ANNALES DE L'I. H. P., SECTION C

K. C. CHANG
J. Q. LIU
M. J. LIU

Nontrivial periodic solutions for strong resonance hamiltonian systems

Annales de l'I. H. P., section C, tome 14, nº 1 (1997), p. 103-117
<http://www.numdam.org/item?id=AIHPC_1997__14_1_103_0>

Ann. Inst. Henri Poincaré,

Vol. 14, n° 1, 1997, p. 103-117 *Analyse non linéaire*

Nontrivial periodic solutions for strong resonance Hamiltonian systems

by

K. C. CHANG*

Institute of Math., Peking University
Bejing 100871, P.R. China.

and

J. Q. LIU* and M. J. LIU

Dept. of Math., Graduate School, Academia Sinica,
Beijing 10039, P. R. China.

ABSTRACT. – For a Hamiltonian system, in which the Hamiltonian is assumed to have an asymptotically linear gradient, the existence of nontrivial periodic solutions is proved under the assumption that the linearized operators have distinct Maslov indices at 0 and at infinity. Both the linearized operators may be degenerate. In particular, the results cover the "strong resonance" case.

RÉSUMÉ. – Pour un système hamiltonien dans lequel l'hamiltonien est supposé avoir un gradient asymptotiquement linéaire, on montre l'existence de solutions périodiques non triviales, sous l'hypothèse que les opérateurs linéarisés ont un indice de Maslov différent en 0 et en l'infini. Les opérateurs linéarisés peuvent même être dégénérés. En particulier, ces résultats comprennent le cas de « résonance forte ».

* Supported by CNSF.

1. INTRODUCTION

We study the following periodic solution problem:

$$-J\,\frac{dx}{dt} = H_x(t,\,x), \tag{1.1}$$

where $H \in C^2([0,\,1] \times \mathbb{R}^{2n},\, \mathbb{R})$ is 1-periodic in t, and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

(1.1) is called asymptotically linear, if there exists a $2n \times 2n$ symmetric matrix function $B_\infty(t)$, which is continuous, 1-periodic such that

$$|H_x(t,\,x) - B_\infty(t)\,x| = 0\,(|x|) \quad \text{as } |x| \to \infty \tag{1.2}$$

where $|\cdot|$ is the $\mathbb{R}^{2n}$ norm.

The following question is raised: Having found one solution, say θ, which is called the trivial solution, can we conclude the existence of a nontrivial solution by assuming conditions on the two linearized systems at θ and at ∞, *i.e.*, on the two matrices:

$$B_0(t) = H_{xx}(t,\,\theta), \tag{1.3}$$

and $B_\infty(t)$.

An important notion in this study is the Maslov index. For a continuous 1-periodic symmetric matrix function $B(t)$, let $W(t)$ be the associate fundamental solution matrix of the linear system: $-J\,\frac{dX}{dt} = B(t)\,X$. B is called nondegenerate if $W(1)$ has no eigenvalue 1, *i.e.*, 1 is not a Floquet multiplier of B. Let $\mathrm{Sp}(n,\,\mathbb{R})$ denote the set of all $2n \times 2n$ symplectic matrices, and let

$$P = \{\gamma \in C([0,\,1],\, \mathrm{Sp}(n,\,\mathbb{R})) | \gamma(0) = I,\ \gamma(1) \text{ has no eigenvalue } 1\}.$$

According to Conley Zehnder [CZ] and Long Zehnder [LZ], there is a map $j : P \to \mathbb{Z}$. For nondegenerate $B(t)$, one defines the Maslov index $i(B) = k$ if $j(W) = k$, where W is the fundamental solution matrix.

If B is degenerate, *i.e.,* 1 is a Floquet multiplier, Long [Lo] extended the definition. A pair $(i_-(B),\, n(B))$ is called the Maslov index of B, if

$$\begin{cases} n(B) = \dim\ker(W(1) - I), \\ i_-(B) = \varliminf\limits_{C \to B} i(C) \quad \text{where } C \text{ is nondegenerate.} \end{cases}$$

In particular, if B is nondegenerate, then $n(B) = 0$, and $i_-(B)$ is the Maslov index $i(B)$.

