

or

$$t^{-1}\ln(||u_\varphi(.,t,\omega)-u_0||_\infty) \geq g'(u_0,0)t^{-1}\left(c_1+\int_0^t d\xi s(\xi,\omega)+G(u_0+c(\omega))\right) \tag{3.51}$$

$\mathbb{P}$-almost surely. Inequality (3.51) along with the Birkhoff-Khintchin pointwise ergodic theorem once again imply that the estimate

$$<s>\, g'(u_0,0) \leq \liminf_{t\to\infty} t^{-1}\ln(||u_\varphi(.,t,\omega)-u_0||_\infty) \tag{3.52}$$

holds $\mathbb{P}$-almost surely. Relation (2.14) then follows from relations (3.50) and (3.52). If $<s>\,> 0$, a similar reasoning based on inequalities (3.45) and Lemma 3.8 leads $\mathbb{P}$-almost surely to the estimates

$$<s>\, g'(u_1,0) \leq \liminf_{t\to\infty} t^{-1}\ln(||u_\varphi(.,t,\omega)-u_1||_\infty)$$
$$\leq \limsup_{t\to\infty} t^{-1}\ln(||u_\varphi(.,t,\omega)-u_1||_\infty) \leq <s>\, g'(u_1,0) \tag{3.53}$$

which give relation (2.15). □

The proof of the corollary concerning the average times is now elementary.

Proof of Corollary 2.4. – Assume that $<s>\,< 0$, $g'(u_0,0) > 0$ and let $\varepsilon \in (0, |<s>|g'(u_0,0))$. Let $T^* \in (0,\infty)$, let $F_{\varepsilon,u_0}(\xi)$ be as in relation (2.18), let $\chi_{\varepsilon,u_0}(\xi)$ be the indicator function of $F_{\varepsilon,u_0}(\xi)$ and let $\delta > 0$. Since the $\mathbb{P}$-almost sure convergence of Theorem 2.3 implies the convergence in probability, there exists $t(\varepsilon,\delta) > 0$ such that the sequence of estimates

$$T^* \geq \mathbb{E}\left(\int_t^{t+T^*} d\xi\chi_{\varepsilon,u_0}(\xi,.)\right) = \int_t^{t+T^*} d\xi \int_X \mathbb{P}(d\omega)\chi_{\varepsilon,u_0}(\xi,\omega)$$
$$= \int_t^{t+T^*} d\xi\mathbb{P}\{\omega\in X : |\xi^{-1}\ln(||u_\varphi(.,\xi,\omega)-u_0||_\infty) - <s>\, g'(u_0,0)| < \varepsilon\}$$
$$\geq (1-\delta)T^*$$

holds for every $t \in (t(\varepsilon, \delta), \infty)$. In order words we have

$$(T^*)^{-1}\mathbb{E}\left(\int_t^{t+T^*} d\xi \chi_{\varepsilon, u_0}(\xi, .) - T^*\right) \in (-\delta, 0] \tag{3.54}$$

for every $t \in (t(\varepsilon, \delta), \infty)$, which proves the first statement of the lemma. The proof of the second statement is similar. □

Lemma 3.8 plays a fundamental role in the remaining part of this section as well. The basic strategy amounts to getting estimates for the probability of various events associated with the random processes $(\hat{u}_\pm(t, .))_{t\in\mathbb{R}}$; such estimates can be readily derived from the hypotheses concerning the random process $(s(t, .))_{t\in\mathbb{R}}$ through the explicit form (2.4). We can then transfer the corresponding information over to the random field u_φ through the comparison Lemma 3.8. We begin with the following

Proof of Theorem 2.5. – Owing to Lemma 3.8, we first notice that

$$\begin{aligned}&\mathbb{P}\{\omega \in X : \hat{u}_+(t, \omega) \leq a(t)\} \leq \mathbb{P}\{\omega \in X : u_\varphi(., t, \omega) < a(t)\}\\ &\leq \mathbb{P}\{\omega \in X : \hat{u}_-(t, \omega) \leq a(t)\}\end{aligned} \tag{3.55}$$

