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B. BONNARD

I. KUPKA

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Generic properties of singular trajectories

by

B. BONNARD *

Université de Bourgogne, Département de Mathématiques,
Laboratoire de Topologie, BP 138, 21004 Dijon Cedex, France.
E-mail: bbonnard@satie.u-bourgogne.fr

and

I. KUPKA

University of Toronto, Department of Mathematics,
M5S1A1 Toronto, Canada.

ABSTRACT. – Let M be a σ-compact C^∞ manifold of dimension $d \geq 3$. Consider on M a single-input control system : $\dot{x}(t) = F_0(x(t)) + u(t)\ F_1(x(t))$, where F_0, F_1 are C^∞ vector fields on M and the set of admissible controls $\mathcal{U}$ is the set of bounded measurable mappings $u : [0, T_u] \longmapsto \mathbf{R}$, $T_u > 0$. A *singular trajectory* is an output corresponding to a control such that the differential of the input-output mapping is not of maximal rank. In this article we show that for an open dense subset of the set of pairs of vector fields (F_0, F_1), endowed with the C^∞-Whitney topology, all the singular trajectories are with *minimal order* and *the corank of the singularity is one.*

Key words: Nonlinear systems, optimal control, singular extremals, generic properties.

* Mailing address.

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RÉSUMÉ. – Soit M une variété C^∞, à base dénombrable et un système mono-entrée sur M : $\dot{x}(t) = F_0(x(t)) + u(t)\ F_1(x(t))$, où F_0 et F_1 sont des champs de vecteurs C^∞, la classe des contrôles admissibles $\mathcal{U}$ étant l'ensemble des applications $u : [0, T(u)] \longmapsto \mathbf{R}$, $T(u) > 0$, mesurables et bornées. L'objet de cette note est de montrer que pour un ensemble ouvert et dense de couples de champs de vecteurs (F_0, F_1), pour la topologie C^∞ de Whitney, toutes les *trajectoires singulières* sont d'*ordre minimal* et la *singularité est de codimension un.*

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0. INTRODUCTION AND NOTATIONS

We shall denote by M a σ-compact manifold of dimension $d \geq 3$. Smooth means either C^∞ or C^ω. We shall use the following notations :

TM : tangent space of M, T_mM : tangent space at $m \in M$.

T^*M : cotangent space of M, T^*_mM : cotangent space at $m \in M$.

The null section of T^*M is denoted by 0 and $(T^*M)_0 = T^*M \backslash \{0\}$.

PT^*M : projectivized cotangent space $(PT^*M = T^*M/\mathbf{R}^*)$.

$[z]$: class of z in PT^*M.

For any integer $N \geq 1$, J^NTM : space of all N-jets of vector fields (*i.e.* : smooth sections of TM over open subsets of M).

Π^N_M : $J^NTM \longmapsto M$: canonical projection.

$VF(M)$: vector space of all vector fields defined on M endowed with the Whitney topology.

$E \times_M F$: fiber product of two fibers spaces (E, Π_E, M) and (F, Π_F, M) on M.

$\vec{H}$: given any smooth function H defined on an open subset $\Omega \subset T^*M$, $\vec{H}$ will denote the Hamiltonian vector field defined by H on Ω.

$\{H_1, H_2\}$: given any two smooth functions on Ω, $\{H_1, H_2\}$ will denote their Poisson bracket : $\{H_1, H_2\} = dH_1(\vec{H}_2)$.

Span A : if A is a subset of a vector space V, it is the vector subspace generated by A.

To each couple (F_0, F_1) of vector fields on M we associate the control system :

$$(0) \qquad \frac{dx}{dt}(t) = F_0(x(t)) + u(t)\ F_1(x(t)), \qquad u(t) \in \mathbf{R}.$$

The study of time-minimal trajectories of (0) leads to the consideration of extremal trajectories : $(z, u) : J \longmapsto T^*M \times \mathbf{R}$, J interval $[T_1, T_2]$, $T_1 < T_2$, is an extremal curve of system (0) if :

1) z is absolutely continuous, u is measurable and bounded;

2) $z(t) \neq 0$ (0 = null section) for all $t \in J$ and;

3) $\frac{dz}{dt}(t) = \overrightarrow{H}_0(z(t)) + u(t)\ \overrightarrow{H}_1(z(t))$ for a.e. $t \in J$, where $H_i : T^*M \longmapsto \mathbf{R}$, $i = 1, 2$, $H_i(z) = \langle F_i(\Pi_{T^*M}(z)), z\rangle$;

4) for a.e $t \in J$, $H_0(z(t)) + u(t)\ H_1(z(t)) = \max_{v\in\mathbf{R}}\{H_0(z(t)) + v\ H_1(z(t))\}$.

More precisely 4) is equivalent to $H_1(z(t)) = 0$ for a.e. $t \in J$. But since $H_1(z)$ is continuous this is equivalent to

4′) $H_1(z(t)) = 0\ \forall t \in J$.

DEFINITION 0. – *A curve* $(z, u) : J \longmapsto T^*M \times \mathbf{R}$ *satisfying the conditions 1-2-3-4′) above will be called a singular extremal and* $(\Pi_{T^*M}(z), u)$ *a singular trajectory.*

MOTIVATION. – Our study is motivated by the following facts. The singular trajectories play an important role in system theory. First of all they are solutions of Pontryagin's maximum principle, for the time-optimal control problem, *see* [BK1] and are the so-called abnormal extremals in subriemannian geometry (and more generally in classical calculus of variations). Secondly, they are invariants for the feedback classification problem, *see* [B2]. Hence they have to be computed and their properties and singularities deeply analyzed. Also similar constrained hamiltonian systems appear in quantum theory, *see* [HT].

1. DETERMINATION OF THE SINGULAR EXTREMALS

Let $(z, u) : J \longmapsto T^*M \times \mathbf{R}$ be such a curve. Using the chain rule and condition 4′) we get:

$$0 = \frac{d}{dt}(H_1(z(t))) = dH_1(z(t))\overrightarrow{H}_0(z(t)) + u(t)\ dH_1\ (z(t))\ \overrightarrow{H}_1\ (z(t))$$

for a.e. $t \in J$.

This implies: $0 = \{H_1, H_0\}(z)$ since the function $\{H_1, H_0\}(z)$ is continuous. Using the chain rule again we get

$$\begin{aligned} 0 &= \frac{d}{dt}\{H_1, H_0\}\ (z(t)) \\ &= \{\{H_1, H_0\}, H_0\}\ (z(t)) + u(t)\ \{\{H_1, H_0\},\ H_1\}\ (z(t)), \\ &\qquad \text{for a.e. } t \in J. \end{aligned}$$

This last relation enables us to compute $u(t)$ in many cases and justifies the following definition.

DEFINITION 1. – *For any singular extremal* $(z, u) : J \longmapsto T^*M \times \mathbf{R}$, $\mathcal{R}(z, u)$ *will denote the set* $\{t/t \in J,\ \{\{H_0, H_1\}, H_1\}\ (z(t)) \neq 0\}$. *The set* $\mathcal{R}(z, u)$, *possibly empty, is always an open subset of* J.

DEFINITION 2. – *A singular extremal* $(z, u) : J \longmapsto T^*M \times \mathbf{R}$ *is called of minimal order if* $\mathcal{R}(z, u)$ *is dense in* J.

The following Proposition is an immediate consequence of Definition 1 and the considerations above.

PROPOSITION 0. – *If* $(z, u) : J \longmapsto T^*M \times \mathbf{R}$ *is a singular extremal and* $\mathcal{R}(z, u)$ *is not empty*

1) z *restricted to* $\mathcal{R}(z, u)$ *is smooth;*

2) $\frac{dz(t)}{dt} = \overrightarrow{H}_0\ (z(t)) + \frac{\{\{H_0, H_1\},\ H_0\}\ (z(t))}{\{\{H_1, H_0\},\ H_1\}\ (z(t))}\ \overrightarrow{H}_1\ (z(t))$ *for all* $t \in \mathcal{R}(z, u)$;

3) $u(t) = \frac{\{\{H_0, H_1\},\ H_0\}}{\{\{H_1, H_0\},\ H_1\}}\ (z(t))$ *for a.e.* $t \in \mathcal{R}(z, u)$.

The minimal order singular extremals are the easiest to compute and there are usually a lot of them as follows from the Proposition:

PROPOSITION 1. – (i) *Let* $(F_0, F_1) \in VF(M) \times VF(M)$ *be a pair such that the open subset* Ω *of all* $z \in (T^*M)_0$, *such that* $\{\{H_0, H_1\},\ H_1\}(z) \neq 0$, *is not empty. If* $H : \Omega \longmapsto \mathbf{R}$ *is the function* $H_0 + \frac{\{\{H_0, H_1\},\ H_0\}}{\{\{H_1, H_0\},\ H_1\}}\ H_1$, *then any trajectory of* $\overrightarrow{H}$, *starting at* $t = 0$ *from the set* $H_1 = \{H_1, H_0\} = 0$ *is a minimal order singular extremal of* (F_0, F_1).

