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Young measure solutions for nonlinear evolutionary systems of mixed type

by

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ABSTRACT. – This is a study of measure-valued solutions for systems of mixed type modelled by a hyperbolic-elliptic and a dispersive-parabolic system in arbitrary dimension. Existence is established by time-discretisation of the equations which is solved by the minimisation of a non-convex functional. By relaxation, a Young measure solution is obtained for every time step. Uniform bounds derived by energy considerations allow passage to the limit of continuous time. The potential gradient and the identity are shown to be independent with respect to the Young measure.

RÉSUMÉ. – Ceci est une étude des mesures, solutions des systèmes de type mixte, modélisés par un système hyperbolique-elliptique et un système dispersif-parabolique, en dimension quelconque. Un résultat d'existence est établi par une discrétisation en la variable temps d'une équation qui est équivalente à la minimisation d'une fonctionnelle non convexe. Par relaxation, une solution, mesure de Young, est obtenue à chaque étape. Des bornes uniformes dérivant de la fonction énergie permettent de passer à la limite en temps continu. Nous prouvons que le gradient du potentiel et l'identité sont indépendants par rapport à la mesure de Young.

Consider the following two systems

$$(1) \qquad \begin{cases} v_t = \nabla w & \text{on } Q_\infty \equiv \Omega \times \mathbb{R}^+ \\ w_t = \nabla \cdot q(v) \end{cases}$$

with initial data $v(x,0) = u_0$ in $H_0^1(\Omega)$ and $w(x,0) = z_0$ in $L^2(\Omega)$ and

$$\begin{cases} v_t = \nabla w & \text{on } Q_\infty \\ w_t = -\Delta q(\nabla \cdot v) \end{cases} \tag{2}$$

with $v(x,0) = u_0$ in $H_0^2(\Omega)$ and $w(x,0) = z_0$ in $L^2(\Omega)$. In both cases $\Omega \subset \mathbb{R}^n$ is open and bounded; q is non-monotone and is the potential gradient of a non-convex energy function ϕ (often referred to as the stored elastic energy function). The lack of convexity of ϕ is associated with the failure of ellipticity of the associated stationary problems and the failure of hyperbolicity or dispersivity in the corresponding dynamical equations; so the above systems are of mixed type, the first hyperbolic-elliptic and the second dispersive-parabolic. In elastodynamics and in three spatial dimensions (1) is known as the anti-plane shear problem (it models the motion of a cylindrical body of a general cross-section undergoing a shear deformation along its cylindrical axis). Also the Riemann problem for (1) in one dimension models dynamics for phase transitions in van der Waals viscoelastic fluids.

For one dimensional, strictly hyperbolic systems (corresponding to the case of a convex energy function) strong solutions have been obtained, either in the class of functions of bounded variation or in the context of compensated compactness, *cf.* [7, 8]. Dynamics in the non-convex case have also been considered. In Ball *et al.* [2] an infinite dimensional dynamical system related to (1) in one space dimension with a viscoelastic term is studied. Existence of strong classical and weak solutions is proved which are unique in each case. Fan and Slemrod [9] construct solutions for the Riemann problem for (1) in one space dimension by a vanishing similarity viscosity term in the special case where hyperbolicity fails on a single (bounded) interval of the real line.

In the case of a non-convex energy strong solutions to (1) or (2), or to their equivalent formulations (3) and (4) respectively, do not exist for general space dimension. Swart and Holmes [16], using a method of Rybka [14], have proved the existence and uniqueness of a strong solution to a regularised version of the anti-plane shear system with an added linear viscoelastic term. In their framework, it should be interesting to investigate the limiting problem as the viscoelastic term vanishes using a method as in [15].

Non-convex energies possess multiple local minima and typically do not admit absolute minimisers. They are associated with the dynamical formation of intricate microstructure and model solid phase transformations in which the co-existence of multiple phases is energetically preferable to a single phase. In equilibrium configurations, microstructure is accounted for by the development of oscillations in minimising sequences which fail to converge to minimisers. The analysis of microstructure formation in dynamical systems with non-convex energies is complicated. For systems modelled by ordinary differential equations with an energy acting as a Lyapunov functional solutions converge to rest points of the energy. For systems modelled by partial differential equations, even in the presence of a dissipative mechanism, it is observed that the dynamical solutions may imitate the behaviour of oscillatory minimising sequences and hence fail to minimise the energy as time tends to infinity. For example in Ball *et al.* [2] it is found for the viscous equation related to (1) mentioned above that time-asymptotically the solution does not minimise the energy (in contrast for example with a solution of the corresponding non-local equation in which the nonlinear term is replaced by a spatial average).

In this article I obtain Young measure solutions to (1) and (2) by the method time-discretisation. This method has been used before to obtain solutions for a variety of evolution problems, including the heat flow of harmonic maps in [3], and for semilinear parabolic systems in [10] to obtain classical weak solutions. By expressing the discretised equations variationally and incorporating the Young measure theory developed in a series of articles by Kinderlehrer and Pedregal in [12], this method has been previously applied to give a Young measure solution for a nonlinear parabolic evolution of forward-backward type in [11].

In the sequel I will discuss Young measure solutions for the systems above by considering the equivalent equations, respectively,

$$(3)\qquad \begin{cases} u_{tt} = \nabla\cdot q(\nabla u) & \text{on } Q_\infty \\ u(\cdot,0) = u_0 & \text{on } \Omega, \quad u_0 \in H_0^1(\Omega) \\ u_t(\cdot,0) = z_0 & \text{on } \Omega, \quad z_0 \in L^2(\Omega) \\ u = 0 & \text{on } \partial\Omega\times\mathbb{R}^+ \end{cases}$$

denoted by $\mathcal{H}$;

$$(4)\qquad \begin{cases} u_{tt} = -\Delta q(\Delta u) & \text{on } Q_\infty \\ u(\cdot,0) = u_0 & \text{on } \Omega, \quad u_0 \in H_0^2(\Omega) \\ u_t(\cdot,0) = z_0 & \text{on } \Omega, \quad z_0 \in L^2(\Omega) \\ u = 0 & \text{on } \partial\Omega\times\mathbb{R}^+ \\ \nabla u = 0 & \text{on } \partial\Omega\times\mathbb{R}^+ \end{cases}$$

denoted by $\mathcal{D}$. Here $\Omega \subset \mathbb{R}^n$ is open and bounded with mildly smooth boundary (the cone or the segment property suffice) and $q = \nabla\phi$ where

$\phi \in C^1(\mathbb{R}^m)$ (where $m = n$ for $\mathcal{H}$ and $m = 1$ for $\mathcal{D}$) and satisfies a quadratic growth condition at infinity. Denote by $W^{k,p}$ the space of functions which together with their k-order weak derivatives belong to (an equivalence class of) L^p. We adopt the convention that $f \in W_0^{k,p}$ means that $D^\alpha f$ has zero trace on $\partial\Omega$ for all multi-indices with $|\alpha| \leq k - 1$.