The above problem was firstly studied by H. Amann and E. Zehnder [AZ 1, 2]. They assumed that both B_0 and B_∞ are constant matrices, where B_∞ is nondegenerate, and $i(B_\infty) \notin [i_-(B_0), i_-(B_0) + n(B_0)]$. Later, C. Conley and E. Zehnder [CZ] studied the case where B_0 and B_∞ are nondegenerate, but not necessarily constant, and $i(B_0) \neq i(B_\infty)$. Other authors followed the study in case where B_∞ is nondegenerate, *see* [LL], [Lo], [LZ], [DL], [Li]. As to degenerate, but constant B_∞, [Ch2] and [Sz] studied the Landesman Lazer type resonance condition; and [Ch3] [Sa] studied the strong resonance condition.

Set

$$h(t, x) = H(t, x) - \frac{1}{2}\langle B_\infty(t)\, x, x\rangle. \tag{1.4}$$

where $\langle\ ,\ \rangle$ is the inner product of $\mathbb{R}^{2n}$. The so called strong resonance condition is defined as follows: (1.1) is asymptotically linear, and $B_\infty(t)$ is degenerate and satisfies:

$$h(t, x) \to 0, \tag{1.5}$$

and

$$|h_x(t, x)| \to 0, \tag{1.6}$$

uniformly in $t \in [0, 1]$ as $|x| \to \infty$.

Our main result reads as

THEOREM 1.1. – *Assume* (1.3), (1.5), (1.6), *where B_∞ is degenerate, and that*

$$|H_{xx}(t, x)| \leq C_1(1 + |x|^s) \tag{1.7}$$

for some $C_1 > 0$, $s \in (1, \infty)$ and all $(t, x) \in [0, 1] \times \mathbb{R}^{2n}$. Then (1.1) *possesses a nontrivial solution if one of the following three cases occurs:*

(1) $\int_0^1 H(t, \theta)\, dt = 0$,

(2) $\int_0^1 H(t, \theta)\, dt > 0$ *and* $i_-(B_\infty) \notin [i_-(B_0), i_-(B_0) + n(B_0)]$,

(3) $\int_0^1 H(t, \theta)\, dt < 0$, *and* $i_-(B_\infty) + n(B_\infty) \notin [i_-(B_0), i_-(B_0) + n(B_0)]$.

For Landesman Lazer type resonance, we have

THEOREM 1.2. – *Assume* (1.3), (1.7) *and the following hypotheses*:

$$|H_x(t, x) - B_\infty(t)\, x| = 0(1) \quad \text{as } |x| \to \infty, \tag{1.8}$$

$$H(t, x) - \frac{1}{2} < B_\infty(t)\, x, x > \to \mp\infty \quad \text{as } |x| \to \infty, \tag{1.9}$$

where $B_\infty(t)$ is a degenerate symmetric continuous matrix function. Then (1.1) possesses a nontrivial solution if

$$i_-(B_\infty) \notin [i_-(B_0),\, i_-(B_0) + n(B_0)],$$

(or $i_-(B_\infty) + n(B_\infty) \notin [i_-(B_0),\, i_-(B_0) + n(B_0)]$ resp.).

As a consequence, we have

COROLLARY 1.3. – *Under the same assumptions in Theorem* 1.1 *or Theorem* 1.2, (1.1) *possesses a nontrivial solution if*

$$[i_-(B_0),\, i_-(B_0) + n(B_0)] \cap [i_-(B_\infty),\, i_-(B_\infty) + n(B_\infty)] = \emptyset.$$

Remark 1.4. – In Theorem 1.2, if further we assume that $B_\infty(t)$ is nondegenerate, and that (1.8) is replaced by (1.2); then the assumption (1.9) can be dropped out.

It seems that the above two theorems and their remark include and extend all known results in literature on this problem.

The novelties in proofs consist of the following three ingredients:

(1) By a variational approach, the Morse inequalities are used to estimate the number of critical points. But, the Palais Smale Condition fails for strong resonance problem. We compactify the kernel of the linear operator by adding an infinity point, and extend our functional to the enlarged manifold, so that the (PS) Condition is gained. (2) We introduce an abstract Maslov index for compact self adjoint operators with respect to a bounded self adjoint operator with finite dimensional kernel. The index relates to the difference of Morse indices of a certain functional. This abstract index coincides with the Maslov index for a matrix function (with respect to $-J\frac{d}{dt}$). (3) The Maslov indices, which replace the critical groups for the strongly indefinite functional, are used to distinguish genuine critical points from the fake.