(ii) *There is an open subset of* $VF(M) \times VF(M)$ *such that for any couple* (F_0, F_1) *in that subset the set* Ω *is open dense in* T^*M.

Remark 0. – The set of all $z \in \Omega$ such that $H_1(z) = \{H_1, H_0\}(z) = 0$, is a codimension 2 symplectic submanifold of Ω.

Our first main result says that for most systems, the only singular extremals are the minimal order ones:

2. MAIN RESULTS

THEOREM 0. – *There exists an open dense subset G of $VF(M) \times VF(M)$ such that for any couple $(F_0, F_1) \in G$, the associated control system (0) has only minimal order singular extremals.*

Our second result shows that these singular extremals are uniquely determined by their projections on M.

THEOREM 1. – *There exists an open dense subset G_1 in G such that for any couple $(F_0, F_1) \in G_1$ if $z_i : J \to (T^*M)_0$, $i = 1, 2$ are two extremals of the system (0) associated to (F_0, F_1) and if $\Pi_{T^*M}(z_1) = \Pi_{T^*M}(z_2)$, then there exists a $\lambda \in \mathbf{R}^*$ such that $z_2 = \lambda z_1$.*

3. THE AD-CONDITIONS AND THE "BAD" SETS

To prove Theorem 0, we are going to define, for each integer N sufficiently large a "bad" set $B(N)$ in $J^N TM \ \times_M \ J^N TM$ having the following property: any couple $(F_0, F_1) \in VF(M) \times VF(M)$ such that $(j_x^N F_0, j_x F_1) \notin B(N) \, \forall x \in M$, has only minimal order singular extremals. Then we shall show using transversality theory that the set G of all couples $(F_0, F_1) \in VF(M) \times VF(M)$ such that $(j_x^N F_0, \ j_x^N F_1) \notin \overline{B(N)}$ for all $x \in M$ is open dense in $VF(M) \times VF(M)$.

To construct the bad set we have to analyze two cases. First, we consider the points x where F_0 and F_1 are *linearly dependent.* This situation can be studied straightforwardly and we show that the bad set has *finite codimension.* When F_0 and F_1 are *linearly independent* the situation is more complicated. The bad set is constructed using the following idea. If there exist singular trajectories which are not of minimal order, they are solutions of a smooth vector field tangent to a surface of codimension one. Differentiating along trajectories we get an infinite number of equations. This defines a bad set of *infinite codimension.*

Now let us define the bad sets.

DEFINITION 3. – *For $N \geq 2d - 1$, let $B_a(N)$ be the subset of $J^N TM \ \times_M \ J^N TM$ of all couples $(j_x^N F_0, \ j_x^N F_1)$ such that*

$$\dim \text{Span} \, \{ad^i \, F_0 \, (F_1)(x) / 0 \leq i \leq 2d - 1\} < d.$$

Here $ad^i\ F_0(F_1) = [ad^{i-1}\ F_0(F_1), F_0]$, $ad^0\ F_0(F_1) = F_1$.

DEFINITION 4. – (i) *For* $N \geq 1$, $B'_\ell(N)$ *is the subset of* $J^NTM \times_M J^NTM$ *of all couples* $(j_x^N\ F_0,\ j_x^N\ F_1)$ *such that* dim Span $\{F_0(x),\ F_1(x),\ [F_0,\ F_1](x)\} \leq 1$.

(ii) *For* $N \geq 2$, *let* $\widehat{B}''_\ell(N)$ *be the subset of* $J^NTM \times_M J^NTM \times \mathbf{R}$ *of all triples* $(j_x^N\ F_0,\ j_x^N\ F_1,\ a)$ *such that:*

1) $F_1(x) \neq 0$;

2) $F_0(x) = aF_1(x)$;

3) dim Span $\{ad^i\ G_a\ (F_1)(x),\ 0 \leq i \leq d-1,\ [[F_0, F_1],\ F_1](x)\} < d$, *where* $G_a = F_0 - aF_1$.

(iii) *Denote by* $B''_\ell(N)$ *the canonical projection of* $\widehat{B}''_\ell(N)$ *onto* $J^NTM \times_M J^NTM$.

(iv) $B_\ell(N) = B'_\ell(N) \cup B''_\ell(N)$.

For the next definition we need some notations.

Notation. – For any multi-index $\alpha \in \{0,1\}^n$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = n$ $|\alpha|_0 =$ card $\{i/\alpha_i = 0\}$, $|\alpha|_1 =$ card $\{i/\alpha_i = 1\}$. The function $H_\alpha : T^*M \longmapsto \mathbf{R}$ is defined inductively by: $H_\alpha = \{H_{(\alpha_1,\cdots,\alpha_{n-1})},\ H_{\alpha_n}\}$.

Remark 1. – 1) If $\alpha_1 = \alpha_2$, $H_\alpha = 0$.

2) $H_{(0,1,\alpha_3,\cdots,\alpha_n)} = -H_{(1,0,\alpha_3,\cdots,\alpha_n)}$.

DEFINITION 5. – (i) For any integers $c \geq 0$, any $\alpha \in \{0,1\}^n$, $n \geq 3$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_1 = 1$, such that $\alpha \neq 10^{n-1}$ (resp. $\alpha \neq 101^{n-2}$), any integer $N \geq n + c - 1$, let $\widehat{B}(N, \alpha, c, 0)$ (resp. $\widehat{B}(N, \alpha, c, 1)$) be the subset of $J^NTM \times J^NTM \times_M PT^*M$ of all triples $(j_x^N\ F_0, j_x^N\ F_1, [z])$ such that:

(1) $F_0(x)$, $F_1(x)$ are linearly independent;

(2) $H_{\alpha 0}(z) \neq 0$, $H_{\alpha 1}(z) \neq 0$;

(3) $\theta(Z_\alpha)^k\ H_{10^{n-1}}(z) = 0$ (resp. $\theta(Z_\alpha)^k\ H_{101^{n-2}}(z) = 0$) for $0 \leq k \leq c$, where Z_α is the vector field $H_{\alpha 1}\ \overrightarrow{H}_0 - H_{\alpha 0}\ \overrightarrow{H}_1$ on T^*M. (Observe that in general Z_α is not Hamiltonian).

(ii) $B(N, \alpha, c, \sigma)(\sigma = 0, 1)$ will denote the canonical projection of $\widehat{B}(N, \alpha, c, \sigma)$ onto $J^NTM \times_M J^NTM$.

DEFINITION 6. – $B(N) = B_a(N) \cup B_\ell(N) \cup \bigcup\{B(N, \alpha, 2d, \sigma)/|3| \leq |\alpha| \leq 2d,\ \sigma \in \{0,1\}\}$.

Now we check that $B(N)$ has the first property stated at the beginning of the present paragraph:

Fundamental Lemma 0. – Let a couple $(F_0, F_1) \in VF(M)^2$ which satisfies the condition: there exists an integer N such that for all $x \in M$,

$(j_x^N F_0, j_x^N F_1) \notin B(N)$. Then the control system associated to (F_0, F_1) has only minimal order singular extremals.

Before proving this basic lemma we shall prove an auxiliary result which would also be useful later.

LEMMA 1. – *Let* $(F_0, F_1) \in VF(M)^2$ *such that for all* $x \in M$, $(j_x^N\ F_0, j_x^N\ F_1) \notin B_\ell(N)$.