The regularisation scheme to obtain existence involves the time-discretisation of the equations which are the Euler-Lagrange conditions for a non-convex functional. In contrast to the case of a gradient flow, estimates for the solutions of the discretisation are derived from the non-increase of the discretised energy rather than from the minimisation. The uniform bounds provided allow passage to continuous time to obtain a weak solution described by a Young measure. The support of the measure is contained in the hyperbolic region for $\mathcal{H}$ and dispersive region for $\mathcal{D}$ but these regions can be *strictly larger* than the support of the measure. The potential gradient q and the identity have the interesting property that they are *independent* with respect to the Young measure. In the parabolic case in [6] this is a key property on which the uniqueness result relies. Not surprisingly, and in contrast to the parabolic case, there are no uniqueness properties for Young measure solutions for $\mathcal{H}$ and $\mathcal{D}$ – at least in the context of this method.

In what follows we first define the Young measure solution and prove its existence for $\mathcal{H}$ and $\mathcal{D}$ and finally show the independence property of the measure mentioned above.

Definition. A Young measure solution to $\mathcal{H}$ is a function

$$(5)\quad u \in W^{2,\infty}_{\text{loc}}(\mathbb{R}^+, H^{-1}(\Omega)) \cap W^{1,\infty}_{\text{loc}}(\mathbb{R}^+, L^2(\Omega)) \cap L^{\infty}_{\text{loc}}(\mathbb{R}^+, H^1_0(\Omega))$$

and a Young measure $\boldsymbol{\nu} = (\nu_{x,t})_{(x.t)\in Q_\infty}$ which satisfy the weak equation

$$(6)\quad \int_0^T \int_\Omega (\langle \boldsymbol{\nu}, q\rangle \cdot \nabla\zeta - u_t\zeta_t)\, dx\, dt = 0 \qquad \forall \zeta \in H^1_0(Q_T),\ \forall T > 0$$

and

$$(7)\qquad \nabla u(x,t) = \int_{\mathbb{R}^n} \lambda d\nu_{(x,t)}(\lambda) \qquad \text{a.e. in } Q_\infty$$

and such that the initial data are obtained in the sense:

$$(8)\qquad u(t) \longrightarrow u_0 \qquad \text{strongly in } L^2(\Omega) \text{ as } t \longrightarrow 0$$

$$(9)\qquad u_t \longrightarrow z_0 \qquad \text{strongly in } H^{-1}(\Omega) \text{ as } t \longrightarrow 0.$$

In the case of $\mathcal{D}$ a similar definition holds: here it is required that

$$u \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^+, H^{-2}(\Omega)) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+, L^2(\Omega)) \cap L^{\infty}_{\mathrm{loc}}(\mathbb{R}^+, H^2_0(\Omega))$$

be paired with a Young measure $\boldsymbol{\nu} = (\nu_{x,t})_{(x.t)\in Q_\infty}$ so that the weak equation

$$\int_0^T \int_\Omega \langle \boldsymbol{\nu}, q \rangle \Delta\zeta - u_t \zeta_t \, dx\, dt = 0 \qquad \forall \zeta \in H^2_0(Q_T),\ \forall T > 0 \tag{10}$$

be satisfied and also

$$\Delta u(x,t) = \int_{\mathbb{R}} \lambda d\nu_{(x,t)}(\lambda) \qquad \text{a.e. in } Q_\infty. \tag{11}$$

The initial data obtained in the sense:

$$\begin{aligned} u(t) &\to u_0 && \text{(strongly) in } L^2(\Omega) \text{ as } t \to 0 \\ u_t &\to z_0 && \text{(strongly) in } H^{-2}(\Omega) \text{ as } t \to 0. \end{aligned}$$

By differentiation the weak equations also hold a.e. in time.

Young measure representation. Recall from the fundamental theorem of Young measures in [1] that a sequence of functions $(f^k)_k$ satisfying a mild boundedness condition will generate a family of probability measures $\boldsymbol{\nu}$ as follows: whenever $(\psi(f^k))_k$ is weakly (sequentially) precompact in L^1 for a continuous function ψ, then $(\psi(f^k))_k$ is in fact convergent (on a bounded domain) in the weak topology in L^1, *i.e.*,

$$\psi(f^k) \overset{k\to\infty}{\longrightarrow} \langle \boldsymbol{\nu}, \psi \rangle \quad \text{in } \sigma(L^1, L^\infty). \tag{12}$$

(Below $\rightharpoonup$ is used to denote weak convergence). That a given Young measure satisfies this L^1 weak limiting property when the $(f^k)_k$ are bounded in L^2 and ψ is of strictly subquadratic growth is immediate: the sequence $(\psi(f^k))_k$ is automatically bounded in L^p for some $1 < p$ and thus weakly (sequentially) precompact. The limiting case is the case of interest here and consists of continuous functions of quadratic growth, namely in the Banach space

$$\mathcal{E} = \left\{ \psi \in C(\mathbb{R}^m) : \sup_{a\in\mathbb{R}^m} \frac{|f(a)|}{1+|a|^2} < \infty \right\}$$

with norm $\|\psi\|_{\mathcal{E}} = \sup_{a\in\mathbb{R}^m} \frac{|\psi(a)|}{1+|a|^2}$. In this case the sequence $(\psi(f^k))_k$ is bounded in L^1 which is not enough to guarantee precompactness and thus more information is required: it suffices to establish (12) in the case of $\tilde{\psi}(\alpha) = |\alpha|^2$ (or any other function bounded quadratically from below) so that

$$|f^k|^2 \longrightarrow \langle \nu, \tilde{\psi} \rangle \quad \text{in } \sigma(L^1, L^\infty).$$

This information alone gives a bound on the generating functions $(f^k)_k$ and guarantees the representation (weak) for all $\psi \in \mathcal{E}$ by a direct application of Dunford-Pettis theorem. As noted in [12] the space $\mathcal{E}$ is not separable, an impediment particularly when duals of spaces such as $L^1(\Omega, X)$, where X is a Banach space, are considered. For this reason in place of $\mathcal{E}$ we consider, when appropriate, its separable subspace, Banach under the same norm,

$$\mathcal{E}_0 = \left\{ \psi \in C(\mathbb{R}^m) : \lim_{|a| \to \infty} \frac{|f(a)|}{1 + |a|^2} \quad \text{exists} \right\}.$$

Furthermore, if ψ has linear growth it is easy to show (using suitable cut-off functions) that the convergence in (12) will be weakly in L^2.