where $a : \mathbb{R}^+ \to (u_0, u_1)$ is the function that appears in the first statement of Theorem 2.5. It is therefore sufficient to estimate the long-time behavior of the probabilities $\mathbb{P}\{\omega \in X : \hat{u}_\pm(t, \omega) < a(t)\}$. Write momentarily $(\hat{u}(t, .))_{t\in\mathbb{R}} = (\hat{u}_\pm(t, .))_{t\in\mathbb{R}}$ for the two random processes of Lemma 3.8 and write $\hat{\varphi} = u_0 + c$, $\hat{\varphi} = u_1 - c$ for their initial non-random conditions. Refering back to the explicit form (2.4) and recalling that the function G is strictly monotone increasing, we obtain

$$\begin{aligned}&\mathbb{P}\{\omega \in X : \hat{u}(t, \omega) \leq a(t)\} = \mathbb{P}\{\omega \in X : G(\hat{u}(t, \omega)) \leq G(a(t))\}\\ &= \mathbb{P}\left\{\omega \in X : t^{-1/2}\int_0^t d\xi(s(\xi, \omega) - <s>)\right.\\ &\quad\left. \leq t^{-1/2}(G(a(t)) - <s> t) - t^{-1/2}G(\hat{\varphi})\right\}\end{aligned} \tag{3.56}$$

for every $t \in \mathbb{R}^+$. Now let a^* be as defined by relation (2.24) ; if $|a^*| < \infty$ then for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that the inequalities

$$\begin{aligned}&\mathbb{P}\left\{\omega \in X : t^{-1/2}\int_0^t d\xi(s(\xi. \omega) - <s>) < a^* - \varepsilon\right\}\\ &\leq \mathbb{P}\left\{\omega \in X : t^{-1/2}\int_0^t d\xi(s(\xi. \omega) - <s>)\right.\end{aligned}$$

$$\leq t^{-1/2}(G(a(t)) - \langle s \rangle\, t) - t^{-1/2}G(\hat{\varphi})\Big\}$$

$$\leq \mathbb{P}\left\{\omega \in X : t^{-1/2}\int_0^t d\xi(s(\xi,\omega) - \langle s \rangle) < a^* + \varepsilon\right\} \tag{3.57}$$

hold for every $t \in (t_\varepsilon, \infty)$. Invoking now hypothesis (CLS) and relations (3.56), (3.57) we obtain

$$(2\pi\sigma^*)^{-1/2}\int_{-\infty}^{a^*-\varepsilon} dx \exp\left[-\frac{x^2}{2\sigma^*}\right] \leq \liminf_{t\to\infty} \mathbb{P}\{\omega \in X : \hat{u}(t,\omega) < a(t)\}$$
$$\leq \limsup_{t\to\infty} \mathbb{P}\{\omega \in X : \hat{u}(t,\omega) < a(t)\}$$
$$\leq (2\pi\sigma^*)^{-1/2}\int_{-\infty}^{a^*+\varepsilon} dx \exp\left[-\frac{x^2}{2\sigma^*}\right] \tag{3.58}$$

for every $\varepsilon > 0$. By observing that a^* does not depend on the initial condition $\hat{\varphi}$ and by letting $\varepsilon \downarrow 0$ we get

$$\lim_{t\to\infty} \mathbb{P}\{\omega \in X : \hat{u}_\pm(t,\omega) < a(t)\} = (2\pi\sigma^*)^{-1/2}\int_{-\infty}^{a^*} dx \exp\left[-\frac{x^2}{2\sigma^*}\right] \tag{3.59}$$

Relations (3.55) and (3.59) then prove the first statement of the theorem when $|a^*| < \infty$. A slight variation of the above argument also shows that relation (3.59) holds when $a^* = \pm\infty$. Finally, we can prove the second statement in a similar way if we notice that

$$\mathbb{P}\{\omega \in X : b(t) < \hat{u}_-(t,\omega)\} \leq \mathbb{P}\{\omega \in X : b(t) < u_\varphi(.,t,\omega)\}$$
$$\leq \mathbb{P}\{\omega \in X : b(t) < \hat{u}_+(t,\omega)\} \tag{3.60}$$

where $b : \mathbb{R}^+ \to (u_0, u_1)$ is the function that appears in the second statement of Theorem 2.5. □

It is now easy to prove Theorem 2.6 by making suitable choices for the functions a and b of Theorem 2.5. For this we need the following

LEMMA 3.9. – *If $g'(u_0, 0) \neq 0$ we have*

$$G(u_0 + \varepsilon) = (g'(u_0, 0))^{-1}\ln(\varepsilon) + 0(1) \tag{3.61}$$

for every $\varepsilon > 0$ sufficiently small. Similarly, if $g'(u_1, 0) \neq 0$ we have

$$G(u_1 - \varepsilon) = (g'(u_1, 0))^{-1}\ln(\varepsilon) + 0(1) \tag{3.62}$$

for every $\varepsilon > 0$ sufficiently small.