1) *Let* $(\overline{z}, \overline{u}) : J \longmapsto T^*M \times \mathbf{R}$ *be a singular extremal such that for all* $t \in J$, dim Span $\{F_0(\overline{x}(t)),\ F_1(\overline{x}(t))\} \leq 1$ *where* $\overline{x} = \Pi_{T^*M}(\overline{z})$. *Then* $\overline{x}(t)$ *is constant.*

2) *Let* $x_0 \in M$. *If* $T^*_{x_0}M$ *contains a singular extremal, then there exist a* $\lambda \in \mathbf{R}$ *and a line* $\ell \in T^*_{x_0}M$, *such that every singular extremal* $(z', u') : J' \to T^*_{x_0}M \times \mathbf{R}$ *is of the form* $z'(t) = e^{\lambda t}\ z_0$, $z_0 \in \ell$ *and* u' *is constant a.e. All these extremals are of minimal order.*

Proof of Lemma 1. – 1) Call S the set of all $t \in J$ such that $F_1(\overline{x}(t)) = 0$. S is closed and has empty interior: otherwise there exists an open non empty interval $J_1 \subset J$ such that $F_1(\overline{x}(t)) = 0$ for all $t \in J_1$. Then $\dot{\overline{x}}(t) = F_0(\overline{x}(t))$ for all $t \in J_1$. This implies that $[F_1, F_0]\ (\overline{x}(t)) = 0$ for all $t \in J_1$. Then dim Span $\{F_0(\overline{x}(t)),\ F_1(\overline{x}(t)),\ [F_1, F_0]\ (\overline{x}(t))\} \leq 1$ for $t \in J_1$. This contradicts the assumption of Lemma 1. Since dim Span $\{F_0(\overline{x}(t)), F_1(\overline{x}(t))\} \leq 1$ for all $t \in J$, there exists an absolutely continuous function $a : J\backslash S \longmapsto \mathbf{R}$ such that $F_0(\overline{x}(t)) = a(t)\ F_1(\overline{x}(t))$ for all $t \in J\backslash S$. This implies that for a.e. $t \in J\backslash S$ $(a(t) + \overline{u}(t))\ [F_0, F_1]\ (\overline{x}(t)) = \dot{a}(t)\ F_1(\overline{x}(t))$ because $\dot{\overline{x}}(t) = F_0(\overline{x}(t)) + \overline{u}(t)\ F_1(\overline{x}(t)) = (a(t) + \overline{u}(t))\ F_1(\overline{x}(t))$ for a.e. $t \in J\backslash S$. Hence $\dot{a}(t) = a(t) + \overline{u}(t) = 0$ for a.e. $t \in J\backslash S$: since by the assumption of Lemma 1, dim Span $\{F_0(\overline{x}(t)),\ F_1(\overline{x}(t)),\ [F_0, F_1]\ (\overline{x}(t))\} \geq 2$ and since $F_0(\overline{x}(t)) = a(t)\ F_1(\overline{x}(t))$, $F_1(\overline{x}(t))$ and $[F_0, F_1]\,(\overline{x}(t))$ are linearly independent for all $t \in J\backslash S$.

Suppose that $S \neq \emptyset$. Then the open set $J\backslash S$ contains an interval $]\alpha, \beta[= \{t/\alpha < t < \beta\}$, where either $\alpha \in S$ or $\beta \in S$. Assume that $\alpha \in S$ ($\beta \in S$ is similar). Since $\dot{a}(t) = a(t) + \overline{u}(t) = 0$ for a.e. $t \in\ ]\alpha, \beta[$, a is constant on $]\alpha, \beta[$ and $\overline{u}(t) = -a$ for a.e. $t \in\ ]\alpha, \beta[$. Hence $\dot{\overline{x}}(t) = 0$ for a.e. $t \in\ ]\alpha, \beta[$. So $\overline{x}(t) = x_0$ for all $t \in]\alpha, \beta[$. This leads to the contradiction $0 = F_1(\overline{x}(\alpha)) = \lim\limits_{\substack{t \to \alpha \\ t \in]\alpha, \beta[}} F_1(\overline{x}(t)) = F_1(x_0) \neq 0$. Hence $S = \emptyset$ and $\overline{x}(t) = x_0$ for all $t \in J$. This proves 1).

2) Let $(\overline{z}, \overline{u}) : J \to T^*M \times \mathbf{R}$ be a singular extremal such that $\overline{z}(t) \in T^*_{x_0}M$ for all $t \in J$. The assumption of Lemma 1 implies that dim Span $\{F_0(x_0),\ F_1(x_0),\ [F_0, F_1]\ (x_0)\} \geq 2$. If $F_1(x_0) = 0$, then $F_0(x_0) \neq 0$ but we have for a.e. $t \in J : 0 = \dot{\overline{x}}(t) = F_0(x_0)$. We have a

contradiction. Since $0 = \dot{\overline{x}}(t) = F_0(x_0) + \overline{u}(t)F_1(x_0)$ for a.e. $t \in J$, there exists an $a \in \mathbf{R}$ such that $F_0(x_0) = aF_1(x_0)$ and $\overline{u}(t) = -a$ for a.e. $t \in J$.

Let $G = F_0 - aF_1 : G(x_0) = 0$ and set $H(z) = \langle G(\Pi_{T^*M}(z), z\rangle$. By definition $\dot{\overline{z}}(t) = \overrightarrow{H}(\overline{z}(t))$ for all $t \in J$ and $H_1(\overline{z}(t)) = 0$. Hence $ad^k\ H(H_1)(\overline{z}(t)) = \langle ad^k G(F_1)\ (\Pi_{T^*M}(\overline{z}(t)),\ \overline{z}(t)\rangle = 0$ for all $t \in J$ and all $k \in N$. Since $G(x_0) = 0$, Span $\{ad^k\ G(F_1)(x_0)\ ;\ k \in N\} = \text{Span}\ \{ad^k\ G(F_1)(x_0),\ 0 \leq k \leq d-1\}$. By assumption x_0 is a singular trajectory, hence this space is at least of codimension one. Since $(j^N_{x_0}\ F_0,\ j^N_{x_0}\ F_1) \notin B_\ell(N)$, it is exactly of codimension one, and moreover $\langle [F_1, [F_0, F_1](x_0)],\ \overline{z}(t)\rangle \neq 0$. Therefore $\overline{z}(t)$ belongs to a line $\ell \in T^*_{x_0}M$ and $(\overline{z}(t), \overline{u}(t))$ is of minimal order. By definition $\overline{z}(t)$ is solution of a linear system and since $G(x_0) = 0$, it is autonomous. Hence 2) is proved.

Proof of Lemma 0. – Assume that $(\overline{z}, \overline{u})\ :\ J \longrightarrow T^*M \times \mathbf{R}$ is a singular extremal not of minimal order. This shows that there exists an open subinterval J_0 of J, J_0 not empty, such that $\{\{H_0, H_1\}, H_1\}\ (\overline{z}(t)) = 0$. Then the closed set $\{t \in J_0/\dim\ \{F_0(\overline{x}(t), F_1(\overline{x}(t))\} \leq 1\}, \overline{x} = \Pi_{T^*M}(\overline{z})$, has an empty interior: otherwise it would contain an open non empty interval $J_{01} \subset J_0$. Then Lemma 1 applies to the restriction (z', u') of $(\overline{z}, \overline{u})$ to J_{01}. But since $\{\{H_0, H_1\}, H_1\}(z') = 0$ we get a contradiction. Replacing J by an open non empty subinterval we can assume that for all $t \in J$:

1) $\{\{H_0, H_1\}, H_1\}(\overline{z}(t)) = 0$;

2) dim Span $\{F_0(\overline{x}(t)),\ F_1(\overline{x}(t))\} = 2$.

Since $\{\{H_0, H_1\}, H_1\}(\overline{z}) = 0$, then

$$\{\{H_0, H_1\}, H_0\}(\overline{z}) = \frac{d}{dt}\ (\{H_0, H_1\}(\overline{z})) = 0.$$

We claim that there exists a multi index $\alpha \in \{0,1\}^n$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_1 = 1$ such that

(i) $3 \leq n \leq 2d$;

(ii) $H_\beta(\overline{z}) = 0$ for all $\beta \in \{0,1\}^k$, $\beta = (\beta_1, \cdots, \beta_k)$, $\beta_1 = 1$, $1 \leq k \leq n$;

(iii) either $H_{\alpha 0}(\overline{z}) \neq 0$ or $H_{\alpha 1}(\overline{z}) \neq 0$.

In fact were there no such α, then $H_\beta(\overline{z}) = 0$ for all $\beta \in \{0,1\}^k$, $\beta = (\beta_1, \cdots, \beta_k)$, $1 \leq |\beta| \leq 2d$. In particular taking $\beta = 10^k$, $0 \leq k \leq 2d-1$, we get $\langle ad^k\ F_0(F_1)\ (\overline{x}(t)),\ \overline{z}(t)\rangle = 0$, $0 \leq k \leq 2d-1$, for all $t \in J$. This shows that dim Span $\{ad^k\ F_0(F_1)\ (\overline{x}(t)),\ 0 \leq k \leq 2d-1\} < d$ for all $t \in J$ and contradicts the assumption that $(j^N_x\ F_0,\ j^N_x\ F_1) \notin B_a(N)$ for all $x \in M$.