As it turns out from the existence scheme below, ν is a *spatial gradient (respectively, Laplacian) Young measure*, that is, ν is generated by gradient (respectively Laplacian) derivatives in the x variable which belong to L^2. (In the present framework one obtains in the case of $\mathcal{H}$ a time-parametrised curve of measures in the space of H_0^1*-gradient Young measures*, a space with a rich structure and, loosely speaking, characterised by a form of Jensen's inequality. Refer to [12] and references therein for an in-depth analysis of gradient generated Young measures. The theory has an analogue to the case of Young measures generated by Laplacian derivatives of functions in $H_0^2(\Omega)$.)

Existence of Young measure solutions

Let ϕ^{**} denote the convexification of ϕ. We assume that $\phi \in C^1(\mathbb{R}^m)$ and impose the growth condition

$$(|a|^2 - 1)^+ \leq \phi(a) \leq c|a|^2 + 1$$

Then, $\phi^{**} \in C^1(\mathbb{R}^m)$ and obeys the same growth condition as ϕ. Clearly, ϕ and ϕ^{**} are in $\mathcal{E}$ (the quadratic growth from above and below is essential to obtain weak precompactness in L^1 as indicated above). Let $p = \nabla \phi^{**}$. We assume that q and p have linear growth: $|q(a)| \leq C|a|$ and a similar bound for p. Corresponding to (3) of $\mathcal{H}$ and (4) of $\mathcal{D}$ are the *relaxed* equations of single type,

$$u_t = \nabla \cdot p(\nabla u) \qquad \text{and} \qquad u_{tt} = -\Delta p(\Delta u)$$

hyperbolic and dispersive respectively. We have the following

Existence theorem. *Given initial data* $u_0 \in H_0^1(\Omega)$ *(respectively, in* $H_0^2(\Omega)$*) and* $z_0 \in L^2(\Omega)$ *there exists a Young measure solution* $(u, \boldsymbol{\nu})$ *for* $\mathcal{H}$ *(respectively,* $\mathcal{D}$*). Moreover, this solution solves the corresponding relaxed equations, respectively,*

$$u_{tt} = \nabla \cdot p(\nabla u)$$

in the case of $\mathcal{H}$ *and*

$$u_{tt} = -\Delta p(\Delta u)$$

in the case of $\mathcal{D}$ *with the same initial-boundary data. In addition, for both* $\mathcal{H}$ *and* $\mathcal{D}$*,*

$$\operatorname{supp} \nu_{x,t} \subset \{a \in \mathbb{R} \,:\, \phi(a) = \phi^{**}(a)\} \qquad a.e. \ in \ Q_\infty.$$

Remark. – The hyperbolic region for $\mathcal{H}$ (or the dispersive region for $\mathcal{D}$) is possibly strictly larger than the support of the Young measure: for example, in one spatial dimension, if ϕ is the double well potential function with wells at ± 1 then the support of the measure lies in the complement of the interval $(-1, 1)$ (where $\phi = \phi^{**}$) whereas the hyperbolic (or dispersive) region includes an interval around zero.

Proof of the theorem. A) The problem $\mathcal{H}$.

Step 1. Discretisation and estimates. – We discretise $\mathcal{H}$ implicitly: for time step $h > 0$ we have the equilibrium problem

$$\frac{u^{h,j} - 2u^{h,j-1} + u^{h,j-2}}{h^2} = \nabla \cdot q(\nabla u^{h,j}) \tag{13}$$

and equivalently the discretised form of the corresponding system is for $j \geq 0$

$$\begin{aligned} \frac{u^{h,j} - u^{h,j-1}}{h} &= w^{h,j} \\ \frac{w^{h,j} - w^{h,j-1}}{h} &= \nabla \cdot q(\nabla u^{h,j}). \end{aligned} \tag{14}$$

In the above we define $u^{h,0} = u_0$ and $u^{h,-1} = u_0 - hz_0$ so that the initial data are attained with interpolation. *A priori* this discretisation obeys estimates obtained by energy considerations: the energy for $\mathcal{H}$ defined as

$$E(t) = \int_\Omega \phi^{**}(\nabla u)(x,t) + \frac{1}{2} u_t^2(x,t) \ dx$$

(the use of ϕ^{**} in place of ϕ is appropriate due to the use of relaxation, as will become evident). The energy is formally conserved:

$$\frac{dE}{dt} = \int_\Omega p(\nabla u) \cdot \nabla u_t + u_t u_{tt}\, dx = 0$$

by (3). Consider the energy discretisation

$$E_{h,j} = \int_\Omega \phi^{**}(\nabla u^{h,j}) + \frac{1}{2}(w^{h,j})^2 \;\; dx \tag{15}$$

and $E_0 := E_{h,0}$ is defined by the convention above. Notice that

$$E_0 \leq c\|u_0\|^2_{H^1_0(\Omega)} + \|z_0\|^2_{L^2(\Omega)}. \tag{16}$$

The crucial estimate in the existence proof is the following one which asserts the non-increase of the disretised energy. Consider the discretised relaxed problem with p in place of q in (13); using (14), the convexity of ϕ^{**} and suppressing temporarily the dependence on h, one obtains

$$\begin{aligned} E_j - E_{j-1} &= \int_\Omega \Big\{ \phi^{**}(\nabla u^j) - \phi^{**}(\nabla u^{j-1}) \\ &\qquad + \frac{1}{2}(w^j + w^{j-1})(w^j - w^{j-1}) \Big\}\, dx \\ &\leq \int_\Omega \Big\{ p(\nabla u^j) \cdot (\nabla u^j - \nabla u^{j-1}) \\ &\qquad + \frac{1}{2} h \nabla \cdot p(\nabla u^j)(w^j + w^{j-1}) \Big\}\, dx \\ &= -\frac{h}{2} \int_\Omega \nabla \cdot p(\nabla u^j)(w^j - w^{j-1})\, dx \\ &= -\frac{h^2}{2} \int_\Omega (\nabla \cdot p(\nabla u^j))^2\, dx \;\leq\; 0. \end{aligned} \tag{17}$$

