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Regularity of solutions for arbitrary order variational inequalities with general convex sets

by

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ABSTRACT. – We consider the regularity of solutions of a system for nonlinear arbitrary order variational inequalities with some general concrete closed convex sets. The maximal function method and the convergence method are used, and a higher integrability of the derivatives is obtained.

RÉSUMÉ. – Nous étudions la régularité des solutions d'un système d'inégalités variationnelles non linéaires d'ordre arbitraire dans des ensembles convexes fermés généraux qui peuvent être représentés sous une forme concrête. On utilise la méthode de la fonction maximale et la méthode de convergence pour obtenir un résultat d'intégrabilité sur les dérivées d'ordre supérieur.

1. INTRODUCTION

Variational inequalities are used in theoretical studies for many free boundary problems. Such problems commonly occur in a variety of disciplines, e.g. elasticity, crystal growth etc.. There are already a lot of results about the existence and the regularity of solutions for second order variational inequalities (see [5], [15], [27] for instance). There are also

This work started when the author visited CMAF, University of Lisbon.

some papers studying the existence of solutions for higher order variational inequalities (e.g. [4]).

Some free boundary problems can be viewed as nonlinear higher order variational inequality problems in specific closed convex sets. A practical example of higher order variational inequalities can be found in [8], and some practical examples of some convex sets, different from the one of the obstacle problem, can be found, e.g., in elasto-plastic theory ([7], [16]), capillarity problems with prescribed volume ([12]), restricted mean curvature of horizontal plate problem ([8]) and others.

Among the different tools available to study the regularity of the solution for a nonlinear problem (see [11], [17] for instance), the so called maximal function method is frequently employed. It uses a quasiconformal map to gain additional regularity (see Gehring, [11]). More precisely, it states that, when a certain reversed maximal function inequality holds for a function in L^1, the function is, in fact, in $L^{1+\varepsilon}$ for some $\varepsilon > 0$. Thus, obtaining this additional regularity reduces the approach to proving a certain reversed maximal function inequality. This method was applied to the many second order nonlinear equation problem (e.g. see [6] and/or Chap. IV in Giaquinta's book [13], as well as references mentioned there). In [23], the method was employed for a higher order nonlinear equation system by Meyers and Elcrat (1975). In [14], Giaquinta and Modica adapted the method to a higher order equation system in a more general form and with a higher growth condition (1979). And in [19], Liang and Santos applied it to study the regularity of solutions of a higher order variational inequality system in a simple case with closed convex sets of obstacle type, i.e.

$$\mathbb{K} = \{\boldsymbol{v} \in [W_0^{M,p}(\Omega)]^L,\ \boldsymbol{v} \geq \boldsymbol{\Psi}\}.$$

The $\boldsymbol{W}^{M,p+\varepsilon}(\Omega)$ regularity of the solution is obtained, for some $\varepsilon > 0$ (1993).

The study of higher order variational inequalities can present closed convex sets of many different types, and a crucial difficulty for the maximal function method will be to find a test function which should be suitable to the maximal function method, and at the same time should be in the closed convex set of the problem. Extending the obstacle $\boldsymbol{v} \geq \boldsymbol{\Psi}$ to an operator inequality in $\mathbb{K}$ brings more troubles in the choice of the test function, which do not show up in equation or second order problems. As the general structures of closed convex sets may be very complicated, it is interesting to study a regularity result, in some more general structures of closed convex sets.

In this paper, we give the first results concerning general arbitrary order problems. The main results we have obtained here concern the $\boldsymbol{W}^{M,p+\varepsilon}$

regularity of solutions for a system of nonlinear arbitrary order variational inequalities in a class of general closed convex sets $\mathbb{K}$ of some concrete types. We will take

$$\mathbb{K} = \{\boldsymbol{v} \in [W^{M,p}(\Omega)]^L,\ T(\boldsymbol{v} - \boldsymbol{\Psi})(x) \in X \quad \text{plus boundary conditions}\},$$

and X is a closed convex set. The cases we consider are, roughly speaking, the following: T may be a differential, integral, or integrodifferential operator. In each of these situations, we need a structure assumption on X, which should contain either a sufficiently large ball, or a cone with vertex at the origin and nonempty interior. In all, there are six cases, which cannot be reduced to a single situation, and in each we obtain higher integrability of high order derivatives (Theorems 3.2, 3.8, 3.11, 3.12, 3.14, 3.16).

These results may be regarded as a considerable development of those presented in [19]. We also use the convergence of closed convex sets results obtained in [18] (see also [20]) to extend the regularity results to cover many nonsmooth cases (Theorem 4.10-4.14).

Since this represents the beginning of the research on the structure of closed convex sets for arbitrary order variational inequalities, there are still many open problems left to be examined, especially the generalization to convex sets $\mathbb{K}$ which do not fall within the cases studied here.

In detail, the problem considered in this paper is described as follows.

Let $\mathbb{B}$ be the vector-valued Banach space $\mathbb{B} = \boldsymbol{W}^{M,p}(\Omega) = [W^{M,p}(\Omega)]^L$, where $p > 1$, Ω is a bounded $C^{M-1,1}$ domain in $\mathbb{R}^N$, $M, N, L \in \mathbb{N}$. Consider the following problem

$$\boldsymbol{u} \in \mathbb{K}, \qquad \langle A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle \geq 0, \qquad \forall\, \boldsymbol{v} \in \mathbb{K}, \tag{1.1}$$

where $\mathbb{K}$ is a nonempty closed convex subset of $\mathbb{B}$ and

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \sum_{l=1}^{L} \sum_{|\gamma| \leq M} A_\gamma^l(x, D^M\boldsymbol{u})\partial_x^\gamma v_l\, dx, \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{B}, \tag{1.2}$$

denoting $\boldsymbol{u} = (u_1, \cdots, u_L)$, $D^M\boldsymbol{u} = \{\partial_x^\alpha u_l\}_{l=1,\ldots,L;|\alpha|\leq M} \in \prod_{|\alpha|\leq M} \mathbb{R}^L$, and $\partial_x^j \boldsymbol{u} = \left\{\frac{\partial^\alpha \boldsymbol{u}}{\partial x^\alpha}\right\}_{|\alpha|=j} \in \prod_{|\alpha|=j} \mathbb{R}^L$, $j = 1, \cdots, M$, where $\alpha = (\alpha_1, \ldots, \alpha_N)$, $|\alpha| = \sum_{i=1}^N \alpha_i$.

We assume that the operator A, with A_γ^l, $l = 1, \ldots, L$, $|\gamma| \leq M$, being Carathéodory functions, satisfies the following assumptions:

- **(A1)**

$$\sum_{l=1}^{L} \sum_{|\gamma|=M} A_\gamma^l(x, D^M\boldsymbol{u})\partial_x^\gamma u_l(x) \geq |\partial_x^M \boldsymbol{u}(x)|^p - a(x), \qquad \text{a.e. in } \Omega,$$

where $a \in L^{\kappa}(\Omega)$, $\kappa > 1$;

• **(A2)** For $|\gamma| = j \leq M$,

$$\sum_{|\gamma|=j} |A_\gamma(x, D^M \boldsymbol{u}(x))| \leq a_j(x)|\partial_x^M \boldsymbol{u}(x)|^{p-\tau_j} + b_j(x), \qquad \text{a.e. in } \Omega,$$

where $a_M \in L^\infty(\Omega)$; $b_j \in L^{\beta_j}(\Omega)$, $\beta_j > \max\left\{1, \frac{Np}{(p-1)N+(M-j)p}\right\}$; $a_j \in L^{\alpha_j}(\Omega)$, $\alpha_j > \max\left\{1, \frac{Np}{(\tau_j-1)N+(M-j)p}\right\}$; $1 \geq \tau_j \geq \begin{cases} \frac{N-(M-j)p}{N}, & \text{if } N > (M-j)p, \\ > 0, & \text{if } N = (M-j)p, \\ \text{no assumption otherwise} \end{cases} \geq 0$; $a_j, b_j \geq 0$; $j = 0, \ldots, M$.

The closed convex subset $\mathbb{K}$, in the variational inequality problem (1.1) considered in this paper, has the following structure

$$\mathbb{K} = \left\{ \boldsymbol{v} \in \mathbb{B} : \ T(\boldsymbol{v} - \boldsymbol{\Psi})(x) \in X, \quad \text{for a.e. } x \in \Omega, \right.$$
$$\left. \frac{d^i \boldsymbol{v}}{d\nu^i}\bigg|_{\partial\Omega} = 0, \quad i = 0, 1, \ldots, M-1 \right\}, \tag{1.3}$$

where ν is the unit outward normal vector at the boundary of Ω. The following conditions in the definition of $\mathbb{K}$ are also required:

• **(C1)** X is a closed convex set in $\boldsymbol{V}$ containing the origin at least, where $\boldsymbol{V}$ is a Banach space;

• **(C2)** $T : \mathbb{B} \to \boldsymbol{V}$ is a continuous linear map;

• **(C3)** $\boldsymbol{\Psi} \in \boldsymbol{W}_0^{M,p}(\Omega)$.

With **(C1)**-**(C3)**, it is also easy to verify that $\mathbb{K}$ is closed and convex. In addition, $\mathbb{K}$ is nonempty since $\boldsymbol{\Psi} \in \mathbb{K}$.

Remark 1.1. – If the boundary condition in the definition of $\mathbb{K}$ is not homogeneous, but still sufficiently smooth, by a translation argument we can reduce the problem to the homogeneous case (see also [19], Remark 4.6).□

We give now some examples of convex sets of (1.3) satisfying conditions **(C1)**-**(C3)**:

Example

• Ex. **a.** If $T = I$, where I is the identity operator, $X = \boldsymbol{V} = \mathbb{B}$, then $\mathbb{K} = \boldsymbol{W}_0^{M,p}(\Omega)$. This is the case of an equation system;

• Ex. **b.** The closed convex set for the obstacle problem is

$$\mathbb{K} = \left\{ \boldsymbol{v} \in \mathbb{B} : \quad \boldsymbol{v} \geq \boldsymbol{\Psi}, \ \text{ a.e. in } \Omega, \quad \frac{d^i \boldsymbol{v}}{d\nu^i}\bigg|_{\partial\Omega} = 0, \quad i = 0, 1, \ldots, M-1 \right\}.$$

Here, $T = I$, $\boldsymbol{V} = \mathbb{B}$ and $X = \{\boldsymbol{x} \in \mathbb{B}, \boldsymbol{x} \geq 0\}$. This is just the case discussed in [19];

• Ex. **c.** In [8], the closed convex set considered is $\mathbb{K} = \{v \in H^2(\Omega) : \alpha \leq \Delta v \leq \beta\}$.

More examples of the closed convex sets of the form of (1.3) can also be found in [18] and [20].

Remark 1.2. – The structures of the closed convex sets should be carefully defined. If the condition for the closed convex set $\mathbb{K}$ is too strong, $\mathbb{K}$ may reduce to $\{0\}$, so that, the problem has only the trivial solution 0 (see Remark 3.7 as well). □

The outline of the paper is as follows. In Section 2, some lemmas, which will be used later, are collected; In Section 3, we discuss the regularity of the solution of problem (1.1) by using the maximal function method. Here, the six different cases are discussed, for three different definitions of T's (differential operator, integral operator or integrodifferential operator) and for two different image sets (containing a big ball or a cone). The key in this section is to look for test functions which are in $\mathbb{K}$ and are suitable for the maximal function method. In Section 4, we discuss the convergence of the solutions with converging convex sets. Then applying these results and the results in [18], we extend the regularity results of the solutions of problem (1.1) of Section 3 to more general closed convex sets.

2. PRELIMINARIES

In this section we present some preliminary lemmas that will be used in the sequel.

First of all, we introduce a symbol "$\lesssim$". Let $a(r)$ be a nonnegative function of r, if

$$a(r) \lesssim c(r),$$

then, $a(r) \leq c(r)$, and $\exists \bar{k} > 0$, $\exists r_0 > 0$ such that $\forall r < r_0$, $a(r) \geq \bar{k} c(r)$.

We denote by $B_{x_0,r}$ the open ball of radius r and center x_0 in any space $\mathbb{R}^\Theta$: the appropriate dimension, either $\Theta = N$, or $\Theta = L$, will always be clear from the context.

LEMMA 2.1. – *Assume that*

$1°$ $1 < p, q < \infty$ *and* $1/p \geq 1/q - 1/N$, *or* $1 \leq p, q \leq \infty$ *and* $1/p > 1/q - 1/N$;

2° $u \in W^{1,q}(\omega)$ *and* $\int_\omega u dx = 0$, *where* $\omega \subseteq B_{x_0,r}$ $(r > 0)$ *has a cone property,* $\text{meas}(\omega) = \bar{c}r^n$, *for some* $\bar{c} > 0$ *independent of* r.
Then,

$$\|u\|_{L^p(\omega)} \lesssim Cr^{N(1/p-1/q)+1}\|grad\ u\|_{L^q(\omega)},$$

where C *depends on* N, p, q, $\bar{c}$ *and the cone condition of* ω, *but does not depend on* r.

Proof. – It is similar to the one of Proposition 1 in [23] when $\omega = B_{x_0,r}$. Notice the proof there, which is also true for $1/p > 1/q - 1/N$, when $p = \infty$, or $q = 1$. □

LEMMA 2.2. – *Assume that*
1° $1 < p, q < \infty$ *and* $1/p \geq 1/q - 1/N$, *or* $1 \leq p, q \leq \infty$ *and* $1/p > 1/q - 1/N$;
2° $u \in W^{1,q}(\omega)$, $u|_\Gamma = 0$, *where* $\omega \subseteq B_{x_0,r}$ $(r > 0)$ *has a cone property,* $\text{meas}_N(\omega) = \bar{c}r^n$, $\Gamma \subset \partial\omega$, $\text{meas}_{N-1}(\Gamma) \geq \bar{c}' r^{N-1}$, *where* $\text{meas}_\Theta(G)$ *means the* Θ*-dimensional Hausdorff measure of* G, $\bar{c}$ *and* $\bar{c}'$ *are positive constants which are independent of* r.
Then,

$$\|u\|_{L^p(\omega)} \lesssim Cr^{N(1/p-1/q)+1}\|\text{grad}\ u\|_{L^q(\omega)},$$

where C *depends on* N, p, q, $\bar{c}$, $\bar{c}'$, *and the cone condition of* ω, *but does not depend on* r.

Proof. – The proof is similar to the one of Lemma 2.1. The argument on the boundary, using its smoothness, can be found in [10]. □

LEMMA 2.3. – *Let* $u \in W^{M,p}(\omega)$, *where* $\omega \subseteq B_{x_0,r}$ $(r > 0)$ *has a cone property,* $\text{meas}(\omega) = \bar{c}r^N$, *where* $\bar{c} > 0$ *is independent of* r, $0 < j \leq M-1$, *then*

$$r^j\left(\int_\omega |\partial_x^j u|^p dx\right)^{1/p} \lesssim C\left[\varepsilon^{1-M/j} r^M \left(\int_\omega |\partial_x^M u|^p dx\right)^{1/p} + \varepsilon\left(\int_\omega |u|^p dx\right)^{1/p}\right], \quad (2.1)$$

for all constant $\varepsilon > 0$, *where* C *depends on* M, N, p, $\bar{c}$ *and the cone condition of* ω, *but does not depend on* r.