The assumption (iii) above can be replaced by the following: (iv) the set $\{t \in J / H_{\alpha 0}(\overline{z}(t)) \neq 0$ and $H_{\alpha 1}\ (\overline{z}(t)) \neq 0\}$ is non empty. In fact, suppose we have either $H_{\alpha 1}(\overline{z}) = 0$ or $H_{\alpha 0}(\overline{z}) = 0$. In the first case we get $0 = \frac{d}{dt}\ H_\alpha(\overline{z}) = H_{\alpha 0}(\overline{z})$. This would contradict (iii). In the second case $H_{\alpha 1}(\overline{z}) \neq 0$ by (iii). On the non empty open set $\mathcal{O} = \{t / t \in J,\ H_{\alpha 1}(\overline{z})) \neq 0\}$, we have almost everywhere: $0 = \frac{d}{dt}(H_\alpha(\overline{z})) = \overline{u}\ H_{\alpha 1}(\overline{z})$. Hence $\overline{u} = 0$ a.e. on $\mathcal{O}$. Then $\dot{\overline{z}} = \overrightarrow{H}_0(\overline{z})$ on $\mathcal{O}$. Since $H_1(\overline{z}) = 0$, we get $0 = \frac{d^k}{dt^k}\ H_1(z(t)) = ad^k\ H_0(H_1)(\overline{z}(t))$ for all $t \in \mathcal{O}$. This implies that for any $t \in \mathcal{O}\langle ad^k F_0(F_1)(\overline{x}(t)),\ \overline{z}(t))\rangle = 0$ for all $k \geq 0$. This contradicts the assumption that $(j_x^N F_0,\ j_x^N\ F_1) \notin B_a(N)$ for all $x \in M$.

Finally replacing J by a subinterval we see that we can assume that:

(1) $3 \leq n = |\alpha| \leq 2d$;

(2) $H_\beta(\overline{z}) = 0$ for all $\beta = (\beta_1, \cdots, \beta_k),\ \beta_1 = 1,\ 1 \leq k \leq n$;

(3) $H_{\alpha 0}(\overline{z}(t)) \neq 0,\ H_{\alpha 1}(\overline{z}(t)) \neq 0$ for all $t \in J$.

Since $H_\alpha(\overline{z}) = 0$ we get for a.e. $t \in J : 0 = \frac{d}{dt}\ (H_\alpha(\overline{z}(t))) = H_{\alpha 0}(\overline{z}(t)) + \overline{u}(t)\ H_{\alpha 1}\ (\overline{z}(t))$.

So $\overline{u}(t) = -\frac{H_{\alpha 0}}{H_{\alpha 1}}\ (\overline{z}(t))$ for a.e. $t \in J$. Since $H_1(\overline{z}) = 0$, this shows that $\overline{z}$ is a trajectory of $\overrightarrow{\mathcal{H}}_\alpha$ where $\mathcal{H}_\alpha : \Omega \longmapsto \mathbf{R}$ is the function $H_0 - \frac{H_{\alpha 0}}{H_{\alpha 1}}\ H_1$ and $\Omega = \{z / H_{\alpha 1}(z) \neq 0\}$. Define now γ to be 10^{n-1} if $\alpha \neq 10^{n-1}$ and 101^{n-2} if $\alpha = 10^{n-1}$. Since $|\gamma| = n,\ H_\gamma(\overline{z}) = 0$ and:

$$0 = \frac{d^k}{dt^k}\ (H_\gamma(\overline{z})) = ad^k\ \mathcal{H}_\alpha(H_\gamma)(\overline{z}) \qquad \text{for all} \quad k \geq 0.$$

It is easily seen that this is equivalent to: $(\theta(Z_\alpha)^k(H_\gamma))(\overline{z}) = 0$ for all $k \geq 0$, since $ad^k\ \mathcal{H}_\alpha(H_\gamma) = \theta(\overrightarrow{\mathcal{H}}_\alpha)^k(H_\gamma)$ and $\overrightarrow{\mathcal{H}}_\alpha$ and Z_α are collinear. This shows that for all $t \in J$, $(j^N_{\overline{x}(t)}\ F_0,\ j^N_{\overline{x}(t)}\ F_1,\ [\overline{z}(t)]) \in \widehat{B}(N, \alpha, c, \sigma), \sigma = 0$ if $\alpha \neq 10^{n-1}$ and $\sigma = 1$ if $\alpha = 10^{n-1}$, where $\overline{x}(t) = \Pi_{T^*M}(\overline{z}(t))$ and $[\overline{z}(t)]$ denotes the class of $\overline{z}(t)$ in PT^*M.

Now we shall prove the second statement in the considerations at the beginning of § 3. To do this we have to study the bad sets and introduce some concepts.

4. PARTIALLY ALGEBRAIC OR SEMI-ALGEBRAIC FIBER BUNDLES

DEFINITION 7. – *A VP bundle on M is a locally trivial fiber bundle on M whose typical fiber is a product $V \times P(W_1) \times \cdots P(W_n), V, W_1, \cdots, W_n$*

finite dimensional vector spaces, $P(W_1), \cdots, P(W_n)$ *the associated projective spaces and whose structural group is* $\mathrm{Aut}\,(V) \times \mathrm{Aut}\,(P(W_1)) \times \cdots \times \mathrm{Aut}\,(P(W_n)) (\mathrm{Aut}\,(V) = GL(V),\ \mathrm{Aut}\,(P(W_i)) = GL(W_i)/R^*)$.

DEFINITION 8. – *A partially algebraic (resp. semi-algebraic) subbundle of a* VP *bundle on* M *is a locally trivial subbundle whose typical fiber* A *is an algebraic (resp. semi-algebraic) subset of the typical fiber* $V \times P(W_1) \times \cdots \times P(W_m)$ *of the* VP*-bundle.*

LEMMA 2. – (i) $J^NTM \times_M J^NTM$, $J^NTM \times_M J^NTM \times \mathbf{R}$, $J^NTM \times_M J^NTM \times_M PT^*M$ are VP bundles on M whose typical fibers are respectively $P(d,N) \times P(d,N)$, $P(d,N) \times P(d,N) \times \mathbf{R}$, $P(d,N) \times P(d,N) \times P(\mathbf{R}^d)$ where $P(d,N)$ denotes the set of all polynomial mappings $P = (P^1, \cdots, P^d) : \mathbf{R}^d \to \mathbf{R}^d$ such that deg $P^i \leq N$ for $1 \leq i \leq d$.

(ii) $B_a(N)$, $B'_\ell(N)$, $\widehat{B}''_\ell(N)$, $\widehat{B}(N, \alpha, c, \sigma)$ are partially algebraic (for the first two) and semi-algebraic (for the last two), subundles of the VP bundles $J^NTM \times_M J^NTM$, $J^NTM \times_M J^NTM$, $J^NTM \times J^NTM \times \mathbf{R}$, $J^NTM \times_M J^NTM \times_M PT^*M$. Their typical fibers $\mathcal{F}_a(N), \mathcal{F}'_\ell(N), \widehat{\mathcal{F}}''_\ell(N), \widehat{\mathcal{F}}(N, \alpha, c, \sigma)$ can be described as follows:

$$\mathcal{F}_a(N) = \{(P_0, P_1)/\mathrm{dim\ Span}\ [ad^k\ P_0(P_1)(0),\ 0 \leq k \leq 2d-1] < d\}.$$

$$\mathcal{F}'_\ell(N) = \{(P_0, P_1)/\mathrm{dim\ Span}\ [P_0(0), P_1(0),\ [P_1, P_0](0)] \leq 1\}.$$

$\mathcal{F}''_\ell(N)$ is the set of all triples $(P_0, P_1, a) \in \mathcal{P}(d,N)^2 \times \mathbf{R}$ such that

(i) $P_1(0) \neq 0$, $P_0(0) = aP_1(0)$

(ii) dim Span $\{\mathrm{ad}^k R_a(P_1)(0), 0 \leq k \leq d-1$ and $[[P_0, P_1], P_1](0)\} < d$ where $R_a = P_0 - aP_1$ and for two vector fields P, Q:

$$[P, Q] = \sum_{i=1}^{d} \frac{\partial P}{\partial x^i}\, Q^i - \frac{\partial Q}{\partial x^i}\, P^i,$$

$x^1, \cdots, x^d$ being the canonical coordinates.