Thus

$$E_{h,j} \;\leq\; E_{h,j-1} \;\leq\; \cdots \;\leq\; E_0. \tag{18}$$

Thus by (17)

$$\sup_{h,j} \int_\Omega \phi^{**}(\nabla u^{h,j})\, dx \;\leq E_0 \tag{19}$$

$$\sup_{h,j} \int_\Omega (w^{h,j})^2 \, dx \ \leq E_0. \tag{20}$$

By the growth condition of ϕ^{**} we have from (4)

$$\sup_{h,j} \|\nabla u^{h,j}\|_{L^2(\Omega;\mathbb{R}^n)} \ \leq E_0$$

and since $u^{h,j} \in H_0^1(\Omega)$ this implies

$$\sup_{h,j} \|u^{h,j}\|_{H_0^1(\Omega)} \ \leq E_0. \tag{21}$$

Next we solve the discretisation. Let $h > 0$ be fixed. For each $j \geq 1$ and $v \in H_0^2(\Omega)$ define the coercive functional

$$\Phi_j(v) = \Phi(v; u^{h,j-1}, u^{h,j-2}) = \int_\Omega \phi(\nabla v) + \frac{(v - 2u^{h,j-1} + u^{h,j-2})^2}{2h^2} \, dx$$

which has Euler-Lagrange equation (13). By relaxation we shall obtain a generalised minimiser $u^{h,j} \in H_0^1(\Omega)$: define $\Phi_j^{**}(v)$ to be the corresponding relaxed functional

$$\Phi_j^{**}(v) = \int_\Omega \phi^{**}(\nabla v) + \frac{(v - 2u^{h,j-1} + u^{h,j-2})^2}{2h^2} \, dx.$$

then Φ^{**} is also coercive and is sequentially weakly lower semicontinuous on $H_0^1(\Omega)$; it attains its infimum at $u^{h,j} \in H_0^1(\Omega)$ and by standard relaxation theorems (*cf.* [5])

$$I_j := \inf \Phi_j(v) = \inf \Phi_j^{**}(v) = \Phi_j^{**}(u^{h,j}) \tag{22}$$

where the infimum is taken over all $v \in H_0^1(\Omega)$. The choice of variational principle is *a priori* a suitable one because it satisfies

$$\begin{aligned} \Phi_j^{**}(u^{h,j}) \leq \ & \Phi_j^{**}(u^{h,j-1}) \\ = & \int_\Omega \phi^{**}(\nabla u^{h,j-1}) + \frac{1}{2}(w^{h,j-1})^2 \, dx \\ = & \ E_{h,j-1} \ \leq \ E_0 \end{aligned} \tag{23}$$

by (18) and thus we obtain bounds (20), (21). The question of what is a good choice of a regularisation and a variational principle to solve it is addressed at the end of the article.

Consider minimising sequences $(u^{h,j,k})_k \subset H_0^1(\Omega)$ such that

$$\lim_{k\to\infty} \Phi_j(u^{h,j,k}) = \lim_{k\to\infty} \Phi_j^{**}(u^{h,j,k}) = \Phi_j^{**}(u^{h,j}) = I_j \tag{24}$$

where we have used the lower semicontinuity of the functional Φ_j^{**}. Also, by the coercivity of Φ^{**} the minimising sequences are bounded in $H_0^1(\Omega)$ uniformly in h by the estimate (23). So we may assume

$$u^{h,j,k} \overset{k\to\infty}{\longrightarrow} u^{h,j} \qquad \text{in } H_0^1(\Omega)\ w-s \tag{25}$$

where the notation $w-s$ is used to imply that the convergence is weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ by compact embedding. Together with (24) this implies

(26)

$$\lim_{k\to\infty}\int_\Omega \phi(\nabla u^{h,j,k})\,dx = \lim_{k\to\infty}\int_\Omega \phi^{**}(\nabla u^{h,j,k})\,dx = \int_\Omega \phi^{**}(\nabla u^{h,j})\,dx.$$

We may now apply a theorem in [12, Theorem 1.1] which asserts that the growth condition and the convexity of ϕ^{**}, (25) and (9) imply that

$$\phi^{**}(\nabla u^{h,j,k}) \overset{k\to\infty}{\longrightarrow} \phi^{**}(\nabla u^{h,j}) \qquad \text{in } L^1(\Omega).$$

Since $\phi \in \mathcal{E}$ has quadratic growth from below we conclude by the remarks on Young measures above that for all $\psi \in \mathcal{E}$ the sequence $\psi(\nabla u^{h,j,k}))_k$ is convergent weakly in $L^1(\Omega)$ to the limit given by the Young measure generated by $(\nabla u^{h,j,k})_k$. To be precise, if for each $j \geq 0$ we let $(\nu_x^{h,j})_{x\in\Omega}$ be the Young measure generated by $(\nabla u^{h,j,k})_k$ we have

$$\phi(\nabla u^{h,j,k}) \overset{k\to\infty}{\rightharpoonup} \langle \boldsymbol{\nu}^{h,j}, \phi\rangle \qquad \text{in } L^1(\Omega) \tag{27}$$

$$\phi^{**}(\nabla u^{h,j,k}) \overset{k\to\infty}{\rightharpoonup} \langle \boldsymbol{\nu}^{h,j}, \phi^{**}\rangle \qquad \text{in } L^1(\Omega) \tag{28}$$

and thus

$$\langle \boldsymbol{\nu}^{h,j}, \phi^{**}\rangle = \phi^{**}(\nabla u^{h,j}).$$

Since q, p have linear growth,

$$q(\nabla u^{h,j,k}) \overset{k\to\infty}{\rightharpoonup} \langle \boldsymbol{\nu}^{h,j}, q\rangle \qquad \text{in } L^2(\Omega)$$

$$p(\nabla u^{h,j,k}) \overset{k\to\infty}{\rightharpoonup} \langle \boldsymbol{\nu}^{h,j}, p\rangle \qquad \text{in } L^2(\Omega).$$

A particular consequence of (26), (27) and (28) is that

$$\int_\Omega \langle \boldsymbol{\nu}^{h,j}, \phi\rangle \, dx = \int_\Omega \langle \boldsymbol{\nu}^{h,j}, \phi^{**}\rangle \, dx$$

and since $\phi^{**} \leq \phi$ this implies

$$\operatorname{supp} \boldsymbol{\nu}^{h,j} \subseteq \{\phi = \phi^{**}\}. \tag{29}$$

Thus,

$$\begin{aligned} \langle \boldsymbol{\nu}^{h,j}, q\rangle &= \langle \boldsymbol{\nu}^{h,j}, p\rangle \qquad \text{a.e. in } \Omega \\ \nabla u^{h,j} &= \langle \boldsymbol{\nu}^{h,j}, id\rangle \qquad \text{a.e. in } \Omega \end{aligned}$$

where id is the identity on $\mathbb{R}^n$. At the minimiser $u^{h,j}$ the derivative of Φ^{**} is zero and we obtain the equation