Proof. – The proof is similar to the one of Lemma 2.3 in [19], noticing that ϵ_0 there can be any positive constant (see [1] 4.17). □

LEMMA 2.4. – *If* X *is a closed convex set of* **V** *containing a cone* **C** *vertexed at the origin with interior* $\overset{o}{\mathbf{C}} \neq \emptyset$, *then for any* $\boldsymbol{\omega} \in \overset{o}{\mathbf{C}}$, *there* *exists* $\theta_0 > 0$ *such that for any* $0 \leq \theta \leq \theta_0$, *and any* z *with* $\|z\|_V \leq 1$, $\omega + \theta z \in \overset{o}{\mathbf{C}}$. *Moreover, for any* $0 \leq \theta \leq \theta_0$ *and any* $v \in X$, $v + \omega + \theta z \in X$, *i.e.,* $v + \omega \in \overset{o}{X}$.

Proof. – (See also [20]). Since X is a closed convex set of V, from $\mathbf{C} \subset X$, for any $\omega \in \overset{o}{\mathbf{C}}$, there exists a $\theta_0 > 0$, such that $\forall\theta,\ 0 \leq \theta \leq \theta_0$, and $\forall z,\ \|z\|_V \leq 1$, we have $\omega + \theta z \in \overset{o}{\mathbf{C}}$.

As $\mathbf{C}$ is a cone vertexed at the origin, $a(\omega + \theta z) \in \mathbf{C}$ for all $a > 0$. Then, from X being a convex set and $\mathbf{C} \subset X$, we have

$$\lambda v + (1 - \lambda)a(\omega + \theta z) \in X, \quad \forall\lambda \in [0,1),\ \forall v \in X.$$

Letting $a = \frac{1}{1-\lambda}$, it is concluded that

$$\lambda v + (\omega + \theta z) \in X, \quad \forall\lambda \in [0,1).$$

And because X is closed, λ going to 1 yields

$$v + \omega + \theta z \in X, \quad \forall\theta,\ 0 \leq \theta \leq \theta_0,$$

that is, $v + \omega \in \overset{o}{X}$. □

LEMMA 2.5. – *Suppose that* $\zeta_{x_0,r}(x)$ *is a cut off function of* $B_{x_0,2r}$ *(i.e.* $\zeta_{x_0,r} \in C_0^\infty(B_{x_0,2r})$, $0 \leq \zeta_{x_0,r} \leq 1$, $\zeta_{x_0,r} \equiv 1$ *on* $B_{x_0,r}$). *Suppose also that*

$$F = \sum_{|\alpha| \leq k} d_\alpha(x)\partial_x^\alpha \tag{2.2}$$

is a differential operator with $d_\alpha \in C^{|\alpha|}(\mathbb{R}^N)$ *and* d_0 *so large that*

$$d_0(x) + \Big(\sum_{0<|\alpha|\leq k} (-1)^\alpha \partial_x^\alpha d_\alpha(x)\Big) \geq 0, \qquad a.e.\ , \tag{2.3}$$

then

$$\int_{B_{x_0,2r}} F\zeta_{x_0,r}(x)dx \geq 0. \tag{2.4}$$

Moreover, if K *is a bounded set in* $\mathbb{R}^N$ *with* $C^{M-1,1}$ *boundary,* $x_0 \in \partial K$, *and* $d_\alpha(x) \in C_0^{|\alpha|}(K)$, *then,*

$$\int_{K \cap B_{x_0,2r}} F\zeta_{x_0,r}(x)dx \geq 0. \tag{2.5}$$

Proof

$$\int_{B_{x_0,2r}} F\zeta_{x_0,r}dx = \int_{B_{x_0,2r}} \sum_{|\alpha|\le k} d_\alpha(x)\partial_x^\alpha \zeta_{x_0,r}(x)dx$$
$$= \int_{B_{x_0,2r}} (\sum_{|\alpha|\le k} (-1)^\alpha \partial_x^\alpha d_\alpha(x))\zeta_{x_0,r}(x)dx \ge 0,$$

by (2.3).

If $x_0 \in \partial K$, because $d_\alpha(x) \in C_0^{|\alpha|}(K)$, the integrals by parts as above formula are still available in $K \cap B_{x_0,2r}$, the argument is similar. □

LEMMA 2.6. – *Assume that* $\boldsymbol{u} \in \boldsymbol{W}^{M,p}(\omega)$, *where* $\omega \subseteq B_{x_0,r}$ $(r > 0)$ *has a uniform cone property,* $x_0 \in \omega$, meas$(\omega) = \bar{c}r^N$, *with* $\bar{c} > 0$ *independent of* r. *There exists a unique polynomial* $\boldsymbol{P}(z)$, *of degree* $\le M - 1$, *such that*

$$\int_\omega \partial_z^\mu \boldsymbol{P} dz = \int_\omega \partial_z^\mu \boldsymbol{u} dz, \qquad \forall\ |\mu| \le M-1,$$

$$\boldsymbol{P}(z) = \sum_{|\mu|\le M-1} \boldsymbol{c}_\mu (z - x_0)^\mu, \tag{2.6}$$

where

$$\boldsymbol{c}_\mu = \sum_{|\alpha|\le \frac{M-1-|\mu|}{2}} c_{\mu,\alpha} r^{-N+2|\alpha|} \int_\omega \partial_z^{\mu+2\alpha} \boldsymbol{u} dz, \tag{2.7}$$

and the $c_{\mu,\alpha}$ *depend only on* M, N, α, μ *and* $\bar{c}$.

Moreover, when $0 < |\alpha| \le k$, $M - k - N/p > 0$, *for any given* $\lambda_0 > 0$, *there exists* $r_0 > 0$, *such that if* $0 < r < r_0$, *the differential* $\partial_x^\alpha \boldsymbol{P}$ *has the estimate*

$$|\partial_x^\alpha \boldsymbol{P}| \le \lambda_0, \qquad \text{on } B_{x_0,2r}, \tag{2.8}$$

where r_0 *depends on the* $\|\boldsymbol{u}\|_{\boldsymbol{W}^{M,p}(\Omega)}$, M, N, L, k, p, $\bar{c}$ *and* λ_0, *but not on* r.

Proof. – A special case of the first part of this lemma, $\omega = B_{x_0,r}$, can be found in [23] and [19]. The proof of the first part of this lemma is similar to the one of that special case.

Now we only need to prove (2.8), in fact, by Lemma 2.3 and imbedding theorem, for any $0 < r < r_0$, $0 < |\alpha| \le k$ and given $\lambda_0 > 0$, when

$x \in B_{x_0,2r}$, we have

$$
\begin{aligned}
|\partial_x^\alpha \boldsymbol{P}(x)| &= r^{-N}\Big|\sum_{\mu\ge\alpha,|\mu|+|\gamma|\le M-1} c_{\mu,\gamma}\frac{\mu!}{\alpha!}r^{|\gamma|}(x-x_0)^{\mu-\alpha}\int_\omega \partial_x^{\mu+\gamma}\boldsymbol{u}dx\Big| \\
&\le C\sum_{\mu\ge\alpha,|\mu|+|\gamma|\le M-1} r^{|\mu-\alpha|+|\gamma|-N}\int_\omega |\partial_x^{\mu+\gamma}\boldsymbol{u}|dx \\
&\le \frac{\epsilon}{\bar{c}r^N}\int_\omega |\boldsymbol{u}|dx + C_\epsilon r^{M-k-N}\int_\omega |\partial_x^M \boldsymbol{u}|dx \\
&\le \epsilon\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(B_{x_0,r})} + C_{p,\epsilon}r^{M-k-N/p}\Big(\int_\omega |\partial_x^M\boldsymbol{u}|^p dx\Big)^{1/p} \\
&\le (\epsilon + C_{p,\epsilon}r^{M-k-N/p})\|\boldsymbol{u}\|_{\boldsymbol{W}^{M,p}(\Omega)} \le \lambda_0,
\end{aligned}
$$

if only taking

$$
0<\epsilon\le\frac{\lambda_0}{2\|\boldsymbol{u}\|_{\boldsymbol{W}^{M,p}(\Omega)}},\qquad \text{then } 0<r_0\le\left(\frac{\lambda_0}{2C_{p,\epsilon}\|\boldsymbol{u}\|_{\boldsymbol{W}^{M,p}(\Omega)}}\right)^{\frac{1}{M-k-N/p}}, \tag{2.9}
$$

which can be done as $M-k-N/p>0$. □

This lemma will be frequently used for the estimates in the sequel. With this polynomial, from Lemma 2.1 and $\partial_x^M \boldsymbol{P}=0$, we have

$$
\Big(\int_\omega |\partial_x^\alpha(\boldsymbol{u}-\boldsymbol{P})|^p dx\Big)^{1/p} \lesssim Cr^{M-|\alpha|}\Big(\int_\omega |\partial_x^M \boldsymbol{u}|^p dx\Big)^{1/p}, \tag{2.10}
$$

where $0\le|\alpha|\le M$, $p>1$, C is independent of r.

LEMMA 2.7. – *Assume that h and g are non-negative measurable functions on $\mathbb{R}^N$ such that*

$$
M(g^\nu)\le b(M^\nu(g)+M(h^\nu)) \qquad a.e. \text{ in } Q,
$$

where $g\in L^\nu(Q)$, $h\in L^s(Q)$ for some $s>\nu>1$; $b>1$ and Q is a compact cube, $M(g):\mathbb{R}^N\to[0,\infty]$ is the maximal function for g, defined as follows:

$$
M(g)(x)=\sup_{R>0}\left\{\frac{1}{\text{meas}(B_{x,R})}\int_{B_{x,R}} g(s)\,ds\right\}. \tag{2.11}
$$

Then, there exists $\mu>\nu$ such that

$$
\frac{1}{\text{meas}(Q)}\int_Q g^\mu dx \le C\Big[\Big(\frac{1}{\text{meas}(Q)}\int_Q g^\nu dx\Big)^{\mu/\nu}+\frac{1}{\text{meas}(Q)}\int_Q h^\mu dx\Big],
$$

where C depends only on ν, μ and b.

Proof. – The proof of this lemma can be found in [11] (see also, e.g., in [6], [14], [23]). □

The following theorem establishes sufficient conditions for existence of solutions of the problem (1.1). The theorem requires more assumptions for the operator A:

- **(A3)**

$$\sum_{l=1}^{L}\sum_{|\gamma|=M}[A_\gamma^l(x,D^{M-1}\boldsymbol{u},\boldsymbol{\xi})-A_\gamma^l(x,D^{M-1}\boldsymbol{u},\boldsymbol{\eta})][\xi_l^\gamma-\eta_l^\gamma]>0,\quad \text{a.e. in } \Omega,$$

for all $\boldsymbol{\xi},\boldsymbol{\eta}\in\prod_{|\gamma|=M}\mathbb{R}^L$.

- **(A4)**

$$\max\left\{\tfrac{j}{M},\tfrac{N-(M-j)p}{N}\right\}<\tau_j\le 1,\quad j=0,1,\ldots,M-1; \tag{2.12}$$

$$0<\tau_0-\max\left\{0,\tfrac{N-Mp}{N}\right\}\le\min_{j\ge 1}\left\{\tau_j-\max\left\{\tfrac{j}{M},\tfrac{N-(M-j)p}{N}\right\}\right\}; \tag{2.13}$$

and

$$\sum_{l=1}^{M}A_0^l(x,D^M\boldsymbol{u})u_l(x)\ge c_0|\boldsymbol{u}|^{p/\tau_0}-\frac{1}{2}|\partial_x^M\boldsymbol{u}(x)|^p-C_0,\qquad \text{a.e. in } \Omega,$$

where $c_0\ge M\overline{C}+1$, being $\overline{C}$ and C_0 sufficiently large positive constants depending only on L, M, N, p, $\|a_j\|_{\alpha_j}$, $\|b_j\|_{\beta_j}$, τ_j, $(j=0,1,\ldots,M-1)$ and Ω.

Remark 2.8. – If τ_j, $j=0,1,\ldots,M-1$ satisfy **(A4)**, then it is not difficult to verify that they satisfy **(A2)**. A special case is that they satisfy the natural condition, $\tau_j=1$, $j=0,1,\cdots,M-1$. □

THEOREM 2.9. – *If the operator A verifies assumptions* (**A1**)-(**A4**), *then problem* (1.1) *admits at least a solution belonging to* $\mathbb{B}$.

Proof. – (See also [4]). From (**A1**)-(**A3**), we know that the operator A is pseudo-monotone in a reflexive Banach space by [21] (the definition of the pseudo-monotone can be found in Definition 4.1 later). Review the proof of Theorem 3.1 in [19] (see also Remark 3.2 in [19]), the test function $\boldsymbol{v}=\boldsymbol{\Psi}$ used in [19] still belongs to $\mathbb{K}$ defined in (1.3). With (**A1**), (**A2**) and (**A4**), we can obtain the *a priori* estimate. Then the existence result is a direct consequence of a result of [21] (p245 Theorem 8.1).

In fact, from **(A2)**, **(A4)** and Hölder inequality, we have,

$$\begin{aligned}
&\sum_{|\gamma|=j}\int_\Omega |A_\gamma(x,D^M\boldsymbol{u})||\partial_x^j\boldsymbol{u}|dx \\
&\le \int_\Omega |a_j||\partial_x^M\boldsymbol{u}|^{p-\tau_j}|\partial_x^j\boldsymbol{u}|dx+\int_\Omega |b_j||\partial_x^j\boldsymbol{u}|dx \\
&\le C[\|a_j\|_{\alpha_j}\|\partial_x^M\boldsymbol{u}\|_p^{p-\tau_j}\|\partial_x^j\boldsymbol{u}\|_{p/\tau_j}+\|b_j\|_{\beta_j}\|\partial_x^j\boldsymbol{u}\|_{\beta_j'}] \\
&\le \frac{1}{8M}\int_\Omega |\partial_x^M\boldsymbol{u}|^p dx + C[\|\partial_x^M\boldsymbol{u}\|_p^{(1-\theta)p/\tau_j}\|\boldsymbol{u}\|_{p/\tau_0}^{\theta p/\tau_j}+1] \\
&\le \frac{1}{4M}\int_\Omega |\partial_x^M\boldsymbol{u}|^p dx + \overline{C}\int_\Omega |\boldsymbol{u}|^{p/\tau_0}dx + C.
\end{aligned}$$

In the above estimate, we have used Nirenberg-Gagliardo inequality

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{j,q}(\Omega)} \le C\|\boldsymbol{u}\|^{1-\theta}_{\boldsymbol{W}^{M,p}(\Omega)}\|\boldsymbol{u}\|^{\theta}_{\boldsymbol{L}^r(\Omega)}, \tag{2.14}$$

for

$$\theta\in\left[0,1-\tfrac{j}{M}\right] \quad\text{and}\quad j-\tfrac{N}{q}\le(1-\theta)\left(M-\tfrac{N}{p}\right)-\theta\tfrac{N}{r}, \tag{2.15}$$

where $1\le p,q,r\le\infty$ (see [22], p69 and [26]).