For the definition of the last fiber we shall use the following notations: for $\alpha \in \{0,1\}^n$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ the function $H_\alpha : \mathbf{R}^d \times \mathbf{R}^{d^*} \to \mathbf{R}$ is defined inductively as follows: if $i = 0$ or $1, H_i(x, \xi) = \langle P_i(x),\ \xi\rangle$. If $\alpha = (\alpha_1, \cdots, \alpha_n)$, $H_\alpha = \{H_{\alpha_1, \cdots, \alpha_{n-1}},\ H_{\alpha_n}\}$ where $\{\ ,\ \}$ denotes the Poisson bracket

$$\{f, g\} = \sum_{k=1}^{d} \frac{\partial f}{\partial x^k}\,\frac{\partial g}{\partial \xi_k} - \frac{\partial f}{\partial \xi_k}\,\frac{\partial g}{\partial x^k}.$$

For any integer $c \geq 0$, any $\alpha \in \{0,1\}^n$, $n \geq 3$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_1 = 1$, $N \geq n + c - 1$, $\widehat{\mathcal{F}}(N, \alpha, c, \sigma)$ is the set of all triples $(P_0, P_1, [\xi])$ in $P(d, N)^2 \times P(\mathbf{R}^d)$ such that:

(i) $P_0(0)$, $P_1(0)$ are lineary independent;

(ii) $H_{\alpha 0}(0, \xi) \neq 0$, $H_{\alpha 1}(0, \xi) \neq 0$;

(iii) $(\theta(Z_\alpha)^k(H_\gamma))(0, \xi) = 0$, $0 \leq k \leq c$;

where $\gamma = 10^{n-1}$, $\sigma = 0$ if $\alpha \neq 10^{n-1}$ and $\gamma = 101^{n-2}$, $\sigma = 1$ if $\alpha = 10^{n-1}$.

5. COORDINATE SYSTEMS ON $\mathcal{P}(d, N)$

First let us explain a few facts about coordinate systems on homogeneous polynomials. For $m \geq 1$, the space $\mathcal{P}_m(d)$ of all homogeneous polynomials of degree m in d variables can be identified with the space of m-multinear symmetric mappings on $\mathbf{R}^d$ as follows. Let $f \in \mathcal{P}_m(d)$, $\xi^{(1)}, \cdots, \xi^{(m)} \in \mathbf{R}^d$, define the total polarization of f as $(\mathcal{P}f)(\xi^{(1)}, \cdots, \xi^{(m)}) = D_{\xi^{(1)}} \cdots D_{\xi^{(m)}} f$ where $D_\xi f = \Sigma \xi^i \frac{\partial f}{\partial x^i}$, $\xi = (\xi^1, \cdots, \xi^d)$. Clearly f and $\mathcal{P}f$ can be identified since $f(x) = \frac{1}{m!} \mathcal{P}f(x, \cdots, x)$. Given a basis $e_1, \cdots, e_d$ of $\mathbf{R}^d$ we define a system of coordinates $\{X_\nu / \nu \in I_m\}$ as follows. The set I_m is the set of sequences $\nu = (i_1, \cdots, i_m), i_k \in [1, d]$ where $(i_1, \cdots, i_m)$ and $(i_{\sigma(1)}, \cdots, i_{\sigma(m)})$ are identified for any permutation σ. Hence we can order with $i_1 \leq \cdots \leq i_m$. Let $|\nu| = m$ denote the length of ν and $|\nu|_i =$ the number of occurences of i. Define X_ν as follows. For $m = 0$, set $I_m = \{0\}$ and define $X_\nu(f) = f(0)$. If $m \geq 1, X_\nu f = (\mathcal{P}f)(e_{i_1}, \cdots, e_{i_m})$.

Now let the couple $(A, B) \in \mathcal{P}(d, N)$. Let U be a neighborhood of (A, B) in $P(d, N)$ and let $e : U \to (\mathbf{R}^d)^d$ be a smooth mapping such that for any $(Q, R) \in U$, $e(Q, R) = (e_1(Q, R)), \cdots, e_d\ (Q, R)$ form a basis of $\mathbf{R}^d$. Then to e we can associate a coordinate system $\{X_\nu^i,\ Y_\nu^i,\ 1 \leq i \leq d,$ $\nu = (i_1, \cdots, i_m) \in I_m, 0 \leq m \leq N\}$ as follows:

$$X_\nu^i(Q, R) = (\mathcal{P}Q_m^i)\ (e_{i_1}(Q, R), \cdots, e_{i_m}(Q, R))$$

$$Y_\nu^i(Q, R) = (\mathcal{P}R_m^i)\ (e_{i_1}(Q, R), \cdots, e_{i_m}(Q, R))$$

where Q_m^i (resp. R_m^i) is the i^{th} component of the homogenous part of degree m of Q (resp. R). This system of coordinate is a curvilinear system of course. We set $X_\nu = (X_\nu^1, \cdots, X_\nu^d)$ and $Y_\nu = (Y_\nu^1, \cdots, Y_\nu^d)$.

6. EVALUATION OF CODIMENSION OF THE $\mathcal{F}(N)$

Each $\mathcal{F}(N)$ being semi-algebraic in their corresponding spaces, the concept of dimension is well defined. We shall estimate their codimensions.

LEMMA 3.

$$\operatorname{codim}\,(\mathcal{F}_a(N);\ \mathcal{P}(d,N)^2) = d+1$$

$$\operatorname{codim}\,(\mathcal{F}'_\ell(N);\ \mathcal{P}(d,N)^2) = 2d-2.$$

Proof.

$$\mathcal{F}_a(N) = \mathcal{F}'_a(N) \cup \mathcal{F}''_a(N) \cup \mathcal{F}'''_a(N)$$

$$\mathcal{F}'_a(N) = \mathcal{F}_a(N) \cap \{(P_0,P_1)/P_0(0) \neq 0\}$$

$$\mathcal{F}''_a(N) = \mathcal{F}_a(N) \cap \{(P_0,P_1)/P_1(0) \neq 0,\ P_0(0)=0\}$$

$$\mathcal{F}'''_a(N) = \mathcal{F}_a(N) \cap \{(P_0,P_1)/P_0(0) = P_1(0) = 0\}.$$

It is easy to see that codim $(\mathcal{F}'''_a(N);\ P(d,N)^2) = 2d$. To compute the codimensions of $\mathcal{F}''_a(N)$, $\mathcal{F}'_a(N)$ let us introduce the following semi-algebraic sets $\mathcal{C} \subset \mathbf{R}^d \times \operatorname{End}\,(\mathbf{R}^d)$, $\mathcal{C} = \{(v,A)/v \neq 0,\ \dim\,\{A^n v/0 \leq n \leq d-1\} < d\}$ and $D \subset (\mathbf{R}^d)^{2d} = \{(v_0,\cdots,v_{2d-1});\ v_i \in \mathbf{R}^d\}$, $\dim D < d$. Then clearly codim $(\mathcal{C},\mathbf{R}^d \times \operatorname{End}\,(\mathbf{R}^d)) = 1$ and it is well known that codim $(D,(\mathbf{R}^d)^{2d}) = d+1$. Consider the mappings: $\lambda : P(d,N)^2 \longmapsto \mathbf{R}^d \times\ \operatorname{End}\,(\mathbf{R}^d) \times \mathbf{R}^d$ and $\mu : P(d,N)^2 \longmapsto (\mathbf{R}^d)^{2d}$ defined as follows:

$$\lambda(P_0,P_1) = (P_1(0),\ P_{01},\ P_0(0))$$

where $P_{01} \in \operatorname{End}\,(\mathbf{R}^d)$ is the linear part of P_0 at 0,

$$\mu(P_0,P_1) = (P_1(0),\ \operatorname{ad}\, P_0(P_1)(0),\cdots,\operatorname{ad}^{\,2d-1} P_0(P_1(0)).$$

Then $\mathcal{F}''_a(N) = \lambda^{-1}(\mathcal{C} \times \{0\})$ and $\mathcal{F}'_a(N) = \mu^{-1}(D)$. Since λ is a projection, it is a submersion and hence codim $(\mathcal{F}''_a(N)\ ;\ \mathcal{P}(d,N)^2) =$ codim $(\mathcal{C} \times \{0\}\ ;\ \mathbf{R}^d \times\ \operatorname{End}\,(\mathbf{R}^d) \times \mathbf{R}^d) = d+1$.

We prove that μ restricted to the open semi-algebraic subset Ω of $\mathcal{P}(d,N)^2$, $\Omega = \{(P_0,P_1)/P_0(0) \neq 0\}$ is a submersion. Since $\mathcal{F}'_a(N) = \mu^{-1}(D \cap \Omega)$ it follows that codim $(\mathcal{F}'_a(N)\ ;\ \mathcal{P}(d,N)^2) = d+1$.