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega p(\nabla u^{h,j}) \cdot \nabla\zeta + \frac{u^{h,j} - 2u^{h,j-1} + u^{h,j-2}}{h^2} \zeta \, dx = 0 \\ \qquad\qquad \forall \zeta \in H_0^1(\Omega), \ \ j = 0, 1, \ldots \end{array} \right. \tag{30}$$

By considering the *stability* of the Young measure minimiser (*cf.* [4, Section 6]), one observes the equilibrium equation

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \langle \boldsymbol{\nu}^{h,j}, q\rangle \cdot \nabla\zeta + \left(\frac{u^{h,j} - 2u^{h,j-1} + u^{h,j-2}}{h^2}\right) \zeta \, dx = 0 \\ \qquad\qquad \forall \zeta \in H_0^1(\Omega), \ \ j = 0, 1, \ldots \end{array} \right. \tag{31}$$

and similarly with p in replacing q. In view of (30) and (31) we have

$$\nabla \cdot \langle \boldsymbol{\nu}^{h,j}, q\rangle = \nabla \cdot \langle \boldsymbol{\nu}^{h,j}, p\rangle = \nabla \cdot p(\nabla u^{h,j}) \qquad \text{in } H^{-1}(\Omega).$$

Step 2. Interpolation. – Let $I^{h,j} = [hj, h(j+1))$ and $\chi^{h,j}$ be the indicator function of $I^{h,j}$. Letting as above

$$w^{h,j} = \frac{u^{h,j} - u^{h,j-1}}{h}$$

and interpolate as follows (explicitly indicating only time-dependence)

$$u_{tt}^h(t) = \sum_j \chi^{h,j}(t) \, \frac{w^{h,j+1} - w^{h,j}}{h} = \sum_j \chi^{h,j} \, \frac{u^{h,j+1} - 2u^{h,j} + u^{h,j-1}}{h^2}$$

and its integral

$$\tilde{u}_t^h(t) = \sum_j \chi^{h,j}(t) \Big\{ w^{h,j} + \frac{w^{h,j+1} - w^{h,j}}{h} (t - hj) \Big\}. \tag{32}$$

Also define

$$
\begin{aligned}
u^h_t(t) &= \sum_j \chi^{h,j}(t)\, w^{h,j+1} \\
\tilde{u}^h(t) &= \sum_j \chi^{h,j}(t)\big(u^{h,j} + w^{h,j+1}(t-hj)\big) \\
u^h(t) &= \sum_j \chi^{h,j}(t) u^{h,j+1}.
\end{aligned}
$$

Interpolate the Young measure

$$
\begin{aligned}
(33) \qquad \boldsymbol{\nu}^h &= (\nu^h_{x,t})_{(x,t)\in Q_\infty} : \\
&= \sum_j \chi^{h,j}(t)\boldsymbol{\nu}^{h,j+1} \quad \in \; L^1_{\text{loc}}(Q_\infty;\mathcal{E}_0') \cap L^2_{\text{loc}}(Q_\infty;\mathcal{F}_0')
\end{aligned}
$$

(where $\mathcal{F}_0$ is the space of continuous functions of linear growth such as q, p); thus $\boldsymbol{\nu}^h$ is a Young measure generated by $\left(\sum_j \chi^{h,j}(t)\nabla u^{h,j,k}\right)_k$ and satisfies

$$
(34) \quad \int_\Omega \langle \nu^h_{\cdot,t}, q\rangle \cdot \nabla\zeta + u^h_{tt}(\cdot,t)\zeta \, dx = 0 \qquad \forall \zeta \in H^1_0(\Omega) \text{ for each } t \geq 0
$$

equivalently,

$$
u^h_{tt} = \nabla \cdot \langle \boldsymbol{\nu}^h, q\rangle \qquad \text{in } H^{-1}(\Omega)\ \forall t \in \mathbb{R}^+ \text{ and in } H^{-1}(Q_T).
$$

Integrating (34) and using (32) we have

$$
(35) \qquad \int_0^T \int_\Omega \langle \boldsymbol{\nu}^h, q\rangle \cdot \nabla\zeta - \tilde{u}^h_t \zeta_t \, dx\, dt = 0 \qquad \forall \zeta \in H^1_0(Q_T)
$$

(the time intergral of a function in $H^2_0(Q_T)$ belongs to $H^2_0(\Omega)$). We may also replace Q_T with Q_∞. In addition,

$$
(36) \qquad \nabla u^h = \langle \boldsymbol{\nu}^h, \text{ id}\rangle \qquad \text{a.e. in } Q_\infty.
$$

With the above definitions $\tilde{u}^h(0) = u_0$ and $\tilde{u}^h_t(0) = z_0$.

Step 3. Passage to the limit as $h \to 0$. – Observe that $(\boldsymbol{\nu}^h)_h$ is bounded in $L^2_{\text{loc}}(Q_\infty;\mathcal{F}_0')$ which is isomorphic to the dual space of $L^2_{\text{loc}}(Q_\infty;\mathcal{F}_0)$

as $\mathcal{F}_0$ is separable. Thus there exists a subsequence (not relabeled) weakly convergent to a parametrised measure $\boldsymbol{\nu}$, that is,

$$\langle \boldsymbol{\nu}^h, f\rangle \overset{k\to\infty}{\rightharpoonup} \langle \nu, f\rangle \qquad \text{in } L^2_{\mathrm{loc}}(Q_\infty)\ \forall f\in\mathcal{F}_0.$$

Using cut-off functions as in [1] it is straightforward to show that the convergence extends to weak L^1 convergence for functions $f(x,t,\cdot)\in\mathcal{E}_0$, so that

$$\boldsymbol{\nu}^h \rightharpoonup \boldsymbol{\nu} \ \text{ weakly in } L^1_{\mathrm{loc}}(Q_\infty;\mathcal{E}_0')\cap L^2_{\mathrm{loc}}(Q_\infty;\mathcal{F}_0'). \tag{37}$$

For details see the proof of [6, Lemma 2.4]. By the same lemma it can be shown that the limiting measure $\boldsymbol{\nu}$ is a Young measure generated by the spatial gradients of a diagonal subsequence of $\left(\sum_j \chi^{h,j}(t)u^{h,j,k}\right)_{h,k}$.