And for our case, we take $q=p/\tau_j$, $r=p/\tau_0$, $1-\theta=j/M$. With (2.12), (2.13), it is not difficult to verify that they satisfy (2.15).

So that, from (2.14), as $\tau_j>1-\theta$, we have

$$C\|\partial_x^j\boldsymbol{u}\|_{\frac{p}{\tau_j}}^{\frac{p}{\tau_j}} \le C\|\partial_x^M\boldsymbol{u}\|_p^{(1-\theta)\frac{p}{\tau_j}}\|\boldsymbol{u}\|_{\frac{p}{\tau_0}}^{\theta\frac{p}{\tau_j}} \le \frac{1}{8M}\int_\Omega|\partial_x^M\boldsymbol{u}|^p dx+\overline{C}\|\boldsymbol{u}\|_{\frac{p}{\tau_0}}^{\theta\frac{\tau_j}{\tau_j-(1-\theta)}\frac{p}{\tau_j}}$$

and $\theta\,\frac{\tau_j}{\tau_j-(1-\theta)}\,\frac{p}{\tau_j}=\frac{\theta p}{\tau_j-(1-\theta)}=\frac{(M-j)p}{M\tau_j-j}\le\frac{p}{\tau_0}$.

In fact, from (2.12), (2.13), if $\frac{N-(M-j)p}{N}\le\frac{j}{M}$, $\frac{(M-j)p}{M\tau_j-j}\le\frac{(M-j)p}{M\tau_0}\le\frac{p}{\tau_0}$; if $\frac{N-(M-j)p}{N}>\frac{j}{M}$, $\frac{N-Mp}{N}>0$, then $\frac{(M-j)p}{M\tau_j-j}\le\frac{(M-j)p}{M(\tau_0-\frac{N-Mp}{N}+\frac{N-(M-j)p}{N})-j}\le\frac{(M-j)p}{M(\tau_0-\frac{N-Mp}{N}+\frac{(M-j)(N-Mp)}{MN})}\le\frac{p}{\tau_0}$.

Then, using **(A1)** and **(A4)** the *a priori* estimate can be obtained as

$$\frac{1}{2}\int_\Omega|\partial_x^M\boldsymbol{u}|^p dx+(c_0-M\overline{C})\int_\Omega|\boldsymbol{u}|^{p/\tau_0}dx\le C, \tag{2.16}$$

where C depends only on the given data. □

Remark 2.10. – It can be seen that if the operator A satisfies the following monotonicity condition as well,

$$\langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle > 0, \qquad a.e.\ in\ \Omega, \quad \forall\, \boldsymbol{u} \not\equiv \boldsymbol{v},$$

then the solution of problem (1.1) is unique. □

Remark 2.11. – The closed convex set $\mathbb{K} \subset \mathbb{B}$ in Theorem 2.9 can, in fact, be more general than the one defined in (1.3) satisfying the conditions **(C1)**-**(C3)**, since we only used that $\mathbb{K}$ is closed, convex and nonempty. □

3. THE REGULARITY BY MAXIMAL FUNCTION METHOD

In this section we want to establish additional regularity for the solution of problem (1.1) by maximal function method. This is more delicate than proving existence.

Throughout, we denote

$$U(x) = |\partial_x^M \boldsymbol{u}(x)|, \tag{3.1}$$

for $x \in \Omega$, where $\boldsymbol{u}$ is the solution of problem (1.1). We intend to use the maximal function method, already used in [23], [14] and [19] for higher order problems. For this purpose, we need to choose carefully a suitable test function in $\mathbb{K}$. More specifically, we are looking for a test function $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\phi}$ (corresponding a test function $\boldsymbol{\phi}$ in equation) verifying the following requests:

1. $\boldsymbol{\phi} \in \boldsymbol{W}_0^{M,p}(B_{x_0,\tilde{c}r} \cap \Omega)$, for any $0 < r < r_0$, where r_0, $\tilde{c}$ are positive constants independent of r;
2. $\sum\limits_{|\gamma|=M} A_\gamma(x, D^M\boldsymbol{u})\partial_x^M \boldsymbol{\phi} \geq \epsilon U^p - h(x, D^{M-1}\boldsymbol{u})$, for $\epsilon > 0$, where h is a function independent of U;
3. $r^{-N/\gamma}\left(\int_{B_{x_0,r}} |\partial_x^\alpha \boldsymbol{\phi}|^\gamma dx\right)^{1/\gamma} \leq C\left[r^{-N/q}\left(\int_{B_{x_0,r}} U^q dx\right)^{1/q} + M(|\boldsymbol{h}|) + 1\right]$, for $|\alpha| \leq M$ and $1 \leq q < p$, where $\boldsymbol{h}$ is a known function in $\boldsymbol{L}^s(\Omega)$ for some $s > 1$, C is a constant independent of r, $\gamma > 1$ is some positive constant depending on α (see [23], [19]);
4. $\boldsymbol{v} \in \mathbb{K}$.

Here, request 1. is set because we study the regularity, which is a local property of the solution; request 2. is for obtaining an estimate, with which we can have the desired positive term in the left hand side of our estimate inequality; request 3. is required to use maximal function method (it implies that the average integral of the differentials of ϕ in B_r should be independent of r except the one which can be bounded by some maximal functions); request 4. is an imposition of the variational inequality.

To meet request 1., it is natural to take some cut off function; it needs to be treated carefully, since each differentiation will produce a r^{-1}. To satisfy request 2., $\boldsymbol{u}$ must appear somewhere in ϕ, (e.g. $\phi = \zeta(-\boldsymbol{u} + Cr^M)$, where ζ is a cut off function, C is a constant). To verify request 3., a function $\boldsymbol{P}$, which is the polynomial defined in Lemma 2.6 (see also [14], [19] and [23]), is introduced. This idea comes from the study of the regularity of second order equation, when $\boldsymbol{u} - \frac{1}{|B_r|}\int_{B_r} \boldsymbol{u}dx$, multiplied by some cut off function ζ, is chosen to be a test function (see [13], Chap. IV); the function $\boldsymbol{u} - \frac{1}{|B_r|}\int_{B_r} \boldsymbol{u}dx$ lets the raise of the differentiation under an integral in B_r become possible to kill the r^{-1} produced by the first differentials of ζ. In our higher order case, corresponding to $\frac{1}{|B_r|}\int_{B_r} \boldsymbol{u}dx$, we have the polynomial $\boldsymbol{P}$. In the equation case, only requests 1.- 3. are needed. Therefore, the main work in this section becomes to verify request 4.

However, in many cases, the general test functions used for equation system will be failed for variational inequality system as they are no longer in the given closed convex set. So we have to find new test functions for our cases studied here. And because of the operators of $\mathbb{K}$, we must overcome particular difficulties which we have not met in the secound order problem.

As mentioned in Remark 1.2, not every case of $\mathbb{K}$ defined in (1.3) is interesting and feasible. In this section, we study six particular cases of three types of T and two types of X for closed convex set $\mathbb{K}$.

For simplicity, we consider the case $\boldsymbol{\Psi} = 0$, noticing that the general case can be treated similarly, replacing $\boldsymbol{u}$ by $\boldsymbol{u} - \boldsymbol{\Psi}$, $\boldsymbol{P}$ by $\boldsymbol{P}_\Psi$, where $\boldsymbol{P}_\Psi$ is the unique polynomial of degree $\leq M - 1$ satisfying

$$\int_{B_{x_0,r}} \partial_x^\mu \boldsymbol{P}_\Psi dx = \int_{B_{x_0,r}} \partial_x^\mu (\boldsymbol{u} - \boldsymbol{\Psi})dx$$

for all $|\mu| \leq M - 1$.

3.1. T is a linear differential operator

In this subsection, we consider that the operator T in $\mathbb{K}$ is F defined by (2.2), with $d_\alpha(x) \in C^\infty(\mathbb{R}^N)$, $k < M - N/p$.

If the map T in (**C2**) is F defined in (2.2), it is easy to verify that T satisfies (**C2**). In fact, $\boldsymbol{V} = \boldsymbol{L}^\infty(\Omega)$, T is continuous,

$$\|T\boldsymbol{\xi}\|_{\boldsymbol{L}^\infty(\Omega)} \leq C\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{M,p}(\Omega)}$$

for $\boldsymbol{\xi} \in \boldsymbol{W}^{M,p}(\Omega)$, and T is a linear operator,

$$T \in \mathcal{L}\Big(\boldsymbol{W}^{M,p}(\Omega), \boldsymbol{L}^\infty(\Omega)\Big).$$

In order to study the regularity of the problem with such an operator T in the definition of $\mathbb{K}$, we seek a test function which is mapped by F into X; to do this, we need more properties on X, that is, we suppose as an additional condition either of the following:

• (**Ca1**) X contains a ball B_{0,R_0} with $R_0 > \max\{\|d_0\|_{\boldsymbol{L}^\infty(\Omega)} \cdot \|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(\Omega)},\ 0\}$, where d_0 is defined in (2.2), $\boldsymbol{u}$ is the $\boldsymbol{W}^{M,p}$-solution of the problem (1.1) with closed convex set $\mathbb{K}$ defined in this case.

• (**Ca2**) k defined in (2.2) equals zero, d_0 in (2.2) is nonnegative, X contains a cone $\mathbf{C}$ vertex at the origin, $\overset{o}{\mathbf{C}} \neq \emptyset$.

We discuss them separately.

3.1.1. case of (Ca1)

THEOREM 3.1. – *Suppose that* $\mathbb{K}$ *is defined by* (1.3) *satisfying* (**C1**) *and* (**Ca1**), T *being a smooth differential operator* F *of order* $k < M - N/p$ *defined in* (2.2), $\boldsymbol{\Psi} = 0$; *and that* A *is an operator satisfying* (**A1**)-(**A2**). *Then the solution* $\boldsymbol{u}$ *in* $\mathbb{B}$ *of problem* (1.1) *with the closed convex set* $\mathbb{K}$ *belongs to* $\boldsymbol{W}^{M,\frac{p}{1-\varepsilon}}(G)$ *for some* $0 < \varepsilon \leq 1$ *in any compact subset* G *of* Ω.

Proof. – If we have found a test function in $\mathbb{K}$, the remain part of the proof, using maximal function method, is similar to the one in the proof of Theorem 2.1 in [14].

Let G be any compact subset contained in Ω. Denote $\lambda =$ dist $(G, \partial\Omega)$. Now choose r_0 such that r_0 satisfies (2.9), (3.3) and $0 < 2r_0 < \min\{1, \lambda\}$.

For any point $x_0 \in G$ and any $0 < r < r_0$, let $\zeta_{x_0,r}(x)$ be a cut off function of $B_{x_0,2r}$, set

$$\boldsymbol{v} = \boldsymbol{u} - \zeta_{x_0,r}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r}), \tag{3.2}$$

where $\boldsymbol{P}_{x_0,2r}$ is defined in Lemma 2.6 with $\omega = B_{x_0,2r}$.

Now, let us check $\boldsymbol{v} \in \mathbb{K}$. In fact, it is obvious that $\boldsymbol{v} \in \mathbb{B}$ and $\frac{d^i\boldsymbol{v}}{d\nu^i}|_{\partial\Omega} = 0$, $i = 0, 1, \ldots, M-1$. To verify $T\boldsymbol{v} \in X$, we need to present now some calculations:

1) In $\Omega\backslash B_{x_0,2r}$,

$$Tv = Tu \in X.$$

2) In $B_{x_0,2r}$,

$$\begin{aligned} Tv &= T(u - \zeta_{x_0,r}(u - P_{x_0,2r})) = Tu - T(\zeta_{x_0,r}(u - P_{x_0,2r})) \\ &= TP_{x_0,2r} + (Tu - TP_{x_0,2r}) \\ &\quad - \sum_{|\beta|\le|\alpha|\le k} d_\alpha \binom{\alpha}{\beta} \partial_x^\beta \zeta_{x_0,r} \cdot \partial_x^{\alpha-\beta}(u - P_{x_0,2r}) \\ &= TP_{x_0,2r} + r^{M-k-N/q}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q} z_0. \end{aligned}$$

Here

$$z_0 = \begin{cases} 0, & \text{if } \int_{B_{x_0,2r}} U^q dx = 0, \\ \dfrac{T(u-P_{x_0,2r}) - \sum_{|\beta|\le|\alpha|\le k} d_\alpha \binom{\alpha}{\beta} \partial_x^\beta \zeta_{x_0,r} \partial_x^{\alpha-\beta}(u-P_{x_0,2r})}{r^{M-k-N/q}(\int_{B_{x_0,2r}} U^q dx)^{1/q}}, & \text{otherwise.} \end{cases}$$

From Lemma 2.6, as well as the definitions of T and $P_{x_0,2r}$, by letting $\lambda_0 = R_0 - \|d_0\|_\infty \cdot \|u\|_\infty$, we have the estimate as follows when r_0 is small enough

$$\begin{aligned} |TP_{x_0,2r}| &= \Bigg| \sum_{|\alpha|\le k} d_\alpha(x) r^{-N} \sum_{\mu\ge\alpha,\ 0<|\mu|+|\gamma|\le M-1} \\ &\qquad c_{\mu,\gamma} \frac{\mu!}{\alpha!} r^{|\gamma|}(x - x_0)^{\mu-\alpha} \int_{B_{x_0,2r}} \partial_x^{\mu+\gamma} u\, dx \\ &\qquad + d_0(x) \frac{1}{\text{meas } (B_{x_0,2r})} \int_{B_{x_0,2r}} u\, dx \Bigg| \\ &\le \frac{\lambda_0}{2} + \frac{|d_0(x)|}{\text{meas } (B_{x_0,2r})} \int_{B_{x_0,2r}} |u| dx \le \frac{\lambda_0}{2} + \|d_0\|_\infty \cdot \|u\|_\infty \\ &\le \frac{\lambda_0}{2} + R_0 - \lambda_0 = R_0 - \frac{\lambda_0}{2}. \end{aligned}$$