Proof. – Choose $y = u_0 + \varepsilon$ in relation (3.44) and $y = u_1 - \varepsilon$ in relation (3.45). □

Then we have the following

Proof of Theorem 2.6. – For any constant $c \in (0, \infty)$ we choose $a(t) = u_0 + c\exp[-\Phi(t)]$. Since $\Phi(t) \to \infty$ when $t \to \infty$ we have $a(t) \in (u_0, u_1)$ for t sufficiently large and

$$\begin{aligned}&\mathbb{P}\{\omega \in X : u_\varphi(.,t,\omega) \leq a(t)\}\\ &\quad = \mathbb{P}\{\omega \in X : ||u_\varphi(.,t,\omega) - u_0||_\infty \leq c\exp[-\Phi(t)]\}\end{aligned} \tag{3.63}$$

Now for the above choice of a and because of relation (3.61) of Lemma 3.9 we have

$$\begin{aligned}a^* &= \lim_{t\to\infty} t^{-1/2}(G(u_0 + c\exp[-\Phi(t)]) - <s>t)\\ &= -\lim_{t\to\infty} t^{-1/2}\left(\frac{\Phi(t)}{g'(u_0,0)} + <s>t\right)\end{aligned} \tag{3.64}$$

Relations (3.63), (3.64) and the second statement of Theorem 2.5 then prove relation (2.29). We can prove the second statement of the theorem in a similar way by choosing $b(t) = u_1 - c\exp[-\Phi(t)]$ and by invoking relation (3.62) of Lemma 3.9. □

Having disposed of Theorem 2.6, we can now prove Corollary 2.7 by making very specific choices for the function Φ, which allow an explicit evaluation of the numbers a^* and b^* given by relations (2.28) and (2.30). For this we have to distinguish the case $<s>= 0$ from the case $<s>\neq 0$.

Proof of Corollary 2.7. – If $<s>< 0$ and $g'(u_0,0) > 0$, let $\varepsilon \in (0, |<s>|g'(u_0,0))$ and choose $\Phi(t) = -(<s> g'(u_0,0) + \varepsilon)t$ in Theorem 2.6. Then $a^* = +\infty$ from relation (2.28) so that

$$\mathbb{P}\{\omega \in X : ||u_\varphi(.,t,\omega) - u_0||_\infty \leq c\exp[(<s> g'(u_0,0) + \varepsilon)t]\} \to 1 \tag{3.65}$$

as $t \to \infty$ because of relation (2.29). But if $\Phi(t) = -(<s> g'(u_0,0) - \varepsilon)t$ then $a^* = -\infty$ so that

$$\mathbb{P}\{\omega \in X : c\exp[(<s> g'(u_0,0) - \varepsilon)t] \leq ||u_\varphi(.,t,\omega) - u_0||_\infty\} \to 1 \tag{3.66}$$

as $t \to \infty$ by switching to the complementary event. Relations (3.65) and (3.66) then immediately imply relation (2.32). In a similar way we can prove relation (2.33) when $<s>> 0$ and $g'(u_1,0) < 0$. Finally, if $<s>= 0$, $g'(u_0,0) > 0$, $g'(u_1,0) < 0$ and if $\Psi, \Psi^* : \mathbb{R}^+ \to \mathbb{R}^+$ are any two continuous functions such that $\Psi(t) \to \infty$, $\Psi^*(t) \to \infty$ and $\frac{\sqrt{t}}{\Psi(t)} \to \infty$ as $t \to \infty$, we choose $\Phi(t) = \frac{\sqrt{t}}{\Psi(t)}$ in relations (2.28) and (2.30). Then $a^* = b^* = 0$ so that the relations

$$\mathbb{P}\left\{\omega \in X : ||u_\varphi(.,t,\omega) - u_0||_\infty \leq c\exp\left[-\frac{\sqrt{t}}{\Psi(t)}\right]\right\} \to 1/2 \qquad (3.67)$$

and

$$\mathbb{P}\left\{\omega \in X : ||u_\varphi(.,t,\omega) - u_1||_\infty \leq c\exp\left[-\frac{\sqrt{t}}{\Psi(t)}\right]\right\} \to 1/2 \qquad (3.68)$$