By the linear growth of q we have $\langle \boldsymbol{\nu}^h, q\rangle \rightharpoonup \langle \boldsymbol{\nu}, q\rangle$ weakly in L^2_{loc} (similarly for p) and passing to the limit as $h\to 0$ in (35) we conclude that $(u^h_{tt})_h$ is bounded and in fact convergent in $H^{-1}(\Omega)$. Together with estimates (16), (17), (23) this implies bounds independently of h and hence the existence of a single weakly$*$ convergent subsequence in h (not relabeled) and limiting functions as indicated below such that

$$\begin{array}{lll}
u^h \overset{*}{\rightharpoonup} u & \text{in } L^\infty(\mathbb{R}^+, H^1_0(\Omega))\\
\tilde u^h \overset{*}{\rightharpoonup} \tilde u & \text{in } W^{1,\infty}(\mathbb{R}^+, L^2(\Omega))\\
u^h_t \overset{*}{\rightharpoonup} v & \text{in } L^\infty(\mathbb{R}^+, L^2(\Omega))\\
\tilde u^h_t \overset{*}{\rightharpoonup} \tilde v & \text{in } W^{1,\infty}(\mathbb{R}^+, H^{-1}(\Omega))\cap L^\infty(\mathbb{R}^+, L^2(\Omega))
\end{array}$$

(where the boundedness of $(u^h_{tt})_h$ in $H^{-1}(\Omega)$ is used for the convergence of $\tilde u^h_t$). Clearly, $\partial_t\tilde u = v$ (as $\partial_t \tilde u^h = u^h_t$). Furthermore, the above convergences hold also weakly in $L^2_{\mathrm{loc}}(Q_\infty)$ and by [12, Lemma 6.3] we conclude that $\tilde u = u$ and $\tilde v = v$ a.e. and thus $\partial_t u = \tilde v$. Hence, u satisfies (5). Also (29) implies

$$\operatorname{supp}\boldsymbol{\nu} \subseteq \{\phi = \phi^{**}\} \tag{38}$$

by choosing test functions with support in the complement of $\{\phi=\phi^{**}\}$ and using the convergence (37). Passing to the limit in h in (35) yields (6) and (7) required in the definition:

$$\int_0^T\int_\Omega \langle \boldsymbol{\nu}, q\rangle\cdot\nabla\zeta - u_t\zeta_t\,dx\,dt = 0 \qquad \forall\zeta\in H^1_0(Q_T)\quad \forall T\ge 0$$

and by (38), $\langle \boldsymbol{\nu}, q \rangle = \langle \boldsymbol{\nu}, p \rangle\ (x,t)$ a.e. in Q_∞ so the same equation as above holds with p replacing q. Equivalently,

$$u_{tt} = \nabla \cdot \langle \boldsymbol{\nu}, q \rangle = \nabla \cdot \langle \boldsymbol{\nu}, p \rangle \quad \text{in } H^{-1}_{\text{loc}}(Q_\infty).$$

Furthermore, passing to the limit in (36) we obtain

$$\nabla u = \langle \boldsymbol{\nu}, id \rangle \quad \text{a.e. in } Q_\infty.$$

The initial data are attained in the sense of the definition: by the embeddings

$$\begin{aligned} &(39) \qquad W^{2,\infty}((0,T), H^{-1}(\Omega)) \hookrightarrow C^1([0,T), H^{-1}(\Omega)) \\ &(40) \qquad W^{1,\infty}((0,T)), L^2(\Omega)) \hookrightarrow C^0([0,T), L^2(\Omega)) \end{aligned}$$

(*see* [13, Lemma 1.2]) and since by construction $\tilde{u}^h(0) = u_0$ and $\tilde{u}^h_t(0) = z_0$ the functions $u(\cdot,t)$ and $u_t(\cdot,t)$ are well defined for each $t \geq 0$ and so (8), (9) are true. Thus the pair $(u, \boldsymbol{\nu})$ is a Young measure solution of $\mathcal{H}$ and the corresponding relaxed problem.

B) The problem $\mathcal{D}$. The above scheme can be modified in the following way to prove existence for $\mathcal{D}$. Discretising similarly,

$$\begin{aligned} \frac{u^j - u^{h,j-1}}{h} &= w^{h,j} \\ \frac{w^{h,j} - w^{h,j-1}}{h} &= -\Delta q(\Delta u^{h,j}) \end{aligned}$$

we obtain the Euler-Lagrange equation of

$$\Phi_{h,j}(v) = \Phi(v; u_{h,j-1}, u_{h,j-2}) = \int_\Omega \phi(\Delta v) + \frac{(v - 2u^{h,j-1} + u^{h,j-2})^2}{2h^2}\, dx.$$

Standard relaxation and weak lower semicontinuity results extend to this case:

$$\inf_{v \in H^2_0(\Omega)} \Phi_{h,j}(v) = \inf_{v \in H^2_0(\Omega)} \Phi^{**}_{h,j}(v)$$

where Φ^{**} is the corresponding relaxed functional which is (sequentially) weakly l.s.c. in $\sigma(H^2_0(\Omega), H^{-2}(\Omega))$ and attains its minimum at $u^{h,j} \in H^2_0(\Omega)$. Consider the energy

$$E(t) = \int_\Omega \phi^{**}(\Delta u) + \frac{1}{2}(u_{tt})^2\, dx$$

and let

$$E_{h,j} = \int_\Omega \phi^{**}(\Delta u^{h,j}) + \frac{1}{2}(w^{h,j})^2\, dx$$

be its time-discretisation. Then $E_{h,j+1} - E_{h,j} \leq 0$. As above, $\Phi_{h,j}(u^{h,j}) \leq E_{h,0}$ whence one obtains the uniform bound

$$\sup_{j\geq 0} \|\Delta u^{h,j}\|_{L^2(\Omega)} < \infty.$$

By the regularity of the Laplacian the sequences

$$\|u^{h,j}\|_{H_0^2(\Omega)}, \ \|\nabla u^{h,j}\|_{H_0^1(\Omega;\mathbb{R}^n)}, \ \|\nabla^2 u^{h,j}\|_{L^2(\Omega;\mathbb{M}^{N\times N})}$$

are bounded independently of h and j. If $(u^{h,j,k})_k \subset H_0^2(\Omega)$ is a minimising sequence then using the coercivity of Φ^{**} we know

$$u^{h,j,k} \overset{k\to\infty}{\longrightarrow} u^{h,j} \quad \text{in } H_0^2(\Omega) w - s - s$$