If we have $|z_0| \le C_0$, C_0 is independent of r, and r_0 is taken small enough as

$$0 < r_0 < \left(\frac{\lambda_0}{2C_0(\int_\Omega U^q dx)^{1/q}}\right)^{\frac{1}{M-k-N/q}}, \tag{3.3}$$

so that when $0 < r < r_0$,

$$r^{M-k-N/q}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q} z_0 < \frac{\lambda_0}{2},$$

then we know that

$$T\boldsymbol{v}(x) \in B_{0,R_0} \subset X.$$

Now, let us verify $|z_0| \leq C_0$. In fact, by Lemma 2.1 and Lemma 2.6, recalling that $\int_{B_{x_0,2r}} \partial_x^\gamma(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})dx = 0$, $|\gamma| \leq M-1$, we have

$$\begin{aligned}
&\left\|\sum_{|\beta|\leq|\alpha|\leq k} d_\alpha \binom{\alpha}{\beta} \partial_x^\beta \zeta_{x_0,r} \cdot \partial_x^{\alpha-\beta}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})\right\|_{\boldsymbol{L}^\infty(B_{x_0,2r})} \\
&\lesssim C \sum_{|\beta|\leq|\alpha|\leq k} r^{-|\beta|} \left\| \partial_x^{\alpha-\beta}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r}) \right. \\
&\qquad\qquad \left. - \int_{B_{x_0,2r}} \partial_x^{\alpha-\beta}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})dx \right\|_{\boldsymbol{L}^\infty(B_{x_0,2r})} \\
&\lesssim C \sum_{|\beta|\leq|\alpha|\leq k} r^{-|\beta|+1} \|\nabla \partial_x^{\alpha-\beta}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})\|_{\boldsymbol{L}^\infty(B_{x_0,2r})} \\
&\quad \cdots \\
&\lesssim C\|D^k(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})\|_{\boldsymbol{L}^\infty(B_{x_0,2r})} \lesssim \frac{C_0}{2} r^{M-k-N/q}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q},
\end{aligned}$$

where

$$\begin{aligned}
&C\|D^k(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})\|_{\boldsymbol{L}^\infty(B_{x_0,2r})} \\
&\lesssim C_1 r^{1-\frac{N}{(M-k)q}} \left(\int_{B_{x_0,2r}} |\partial_x^{k+1}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})|^{(M-k)q} dx\right)^{\frac{1}{(M-k)q}} \\
&\lesssim C_2 r^{2-\frac{2N}{(M-k)q}} \left(\int_{B_{x_0,2r}} |\partial_x^{k+2}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})|^{\frac{(M-k)q}{2}} dx\right)^{\frac{2}{(M-k)q}} \\
&\quad \cdots \\
&\lesssim C_{M-k} r^{M-k-\frac{(M-k)N}{(M-k)q}} \left(\int_{B_{x_0,2r}} |\partial_x^{k+(M-k)}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})|^{\frac{(M-k)q}{M-k}} dx\right)^{\frac{M-k}{(M-k)q}} \\
&\lesssim \frac{C_0}{2} r^{M-k-N/q} \left(\int_{B_{x_0,2r}} |\partial_x^M \boldsymbol{u}|^q dx\right)^{1/q} \\
&= \frac{C_0}{2} r^{M-k-N/q} \left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q},
\end{aligned}$$

where q is chosen such that $p > q > \max\{1, \frac{N}{M-k}\}$, which can be done because $k < M - N/p$ and $p > 1$; C_0 is chosen to satisfy the last inequality of above formula and (3.4) below, which depends only on L, M, N, p, L^∞-norm of d_α, $|\alpha| \leq k$, it is independent of r. In the same reason

$$\max_{x \in B_{x_0,2r}} |T\boldsymbol{u} - T\boldsymbol{P}_{x_0,2r}| \lesssim \frac{C_0}{2} r^{M-k-N/q} \left(\int_{B_{x_0,2r}} U^q dx \right)^{1/q}. \tag{3.4}$$

Therefore, $|z_0| \lesssim C_0$, so that

$$\boldsymbol{v} \in \mathbb{K}.$$

Then, using this test function, we can obtain a Caccioppoli type reversed inequality, by it, we can use maximal function method to obtain the maximal function inequality. The steps are almost the same as in [14] (see also [19] and [23]). Therefore, we can prove that the solution of problem (1.1) belongs to $\boldsymbol{W}^{M,\frac{p}{1-\varepsilon}}(B_{x_0,r})$, for some constant $0 < \varepsilon \leq 1$. Since G can be covered by a finite number of such balls, $\boldsymbol{u}$ belongs to $\boldsymbol{W}^{M,\frac{p}{1-\varepsilon}}(G)$, with norm depending on r_0 as well.

In fact, using $\boldsymbol{v}$ as a test function, we have

$$\sum_{j=0}^{M} S_j \leq 0$$

where

$$S_j = \sum_{l=1}^{L} \sum_{|\gamma|=j} \int_\Omega A_\gamma^l(x, D^M \boldsymbol{u}) \partial_x^\gamma \phi_l dx,$$

where $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_L) = \zeta_{x_0,r}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})$ having a support in $B_{x_0,2r}$.

Now let us estimate S_j, $j \leq M$. First, we have

$$\begin{aligned} S_M \geq & \int_{B_{x_0,r}} |\partial_x^M \boldsymbol{u}|^p dx - \int_{B_{x_0,2r}} a dx - C \sum_{|\gamma|=M} \sum_{0<\beta\leq\gamma} r^{-|\beta|} \\ & \cdot \left[a_M \int_{B_{x_0,2r}} |\partial_x^M \boldsymbol{u}|^{p-1} |\partial_x^{\gamma-\beta}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})| dx \right. \\ & \left. + \int_{B_{x_0,2r}} b_M |\partial_x^{\gamma-\beta}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})| dx \right] \\ \geq & \int_{B_{x_0,r}} U^p dx - \epsilon \int_{B_{x_0,2r}} U^p dx \\ & - C \left(\int_{B_{x_0,2r}} |\boldsymbol{h}| dx + r^{N(1-p/q)} \left(\int_{B_{x_0,2r}} U^q dx \right)^{p/q} \right) \end{aligned}$$

where $\boldsymbol{h}$ is a function of known terms in $\boldsymbol{L}^s(\Omega)$ for some $s > 1$, which depends on given functions a, a_M, b_M, etc., $0 < \epsilon < 1$, $\max\{1, \frac{N}{M-k}\} < q < p$, C depends on given data and ϵ, it is independent of r. Notice here, we utilise the same estimate of $D(\zeta_{x_0,r}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r}))$ as (2.10), so that for $|\gamma| = M$,

$$\begin{aligned}\int_{B_{x_0,2r}} |\partial_x^\gamma \phi|^p dx &\leq C \sum_{0<\beta\leq\gamma} r^{-|\beta|p} \int_{B_{x_0,2r}} |\partial_x^{\gamma-\beta}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})|^p dx \\ &\leq C \int_{B_{x_0,2r}} U^p dx.\end{aligned}$$

Next, for $0 \leq j < M$,

$$\begin{aligned}|S_j| &\leq C \sum_{|\gamma|=j} \sum_{\beta\leq\gamma} \left[r^{-|\beta|} \int_{B_{x_0,2r}} a_j |\partial_x^M \boldsymbol{u}|^{p-\tau_j} |\partial_x^{\gamma-\beta}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})| dx \right. \\ &\qquad \left. + r^{-|\beta|} \int_{B_{x_0,2r}} b_j |\partial_x^{\gamma-\beta}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})| dx \right] \\ &\leq C \left[\int_{B_{x_0,2r}} |\boldsymbol{h}| dx + \left(\int_{B_{x_0,2r}} U^p dx \right)^{1+\frac{1-\tau_j}{p}} \left(\int_{B_{x_0,2r}} |a_j|^{\alpha'_j} dx \right)^{\frac{1}{\alpha'_j}} \right. \\ &\qquad \left. + r^{N(1-\frac{p}{q})} \left(\int_{B_{x_0,2r}} U^q dx \right)^{\frac{p}{q}} \right],\end{aligned}$$

where $\boldsymbol{h}$ is a function of known terms in $\boldsymbol{L}^s(\Omega)$ for some $s > 1$, which depends on given functions a_j, b_j, etc., $\alpha_j > \alpha'_j \max\left\{1, \frac{Np}{(\tau_j-1)N+(M-j)p}\right\}$, C is independent of r.

As $\boldsymbol{u} \in \boldsymbol{W}^{M,p}(\Omega)$, $a_j \in L^{\alpha_j}(\Omega)$, $\tau_j \leq 1$, there exists a $r_0 > 0$, which depends only on ϵ, $\|\boldsymbol{u}\|_{\boldsymbol{W}^{M,p}(\Omega)}$ and $\|a_j\|_{L^{\alpha_j}(\Omega)}$, $j = 0, 1, \ldots, M$, such that when $0 < r < r_0$,

$$\left(\int_{B_{x_0,2r}} U^p dx \right)^{1+\frac{1-\tau_j}{p}} \left(\int_{B_{x_0,2r}} |a_j|^{\alpha'_j} dx \right)^{\frac{1}{\alpha'_j}} \leq \epsilon \int_{B_{x_0,2r}} U^p dx, \qquad (3.5)$$

for any given $0 < \epsilon < 1$.

Then, we can obtain the following Caccioppoli type reversed inequality, for $x_0 \in \Omega$ and $0 < r < r_0$ such that $B_{x_0,2r} \subset \Omega$, and r_0 is small enough satisfying (2.9), (3.3), (3.5) etc.

$$\begin{aligned}\int_{B_{x_0,r}} |\partial_x^M \boldsymbol{u}|^p dx &\leq \delta \int_{B_{x_0,2r}} |\partial_x^M \boldsymbol{u}|^p dx \\ &\quad + C \left[r^{N(1-p/q)} \left(\int_{B_{x_0,2r}} |\partial_x^M \boldsymbol{u}|^q dx \right)^{p/q} + \int_{B_{x_0,2r}} |\boldsymbol{h}| dx \right], \qquad (3.6)\end{aligned}$$

where $0 < \delta < 1$ is a constant, $1 < q < p$, $\boldsymbol{h}$ is a function of known terms in $\boldsymbol{L}^s(\Omega)$ for some $s > 1$, C and δ depends only on the given data and $\|\boldsymbol{u}\|_{\boldsymbol{W}^{M,p}(\Omega)}$.

Multiplying the both sides of (3.6) by r^{-N} and using standard arguments, we have

$$M(|\partial_x^M \boldsymbol{u}|^p) \leq CM^{p/q}(|\partial_x^M \boldsymbol{u}|^q) + CM(|\boldsymbol{h}|). \tag{3.7}$$

Then, by Lemma 2.7, we have proved the result of this theorem. □

THEOREM 3.2. – *Suppose that* $\mathbb{K}$ *is defined by* (1.3) *satisfying* **(C1)** *and* **(Ca1)**, T *being a smooth linear differential operator* F *of order* $k < M - N/p$ *defined in* (2.2), *and that the operator* A *verifies the assumptions* **(A1)**-**(A2)**.

Then the solution $\boldsymbol{u}$ *of problem* (1.1) *with the closed convex set* $\mathbb{K}$ *has the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{M,\frac{p}{1-\varepsilon}}(\Omega)} \leq C, \tag{3.8}$$

where $0 < \varepsilon < 1$ *and* $C > 0$ *depend only on*

$$Q = \{N, M, L, p, \|a\|_\kappa, \|a_j\|_{\alpha_j}, \|b_j\|_{\beta_j}, \tau_j, \Omega, \|d_\alpha\|_{L^\infty}, R_0, \|\boldsymbol{u}\|_{\boldsymbol{W}^{M,p}(\Omega)}\},$$

for $j \leq M$.

Proof. – We only need to prove that the Caccioppoli type reversed inequality (3.6) holds also on the boundary of Ω. That again reduces to find a suitable test function on the boundary.

We extend $\boldsymbol{u}$ to $\mathbb{R}^N$ by defining it to be valued 0 in $\mathbb{R}^N \backslash \Omega$. For $x_0 \in \partial\Omega$ and $0 < r \leq r_0$, choose

$$\boldsymbol{v} = \boldsymbol{u} - \zeta_{x_0,r}\boldsymbol{u} \tag{3.9}$$

as a test function in (1.1), where $r_0 > 0$ will be chosen later. $\zeta_{x_0,r}$ is a cut off function of $B_{x_0,2r}$. One can show that $\boldsymbol{v} \in \mathbb{B}$, $\frac{\partial^i \boldsymbol{v}}{\partial \nu^i}|_{\partial\Omega} = 0$, $i = 1, \ldots M-1$.

If $x \in \overline{\Omega} \backslash B_{x_0,2r}$, then

$$T\boldsymbol{v} = T\boldsymbol{u} \in X.$$

Let us see the image of $T\boldsymbol{v}$ when $x \in \overline{\Omega} \cap B_{x_0,2r}$. Since $\partial\Omega \in C^{M-1,1}$ and $\boldsymbol{u} \in \boldsymbol{W}_0^{M,p}(\Omega)$, we can verify $D^{M-1}\boldsymbol{u} = 0$ on $\partial\Omega$ in distribution sense, so that $Tu = 0$, on $\partial\Omega$, and $\partial_x^j u$ is continuous, for any $0 \le j \le k$, as $M - k > N/p$. That means, there exists $r_1 > 0$, such that in B_{x_0, r_1},

$$Tu \in B_{0,R_0/2};$$

and by Lemma 2.2, there exists $r_2 > 0$, such that when $0 < r < r_2$,

$$|T(\zeta_{x_0,r}u)| \le C_0 r^{M-k-N/p} \left(\int_{B_{x_0,r} \cap \Omega} U^p dx \right)^{1/p} \le \frac{R_0}{2}.$$

Set

$$r_0 = \frac{1}{4} \min\{r_1, r_2\} \tag{3.10}$$

and let $0 < r < r_0$, which depends only on the bounded norm of the coefficients of F (i.e. L^∞-norm of d_α) and other given data, that is,

$$Tv = Tu - T(\zeta_{x_0,r}u) \in B_{0,R_0} \in X,$$

namely

$$v \in \mathbb{K}.$$

Then, if we work in almost the same way as in the proof of Theorem 3.1, we can obtain that the maximal function $M(U)$ has the same estimate as (3.7) in Theorem 3.1 on the boundary. By finite covering, the result of this theorem follows.

Now, we see that ε and $||u||_{W^{M,\frac{p}{1-\varepsilon}}(\Omega)}$ depends on the C and h in Caccioppoli type inequality (3.6), and r_0 as well because of the finite covering. Review all the conditions we set on C, h and r_0, we have the estimate (3.8). □

COROLLARY 3.3. – *Suppose that* $\mathbb{K}$ *defined in* (1.3) *verifies* (**C1**)-(**C3**) *and* (**Ca1**), T *being a smooth linear differential operator* F *of order* $k < M - N/p$ *defined in* (2.2); $\Psi \in W_0^{M,p+\epsilon}(\Omega)$, *with* $\epsilon > 0$, *and* A *is an operator satisfying* (**A1**)-(**A2**). *Then, there exists* $0 < \varepsilon' \le \epsilon$ *such that the solution* u *of problem* (1.1) *with the closed convex set* $\mathbb{K}$ *satisfies*

$$||u||_{W^{M,p+\varepsilon'}(\Omega)} \le C,$$

where ε' *and* C *depend only on* $||\Psi||_{W^{M,p+\epsilon}(\Omega)}$ *and the constants* Q *defined in Theorem* 3.2. □

COROLLARY 3.4. – *Suppose that* $\mathbb{K}$ *defined in* (1.3) *verifies* (**C1**)-(**C3**) *and* (**Ca1**), T *being a smooth linear differential operator* F *of order*

$k < M - N/p$ defined in (2.2); $\boldsymbol{\Psi} \in \boldsymbol{W}_0^{M,p+\epsilon}(\Omega)$, *with* $\epsilon > 0$, *and* A *is an operator satisfying* (**A1**)-(**A2**). *Suppose also that* $(M - j)p = N$ *for some* $0 \leq j < M$. *Then the solution* $\boldsymbol{u}$ *of problem* (1.1) *with the closed convex set* $\mathbb{K}$ *satisfies*

$$\left\|\partial_x^j \boldsymbol{u}\right\|_{\boldsymbol{L}^\infty(\Omega)} \leq C,$$

$$\left|\partial_x^j \boldsymbol{u}(x) - \partial_x^j \boldsymbol{u}(y)\right| \leq C|x - y|^\delta, \qquad for\, x, y \in \Omega,$$

where $0 < \delta < 1$, δ *and* C *depend only on* $\|\boldsymbol{\Psi}\|_{\boldsymbol{W}^{M,p+\epsilon}(\Omega)}$ *and the constants* Q *defined in Theorem* 3.2. □

Remark 3.5. – If the coefficients $d_\alpha(x)$, $|\alpha| \leq k$, in the operator T only belong to $C^0(\mathbb{R}^N)$, all steps of the proof still hold. So under this weaker condition, the results of Theorem 3.2 and the corollaries above hold, too, but we can obtain a similar result for some further weak conditions on T, by using the convergence results of closed convex sets (see next section).□

Remark 3.6. – If we already have that the solution $\boldsymbol{u}$ of (1.1) belongs to $\boldsymbol{W}_0^{M,p}(\Omega)$, the assumption of $\partial\Omega \in C^{M-1,1}$ for the boundary of Ω can be weakened to the following assumption,

$$\inf_{r \geq 2\mathrm{dist}(x,\partial\Omega)} \mathcal{B}_{1,q_0}(\mathcal{E}_{x,r}) > 0, \quad \text{for some } 1 < q_0 < p, \tag{3.11}$$

where $\mathcal{B}_{1,q_0}(A)$ is the Bessel capacity (see [2]) and $\mathcal{E}_{x,r} = r^{-1}(B_{x,r} \backslash \Omega)$ (the condition above is always satisfied for $p > N$). (See [2] and the proof of Theorem 2 in [23]).