To study μ, take a couple $(Q_0,Q_1) \in \Omega$. There exist vectors $e_2,\cdots,e_d \in \mathbf{R}^d$ such that $(Q_0(0) = e_1,\ e_2,\cdots,e_d)$ is a basis of $\mathbf{R}^d$. Then on a neighborhood V of (Q_0,Q_1) contained in Ω the mapping

$$e : V \longmapsto \mathbf{R}^d \times \cdots \times \mathbf{R}^d, \qquad e(P_0,P_1) = (e_1(P_0,P_1),\cdots,\ e_d(P_0,P_1))$$

$$e_1(P_0,P_1) = P_0(0), \qquad e_i(P_0,P_1) = e_i, \qquad 2 \leq i \leq d$$

takes its values in the basis of $\mathbf{R}^d$. As we explained in 5), let $(X^i_\nu,\ Y^i_\nu)$ be the coordinate system associated to e on V. For any $(P_0, P_1) \in V$, the i^{th} component of $\mathrm{ad}^k P_0(P_1)(0)$ has the form: $Y^i_{1^k} + R^i_k$, where R^i_k is a polynomial function in the variables X^j_ν, $|\nu| \leq k$, Y^j_ν, $|\nu| \leq k-1$, $1 \leq j \leq d$.

Indeed ad $P_0(P_1)(0) = \theta_{P_0}(P_1)(0) - \theta_{P_1}(P_0)(0) = Y_1 - X_1$ and by induction ad ${}^k P_0(P_1)(0) = Y_{1^k} + R_k$ where R_k is a function of X^i_ν, $|\nu| \leq k$, Y^i_ν, $|\nu| \leq k-1$. This shows immediately that μ is a submersion at each point in V. As (Q_0, Q_1) is arbitrary in Ω, $\mu_{|\Omega}$ is a submersion.

The proof that codim $(\mathcal{F}'_\ell(N);\ \mathcal{P}(d;\ N)^2) = 2d - 2$ is very similar.

$$\mathcal{F}'_\ell(N) = \mathcal{F}^{(3)}_\ell(N) \cup \mathcal{F}^{(4)}_\ell(N),$$

$$\mathcal{F}^{(3)}_\ell(N) = \mathcal{F}'_\ell(N) \cap \{(P_0, P_1)/P_0(0) = P_1(0) = 0\}$$

$$\mathcal{F}^4_\ell(N) = \mathcal{F}'_\ell \backslash \mathcal{F}^{(3)}_\ell(N).$$

Clearly codim $(\mathcal{F}^{(3)}_\ell(N);\ \mathcal{P}(d, N)^2) = 2d$. Let $\Omega_{0,1}$ be the open set of $\mathcal{P}(d, N)^2$ of all couples (P_0, P_1) such that $P_0(0) \neq 0$ or $P_1(0) \neq 0$. The mapping $\nu : \Omega_{0,1} \longmapsto (\mathbf{R}^d)^3$, $\nu(P_0, P_1) = (P_0(0),\ P_1(0),\ P_{01}(P_1(0)) - P_{11}(P_0(0)))$ is a submersion and $\mathcal{F}^{(4)}(N) = \nu^{-1}(D_3)$, where $D_3 = \{(v_0, v_1, v_2) | v_i \in \mathbf{R}^d,\ i = 0, 1, 2,\ \dim \mathrm{Span}\ (v_0, v_1, v_2) \leq 1\}$. Clearly codim $(D_3\ ;\ (\mathbf{R}^d)^3) = 2d - 2$. This gives the second result of Lemma 3.

LEMMA 4.

$$\mathrm{codim}\ (\widehat{\mathcal{F}}''_\ell(N);\ \mathcal{P}(d, N)^2 \times \mathbf{R}) = d + 2$$

$$\mathrm{codim}\ (\widehat{\mathcal{F}}(N, \alpha, c, \sigma);\ \mathcal{P}(d, N)^2 \times P(\mathbf{R}^d)) = c + 1.$$

Proof. – Let $Z_{01} = \{(P_0, P_1, a)/P_1(0) \neq 0,\ P_0(0) = aP_1(0)\}$. Define the mapping $\chi : Z_{01} \longmapsto \mathbf{R}^d \times \mathrm{End}\ (\mathbf{R}^d) \times \mathbf{R}^d$ as follows $\chi(P_0, P_1, a) = (P_1(0),\ P_{01} - aP_{11},\ [[P_0, P_1],\ P_1](0))$.

Clearly χ is a submersion and $\widehat{\mathcal{F}}''_\ell(N) = \chi^{-1}(\mathcal{C}_1)$ where

$$\begin{aligned}\mathcal{C}_1 = \{(v, A, w)/v, w \in \mathbf{R}^d,\ A \in \mathrm{End}\ (\mathbf{R}^d),\ v \neq 0,\\ \dim \mathrm{Span}\ \{A^n(v),\ 0 \leq n \leq d-1,\ w\} < d\}.\end{aligned}$$

Then

$$\mathrm{codim}\ (\mathcal{C}_1,\ \mathbf{R}^d \times \mathrm{End}\ (\mathbf{R}^d) \times \mathbf{R}^d) = 2.$$

So

$$\mathrm{codim}\ (\widehat{\mathcal{F}}''_\ell(N);\ Z_{01}) = 2$$

and

$$\begin{aligned}&\mathrm{codim}\ (\widehat{\mathcal{F}}''_\ell(N); \mathcal{P}(d, N)^2 \times \mathbf{R})\\ &\quad = \mathrm{cod}\ (\widehat{\mathcal{F}}''_\ell(N);\ Z_{01}) + \mathrm{codim}\ (Z_{01};\ \mathcal{P}(d, N)^2 \times \mathbf{R}) = 2 + d.\end{aligned}$$

Now we shall consider the case of $\widehat{\mathcal{F}}(N, \alpha, c, 0)$. The case of $\widehat{\mathcal{F}}(N, \alpha, c, 1)$ is similar. Let $\Omega_{01} = \{(P_0, P_1, \xi)/P_0(0),\ P_1(0)$ are linearly independent, $\langle P_{\alpha 0}(0),\ \xi\rangle \neq 0$, $\langle P_{\alpha 1}(0),\ \xi\rangle \neq 0$, $[\xi] \in P(\mathbf{R}^d)\}$ and where

$$P_\alpha = [P_{(\alpha_1, \cdots, \alpha_{n-1})},\ P_{\alpha_n}]$$

Let $\zeta : \Omega_{01} \times \mathbf{R}^d \longmapsto \mathbf{R}^{c+1}$ be the mapping:

$$\zeta(P_0, P_1, \xi) = (H_\gamma(0, \xi),\ \theta(Z_\alpha)\ H_\gamma\ (\theta, \xi), \cdots,\ \theta(Z_\alpha)^c\ H_\gamma(0, \xi)),\ \xi \neq 0,$$

where $\gamma = 10^{n-1}$. Then $\widehat{\mathcal{F}}\ (N, \alpha, c, 0) = \zeta^{-1}(0)$. If we show that ζ is a submersion it will follow that codim $(\widehat{\mathcal{F}}(N, \alpha, c, 0);\ P(d, N)^2 \times P(\mathbf{R}^d)) = c + 1$.

Using the rule

$$\begin{aligned}\theta(Z_\alpha)(FG) = &F[H_{\alpha 1}\{G, H_0\} - H_{\alpha 0}\{G, H_1\}]\\ &+ G[H_{\alpha 1}\{F, H_0\} - H_{\alpha 0}\{F, H_1\}]\end{aligned}$$

an easy induction shows that: $\theta(Z_\alpha)^k\ H_\gamma = H_{\alpha 1}^k\ H_{10^{n+k-1}} + \Pi_{\alpha,k}$, where $\Pi_{\alpha,k}$ is a polynomial in H_δ, where either $|\delta| < n + k$ or $|\delta| = n + k$ but $\delta \neq 10^{n+k-1}$.

Take a $(P_0', P_1') \in \Omega_{01}$. There exist $e_3', \cdots, e_d'$ in $\mathbf{R}^d$ such that $(P_0'(0), P_1'(0), e_3', \cdots, e_d')$ is a basis of $\mathbf{R}^d$. Then one can find a neighborhood V of (P_0', P_1') such that for all $(P_0, P_1) \in V$, the d vectors $e_1(P_0, P_1) = P_0(0)$, $e_2(P_0, P_1) = P_1(0)$, $e_i(P_0, P_1) = e_i'$, $3 \leq i \leq d$ form a basis of $\mathbf{R}^d$. Let X_ν^i, Y_ν^i be the coordinate system on V associated to the mapping $e = (e_1, \cdots, e_d)$.

Now $[F_1, F_0](0) = \theta_{F_0}(F_1)(0) - \theta_{F_1}(F_0)(0)$. Hence $H_{10}(0, \xi) = \langle\xi, Y_1 - X_2\rangle$. Therefore computing by induction we get $H_{10\beta}\ (0, \xi) = \langle\xi, Y_{1^{k+1}2^{k'}} - X_{2^{k'+1}1^k}\rangle + R_n$ where $n = |\beta|$, $k = |\beta|_1$, $k' = |\beta|_2$, $k + k' = n$, and R_n is a polynomial in ξ, X_ν^i, Y_ν^i, $|\nu| \leq n$.