where the notation $w - s - s$ is used to imply the triple convergence weakly in $H_0^2(\Omega)$ and strongly in $H_0^1(\Omega)$ and $L^2(\Omega)$ by compact embedding. We let $(\Delta u^{h,j,k})_k$ generate the Young measure $\boldsymbol{\nu}^{h,j} = (\nu_x^{h,j})_{x\in\Omega}$. As in the case of gradient generated Young measures, $\boldsymbol{\nu}^{h,j} \in \mathcal{E}_0'$ and characterises weak limits in $L^1(\Omega)$ of $\psi(\Delta u^{h,j,k})$ for all $\psi \in \mathcal{E}$ (generalising to the space $\mathcal{E}$ was justified earlier). Letting $k \to \infty$ one can deduce as before that

$$\text{supp } \boldsymbol{\nu} \subseteq \{\phi = \phi^{**}\}$$

$$\Delta u^{h,j} = \langle \boldsymbol{\nu}^{h,j}, \text{ id} \rangle \quad \text{a.e. in } \Omega$$

$$\int_\Omega p(\Delta u^{h,j})\Delta\zeta + \left(\frac{u^{h,j} - 2u^{h,j-1} + u^{h,j-2}}{h^2}\right)\zeta \, dx = 0 \qquad \forall \zeta \in H_0^2(\Omega)$$

and by the stability of the Young measure minimiser

$$\Delta\langle \boldsymbol{\nu}^{h,j}, q\rangle = \Delta\langle \boldsymbol{\nu}^{h,j}, p\rangle = \Delta p(\Delta u^{h,j}) \quad \text{in } H^{-1}(\Omega).$$

Following this, interpolate as for $\mathcal{H}$ and use similar uniform estimates (with $H_0^2(\Omega)$ and $H^{-2}(\Omega)$ replacing $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ respectively) to pass to the limit as $h \to 0$. This yields a Young measure solution $(u, \boldsymbol{\nu})$ satisfying equations (10) and (11) in accordance with the definition and which also solves the corresponding relaxed problem. The initial data are attained by the embeddings (39), (40) with $H^{-2}(\Omega)$ replacing $H^{-1}(\Omega)$. This completes the existence proof. □

Independence

The Young measure solutions for $\mathcal{H}$ and $\mathcal{D}$ have the property that the measure q and the identity function are independent with respect to the measure $\boldsymbol{\nu}$. In the case of the parabolic equation $u_t = \nabla \cdot q(\nabla u)$ this independence is a key property for proving the uniqueness of a Young measure solution in [6].

Independence lemma. – *Let $(u, \boldsymbol{\nu})$ be a solution to $\mathcal{H}$ or $\mathcal{D}$. Then the equality below holds,*

$$\langle \boldsymbol{\nu},\, q \cdot id \rangle = \langle \boldsymbol{\nu}, q \rangle \cdot \langle \boldsymbol{\nu}, id \rangle \qquad (x,t) \quad a.e.\ in\ Q_\infty.$$

Proof. – The cases $\mathcal{H}$ and $\mathcal{D}$ are treated similarly. Here the proof is given for $\mathcal{H}$. Let $(u^{h,j,k})_k$ be the minimising sequence as in (25) generating $\boldsymbol{\nu}^{h,j}$. Taking the limit as $k \to \infty$ and using (31) one has

$$\int_\Omega q(\nabla u^{h,j,k}) \cdot \nabla \zeta + \left(\frac{u^{h,j,k} - 2u^{h,j-1} + u^{h,j-2}}{h^2} \right) \zeta \, dx$$
$$\longrightarrow \int_\Omega \langle \boldsymbol{\nu}^{h,j}, q \rangle \cdot \nabla \zeta + \frac{u^{h,j} - 2u^{h,j-1} + u^{h,j-2}}{h^2} \zeta \, dx = 0$$

and by the strong L^2 convergence of $(u^{h,j,k})_k$,

$$\nabla \cdot q(\nabla u^{h,j,k}) \quad \longrightarrow \quad \nabla \cdot \langle \boldsymbol{\nu}^{h,j}, q \rangle \quad \text{(strongly) in } H^{-1}(\Omega).$$

Furthermore, since q, $id \in \mathcal{F}$ and $q \cdot id \in \mathcal{E}$, we have as $k \to \infty$

$$\begin{aligned} q(\nabla u^{h,j,k}) &\rightharpoonup \langle \boldsymbol{\nu}^{h,j}, q \rangle \quad \text{in } L^2(\Omega) \\ \nabla u^{h,j,k} &\rightharpoonup \langle \boldsymbol{\nu}^{h,j}, id \rangle \quad \text{in } L^2(\Omega) \\ q(\nabla u^{h,j,k}) \cdot \nabla u^{h,j,k} &\rightharpoonup \langle \boldsymbol{\nu}^{h,j}, q \cdot id \rangle \quad \text{in } L^1(\Omega). \end{aligned}$$

By the div-curl lemma (or by direct computation using the strong H^{-1} convergence) one concludes

$$q(\nabla u^{h,j,k}) \cdot \nabla u^{h,j,k} \overset{k\to\infty}{\rightharpoonup} \langle \boldsymbol{\nu}^{h,j}, q \rangle \cdot \langle \boldsymbol{\nu}^{h,j}, id \rangle \quad \text{in } \mathcal{D}'(\Omega)$$

(that is, in the sense of distributions). Therefore,

$$(41) \qquad \langle \boldsymbol{\nu}^{h,j},\, q \cdot id \rangle = \langle \boldsymbol{\nu}^{h,j},\, q \rangle \cdot \langle \boldsymbol{\nu}^{h,j},\, \mathrm{id} \rangle \ x \quad \text{a.e. in } \Omega$$

By the interpolation in (33)

$$(42) \qquad \langle \boldsymbol{\nu}^h, q \cdot id \rangle = \langle \boldsymbol{\nu}^h, q \rangle \cdot \langle \boldsymbol{\nu}^h,\, id \rangle \ x \quad \text{a.e. in } \Omega, \qquad \forall t \geq 0.$$

We now wish to take the limit as $h \to 0$. By the existence theorem $\tilde{u}_t^h \rightharpoonup u_t$ weakly in $L^\infty(\mathbb{R}^+, L^2(\Omega))$ and thus also weakly in $L^2_{\text{loc}}(Q_\infty)$.

By the embedding $L^2(Q_T) \hookrightarrow H^{-1}(Q_T)$ and by taking into account (35) and (37) this implies

$$\nabla \cdot \langle \boldsymbol{\nu}^h, q \rangle \longrightarrow \nabla \cdot \langle \boldsymbol{\nu}, q \rangle \quad \text{in } H^{-1}(Q_T).$$

By applying the div-curl lemma once more we may pass to the limit in (42) above and the lemma follows. □

Alternative discretisations and variational principles

It is clear that the way of discretising the systems above is not unique and neither is the variational principle used to solve a discretisation. I try to illustrate here two aspects, namely, (i) as long as the discretisation obeys energy non-increase, any variational principle which solves it (and can be minimised by relaxation) is admissible; (ii) if the requirements on the discretisation are too severe, (if for example the energy were minimised at each iteration), the dynamics may disappear in the process completely.