In the same way, in [19], with $\partial\Omega \in C^{M-1,1}$, the condition similar to (3.11) in Theorem 4.3 is not necessary. □

3.1.2. case of (Ca2)

We also can consider the case when X does not necessarily contain a large ball centered at the origin, but instead it contains a cone with nonempty interior, and with vertex at the origin. The cone can pull the other vectors into X as shown in Lemma 2.4. We can look for a function in $\boldsymbol{W}_0^{M,p}(B_{x_0,r})$ such that the image of F is in $\mathbf{C}$, but

Remark 3.7. – In general, there exists no such function $\boldsymbol{\xi}$ that $\boldsymbol{\xi} \in \boldsymbol{W}_0^{M,p}(B_{x_0,r})$ and $F\boldsymbol{\xi} \in \mathbf{C}$ when the order k of F is positive, except $\boldsymbol{\xi} \equiv 0$.

We show this remark by a counter-example in a very simple case. If $N = 1$, $\Omega = (0, 1)$, $M = 2$, $k = 1$ we can not find a function ξ such that $\xi'' \geq 0$ and $\xi(0) = \xi(1) = \xi'(0) = \xi'(1) = 0$, except $\xi \equiv 0$ by a simple analysis. □

However, if the order k of F is 0, we can have the same results as the previous theorems and corollaries with this different condition on X. That is,

THEOREM 3.8. – *If T is the differential operator defined in* (2.2) *with* $k = 0$ *and* $d_0(x) \geq 0$ *a.e. in* Ω; $M > N/p$; X *satisfies* **(C1)** *and* **(Ca2)**; *then the results of Theorem* **3.1, 3.2,** *Corollary* **3.3, 3.4** *still hold, with the norm* $\|\boldsymbol{u}\|_{\boldsymbol{W}^{M,\frac{p}{1-\varepsilon}}(\Omega)}$ *estimated by* **(3.8)** *of* C *depending on* $\mathbf{C}$ *in addition, but independent of* R_0.

Proof. – We can use

$$\boldsymbol{v} = \boldsymbol{u} + \zeta_{x_0,r}\left[C_0 r^{M-N/q}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q}\boldsymbol{\omega} - (\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})\right], \quad (3.12)$$

as a test function in the interior point x_0, where $\max\{1, \frac{N}{M}\} < q < p$, $\boldsymbol{\omega} \in \overset{o}{\mathbf{C}}$, $\zeta_{x_0,r}$ is a cut off function of $B_{x_0,2r}$, C_0 will be determined later; and $\boldsymbol{v}$ defined in (3.9) for boundary point x_0.

The regularity result for this problem can be obtained in the same way as Theorem 3.2 etc. by these test functions.

Noticing that here, the constant term in the square bracket in (3.12), different from previous case, has a factor r^M with a maximal term, which can kill the troublesome r^{-j}, $j \leq M$, caused by the differentials of $\zeta_{x_0,r}$ when we do the estimates for maximal function, so that this test function is qualified for maximal function method (checking requests 1.- 3. in the beginning of this section and reviewing the proof of Theorem 3.1).

So we only need to show that in both cases $\boldsymbol{v} \in \mathbb{K}$. In fact, $\boldsymbol{v} \in \boldsymbol{W}_0^{M,p}(\Omega)$ is obvious. In $\Omega \backslash B_{x_0,2r}$, $T\boldsymbol{v} = T\boldsymbol{u} \in X$ and in $B_{x_0,2r}$,

$$\begin{aligned} T\boldsymbol{v} &= T\left[\boldsymbol{u} + \zeta_{x_0,r}\left(C_0\boldsymbol{\omega} r^{M-N/q}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q} - (\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})\right)\right] \\ &= d_0(x)\boldsymbol{u} + d_0(x)\zeta_{x_0,r}\left(C_0\boldsymbol{\omega} r^{M-N/q}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q} - (\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})\right) \\ &= d_0(x)\boldsymbol{u} + d_0(x)\zeta_{x_0,r} C_0 r^{M-N/q}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q}(\boldsymbol{\omega} + \theta_0 \boldsymbol{z}_0) \in X, \end{aligned}$$

by Lemma 2.4 $\Big($noticing $d_0(x) \geq 0$, a.e. and $\zeta_{x_0,r} \geq 0\Big)$, if only $\|\boldsymbol{z}_0\|_{\boldsymbol{L}^\infty} \leq 1$, a.e., where θ_0 is defined in Lemma 2.4. But

$$\boldsymbol{z}_0 = \begin{cases} 0, & \text{if } \int_{B_{x_0,2r}} U^q dx = 0 \\ \dfrac{-(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})}{C_0\theta_0 r^{M-N/q}(\int_{B_{x_0,2r}} U^q dx)^{1/q}}, & \text{otherwise.} \end{cases}$$

C_0 can be chosen big enough such that $\|z_0\|_{\boldsymbol{L}^\infty} \le 1$ by the estimate

$$\|\boldsymbol{u} - \boldsymbol{P}_{x_0,2r}\|_{\boldsymbol{L}^\infty(B_{x_0,2r})} \le C_0\theta_0 r^{M-N/q}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q},$$

which as we have done before in the proof of Theorem 3.1, provided $M > N/q$.

On the boundary, as $0 \le \zeta_{x_0,r} \le 1$, we have, for $\boldsymbol{v}$ as (3.9),

$$T\boldsymbol{v} = (1 - \zeta_{x_0,r})d_0\boldsymbol{u} = (1 - \zeta_{x_0,r})T\boldsymbol{u} \in X,$$

for the convexity of X and $0 \in X$.

So that

$$\boldsymbol{v} \in \mathbb{K}.$$

□

Even though,

Remark 3.9. – the problem discussed in [19] is not included in Theorem 3.8 above. When N is large enough, the condition $M > N/p$ might fail. However, if X is only a cone vertexed at the origin, the condition $M > N/p$ may be skipped too. With some technical transformation, the proof is similar to the one in [19]. □

Remark 3.10. – Having a suitable test function, using the same method as in [14], we also can discuss the almost everywhere regularity of the solution for this problem in all the cases discussed in this paper. □

3.2. T is an integral operator

In this subsection, we consider the closed convex set $\mathbb{K}$ defined in (1.3) when T is an integral operator

$$T\boldsymbol{u}(x) = \int_K \Gamma(x,y)\boldsymbol{u}(y)dy, \tag{3.13}$$

where K is any open subset of Ω with boundary belonging to $C^{0,1}$, $\Gamma(x,y)$ is the kernel of the integral.

It is easy to verify that T defined in (3.13) satisfies (**C2**). In fact, $\boldsymbol{V} = \boldsymbol{L}^\infty(\Omega)$,

$$\|T\boldsymbol{\xi}\|_{\boldsymbol{L}^\infty(\Omega)} \le \max_{x\in\Omega}\left(\int_K |\Gamma(x,y)|^\theta dy\right)^{1/\theta} \|\boldsymbol{\xi}\|_{\boldsymbol{L}^{\theta'}(\Omega)} \le C\|\boldsymbol{\xi}\|_{\boldsymbol{W}^{M,p}(\Omega)},$$

for $\boldsymbol{\xi} \in \boldsymbol{W}^{M,p}(\Omega)$, if only

$$\int_K |\Gamma(x,y)|^\theta dy \le C, \ \forall x \in \Omega \text{ and some positive constant } C, \qquad (3.14)$$

where $\theta = \begin{cases} \frac{Np}{Mp+N(p-1)}, & \text{if } N > Mp, \\ > 1, & \text{if } N = Mp, \\ 1, & \text{if } N < Mp. \end{cases}$ That is, T is continuous. And T is a linear operator.

For this problem, besides **(C1)**, **(C2)**, we need more conditions for X and T. That is, either of the following conditions is additionally required:

• **(Cb1)** X contains a cone $\mathbf{C}$ vertexed at the origin with $\overset{o}{\mathbf{C}} \neq \emptyset$; $M > N/p$; $\Gamma(x,y)$ satisfies

$$\Gamma(x,y) \ge 0, \ \forall x \in \Omega, \ y \in K; \qquad \Gamma(x,y) \in C^0(\Omega, L^1(\Omega)).$$

• **(Cb2)** X contains a ball B_{0,R_0}, $\Gamma \in C^0(\Omega, L^\theta(\Omega))$, and $R_0 > \|\Gamma\|_{L^\infty(\Omega, L^\theta(\Omega))} \cdot \|\boldsymbol{u}\|_{\theta'}$, where θ is defined in (3.14) and $\boldsymbol{u}$ is the $\boldsymbol{W}^{M,p}$-solution of the problem (1.1) with closed convex set $\mathbb{K}$ defined in this case.

3.2.1. case of (Cb1)

Here, the cone is used for pulling the test function into X.

As the discussion of Theorem 3.8, for the interior point, we can use the same test function $\boldsymbol{v}$ as in (3.12).

It is obvious that this function $\boldsymbol{v}$ belongs to $\boldsymbol{W}_0^{M,p}(\Omega)$, and as we discussed before, it works for maximal function method. Also,

$$T\boldsymbol{v} = \int_K \Gamma(x,y)\boldsymbol{u}(y)dy + C_0 r^{M-N/q}\Big(\int_{B_{x_0,2r}} U^q dx\Big)^{1/q} \int_K \Gamma(x,y)\zeta_{x_0,r}(y)dy(\boldsymbol{\omega} + \theta_0 \boldsymbol{z}_0),$$

where

$$\boldsymbol{z}_0 = \begin{cases} 0, & \text{if } \int_{B_{x_0,2r}} U^q dx \cdot \int_K \Gamma\zeta_{x_0,r} dy = 0 \\ \dfrac{-\int_K \Gamma(x,y)\zeta_{x_0,r}(y)(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})(y)dy}{C_0\theta_0 r^{M-N/q}(\int_{B_{x_0,2r}} U^q dx)^{1/q} \int_K \Gamma(x,y)\zeta_{x_0,r}(y)dy}, & \text{otherwise.} \end{cases}$$

We have, similar as in the proof of Theorem 3.8, $|z_0| \leq 1$, if only C_0 is chosen properly. That is, taking count of $\int_K \Gamma(x,y)\boldsymbol{u}(y)dy \in X$, $T\boldsymbol{v} \in X$, then

$$\boldsymbol{v} \in \mathbb{K}.$$

Therefore, we have a local Caccioppoli type reversed inequality for maximal function as (3.6) and a finite cover. As almost the same as we have done before, we have the interior estimate.

On the boundary, we should consider it carefully in another way. Notice here, the test function defined in (3.12) no longer satisfies the boundary condition of $\mathbb{K}$ if x_0 is on the boundary of Ω.

Let $x_0 \in \partial\Omega$. Without any loss in generality we may suppose that the boundary near x_0 is $\{(x_1, \cdots, x_N),\ x_1 = 0\}$ and $\{x_1 > 0\}$ corresponds to Ω. We choose

$$\boldsymbol{v} = \begin{cases} \boldsymbol{u} + \zeta_{x_0,r}\left(C_0 x_1^{M-N/q}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q}\boldsymbol{\omega} - \boldsymbol{u}\right), & \text{if } x_1 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

as a test function, where $\zeta_{x_0,r}$, $\boldsymbol{\omega}$ are as before.

Then, $\boldsymbol{v} \in \boldsymbol{W}_0^{M,p}(\Omega)$ and

$$\begin{aligned} T\boldsymbol{v} &= \int_K \Gamma(x,y)\boldsymbol{u}dx \\ &\quad + C_0\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q} \int_K \Gamma(x,y) y_1^{M-N/q}\zeta_{x_0,r}(y)dy\boldsymbol{\omega} \\ &\quad - \int_K \Gamma(x,y)\zeta_{x_0,r}(y)\boldsymbol{u}(y)dy \\ &= \int_K \Gamma(x,y)\boldsymbol{u}dx \\ &\quad + C_0\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q} \int_K \Gamma(x,y) y_1^{M-N/q}\zeta_{x_0,r}(y)dy(\boldsymbol{\omega} + \theta_0 \boldsymbol{z}_0) \in X, \end{aligned}$$

if only $\|\boldsymbol{z}_0\|_{\boldsymbol{L}^\infty(\Omega)} \leq 1$, where

$$\boldsymbol{z}_0 = \begin{cases} 0, \quad \text{if } \displaystyle\int_{B_{x_0,2r}} U^q dx = 0 \text{ or } \int_K \Gamma y_1^{M-N/q}\zeta_{x_0,r}dy = 0, \\ \dfrac{-\int_K \Gamma(x,y)\zeta_{x_0,r}(y)\boldsymbol{u}(y)dy}{C_0\theta_0(\int_{B_{x_0,2r}} U^q dx)^{1/q}\int_K \Gamma(x,y)y_1^{M-N/q}\zeta_{x_0,r}(y)dy}, & \text{otherwise.} \end{cases}$$

But,

$$\left|\int_K \Gamma(x,y)\zeta_{x_0,r}(y)\boldsymbol{u}(y)dy\right|$$
$$\leq C\int_K \Gamma(x,y)y_1^{M-N/q}\left(\int_{B_{x_0,2r}} |\partial_{y_1}^M \boldsymbol{u}|^q dx\right)^{1/q}\zeta_{x_0,r}(y)dy$$
$$= C_0\theta_0\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q}\int_K \Gamma(x,y)y_1^{M-N/q}\zeta_{x_0,r}(y)dy, \quad (3.16)$$

where, by noticing

$$\partial_{y_1}^j \boldsymbol{u}\Big|_{y_1=0} = 0, \quad j = 0, 1\cdots, M-1,$$

we have

$$\max_{y\in B_{x_0,2r}\cap\{y_1>0\}} |\boldsymbol{u}(y_1, y_2, \cdots, y_n)| \leq C y_1^{M-N/p}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q}, \quad (3.17)$$

with C being independent of r.