hold simultaneously as $t \to \infty$. But if $\Phi(t) = \sqrt{t}\Psi^*(t)$ then $a^* = -\infty$ and $b^* = \infty$ so that

$$\mathbb{P}\{\omega \in X : c\exp[-\sqrt{t}\Psi^*(t)] \leq ||u_\varphi(.,t,\omega) - u_0||_\infty\} \to 1 \qquad (3.69)$$

and

$$\mathbb{P}\{\omega \in X : c\exp[-\sqrt{t}\Psi^*(t)] \leq ||u_\varphi(.,t,\omega) - u_1||_\infty\} \to 1 \qquad (3.70)$$

as $t \to \infty$, again by switching to complementary events. Relations (3.67) and (3.69) then imply relation (2.34), while relations (3.68) and (3.70) imply relation (2.35). □

We next observe that there is no need to prove Corollary 2.8 directly, for the preceding considerations and the above choices of Φ for the case $<s>= 0$ show that Corollary 2.8 is a simple consequence of Theorem 2.9. Therefore, we now turn to the proof of that theorem.

Proof of Theorem 2.9. – We begin by observing that

$$\mathbb{E}\left(\int_t^{t+T^*} d\xi \chi_{\Phi,u_0}(\xi,.)\right) = \int_t^{t+T^*} d\xi \int_X \mathbb{P}(d\omega)\chi_{\Phi,u_0}(\xi,\omega)$$

$$= \int_t^{t+T^*} d\xi \mathbb{P}\{\omega \in X : ||u_\varphi(.,\xi,\omega) - u_0||_\infty \leq c\exp[-\Phi(\xi)]\} \quad (3.71)$$

From relation (3.71) and the first statement of Theorem 2.6 we then infer that

$$\left|\mathbb{E}\left(\int_t^{t+T^*} d\xi \chi_{\Phi,u_0}(\xi,.)\right) - T^*(2\pi\sigma^*)^{-1/2}\int_{-\infty}^{a^*} dx \exp\left[-\frac{x^2}{2\sigma^*}\right]\right|$$

$$\leq \int_t^{t+T^*} d\xi \Big|\mathbb{P}\{\omega \in X : ||u_\varphi(.,\xi,\omega) - u_0||_\infty \leq c\exp[-\Phi(\xi)]\}$$

$$- (2\pi\sigma^*)^{-1/2}\int_{-\infty}^{a^*} dx \exp\left[-\frac{x^2}{2\sigma^2}\right]\Big| \to 0$$

as $t \to \infty$ for every $T^* \in (0,\infty)$. The proof of the second statement of Theorem 2.9 is similar. □

Finally, we devote the remaining part of this section to proving the results of Section 2 that are relative to the case where the random process $(s(t,.))_{t\in\mathbb{R}}$ satisfies hypothesis (NS). Lemma 3.8 plays an essential role here as well. We begin with the following.

Proof of Theorem 2.10. – As in the proof of Theorem 2.5, it is sufficient to estimate the long-time behavior of the probabilities $\mathbb{P}\{\omega \in X : \hat{u}_{\pm}(t,\omega) \leq a(t)\}$ where $\{\hat{u}_{\pm}(t,.)\}_{t\in\mathbb{R}}$ are the two random processes of Lemma 3.8, and where $a : \mathbb{R}^+ \to (u_0, u_1)$ is the function that appears in the first statement of Theorem 2.10. By using the same notation as in the proof of Theorem 2.5 and by noticing that the integrated process $t \to \int_0^t d\xi s(\xi,\cdot)$ is Gaussian as well, with average $<s>\, t$ and variance $\sigma(t)$ given by relation (2.47), we obtain

$$\begin{aligned}
&\mathbb{P}\{\omega \in X : \hat{u}(t,\omega) \leq a(t)\} = \mathbb{P}\{\omega \in X : G(\hat{u}(t,\omega)) \leq G(a(t))\} \\
&= \mathbb{P}\left\{\omega \in X : \int_0^t d\xi(s(\xi,\omega) - <s>) \leq G(a(t)) - <s>\, t - G(\hat{\varphi})\right\} \\
&= (2\pi\sigma(t))^{-1/2} \int_{-\infty}^{G(a(t)) - <s> t - G(\hat{\varphi})} dx \exp\left[-\frac{x^2}{2\sigma(t)}\right] \\
&= (2\pi)^{-1/2} \int_{-\infty}^{(\sigma(t))^{-1/2}(G(a(t)) - <s> t) - (\sigma(t))^{-1/2} G(\hat{\varphi})} dx \exp\left[-\frac{x^2}{2}\right] \qquad (3.72)
\end{aligned}$$