2. ABSTRACT THEORY

We would study the above problems in an abstract framework. Let $\mathcal{H}$ be a separable Hilbert space with inner product (,) and norm $\|\cdot\|$. Assume

(A) A is a bounded self adjoint operator with a finite dimensional kernel N, and the restriction $A|_{N^\perp}$ is invertible. (Denote by P the orthogonal projection $\mathcal{H} \to N$.)

(G) $G : \mathcal{H} \to \mathbb{R}^1$ is a C^1-functional with a compact differential G'. And there is a linear compact symmetric operator B_∞ such that

$$G(x) - \frac{1}{2}(B_\infty x, x) \to 0,$$
$$\|G'(x) - B_\infty x\| \to 0,$$

as $\|P_\infty x\| \to \infty$ uniformly on any subset in which Qx is bounded, where P_∞ is the orthogonal projection from $\mathcal{H}$ to $\mathcal{H}_\infty$, the kernel of $A + B_\infty$, and $Q = I - P_\infty$.

We consider the functional

$$f(x) = \frac{1}{2}(Ax, x) + G(x). \tag{2.1}$$

Suppose $G'(\theta) = \theta$, then θ is a critical point of f, we are looking for nontrivial critical points of f. Since f is strong indefinite, the critical point θ has ∞ as its Morse index. In order to go around the infinity of Morse indices, we introduce the abstract Maslov index with the aid of a Galërkin approximation procedure.

DEFINITION 2.1. – Let $\Gamma = \{P_n | n = 1, 2, ...\}$ be a sequence of orthogonal projections. We call Γ an approximation scheme w.r.t. A, if the following properties hold:

(1) $\mathcal{H}_n := P_n \mathcal{H}$ is finite dimensional $\forall n$,

(2) $P_n \to I$ strongly as $n \to \infty$,

(3) $[P_n, A] = P_n A - A P_n \to 0$ in the operator norm.

For a self adjoint bounded operator C, denote by $m(C)$ the Morse index of C.

LEMMA 2.1. – *If T is a compact linear operator defined on $\mathcal{H}$, and if $\{P_n\}$ is a sequence of orthogonal projections satisfying* (2) *in Definition* 2.1, *then $\forall \varepsilon > 0$ there exists n_0 an integer, such that $\|T(I - P_n)\|$ and $\|(I - P_n)T\| < \varepsilon$, $\forall n \geq n_0$.*

Proof. – We only prove the first one; the second is proved similarly. If not, there exist $\varepsilon_0 > 0$, and a sequence x_n with $\|x_n\| \leq 1$ such that $\|T(I - P_n)x_n\| \geq \varepsilon_0$. Substracting a subsequence, denoting again by x_n, we have $x_n \rightharpoonup x$, and then $Tx_n \to Tx$. Since $P_n x_n \rightharpoonup x$, $TP_n x_n \to Tx$. This is a contradiction.

Now we prove

THEOREM 2.2. – *Let B be a linear symmetric compact operator. Suppose that $A + B$ has a bounded inverse. Then the difference of Morse indices*

$$m(P_n(A + B)P_n) - m(P_n(A + P)P_n)$$

eventually becomes a constant independent of n, where A satisfies (A), P is the orthogonal projection onto the kernel of A, and Γ is an approximation scheme w.r.t. A.

Proof. 1° $P_n(A+B)P_n + (I-P_n)$ is invertible for n large. Indeed, we only need to verify that

$$\|P_n(A+B)P_n x\| \geq \varepsilon \|P_n x\| \qquad \forall n \geq n_0 \tag{2.2}$$

for some $\varepsilon > 0$ and n_0. However, for n large

$$\begin{aligned} &\|P_n(A+B)P_n x\| \\ &\quad \geq \|(A+B)P_n x\| - \|(I-P_n)BP_n x\| - \|[P_n, A]P_n x\| \\ &\quad \geq (C_1 - \|(I-P_n)B\| - \|[P_n, A]\|)\|P_n x\| \end{aligned}$$

where $C_1 = \|(A+B)^{-1}\|^{-1}$. By virtue of lemma 2.1 and (3) in Definition 2.1, $\|(I-P_n)B\| < C_1/3$ and $\|[P_n, A]\| < C_1/3$, our conclusion follows.