The inequality (3.17) comes from the imbedding theorem on the boundary presented in p433, Corollary 2. of [22] (q can be chosen properly).

When C_0 in (3.16) is properly chosen, it leads to

$$\|z_0\|_{\boldsymbol{L}^\infty(\Omega)} \leq 1.$$

Namely

$$\boldsymbol{v} \in \mathbb{K}.$$

Then we have a Caccioppoli type reversed inequality as (3.6) over Ω, by the finite covering, the regularity results run over the domain Ω. The estimate (3.8) then holds but depending on r_0, which can be determined by given data, and $\|\Gamma\|_{L^\infty(\Omega, L^1(K))}$ as well.

Now, there comes

THEOREM 3.11. – *If the operator A satisfies the assumptions* (**A1**)-(**A2**); $\mathbb{K}$ *is defined by* (1.3) verifying (**C1**) *and* (**Cb1**), *T being an integral operator defined in* (3.13), *where K is a subset of Ω with $\partial K \in C^{0,1}$; $\boldsymbol{\Psi} = 0$; then the solution $\boldsymbol{u}$ of problem* (1.1) *with the closed convex set $\mathbb{K}$, belongs to $\boldsymbol{W}^{M,\frac{p}{1-\varepsilon}}(\Omega)$ for some $0 < \varepsilon \leq 1$. More precisely,* (3.8) *holds with C depending on $\|\Gamma\|_{L^\infty(\Omega, L^1(K))}$ and the cone* **C** *in addition, but independent of d_α and R_0.*

Similarly, Corollaries 3.3, 3.4 *also hold for this case.* □

3.2.2. case of (Cb2)

On the boundary, we can choose $\boldsymbol{v}$ as (3.9), while in the interior we can choose $\boldsymbol{v}$ as (3.2) to be test functions, which work for maximal function method and are in $\mathbb{K}$.

On the boundary, we can use Lemma 2.2 and the fact $D^j u|_{\partial\Omega} = 0$ for all $0 \le j \le M-1$. The other discussion is similar to the one in the interior, we only write the details about interior points here.

In fact,

$$\begin{aligned}
|T\boldsymbol{v}| &= |T(\boldsymbol{u} - \zeta_{x_0,r}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r}))| \\
&\le \int_K |\Gamma(x,y)[\boldsymbol{u}(y) - \zeta_{x_0,r}(y)(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})(y)]|dy \\
&\le \|\Gamma\|_{L^\infty(\Omega, L^\theta(K))} \cdot \|\boldsymbol{u}\|_{\theta'} \\
&\quad + C_0 r^{M+N(\frac{1}{\theta'}-\frac{1}{p})} \int_K |\Gamma(x,y)|^\theta \zeta_{x_0,r}(y)dy \left(\int_{B_{x_0,2r}} U^p dx \right)^{1/p} \\
&\le R_0,
\end{aligned} \tag{3.18}$$

if only $0 < r < r_0$ and r_0 is small enough such that

$$\begin{aligned}
&C_0 r^{M+N(\frac{1}{\theta'}-\frac{1}{p})} \int_K |\Gamma(x,y)|^\theta \zeta_{x_0,r}(y)dy \left(\int_{B_{x_0,2r}} U^p dx \right)^{1/p} \\
&\quad \le R_0 - \|\Gamma\|_{L^\infty(\Omega, L^\theta(K))} \cdot \|\boldsymbol{u}\|_{\theta'},
\end{aligned}$$

which can be done since $M + N(1/\theta' - 1/p) \ge 0$ and $\boldsymbol{u} \in \mathbf{W}^{M,p}(\Omega)$.

Therefore,

$$T\boldsymbol{v} \in B_{0,R_0} \subset X$$

by **(Cb2)**.

Then we have

THEOREM 3.12. – *If the operator A satisfies the assumptions* **(A1)**-**(A2)**; $\mathbb{K}$ *is defined by* (1.3) *verifying* **(C1)** *and* **(Cb2)**, *T being an integral operator defined in* (3.13), *where K is a subset of Ω with $\partial K \in C^{0,1}$; $\boldsymbol{\Psi} = 0$; then the solution $\boldsymbol{u}$ of problem* (1.1) *with the closed convex set $\mathbb{K}$, belongs to $\mathbf{W}^{M,\frac{p}{1-\varepsilon}}(\Omega)$ for some $0 < \varepsilon \le 1$. More precisely,* (3.8) *holds with C depending on $\|\Gamma\|_{L^\infty(\Omega, L^\theta(\Omega))}$ in addition, but independent of d_α.*

Similarly, Corollaries 3.3, 3.4 *also hold for this case.* □

3.3. T is an integrodifferential operator

Now let the operator T in $\mathbb{K}$ be defined as

$$Tu = \int_K \Gamma(x,y)Fu(y)dy, \tag{3.19}$$

where K is any open subset of Ω with boundary belonging to $C^{M-1,1}$; F is a linear differential operator as (2.2), with $d_\alpha(x) \in C^\infty(\mathbb{R}^N)$, $k \le M$; and the kernel Γ satisfies

$$\int_\Omega |\Gamma(x,y)|^\tau dy \le C, \quad \forall x \in \Omega \text{ and some positive constant } C, \tag{3.20}$$

where $\tau = \begin{cases} \frac{Np}{(M-k+N)p-N}, & \text{if } N > (M-k)p, \\ > 1, & \text{if } N = (M-k)p, \\ 1, & \text{if } N < (M-k)p. \end{cases}$

It is easy to verify that T defined in (3.19) satisfies (**C2**).

As in last subsection, for this problem, besides (**C1**), (**C2**), we also need more conditions for X and T. That is, one of the follows is additionally required:

- (**Cc1**) X contains a cone $\mathbf{C}$ vertexed at the origin with $\overset{o}{\mathbf{C}} \neq \emptyset$; $\Gamma(x,y)$=constant, F in (3.19) defined by (2.2) verifies (2.3), with $k \le M$, $M > N/p$; $K = \Omega$ or $d_\alpha(x) \in C_0^{|\alpha|}(K)$.
- (**Cc2**) X contains a ball B_{0,R_0}, $\Gamma \in C^0(\Omega, L^\tau(\Omega))$, and $R_0 > \|\Gamma(x,y)\|_{L^\infty(\Omega,L^\tau(\Omega))} \cdot \|Fu\|_{\mathbf{L}^{\tau'}(\Omega)}$, where τ is defined in (3.20) and u is the $W^{M,p}$-solution of the problem (1.1) with closed convex set $\mathbb{K}$ defined in this case.

Remark 3.13. – Under (**Cc1**), if $k = 0$ and $d_0 = 1$, it is the case we discussed in last subsection, when we need not require Γ to be a constant but a $C^0(\Omega, L^1(K))$-function. Also, the set K may be any subset of Ω with $C^{0,1}$ boundary. □

3.3.1. case of (Cc1)

We do not lose any generality if we assume that $\Gamma(x,y) = 1$.

We denote

$$b_0(x) = \sum_{|\alpha|\le k} (-1)^\alpha \partial_x^\alpha d_\alpha(x),$$

thus, by (2.3) we have

$$b_0(x) \ge 0, \quad \text{a.e. } x \in \Omega. \tag{3.21}$$

As in previous subsections, the cone is used for pulling the test function into X. It should be careful to deal with the point in Ω but on the boundary of K.

Let G and λ as before in subsection 3.1.1, $0 < 2r_0 < \min\{1, \lambda\}$. Considering $x_0 \in G$ and $0 < r < r_0$, let us define a test function $\boldsymbol{v}$ in $\mathbb{K}$ for maximal function method.

If $K = \Omega$, take any $x_0 \in G$, there exists $0 < r < r_0$ small enough such that $B_{x_0,2r} \subset \Omega$, define a test function $\boldsymbol{v}$ as (3.12).

Let us show that $\boldsymbol{v} \in \mathbb{K}$. In fact, it is obvious that $\boldsymbol{v} \in \mathbb{B}$, and $\boldsymbol{v}$ satisfies the boundary condition of $\mathbb{K}$, and by Lemma 2.5,

$$\begin{aligned} T\boldsymbol{v} &= \int_K F\boldsymbol{v}dx \\ &= \int_K \left[F\boldsymbol{u} + F\left(\zeta_{x_0,r}\left[C_0\boldsymbol{\omega}r^{M-N/q}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q} - (\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})\right]\right)\right]dx \\ &= \int_K F\boldsymbol{u}dx + \sum_{0\le|\alpha|\le k} \int_{B_{x_0,2r}} d_\alpha(x) \\ &\quad \cdot \partial_x^\alpha\left(\zeta_{x_0,r}\left[C_0\boldsymbol{\omega}r^{M-N/q}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q} - (\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})\right]\right)dx \\ &= \int_K F\boldsymbol{u}dx + \int_{B_{x_0,2r}} \Big(\sum_{|\alpha|\le k}(-1)^\alpha \partial_x^\alpha d_\alpha(x)\Big)\zeta_{x_0,r} \\ &\quad \cdot \left[C_0\boldsymbol{\omega}r^{M-N/q}\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q} - (\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})\right]dx \\ &= \int_K F\boldsymbol{u}dx \\ &\quad + C_0 r^{M-N/q}\int_{B_{x_0,2r}} b_0(x)\zeta_{x_0,r}dx\left(\int_{B_{x_0,2r}} U^q dx\right)^{1/q}(\boldsymbol{\omega} + \theta_0 \boldsymbol{z}_0), \end{aligned}$$

where θ_0 is a constant chosen as in Lemma 2.4, and

$$\boldsymbol{z}_0 = \begin{cases} 0, & \text{if } \int_{B_{x_0,2r}} U^q dx \cdot \int_{B_{x_0,2r}} b_0\zeta_{x_0,r}dx = 0 \\ \dfrac{-\int_{B_{x_0,2r}} b_0\zeta_{x_0,r}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r})dx}{C_0\theta_0 r^{M-N/q}\int_{B_{x_0,2r}} b_0\zeta_{x_0,r}dx(\int_{B_{x_0,2r}} U^q dx)^{1/q}}, & \text{otherwise.} \end{cases}$$

Notice, the coefficient of the $\boldsymbol{\omega} + \theta_0\boldsymbol{z}_0$ in the above formula is not negative, and $\int_K F\boldsymbol{u}dx \in X$, so that by Lemma 2.4, $\int_K F\boldsymbol{v}dx \in X$, as long as $\|\boldsymbol{z}_0\|_{\boldsymbol{L}^\infty(\Omega)} \le 1$, which can be verified as before by choosing proper C_0.

That is, this test function $\boldsymbol{v}$ is the one we wanted.

If $K \subset \Omega$ but $K \neq \Omega$, then, the following cases are considered separately:

a.) If $x_0 \in \overset{o}{K} \cap G$, we can discuss it in the same way as the case of $K = \Omega$ above by choosing $0 < r < r_0$ small enough, such that $B_{x_0,2r} \subset K$.

b.) If $x_0 \in \partial K \cap G$, in a similar way of the case $K = \Omega$, we also can discuss it, but by use of the second part of Lemma 2.5, and the fact $d_\alpha \in C_0^{|\alpha|}(K)$.

c.) If $x_0 \in G \backslash \overline{K}$, then, when $0 < r < r_0$ small enough, there exists a cut off function $\zeta_{x_0,r}$, such that $\mathrm{supp}(\zeta_{x_0,r}) \subset G \backslash \overline{K}$. Set

$$\boldsymbol{v} = \boldsymbol{u} - \zeta_{x_0,r}(\boldsymbol{u} - \boldsymbol{P}_{x_0,2r}),$$

therefore

$$\int_K F\boldsymbol{v}dx = \int_K F\boldsymbol{u}dx \in X.$$

Hence, combining **a.)-c.)**, as well as having the discussion on case $K = \Omega$, we have,

$$\boldsymbol{v} \in \mathbb{K}$$

holds in any case presented in the theorem for this operator.

Thus, we have a local Caccioppoli type reversed inequality for maximal function as (3.6) and a finite cover in Ω. By the same steps as we have done before, we have the interior estimate.

As in the last subsection, without losing the generality, for $x_0 \in \partial\Omega$, we suppose that the boundary near x_0 is $\{(x_1, \cdots, x_N),\ x_1 = 0\}$, and $\{x_1 > 0\}$ corresponds to Ω, we can choose the test function $\boldsymbol{v}$ as (3.15) for x_0. It is not difficult to see, this function is in $\boldsymbol{W}_0^{M,p}(\Omega)$. Then using Lemma 2.2, in a similar way as the discussion on the interior points, by noticing (3.17), we also can have the same boundary estimate as the interior one.

So that

THEOREM 3.14. – *If the operator A satisfies the assumptions* (**A1**)-(**A2**); $\mathbb{K}$ *is defined as* (1.3) *verifying* (**C1**) *and* (**Cc1**), *T being an integrodifferential operator defined as* (3.19), *where F is defined in* (2.2) *verifying* (2.3) *for $M \geq k$, $M > N/p$, $K = \Omega$ or K is a subset of Ω with $\partial K \in C^{M-1,1}$ and $d_\alpha \in C_0^{|\alpha|}(K)$; $\boldsymbol{\Psi} = 0$; then the solution $\boldsymbol{u}$ of problem* (1.1) *with the closed convex set $\mathbb{K}$, belongs to $\boldsymbol{W}^{M,\frac{p}{1-\varepsilon}}(\Omega)$ for some $0 < \varepsilon \leq 1$. More precisely,* (3.8) *holds with C depending on* **C** *in addition, but independent of R_0.*

Similarly, Corollaries 3.3, 3.4 *also hold for this case.* □

Remark 3.15. – It is not difficult to see that, under **(Cc1)**, if there is a kernel $\Gamma(x, y)$ not being a constant but a function smooth enough and satisfying that

$$F_x^* \Gamma(x, y) \geq 0, \quad \text{for a.e. } x, y \in \Omega,$$

where F^* is the adjoint of F defined in (2.2), and F_x^* means the differentials with respect to x, then the results obtained in this subsection can be extended to the set $\mathbb{K}$ with the operator

$$T\boldsymbol{u} = \int_K \Gamma(x, y) F \boldsymbol{u} dx,$$

where $K = \Omega$ or $K \subset \Omega$ with $\partial K \in C^{M-1,1}$ and $d_\alpha \in C_0^{|\alpha|}(K)$. □

3.3.2. case of (Cc2)

In this case, the discussion is almost the same as in **(Cb2)**. So that,

THEOREM 3.16. – *If the operator A satisfies the assumptions* **(A1)**-**(A2)**; $\mathbb{K}$ *is defined in* (1.3) *verifying* **(C1)** *and* **(Cc2)**, *T being an integrodifferential operator defined as* (3.19), *where F is defined in* (2.2) *verifying $M \geq k$, and K is a subset of Ω with $\partial K \in C^{M-1,1}$; $\boldsymbol{\Psi} = 0$; then the solution $\boldsymbol{u}$ of problem* (1.1) *with the closed convex set $\mathbb{K}$ belongs to $\mathbf{W}^{M, \frac{p}{1-\varepsilon}}(\Omega)$ for some $0 < \varepsilon \leq 1$. More precisely,* (3.8) *holds with C depending on $||\Gamma||_{L^\infty(\Omega, L^\tau(K))}$ in addition.*

Similarly, Corollaries 3.3, 3.4 *also hold for this case.* □

4. THE REGULARITY BY CONVERGENCE RESULTS

In this section, we use the results obtained in [18] to get some more regularity results by a limit process.