for every $t \in \mathbb{R}^+$. Invoking then relation (2.49) and the fact that $\sigma(t) \to \infty$ as $t \to \infty$ we obtain

$$\lim_{t\to\infty} \mathbb{P}\{\omega \in X : \hat{u}(t,\omega) \leq a(t)\} = (2\pi)^{-1/2} \int_{-\infty}^{a^*} dx \exp\left[-\frac{x^2}{2}\right] \qquad (3.73)$$

independently of the initial condition $\hat{\varphi}$. This and relation (3.55) then imply relation (2.50). We can carry out the proof of relation (2.52) in a similar way. □

It is now clear that Theorem 2.11 follows from Theorem 2.10 in exactly the same way as Theorem 2.6 follows from Theorem 2.5, and that Theorem 2.12 follows from Theorem 2.11 exactly as Theorem 2.9 follows from Theorem 2.6.

4. SOME CONCLUDING REMARKS

In this paper we have investigated the long-time behavior of random fields that are $\mathbb{P}$-almost surely classical solutions to quasilinear parabolic problems with random coefficients of the form (1.1) when hypotheses (K), (S) and (G) hold. We have obtained the most complete and precise results for the case where the lower-order coefficients $(s(t,.))_{t\in\mathbb{R}}$ are either stationary random processes whose statistics obey the central limit theorem, or stationary Gaussian processes such as the Ornstein-Uhlenbeck process. In both cases, we have shown that a $\mathbb{P}$-almost sure solution to (1.1) first homogeneizes over the region Ω to identify eventually with an x-independent random process $(\hat{u}(t,.))_{t\in\mathbb{R}}$ of the form (2.4), and then either converges to a spatially and temporally homogeneous asymptotic state or undergoes oscillations between two such asymptotic states. In both cases we have also determined the corresponding rates of stabilization along with the average times that the random fields spend in small neighborhoods of the asymptotic states. Related results hold in case $(s(t,.))_{t\in\mathbb{R}}$ is homogeneous multiplicative white noise, provided that Problem (1.1) be semilinear and that the nonlinearity g does not depend on ∇u. We develop and present these results in [8] and [9].

ACKNOWLEDGEMENTS

The final version of this paper was written while the first author was visiting the Institut de Mathématiques de Jussieu (Paris-VI and Paris-VII) and the Institut Elie Cartan (Nancy-I). He would like to thank the people at both institutions for their very kind hospitality. He also gratefully acknowledges the financial support of the CNRS and of the University Henri-Poincaré (Nancy-I). The research of the second author was supported in part by the Institute for Mathematics and its Applications in Minneapolis with funds from the National Science Foundation of the United States of America. He would like the thank Professor A. Friedman for his invitation.

REFERENCES

[1] R. A. Adams, "Sobolev Spaces", Academic Press, New York-London, 1975.

[2] D. G. Aronson and H. F. Weinberger, Nonlinear Dynamics in Population Genetics, Combustion and Nerve Pulse Propagation. *In* Partial Differential Equations and Related Topics, J. A. Goldstein, Ed., pp. 5-49, *Lecture Notes in Mathematics*, Vol. **446**, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[3] S. R. Bernfeld, Y. Y. Hu and P. Vuillermot, Homogénéisation Spatiale et Équivalence Asymptotique pour une Classe d'Équations Paraboliques Semilinéaires Non Autonomes, *C. R. Acad. Sci. Paris*, Vol. **320**, Série I, 1995, pp. 859-862.

[4] S. R. Bernfeld, Y. Y. Hu and P. Vuillermot, Large-Time Asymptotic Equivalence for a Class of Non-Autonomous Semilinear Parabolic Equations, *Bull. Sci. Math.*, 1998 (in press).

[5] H. Brézis, Analyse Fonctionnelle : Théorie et Applications, Masson, Paris-New York, 1983.

[6] P. Brunowski, P. Polacik and B. Santsede, Convergence in General Periodic Parabolic Equations in One-Space Dimension, *Nonlinear Analysis TMA*, Vol. **18**, No. 3, 1992, pp. 209-215.

[7] I. D. CHUESHOV and P. VUILLERMOT, On the Large-Time Dynamics of a Class of Random Parabolic Equations, *C. R. Acad. Sci. Paris*, Vol. **322**, Série I, 1996, pp. 1181-1186.