Then the functions $\theta(Z_\alpha)^k(H_\gamma)(0, \xi)$ can be expressed as follows in these coordinates:

$$\theta(Z_\alpha)^k\ H_\gamma(0, \xi) = H_{\alpha 1}^k(0, \xi) \sum_{i=1}^{d} Y_{1^{n+k-1}}^i\ \xi^i + R_k,$$
$$0 \leq k \leq c, \qquad \xi = (\xi_1, \cdots, \xi_d)$$

and R_k is a polynomial in ξ, X_ν^i, $|\nu| \leq n + k - 1$, Y_ν^i with $|\nu| \leq n + k - 1$, $|\nu|_1 < n + k - 1$. Hence ζ is a submersion.

COROLLARY.

$$\operatorname{codim}\,(\mathcal{F}'_\ell(N);\ P(d,N)^2) \geq d+1$$

$$\operatorname{codim}\,(\mathcal{F}(N,\alpha,c,\alpha);\ \mathcal{P}(d,N)^2) \geq c+1-d.$$

LEMMA 5. – codim $(B(N);\ J^NTM\ \times_M J^NTM) \geq \min\,(d+1,\,2d-2,\,c+1-d) = \min\,(d+1,\ 2d-2)$ *(Since we have chosen $c = 2d$ in the definition of $B(N)$.)*

7. END OF THE PROOF OF THEOREM 0

For $d \geq 3$, codim $(B(N);\ J^NTM \times_M J^NTM) \geq d+1$. Hence $\overline{B(N)}$ is a partially semi-algebraic closed subbundle of the vector bundle $J^NTM \times J^NTM$ of codimension $\geq d+1$. The theorem in $[GM]$ shows that the set of all $(F_0, F_1) \in VF^2(M)$ such that $(j_x^N\ F_0,\ j_x^N F_1) \notin \overline{B(N)}$ for all $x \in M$ is open dense. This ends the proof of Theorem 0.

8. INPUT-OUTPUT MAPPING AND THEIR SINGULARITIES

Let $(F_0, F_1) \in VF(M)^2$, $m \in M$, $T \in \mathbf{R}$, $T > 0$.

DEFINITION 9. – *The input-output mapping associated to the quadruple $(F_0,\ F_1,\ m,\ T)$ is the mapping $E_{m,T}\ : U(m,T) \longmapsto M$, defined as follows: its domain $U(m,T)$ is the set of all $\overline{u} \in L^\infty([0,T];\ \mathbf{R})$ such that the solution $\bar{x}$, of the Cauchy problem:*

$$\frac{d\bar{x}}{dt}(t) = F_0(\bar{x}(t)) + \bar{u}(t)F_1(\bar{x}(t)), \qquad \overline{x}(0) = m,$$

is defined on $[0,T]$. Then $E_{m,T}\ (\bar{u}) = \bar{x}(T) \in M$.

Then we have the result (for the proof *see* $([BK1])$).

PROPOSITION 2. – (i) *$U(m,T)$ is open in $L^\infty([0,T]\ ;\ \mathbf{R})$ and $E_{m,T}$ is a smooth mapping.*

(ii) *A point $\overline{m} \in M$ is a critical value of $E_{m,T}$ if and only if there exists a singular extremal $(\overline{z},\overline{u}) : [0,T] \to T^*M \times \mathbf{R}$ such that $\Pi_{T^*M}(\overline{z}(0)) = m$ and $\Pi_{T^*M}(\overline{z}(T)) = \overline{m}$.*

9. THE "BAD" SETS FOR THEOREM 1

Let $(F_0, F_1) \in VF(M)^2$. We shall use the notations of §3: $H_i : T^*M \to \mathbf{R}$, $i = 1, 2$ is the function $H_i(z) = \langle F_i(\Pi_{T^*M}(z)), z\rangle$. If $\alpha \in \{0,1\}^n$, $H_\alpha : T^*M \to \mathbf{R}$ is defined inductively by $H_\alpha = \{H_{(\alpha_1,\cdots,\alpha_{n-1})}, H_{\alpha_n}\}$. The set Ω_{011} is the open subset of all $z \in T^*M$ such that $H_{011}(z) \neq 0$. Let Z be the field $\overrightarrow{H}_0 + \frac{H_{100}}{H_{011}}\overrightarrow{H}_1$ on Ω_{011}.

DEFINITION 10. – (i) *For any integer $q \geq 0$, any integer $N \geq q + 2$, let $\widehat{B}_c(N, q)$ be the subset of $J^N TM \times_M J^N TM \times_M PT^*M \times_M PT^*M$ of all quadruples $(j_x^N F_0, j_x^N F_1, [z_1], [z_2])$ such that:*

1) $[z_1] \neq [z_2]$,

2) $H_{011}(z_i) \neq 0$, $i = 1, 2$,

3) $\theta(Z)^k(\frac{H_{100}}{H_{011}})(z_1) = \theta(Z)^k(\frac{H_{100}}{H_{011}})(z_2)$, $0 \leq k \leq q$, $i = 1, 2$,

4) $F_0(x)$, $F_1(x)$ *are linearly independent.*

(ii) *$B_c(N, q)$ will denote the canonical projection of $\widehat{B}_c(N, q)$ onto $J^N TM \times_M J^N TM$.*

Fundamental Lemma 6. – Let $(F_0, F_1) \in VF(M)^2$ be a couple such that for any $x \in M$, $(j_x F_0, j_x F_1) \notin B(N) \cup B_c(N, q)$. Then every singular extremal of (F_0, F_1) is of minimal order and there does not exist any two singular extremals $(\overline{z}_i, \overline{u}_i) : J \to T^*M \times \mathbf{R}$ such that $\Pi_{T^*M}(\overline{z}_1) = \Pi_{T^*M}(\overline{z}_2)$ and $[\overline{z}_1] \neq [\overline{z}_2]$.

Proof. – The first part is just a restatement of Lemma 0. As for the second part let $(\overline{z}_i, \overline{u}_i) : J \to T^*M \times \mathbf{R}$, $i = 1, 2$, be two singular extremals of (F_0, F_1) such that $\Pi_{T^*M}(\overline{z}_1) = \Pi_{T^*M}(\overline{z}_2)$ and $[\overline{z}_1] \neq [\overline{z}_2]$. By the first statement both $(\overline{z}_i, \overline{u}_i)$, $i = 1, 2$, are of minimal order. The set $\{t/[z_1(t)] \neq [z_2(t)]$ is open and non empty. Since the sets $\mathcal{R}(\overline{z}_i, \overline{u}_i)$ are both open and dense (*see* definition 1) there exists an open non empty subinterval J' of J such that:

(1) $[\overline{z}_1(t)] \neq [z_2(t)]$ for all $t \in J'$,

(2) $H_{011}(\overline{z}_i(t)) \neq 0$, $i = 1, 2$ for all $t \in J'$.

The closed subset

$$\{t \in J'/\dim \text{Span}\,\{F_0(\overline{x}(t)), F_1(\overline{x}(t))\} \leq 1,$$

$$\overline{x} = \Pi_{T^*M}(\overline{z}_1) = \Pi_{T^*M}(\overline{z}_2),$$

has an empty interior: otherwise on an open non empty subinterval J'' of J' we would have $\dim \text{Span}\,\{F_0(\overline{x}(t)), F_1(\overline{x}(t))\} \leq 1$. Applying Lemma 1 to the restrictions of $(\overline{z}_i, \overline{u}_i)$ $i = 1, 2$ to J'' we get that $[z_1(t)] = [z_2(t)]$ for all $t \in J''$. This contradicts 1) above. Replacing J by an open subinterval we can assume that:

(3) $[\overline{z}_1(t)] \neq [\overline{z}_2(t)]$ for all $t \in J$,

(4) $H_{011}(\overline{z}_i(t)) \neq 0$, $i = 1, 2$, for all $t \in J$,

(5) $F_0(\overline{x}(t))$, $F_1(\overline{x}(t))$ are linearly independent for all $t \in J$.