In the discussions of the last section, we see that a certain smoothness of $\mathbb{K}$ is necessary, but, in general, the regularity results do not depend so strongly on this assumption on $\mathbb{K}$. So that, there is an expectation to weaken this smoothness assumption. The discussions on the convergence of closed convex sets (see [18], [20]) make this expectation possible.

We give some applications of the convergence of the closed convex sets here, thus we extend the regularity results obtained in the last section.

The convergence of the closed convex sets we discuss here is called the convergence in Mosco sense and/or in local gap. It is stated in [24] and

[25] that the convergence of the closed convex set in Mosco sense will imply the convergence of the solutions of problem (1.1) if A is monotone (so that the solution is unique). Also discussed there is the convergence of the solutions for a sequence of operators $\{A_n\}_n$.

To study our problem, first of all, we list some definitions and theorems about the convergences we will use and we already have. More results about Mosco convergences can also be found in [3] and its references, as well as in [18].

DEFINITION 4.1 (see also [9], [21]). – *An operator $A : V \to V'$ is called pseudo-monotone if for every w_n such that $w_n \rightharpoonup w$ weakly in V when $n \to \infty$, and $\limsup_n \langle Aw_n, w_n - w\rangle \leq 0$, then*

$$\liminf_{n\to\infty} \langle Aw_n, w_n - v\rangle \geq \langle Aw, w - v\rangle, \qquad \forall v \in V. \qquad \square$$

Remark 4.2. – We notice that when the operator A of problem (1.1) satisfies the assumptions **(A1)**-**(A3)**, A is an operator of the Calculus of Variations *(*in the sense of Leray-Lions*)*, so that it is pseudo-monotone by a result of [21] (pp182-183). □

DEFINITION 4.3. – *Let V be a Banach space, $\{\mathbb{K}_n\}_n$ a sequence of closed convex sets of V. We say that $\mathbb{K}_n$ converges to $\mathbb{K}_\infty$ in the Mosco sense, if*

1] $\forall v \in \mathbb{K}_\infty$, $\exists v_n \in \mathbb{K}_n$, $v_n \to v$ *as* $n \to \infty$, *in* V,
2] $v_{n_i} \in \mathbb{K}_{n_i}$, $v_{n_i} \rightharpoonup v$ *as* $n_i \to \infty$, *weakly in* $V \Longrightarrow v \in \mathbb{K}_\infty$, *where* $\{\mathbb{K}_{n_i}\}$ *is any subsequence of* $\{\mathbb{K}_n\}$. □

DEFINITION 4.4 (See Section 4 in [25]). – *Let $\{\mathbb{K}_n\}$ be a sequence of nonempty closed convex sets of a Banach space V. We say that $\mathbb{K}_n$ converges to $\mathbb{K}_\infty$ in local gap in V, if there exists $\bar{k}_0 > 0$ such that*

$$\forall k_0 > \bar{k}_0, \qquad \sigma_{k_0}(\mathbb{K}_n, \mathbb{K}_\infty) \to 0,$$

where $\sigma_{k_0}(X, Y) = \max\{\sigma(X^{k_0}, Y), \sigma(Y^{k_0}, X)\}$, *with* $X^{k_0} = \{x \in X,\ \|x\|_V \leq k_0\}$, *and* $\sigma(X, Y) = \sup\{\text{dist}(x, Y),\ x \in X\}$. □

Convergence in local gap results in the convergence in Mosco sense. More discussions on these two convergences can be found in [18].

Set

$$\mathbb{K}_n = \Big\{ \boldsymbol{v} \in \mathbb{B} : \ T_n\boldsymbol{v}(x) \in X, \quad \text{for a.e. } x \in \Omega, \\ \left.\frac{d^i\boldsymbol{v}}{d\nu^i}\right|_{\partial\Omega} = 0, \quad i = 0, 1, \ldots, M-1 \Big\}, \qquad n \in \mathbb{N} \cup \{\infty\}, \quad (4.1)$$

The following two theorems can be found in [18]:

THEOREM 4.5. – *Suppose that* **(C1)-(C3)** *are verified for* $\mathbb{K}_n$ *defined in* (4.1), X *also contains a small ball* B_{0,R_0} *with* $R_0 > 0$; $T_n = F_n$ *is a sequence of differential operator of order* $k < M - N/p$ *defined by*

$$F_n = \sum_{|\alpha| \leq k} d_{\alpha n}(x)\partial_x^\alpha, \qquad n \in \mathbb{N} \cup \{\infty\}, \tag{4.2}$$

such that $d_{\alpha n} \in C^\infty(\mathbb{R}^N)$, $\forall n$, *and* $d_{\alpha n}(x) \to d_{\alpha\infty}(x)$ *in* $L^\infty(\Omega)$, *as* $n \to \infty$, $|\alpha| \leq k$. *Then,* $\{\mathbb{K}_n\}_n$ *converges to* $\mathbb{K}_\infty$ *in Mosco sense, when* $n \to \infty$. □

THEOREM 4.6. – *Suppose that* **(C1)-(C3)** *are verified for* $\mathbb{K}_n$ *defined in* (4.1), X *also contains a small ball* B_{0,R_0} *with* $R_0 > 0$; T_n *is the integrodifferential operator*

$$T_n\boldsymbol{u} = \int_K \Gamma_n(x,y)F_n\boldsymbol{u}(y)dy,$$

where the integral area $K \subseteq \Omega$ *with* $\partial K \in C^{M-1,1}$, Γ_n *satisfies* (3.20), F_n *is defined by* (4.2) *with* $k \leq M$, *and* $\Gamma_n \to \Gamma$ *in* $L^\infty(\Omega, L^\tau(\Omega))$, *where* τ *is defined in* (3.20), $d_{\alpha n}(x) \to d_{\alpha\infty}(x)$, in $L^\infty(\Omega)$, $|\alpha| \leq k$, as $n \to \infty$. *Then the result of Theorem* 4.5 *holds.* □

Remark 4.7. – The convergences discussed in Theorem 4.5, 4.6 are in fact, in local gap. (See [18]). □

Now, let us see a theorem about a convergence of the solution of problem (1.1) with a converging sequence of closed convex sets as follows.

THEOREM 4.8. – *Let* $\{\mathbb{K}_n\}_n$, $n \in \mathbb{N} \cup \{\infty\}$, *be closed convex sets of* $\mathbb{B}$ *such that* $\mathbb{K}_n \to \mathbb{K}_\infty$, *in Mosco sense, when* $n \to \infty$. *If* A *satisfies the assumptions* **(A1)-(A4)**, *let* $\{\boldsymbol{u}_n\}_n$ *be any sequence of solutions of problem* (1.1) *with the closed convex set* $\{\mathbb{K}_n\}_n$, $n \in \mathbb{N}$. *Then there exists at least a solution* $\boldsymbol{u}_\infty$ *of the problem* (1.1) *with closed convex set* $\mathbb{K}_\infty$ *which can be approximated weakly by a subsequence of* $\{\boldsymbol{u}_n\}_n$ *in* $\mathbb{B}$. *Moreover, if* A *satisfies*

$$\langle A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v}\rangle \geq \gamma(\|\boldsymbol{u} - \boldsymbol{v}\|_{\mathbb{B}}), \tag{4.3}$$

where $\gamma(\cdot)$ *is a continuous strictly increasing function from* $[0, +\infty)$ *to* $[0, +\infty)$ *with* $\gamma(0) = 0$ *and* $\lim_{r\to+\infty} \gamma(r) = +\infty$, *then the solutions are unique and the sequence* $\{\boldsymbol{u}_n\}_n$ *converges to* $\boldsymbol{u}_\infty$ *strongly in* $\mathbb{B}$.

Proof. – (See also [24], [25]. Since the definition of A in our case here, which is only pseudo-monotone but has a concrete structure, is different from Mosco's, the proof is different as well.)

From Remark 4.2, A satisfies (**A1**)-(**A3**), then A is a pseudo-monotone operator.

If $\boldsymbol{u}_n$ is the solution of problem (1.1) with $\mathbb{K}_n$, then by (**A1**), (**A2**) and (**A4**), reviewing (2.16), we have

$$\int_\Omega |\partial_x^M \boldsymbol{u}_n|^p dx + (c_0 - M\overline{C}) \int_\Omega |\boldsymbol{u}_n|^{p/\tau_0} dx \le C,$$

where C depends on given data and is independent of n; τ_0, c_0 and $\overline{C}$ are defined in (**A4**). i.e. there exists $R > 0$ independent of n, such that

$$||\boldsymbol{u}_n||_{\mathbb{B}} \le R. \tag{4.4}$$

Thus, there exists a subsequence of $\{\boldsymbol{u}_n\}_n$ converging weakly in $\mathbb{B}$ to $\overline{\boldsymbol{u}}_\infty$.

As $\mathbb{K}_n$ converges to $\mathbb{K}_\infty$ in Mosco sense, from **2]** of Definition 4.3, we have, $\overline{\boldsymbol{u}}_\infty \in \mathbb{K}_\infty$.

Also, from **1]** of Definition 4.3, we know that there exists $\overline{\boldsymbol{u}}_n \in \mathbb{K}_n$ such that $\overline{\boldsymbol{u}}_n \to \overline{\boldsymbol{u}}_\infty$ in $\mathbb{B}$. Then for $\boldsymbol{u}_n$ being a solution,

$$\langle A\boldsymbol{u}_n, \overline{\boldsymbol{u}}_n - \boldsymbol{u}_n \rangle \ge 0$$

and

$$\liminf_{n\to\infty} \langle A\boldsymbol{u}_n, \overline{\boldsymbol{u}}_\infty - \boldsymbol{u}_n \rangle = \liminf_{n\to\infty} \langle A\boldsymbol{u}_n, \overline{\boldsymbol{u}}_n - \boldsymbol{u}_n \rangle + \liminf_{n\to\infty} \langle A\boldsymbol{u}_n, \overline{\boldsymbol{u}}_\infty - \overline{\boldsymbol{u}}_n \rangle \ge 0. \tag{4.5}$$

Now, we are going to show that this $\overline{\boldsymbol{u}}_\infty$ is the solution of problem (1.1) in $\mathbb{K}_\infty$, i.e. one should be hold

$$\overline{\boldsymbol{u}}_\infty \in \mathbb{K}_\infty, \qquad \langle A\overline{\boldsymbol{u}}_\infty, \boldsymbol{v}_\infty - \overline{\boldsymbol{u}}_\infty \rangle \ge 0, \qquad \forall \boldsymbol{v}_\infty \in \mathbb{K}_\infty. \tag{4.6}$$

In fact, for any $\boldsymbol{v}_\infty \in \mathbb{K}_\infty$, from **1]**, there exists $\boldsymbol{v}_n \in \mathbb{K}_n$ such that $\boldsymbol{v}_n \to \boldsymbol{v}_\infty$ in $\mathbb{B}$. Since $\boldsymbol{u}_n$ is the solution in $\mathbb{K}_n$, for these $\boldsymbol{v}_n \in \mathbb{K}_n$, one holds

$$\boldsymbol{u}_n \in \mathbb{K}_n, \qquad \langle A\boldsymbol{u}_n, \boldsymbol{v}_n - \boldsymbol{u}_n \rangle \ge 0.$$

Hence,

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \langle A\boldsymbol{u}_n, \boldsymbol{v}_n - \boldsymbol{u}_n \rangle \\ &= \limsup_{n\to\infty} \langle A\boldsymbol{u}_n, \boldsymbol{v}_\infty - \boldsymbol{u}_n \rangle + \limsup_{n\to\infty} \langle A\boldsymbol{u}_n, \boldsymbol{v}_n - \boldsymbol{v}_\infty \rangle \le \langle A\overline{\boldsymbol{u}}_\infty, \boldsymbol{v}_\infty - \overline{\boldsymbol{u}}_\infty \rangle, \end{aligned}$$

because:
1^0 $v_n \to v_\infty$ in $\mathbb{B}$ as $n \to \infty$,

$$\limsup_{n\to\infty} \langle A\boldsymbol{u}_n, \boldsymbol{v}_\infty - \boldsymbol{v}_n\rangle \to 0;$$

2^0 A is pseudo-monotone with $\boldsymbol{u}_n \rightharpoonup \overline{\boldsymbol{u}}_\infty$, weakly in $\mathbb{B}$ and $\liminf_n \langle A\boldsymbol{u}_n, \overline{\boldsymbol{u}}_\infty - \boldsymbol{u}_n\rangle \le 0$ by (4.5), as $n \to \infty$, then from Definition 4.1

$$\limsup_{n\to\infty} \langle A\boldsymbol{u}_n, \boldsymbol{v}_\infty - \boldsymbol{u}_n\rangle \le \langle A\overline{\boldsymbol{u}}_\infty, \boldsymbol{v}_\infty - \overline{\boldsymbol{u}}_\infty\rangle.$$

That is, $\overline{\boldsymbol{u}}_\infty$ is a solution of problem (1.1) with $\mathbb{K}_\infty$.