(i) In the existence proof the uniform estimates on the $u^{h,j}$ come from the *energy* and cannot be expected in general to come from the variational principle. In particular, (in the case of $\mathcal{H}$ with similar considerations in the case of $\mathcal{D}$), the estimate

$$\int_\Omega \phi^{**}(\nabla u^{h,j}) dx \;\leq\; \Phi_j^{**}(u^{h,j})$$

which is the crucial estimate in the case of a gradient flow (*see* [12]), does not suffice to infer (21). The criteria for choice of a suitable variational principle are that it possess the correct Euler-Lagrange equation (agreeing with the discretisation (14)), and that the discretised energy dominate the minimum at each level j so that the uniform bounds (20), (21) hold. For example, an equally appropriate choice of a functional to solve the discretisation (14) is

$$\Psi_j(v) = \int_\Omega \phi(\nabla v) + \frac{(v - 2u^{h,j-1} + u^{h,j-2})(v - u^{h,j-1})}{h^2} - \frac{(v - u^{h,j-1})^2}{2h^2}\, dx$$

for all $v \in H_0^2(\Omega)$, which is also coercive:

$$\begin{aligned}\Psi_j(v) &= \int_\Omega \phi(\nabla v) - \frac{(v-u^{h,j-1})(u^{h,j-1}-u^{h,j-2})}{h^2} + \frac{(v-u^{h,j-1})^2}{2h^2}\,dx \\ &\geq c\int_\Omega (|\nabla v|^2 - k)^+ + \frac{(1-\delta)(v-u^{h,j-1})^2}{2h^2} \\ &\quad - \frac{(u^{h,j-1}-u^{h,j-2})^2}{2\delta h^2}\,dx \\ &\geq c\int_\Omega (|\nabla v|^2 - \beta_j)\,dx\end{aligned}$$

where $0 < c$, $0 < k$, $0 < \delta < 1$ and β_j are constants. Thus Ψ^{**} attains its infimum in $H_0^1(\Omega)$, say at $z^{h,j}$. Notice that in this case also the variational principle yields no uniform estimates by itself but through its domination from above by the energy:

$$\Psi_j^{**}(z^{h,j}) \leq \Psi_j^{**}(z^{h,j-1}) = \int_\Omega \phi^{**}(\nabla z^{h,j-1})dx \;\leq\; E_{h,j-1} \;\leq\; E_0.$$

Both functionals solve the same discretisation (14) for which the energy is given by (15). The present method of construction of solution to $\mathcal{H}$ has no way of discriminating between the two functionals. In what follows we continue to work with Φ_j. (ii) A rather subtle modification of the existence scheme results in a regularisation for which the *dynamics of the equations are lost* and the scheme approaches the solution of the equilibrium equations $\nabla \cdot q(\nabla u) = 0$ (respectively, $\Delta q(\Delta u) = 0$). Assume Φ_j^{**} and $E_{h,j}$ are as above in the case of $\mathcal{H}$. For $j \geq 0$ we let $u^{h,2j+1}$ be the minimiser of $E_{h,2j+1}$ (which is possible since $E_{h,j}$ is convex and coercive) and $u^{h,2j+2}$ the minimiser of $\Phi^{**}(v; u^{h,2j+1}, u^{h,2j})$. That is, the energy is minimised at each step rather than being evaluated at the minimiser of Φ^{**}. Clearly the energy will fail to be conserved in the limit as $h \to 0$ and it approaches its equilibrium value. In fact, u_t^h vanishes as $h \to 0$. To see this, consider the estimate

$$\begin{aligned}(43)\quad &\int_\Omega \phi^{**}(\nabla u^{h,2j+2})\,dx \\ &\leq \int_\Omega \phi^{**}(\nabla u^{h,2j+2}) + \frac{(u^{h,2j+2} - 2u^{h,2j+1} + u^{h,2j})^2}{2h^2}\,dx \\ &\leq \int_\Omega \phi^{**}(\nabla u^{h,2j+1}) + \frac{(u^{h,2j+1} - u^{h,2j})^2}{2h^2}\,dx \\ &\leq \int_\Omega \phi^{**}(\nabla u^{h,2j})\,dx\end{aligned}$$

$$
\begin{aligned}
&\leq \int_\Omega \phi^{**}(\nabla u^{h,2j}) + \frac{(u^{h,2j} - 2u^{h,2j-1} + u^{h,2j-2})^2}{2h^2}\, dx \\
&\leq \dots \\
&\leq \int_\Omega \phi^{**}(\nabla u_0)\, dx
\end{aligned}
$$

independently of h, j. That is,

$$
\begin{aligned}
\Phi^{**}(u^{h,2j+2}; u^{h,2j+1}, u^{h,2j}) \leq E_{h,2j+1} \leq \int_\Omega \phi^{**}(\nabla u^{h,2j})\, dx \\
\leq \int_\Omega \phi^{**}(\nabla u_0)\, dx.
\end{aligned}
$$

Consider the sequence $(\nabla u^{h,j})_{j\geq 1}$ which we use to generate Young measures $(\boldsymbol{\nu}^{h,j})_j$ and following interpolate and obtain a Young measure solution as above. It is easy to see that (43) implies

$$
\begin{aligned}
\|u^h_{tt}\|^2_{L^2_{(Q_\infty)}} &= \sum_{j=0}^{\infty} \int_{2jh}^{(2j+2)h} \int_\Omega \frac{(u^{h,2j+2} - 2u^{h,2j+1} - u^{h,2j})^2}{2h^2}\, dx \\
&\leq \frac{2}{h} \int_\Omega \phi^{**}(\nabla u_0)\, dx
\end{aligned}
$$

so that $\|u^h_{tt}\|_{L^2(Q_\infty)} = O(\frac{1}{\sqrt{h}})$. Similarly by (32),

$$
\|u^h_t\|^2_{L^2_{(Q_\infty)}} \leq Ch \int_\Omega \phi^{**}(\nabla u_0)\, dx
$$

and so $\|u^h_t\|_{L^2(Q_\infty)} = O(\sqrt{h})$ as $h \to 0$, *i.e.*, $u^h_t \to 0$ as $h \to 0$ and so $u_t = 0$.

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