[8] I. D. CHUESHOV and P. VUILLERMOT, On the Large-Time Dynamics of a Class of Parabolic Equations Subjected to Homogeneous White Noise: Stratonovitch's Case, *C. R. Acad. Sci. Paris*, Vol. **323**, Série I, 1996, pp. 29-33.

[9] I. D. CHUESHOV and P. VUILLERMOT, Long-Time Behavior of Solutions to a Class of Stochastic Parabolic Equations with Homogeneous White Noise: Stratonovitch's Case, 1997 (preprint).

[10] I. P. CORNFELD, S. V. FOMIN and Ya. G. SINAI, Ergodic Theory, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[11] E. N. DANCER and P. HESS, Stable Subharmonic Solutions in Periodic Reaction-Diffusion Equations, *J. Differential Equations*, Vol. **108**, No 1, 1994, pp. 190-200.

[12] E. N. DANCER and P. HESS, Stability of Fixed Points for Order Preserving Discrete-Time Dynamical Systems, *J. Reine Angew. Math.*, Vol. **419**, 1991, pp. 125-139.

[13] D. DANERS and P. KOCH MEDINA, Abstract Evolution Equations, Periodic Problems and Applications, Pitman Research Notes in Mathematics Series, *Langman Sci. Techn.*, Harlaw, Vol. **279**, 1992.

[14] A. FIGOTIN and L. PASTUR, Spectra of Random and Almost-Periodic Schrödinger Operators, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[15] A. FRIEDMAN, Partial Differential Equations of Parabolic Type, Prentice Hall, Inc., Englewood Cliffs, N.J., 1964.

[16] P. HESS and H. F. WEINBERGER, Convergence to Spatial-Temporal Clines in the Fisher Equations with Time-Periodic Fitnesses, *J. Math. Biol.*, Vol. **28**, No. 1, 1990, pp. 83-98.

[17] P. HESS, Periodic Parabolic Boundary-Value Problems and Positivity, Pitman Research Notes in Mathematics Series, Vol. **247**, *Langman Sci. Tech.*, Harlaw, 1990.

[18] T. KATO, Abstract Evolution Equations of Parabolic Type in Banach and Hilbert Spaces, *Nagoya Math. Journal*, Vol. **19**, 1961, pp. 93-125.

[19] T. KATO Perturbation Theory for Linear Operators, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[20] O. A. LADYZENSKAYA, N. N. URALTCEVA and V. A. SOLONNIKOV, Linear and Quasilinear Equations of Parabolic Type, *Amer. Math. Soc. Transl. of Math. Monographs*, Vol. **23**, 1968.

[21] J. NEÇAS, Introduction to the Theory of Nonlinear Elliptic Equations, John Wiley and Sons, New York, 1986.

[22] A. PAZY, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, Berlin-Heidelberg-New York, 1983.

[23] B. SIMON, Functional Integration and Quantum Physics, Academic Press, New York-London, 1979.

[24] P. TAKAC, Linearly Stable Subharmonics in Strongly Monotone Time-Periodic Dynamical Systems, *Proc. Am. Math. Soc.*, Vol. **115**, No. 3, 1992, pp. 691-698.

[25] P. VUILLERMOT, Almost-Periodic Attractors for a Class of Non-Autonomous Reaction-Diffusion Equations on $\mathbb{R}^N$, I. Global Stabilization Processes, *J. Differential Equations*, Vol. **94**, No. 2, 1991, pp. 228-253.

[26] P. VUILLERMOT, Almost-Periodic Attractors for a Class of Non-Autonomous Reaction-Diffusion Equations on $\mathbb{R}^N$, II. Codimension-One Stable Manifolds, *Differential and Integral Equations*, Vol. **5**, No. 3, 1992, pp. 693-720.

[27] P. VUILLERMOT, Global Exponential Attractors for a Class of Almost-Periodic Parabolic Equations on $\mathbb{R}^N$, *Proc. Am. Math. Soc.*, Vol. **116**, No. 3, 1992, pp. 775-782.

[28] P. VUILLERMOT, Almost-Periodic Attractors for a Class of Non-Autonomous Reaction-Diffusion Equations on $\mathbb{R}^N$, III. Center Curves and Liapounov Stability, *Nonlinear Analysis TMA*, Vol. **22**, No. 5, 1994, pp. 533-559.

(Manuscript received January 22, 1996.)