Finally, if A also satisfies (4.3), then from Remark 2.10, the solution $\boldsymbol{u}_n$ for problem (1.1) with closed convex set $\mathbb{K}_n$ is unique, $n \in \mathbb{N} \cap \{\infty\}$. Thus, all subsequences of $\{\boldsymbol{u}_n\}_n$ converges to $\boldsymbol{u}_\infty$ weakly in $\mathbb{B}$, that means, the sequence converge to $\boldsymbol{u}_\infty$ weakly in $\mathbb{B}$. Moreover, from (4.3) and Definition 4.3, there exists $\boldsymbol{u}'_n \in \mathbb{K}_n$ and $\boldsymbol{u}'_n \to \boldsymbol{u}_\infty$ in $\mathbb{B}$. So

$$\begin{aligned}\gamma(\|\boldsymbol{u}_n - \boldsymbol{u}_\infty\|_{\mathbb{B}}) &\le \langle A\boldsymbol{u}_n - A\boldsymbol{u}_\infty, \boldsymbol{u}_n - \boldsymbol{u}_\infty\rangle \\ &= \langle A\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}_\infty\rangle - \langle A\boldsymbol{u}_\infty, \boldsymbol{u}_n - \boldsymbol{u}_\infty\rangle \\ &= \langle A\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u}'_n\rangle + \langle A\boldsymbol{u}_n, \boldsymbol{u}'_n - \boldsymbol{u}_\infty\rangle - \langle A\boldsymbol{u}_\infty, \boldsymbol{u}_n - \boldsymbol{u}_\infty\rangle \\ &\le \langle A\boldsymbol{u}_n, \boldsymbol{u}'_n - \boldsymbol{u}_\infty\rangle - \langle A\boldsymbol{u}_\infty, \boldsymbol{u}_n - \boldsymbol{u}_\infty\rangle \to 0,\end{aligned}$$

as $n \to \infty$. That means

$$\boldsymbol{u}_n \to \boldsymbol{u}_\infty, \quad \text{strongly in } \mathbb{B}.$$

Therefore, we have proved the theorem. □

From this theorem and Mazur Theorem (see [28] p120), we immediately have

COROLLARY 4.9. – *Let* $\{\mathbb{K}_n\}_n$, $n \in \mathbb{N} \cup \{\infty\}$, *be closed convex sets of* $\mathbb{B}$ *such that* $\mathbb{K}_n \to \mathbb{K}_\infty$, *in Mosco sense, when* $n \to \infty$. *If* A *satisfies the assumptions* (**A1**)-(**A4**), $\{\boldsymbol{u}_n\}_n$ *be any solution sequence of the problem* (1.1) *with the closed convex set* $\{\mathbb{K}_n\}_n$, $n \in \mathbb{N}$, *then there exists at least a solution* $\boldsymbol{u}_\infty$ *of the problem* (1.1) *with closed convex set* $\mathbb{K}_\infty$ *which can be approximated strongly by a convex combination of the sequence* $\{\boldsymbol{u}_n\}_n$ *in* $\mathbb{B}$, *i.e.*

$$\left\|\boldsymbol{u}_\infty - \sum_{j=1}^{n} \lambda_j \boldsymbol{u}_j\right\|_{\mathbb{B}} \to 0, \quad \textit{as } n \to \infty,$$

for some $\lambda_j \ge 0, \sum_{j=1}^n \lambda_j = 1$. □

Now, we give some applications of the convergences on the regularities. In Subsection 3.1, we have considered the regularity of the solution of problem (1.1) being $\mathbb{K}$ defined through a C^∞ (in fact C^0) operator T defined in $\mathbb{R}^N$. Since in the proof, we need T to be smooth over Ω. However, the regularity and the converging result only depend on the L^∞-norm of the coefficients of T. Also, T definded on Ω is enough. This means we can improve this result by supposing $d_\alpha \in L^\infty(\Omega)$ only. The discussion on the convergence of the closed convex sets in Theorem 4.5 and Theorem 4.8 make it possible.

THEOREM 4.10. – *Suppose that A satisfies* **(A1)**-**(A4)**; $\mathbb{K}$ *is defined in* (1.3) *verifying* **(C1)**-**(C3)**, *where X contains a ball B_{0,R_0}, T is a differential operator F defined in* (2.2) *with* $R_0 > \max\left\{\|d_0\|_\infty \cdot \max_{solution}\{\|\boldsymbol{u}\|_\infty\},\ 0\right\}$, $d_\alpha \in L^\infty(\Omega)$ *only,* $|\alpha| \leq k$ *with* $M - k > N/p$. *Then there exists a solution* $\boldsymbol{u}$ *of the problem* (1.1) *with closed convex set* $\mathbb{K}$ *such that the results of Theorem* 3.1, 3.2, *Corollary* 3.3, 3.4 *still hold for this solution.*

Proof. – Let

$$T_\infty = \sum_{|\alpha|\leq k} d_{\alpha\infty}\partial_x^\alpha$$

be the operator satisfying the assumptions of this theorem, $d_{\alpha\infty}(x)$ is defined in $\mathbb{R}^N$, $d_{\alpha\infty} \equiv d_\alpha$ on Ω, $d_{\alpha\infty} \in L^\infty(\mathbb{R}^N)$, and $\|d_{\alpha\infty}\|_{L^\infty(\mathbb{R}^N)} \leq \|d_\alpha\|_{L^\infty(\Omega)}$. Then $T_\infty \equiv T$ on Ω. We can find a sequence of $\{T_n\}_n$ such that T_n is defined as

$$T_n = \sum_{|\alpha|\leq k} d_{\alpha n}\partial_x^\alpha, \qquad n \in \mathbb{N} \tag{4.7}$$

with $d_{\alpha n} \in C^\infty(\mathbb{R}^N)$, and

$$d_{\alpha n}(x) \to d_{\alpha\infty}(x), \text{ in } L^\infty(\Omega), \text{ as } n \to \infty.$$

Then,

$$T_n \to T_\infty, \quad \text{in } \mathcal{L}\big(\boldsymbol{W}^{M,p}(\Omega), \boldsymbol{L}^\infty(\Omega)\big), \quad \text{as } n \to \infty. \tag{4.8}$$

Letting $\mathbb{K}_n$, $n \in \mathbb{N}\cup\{\infty\}$, be defined as in (4.1) with operator T_n defined in (4.7), by Theorem 4.5 and (4.8), we know that $\{\mathbb{K}_n\}_n$ converges, in Mosco sense, to $\mathbb{K}_\infty$, where $\mathbb{K}_\infty$ is the $\mathbb{K}$ considered in this theorem.

Notice that in the converging process, the corresponding the solution sequence depends only on $\max_{|\alpha|\le k} \|d_{\alpha n}\|_{L^\infty(\mathbb{R}^N)}$, which is bounded by $\max_{|\alpha|\le k} \|d_{\alpha\infty}\|_{L^\infty(\mathbb{R}^N)} + 1$.

That is, by Theorem 4.8 and Corollary 4.9, there exists a solution $\boldsymbol{u}_\infty$ of problem (1.1) with a closed convex set $\mathbb{K}_\infty$ and a convex combination of the sequence of $\{\boldsymbol{u}_n\}_n$, - the solutions of problem (1.1) with the closed convex set $\mathbb{K}_n$, such that

$$\sum_{j=1}^{n} \lambda_j \boldsymbol{u}_j \to \boldsymbol{u}_\infty = \boldsymbol{u}, \qquad \text{in } \mathbb{B},$$

as $n \to \infty$.

By Theorems 3.1 and 3.2, there exists $0 < \varepsilon_n \le 1$, such that

$$\boldsymbol{u}_n \in \boldsymbol{W}^{M,\frac{p}{1-\varepsilon_n}}(\Omega).$$

Reviewing the proof of Theorem 3.1 and 3.2, we find out that the norm $\|\boldsymbol{u}_n\|_{\boldsymbol{W}^{M,\frac{p}{1-\varepsilon_n}}(\Omega)}$ and ε_n depend uniformly on the L^∞-norms of $d_{\alpha n}$, for $|\alpha| \le k$, with respect to n, so are the norm $\|\sum_{j=1}^n \lambda_j \boldsymbol{u}_j\|_{\boldsymbol{W}^{M,\frac{p}{1-\varepsilon'_n}}(\Omega)}$, where $\varepsilon'_n = \min_{0\le j\le n}\{\varepsilon_j\}$. Then taking $\varepsilon_0 = \min_{j\ge 0}\{\varepsilon_j\}$, which is positive by the uniform argument, from Vitali theorem (see [27] p59), we have

$$\boldsymbol{u} \in \boldsymbol{W}^{M,\frac{p}{1-\varepsilon_0}}(\Omega),$$

for some $0 < \varepsilon_0 < 1$. The other results then follow. □

THEOREM 4.11. – *Suppose that A satisfies* (**A1**)-(**A4**)*; $\mathbb{K}$ is defined in* (1.3) *verifying* (**C1**)-(**C3**)*, where X contains a cone* **C** *vertexed at the origin with $\overset{o}{\mathbf{C}} \ne \emptyset$ and a ball B_{0,R_0} with $R_0 > 0$, T is an integral operator defined by* (3.13)*, with $K \subseteq \Omega$, $\partial K \in C^{0,1}$, $\Gamma \ge 0$ is only in $L^\infty(\Omega, L^1(K))$, $M > N/p$. Then there exists a solution* $\boldsymbol{u}$ *of the problem* (1.1) *with closed convex set $\mathbb{K}$, such that the results of Theorem* 3.11 *and corresponding Corollary* 3.3, 3.4 *still hold for this solution.*

Proof. – By using Theorems 4.6 (when the order k of F_n equal 0), choose $T_n = \int_K dy\Gamma_n(x,y)$, with $\Gamma_n \in C^\infty(\Omega \times \Omega)$ converging to Γ in $L^\infty(\Omega, L^1(K))$, so that T_n converging to $T = \int_K dy\Gamma(x,y)$, in $\mathcal{L}(\boldsymbol{W}^{M,p}(\Omega), \boldsymbol{L}^\infty(\Omega))$.

Then noticing the uniform estimate (3.14), by Theorem 4.6 (when $k = 0$), we have $\mathbb{K}_n \to \mathbb{K}$ in Mosco sense, where $\mathbb{K}_n$ and $\mathbb{K}$ defined by (4.1) and (1.3) respectively, with operator T_n and T defined above respectively. So, by Corollary 4.9 there exists a solution $\boldsymbol{u}$ of problem (1.1) with closed convex set $\mathbb{K}$ and a convex combination of the sequence of $\{\boldsymbol{u}_n\}_n$, - the solution of problem (1.1) with closed convex set $\mathbb{K}_n$, such that $\sum_{j=1}^{n} \lambda_j \boldsymbol{u}_j \to \boldsymbol{u}$, in $\mathbb{B}$, but by Theorem 3.11, $\boldsymbol{u}_n \in \boldsymbol{W}^{M,\frac{p}{1-\varepsilon_n}}(\Omega)$ uniformly with respect to n, so are $\sum_{j=1}^{n} \lambda_j \boldsymbol{u}_j$. Then by Vitali theorem, $\boldsymbol{u} \in \boldsymbol{W}^{M,\frac{p}{1-\varepsilon_0}}(\Omega)$, for some $0 < \varepsilon_0 < 1$. Thus, the results follow. □

Similarly, we have

THEOREM 4.12. – *Suppose that A satisfies* (**A1**)-(**A4**)*; $\mathbb{K}$ is defined in* (1.3) *verifying* (**C1**)-(**C3**)*, where X contains a ball B_{0,R_0}, T is an integral operator defined by* (3.13)*, with $K \subseteq \Omega$, $\partial K \in C^{0,1}$, Γ being only in $L^\infty(\Omega, L^\theta(K))$, θ defined in* (3.14)*; $R_0 > \|\Gamma\|_{L^\infty(\Omega,L^\theta(K))} \cdot \max_{solution}\{\|\boldsymbol{u}\|_{\theta'}\}$. Then there exists a solution $\boldsymbol{u}$ of the problem* (1.1) *with closed convex set $\mathbb{K}$, such that the results of Theorem* 3.12 *and corresponding Corollary* 3.3, 3.4 *still hold for this solution.* □

The following result is much more advanced in a certain sense than the one obtained in Subsection 3.3. Review the proof, we can see that the assumption on the coefficients of d_α in F is at least $C^{|\alpha|}(\mathbb{R}^N)$, but the regularity result does not depend on the $C^{|\alpha|}$-norm but $L^\infty(K)$-norm (or $W_0^{|\alpha|,\infty}(K)$-norm if $K \neq \Omega$) of d_α. That is,

THEOREM 4.13. – *Suppose that A satisfies* (**A1**)-(**A4**)*; $\mathbb{K}$ is defined in* (1.3) *verifying* (**C1**)-(**C3**)*, where X contains a cone $\mathbf{C}$ vertexed at the origin with $\overset{o}{\mathbf{C}} \neq \emptyset$ and a ball B_{0,R_0} with $R_0 > 0$, T is an integrodifferential operator defined in* (3.19) *satisfying*

$$\int_{\mathbb{R}^N} \sum_{|\alpha| \le k} d_\alpha(x) \partial_x^\alpha \zeta(x) dx > 0, \quad 0 \le \forall \zeta(x) \in C_0^\infty(\mathbb{R}^N), \tag{4.9}$$

with Γ being a constant, $K = \Omega$ and $d_\alpha \in L^\infty(K)$ only; or $K \subset \Omega$, $K \neq \Omega$, $\partial K \in C^{M-1,1}$ and $d_\alpha \in W_0^{|\alpha|,\infty}(K)$, for $M - k \ge 0$, $M > N/p$. Then there exists a solution $\boldsymbol{u}$ of the problem (1.1) *with closed convex set $\mathbb{K}$, such that the results of Theorem* 3.14 *and corresponding Corollary* 3.3, 3.4 *still hold for this solution.*

Proof. – The proof is almost the same as the one of Theorem 4.10. By using Theorems 4.6, we can choose

$$T_n = \int_K dx F_n = \int_K dx \sum_{|\alpha| \le k} d_{\alpha n}(x) \partial_x^\alpha,$$

with $d_{\alpha n} \in C_0^\infty(\mathbb{R}^N)$ converges to d_α in $L^\infty(K)$ (or in $W_0^{|\alpha|,\infty}(K)$, if $K \subset \Omega$ and $K \neq \Omega$), so that, by (4.9), when n is big enough, F_n

verifies (2.3). Therefore T_n converges to

$$T = \int_K dx F = \int_K dx \sum_{|\alpha|\leq k} d_\alpha(x)\partial_x^\alpha,$$

in $\mathcal{L}(\boldsymbol{W}^{M,p}(\Omega), \boldsymbol{L}^\infty(\Omega))$, and T_n satisfies the conditions of Theorem 3.14. By Theorem 4.6, $\mathbb{K}_n$ with operator T_n converges to $\mathbb{K}$ with operator T in Mosco sense. Then by Corollary 4.9, there exists a solution $\boldsymbol{u}$ of problem (1.1) with closed convex set $\mathbb{K}$ and a convex combination of the sequence of $\{\boldsymbol{u}_n\}_n$, - the solution of problem (1.1) with closed convex set $\mathbb{K}_n$, such that $\sum_{j=1}^n \lambda_j \boldsymbol{u}_j \to \boldsymbol{u}$, in $\mathbb{B}$, but by Theorem 3.14, $\boldsymbol{u}_n \in \boldsymbol{W}^{M,\frac{p}{1-\varepsilon_n}}(\Omega)$ uniformly with respect to n, so are $\sum_{j=1}^n \lambda_j \boldsymbol{u}_j$. Then by Vitali theorem, $\boldsymbol{u} \in \boldsymbol{W}^{M,\frac{p}{1-\varepsilon_0}}(\Omega)$, for some $0 < \varepsilon_0 < 1$. Thus, the results follow. □

Similarly, consider also $\Gamma_n \to \Gamma$ in $L^\infty(\Omega, L^\tau(K))$ and $d_{\alpha n} \to d_\alpha$ in $L^\infty(K)$, we have

THEOREM 4.14. – *Suppose that A satisfies* (**A1**)-(**A4**)*; $\mathbb{K}$ is defined in* (1.3) *verifying* (**C1**)-(**C3**)*, where X also containing a ball B_{0,R_0}, T is an integrodifferential operator defined in* (3.19) *with order of F, $k \leq M$, and $K \subseteq \Omega$, $\partial K \in C^{M-1,1}$, $\Gamma \in L^\infty(\Omega, L^\tau(K))$, $d_\alpha \in L^\infty(\Omega)$ only, for $|\alpha| \leq k$; $R_0 > \|\Gamma\|_{L^\infty(\Omega, L^\tau(K))} \cdot \max_{solution} \|F\boldsymbol{u}\|_{\tau'}$. Then there exists a solution $\boldsymbol{u}$ of the problem* (1.1) *with closed convex set $\mathbb{K}$, such that the results of Theorem* 3.16 *and corresponding Corollary* 3.3, 3.4 *still hold for this solution.* □

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