It follows from proposition 0 that $\dot{\overline{z}}_i = Z(\overline{z}_i)$ a.e., $i = 1, 2$, and $\overline{u}_i(t) = \frac{H_{100}}{H_{011}}(\overline{z}_i(t))$ a.e. Projecting on M we get: $\dot{\overline{x}}(t) = F_0(\bar{x}(t)) + \bar{u}_1(t) F_1(\overline{x}(t)) = F_0(\bar{x}(t)) + \bar{u}_2(t)\ F_1(\bar{x}(t))$ for a.e. $t \in J$. Since by (5) above $F_1(\bar{x}(t)) \neq 0$ for all $t \in J$, $u_1(t) = u_2(t)$ for a.e. $t \in J$. This implies that

$$\frac{H_{100}}{H_{011}}(\bar{z}_1) = \frac{H_{100}}{H_{011}}(\bar{z}_2).$$

Deriving this relation with respect to t we get that for all $k \in \mathbf{N}$: $\left(\theta(Z)^k\left(\frac{H_{100}}{H_{011}}\right)\right)(\overline{z}_1) = \left(\theta(Z)^k\left(\frac{H_{100}}{H_{011}}\right)\right)(\bar{z}_2)$. Hence for all $t \in J$ the quadruple $(j^N_{\bar{x}(t)}F_0, j^N_{\bar{x}(t)}F_1, [\overline{z}_1(t)], [\overline{z}_2(t)])$ belongs to $\widehat{B}_i(N, q)$. A contradiction.

10. EVALUATION OF THE CODIMENSION OF $B_c(N, q)$

It is clear that $\widehat{B}_c(N, q)$ is a partially semi-algebraic subbundle of the VP-bundle $J^N TM \times_M J^N T_2 M \times_M PT^*M \times_M PT^*M$, $N \geq q+2$. Its typical fiber $\widehat{\mathcal{F}}_c(N, q)$ in $\mathcal{P}(d, N)^2 \times P(\mathbf{R}^d)^2$ is the set of all $(P_0, P_1, [\xi_1], [\xi_2])$

(i) $[\xi_1] \neq [\xi_2]$,

(ii) dim Span $\{P_0(0), \ P_1(0)\} = 2$,

(iii) $H_{011}\ (0, \xi_i) \neq 0$, $i = 1, 2$,

(iv) $\left(\theta(Z)^k\ \left(\frac{H_{100}}{H_{011}}\right)\right)\ (0, \xi_1) = \left(\theta(Z)^k\ \left(\frac{H_{100}}{H_{011}}\right)\right)\ (0, \xi_2)$, $0 \leq k \leq q$.

where

$$H_i(x, \xi) = \langle P_i(x),\ \xi\rangle, \qquad i = 1,\ 2,$$

$$H_{100}(x, \xi) = \langle [[P_1, P_0],\ P_0](x),\ \xi\rangle, \quad H_{011}(x, \xi) = \langle [[P_0,\ P_1],\ P_1](x),\ \xi\rangle,$$

$Z = \overrightarrow{H}_0 + \frac{H_{100}}{H_{011}}\ \overrightarrow{H}_1$ on the open subset Ω_{011} in $\mathcal{P}(d, N)^2 \times \mathbf{R}^d$ of all (P_0, P_1, ξ) such that $H_{011}(0, \xi) \neq 0$.

LEMMA 7. – (i) For every $k \geq 0$, there exists a polynomial function $\Phi_k : \mathcal{P}(d, N)^2 \times \mathbf{R}^d \longmapsto \mathbf{R}$ such that on Ω_{011}

$$\left(\theta(Z)^k\left(\frac{H_{100}}{H_{011}}\right)\right)(0, \xi) = \frac{\Phi_k(P_0, P_1, \xi)}{H_{011}(0, \xi)},$$

(ii) $\Phi_0 = H_{100}$ and Φ_k, $k \geq 1$, has the form

$$\Phi_k = H_{10^{k+2}} - (H_{100}/\ H_{011})^{k+1}\ H_{01^{k+2}} + \psi_k$$

where ψ_k is a polynomial in the H_α such that $|\alpha| < k+3$ or $|\alpha| = k+3$ and $|\alpha|_0 > 1$, $|\alpha|_1 > 1$. The proof is an easy induction on k.

COROLLARY. – *$\widehat{\mathcal{F}}_c(N, d)$ is the set of all quadruples $(P_0, P_1,\ [\xi_1],\ [\xi_2])$ such that*

(i) $[\xi_1] \neq [\xi_2]$,

(ii) dim Span $\{P_0(0),\ P_1(0)\} = 2$,

(iii) $H_{011}(0, \xi_i) \neq 0$, $i = 1, 2$,

(iv) $H_{011}(0, \xi_2)\, \Phi_k(P_0, P_1, \xi_1) = H_{011}(0, \xi_1)\ \Phi_k(P_0, P_1, \xi_2)$, $0 \leq k \leq q$.

Take any $(P'_0,\ P'_1,\ [\xi'_1],\ [\xi'_2])$ in the open set $\mathcal{O} \subset \mathcal{P}(d, N)^2 \times P(\mathbf{R}^d)$, $\mathcal{O} = \{(P_0, P_1, [\xi_1], [\xi_2])/(i)\ [\xi_1] \neq [\xi_2]$, (ii) dim Span $\{P_0(0),\ P_1(0)\} = 2$, (iii) $H_{011}(0, \xi_i) \neq 0,\ i = 1, 2\}$. Complete $P'_0(0),\ P'_1(0)$ into a basis $P'_0(0),\ P'_1(0),\ e'_3, \cdots, e'_d$ of $\mathbf{R}^d$ and define the mapping $e : \mathcal{O} \longmapsto (\mathbf{R}^d)^d$, $e(P, Q) = (e_1(P, Q), \cdots, e_d(P, Q))$, $e_1(P, Q) = P_0(0)$, $e_2(P, Q) = P_1(0)$, $e_i(P, Q) = e'_i$, $3 \leq i \leq d$. For a small neighborhood V of (P'_0, P'_1) in $\mathcal{P}(d, N)^2$, $e_{|V}$ is basis valued and we can associate to e, a coordinate system X^i_ν, Y^i_ν as in § 5. We get for $k \geq 1$:

$$\Phi_k(P_0, P_1, \xi) = \sum_{i=1}^{d} \xi^i\ Y^i_{1^{k+2}} + R_k$$

where R_k is a polynomial in ξ, X^i_ν, $|\nu| \leq k+2$, Y^i_ν $|\nu| \leq k+2$ and $\nu \neq 1^{k+2}$.

Hence for $k \geq 1$, (iv) can be written:

$$H_{011}\ (0, \xi_2)\ \sum_{i=1}^{d} \xi^i_1\ Y^i_{1^{k+2}} = H_{011}\ (0, \xi_1)\ \sum_{i=1}^{d} \xi^i_2\ Y^i_{1^{k+2}} + R'_k$$

where R'_k is not depending upon $Y^i_{1^{k+2}}$. Therefore we have

$$\langle H_{011}(0, \xi_2)\xi_1 - H_{011}(0, \xi_1)\ \xi_2,\ Y_{1^{k+2}}\rangle + R''_k = 0$$

where R''_k is not depending upon $Y^i_{1^{k+2}}$.

For $1 \leq k \leq q$, these q relations define on $\mathcal{O} \cap V$ a smooth submanifold of codimension q, since ξ_1 and ξ_2 are not collinear. This shows that in $\mathcal{O} \cap V$,

$\widehat{\mathcal{F}}_c(N, q)$ is of codimension at least q. Since the $V's$ for different choices of (P_0', P_1') cover $\mathcal{O}$, $\widehat{\mathcal{F}}_c(N, q)$ is at least of codimension q. Consequently its projection $\mathcal{F}_c(N, q)$ into $\mathcal{P}(d, N)^2$ is a semi-algebraic subset of codimension $\geq q + 2 - 2d$.

LEMMA 8. – (i) *codimension* $(\mathcal{F}_c(N, q);\ \mathcal{P}(d, N)^2) \geq q + 2 - 2d$.

(ii) *The set of all couples* $(F_0, F_1) \in VF(M)^2$ *such that* $(j_x^N\ F_0, j_x^N\ F_1) \notin \overline{B(N) \cup B_c\ (N,\ 3d - 1)}$ *is open dense in* $VF(M)^2$.

Lemma 6 and Lemma 8 prove Theorem 1.

11. CONCLUSION

Similar results have been obtained in the multi-inputs case and can be applied to systems without drift (subriemannian geometry).

Two important questions are still open.

1) Are the two properties studied in this article open ?

2) For a singular extremal (z, u) with minimal order let $S = \{t \in J, \{\{H_0, H_1\}, H_1\}(z(t)) = 0\}$. Can the Lebesgue measure of S be non zero ?

Both questions are connected to the analysis of the *behaviors of singular trajectories* near the previous set. Reference [B1] contains some results in this direction. Moreover it was pointed out by A. Agracev, that some additional *regularity properties* could be obtained by dealing only with *optimal singular extremals.*

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