2° We define a finite dimensional orthogonal projection S satisfying

$$[S, A] = 0, \qquad \text{and} \quad \|(I-S)B\| < \varepsilon/6. \tag{2.3}$$

as follows: Let $y_1, y_2, \ldots, y_l$ be a $\varepsilon/18$ net of the image of B acting on the unit ball U, *i.e.*, $\forall x \in U$, there exists $i \in [1, l]$ such that $\|Bx - y_i\| < \varepsilon/18$. There exists a finite dimensional orthogonal projection S satisfying $[S, A] = 0$, and $\|Sy_j - y_j\| < \varepsilon/18\ \forall j$, according to the Spectral Decomposition Theorem. It follows

$$\|(I-S)Bx\| \leq \|Bx-y_i\|+\|y_i-Sy_i\|+\|S(Bx-y_i)\| < \varepsilon/6, \qquad \forall x \in U.$$

Set $S_n = P_n S P_n$, we shall prove

$$\begin{aligned} m(P_n(A+B)P_n) =& m(S(A+B)S) \\ &+ m(P_n(I-S_n)(A+B)(I-S_n)P_n), \end{aligned} \tag{2.4}$$

for large n. Indeed,

$$\begin{aligned} P_n(A+B)P_n =& S_n(A+B)S_n + P_n(I-S_n)(A+B)(I-S_n)P_n \\ &+ S_n(A+B)(I-S_n)P_n + P_n(I-S_n)(A+B)S_n. \end{aligned}$$

Applying lemma 2.1 to $T = S$ and $P_n S$ respectively, we obtain

$$\|S_n - S\| \leq \|S_n - P_n S\| + \|(I-P_n)S\| < \varepsilon/(6M), \tag{2.5}$$

where $M = \|A + B\|$, for large n, and then

$$\begin{aligned}
&\|S_n (A + B) (I - S_n)\| \\
&\le \|(S_n - S) (A + B) (I - S_n)\| + \|S (A + B) (S_n - S)\| \\
&+ \|S (A + B) (I - S)\| \\
&< \varepsilon/2.
\end{aligned}$$

Similarly, we have the same estimates for $(I - S_n) (A + B) S_n$. Thus

$$\begin{aligned}
m (P_n (A + B) P_n) =& m (S_n (A + B) S_n) \\
&+ m (P_n (I - S_n) (A + B) (I - S_n) P_n),
\end{aligned}$$

because $x = S_n x + P_n (I - S_n) x$ is a direct sum in $\mathcal{H}_n$. By the same argument in 1°, one may choose S satisfying (2.3), such that $S (A + B) S + (I - S)$ is invertible. Again, by (2.5) for n large,

$$\begin{aligned}
m (S (A + B) S) &= m (S (A + B) S) + (I - S)) \\
&= m (S_n (A + B) S_n + (I - S_n)) = m (S_n (A + B) S_n).
\end{aligned}$$

This proves (2.4).

3° Recall P is the orthogonal projection onto N. Again by (2.5),

$$m (P_n (I-S_n) (A+B) (I - S_n) P_n) = m (P_n (I-S_n) (A+P) (I - S_n) P_n)$$

and then (2.4) becomes

$$\begin{aligned}
m (P_n (A + B) P_n) =& m (S (A + B) S) \\
&+ m (P_n (I - S_n) (A + P) (I - S_n) P_n).
\end{aligned}$$

for n large. Similarly we have

$$\begin{aligned}
m (P_n (A + P) P_n) =& m (S (A + P) S) \\
&+ m (P_n(I - S_n) (A + P) (I - S_n) P_n).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&m (P_n (A + B) P_n) - m (P_n (A + P) P_n) \\
&\quad = m (S (A + B) S) - m (S (A + P) S)
\end{aligned} \tag{2.6}$$

for n large. And the right hand side of (2.6) is independent of n.

Given an invertible $A+B$, with compact symmetric B, we define an index

$$I(B) = \lim_{n\to\infty} (m(P_n(A+B)P_n) - m(P_n(A+P)P_n)). \tag{2.7}$$

It is easily seen that the index I does not depend on the special choice of the approximation scheme. In fact, let $\tilde{\Gamma} = \{\tilde{P}_n | n = 1, 2, \ldots\}$ be another scheme different from Γ, we define a new scheme $\Gamma \vee \tilde{\Gamma} = \{P'_n | n = 1, 2, \ldots\}$ where

$$P'_n = \begin{cases} P_k & n = 2k-1, \\ \tilde{P}_k & n = 2k, \end{cases}$$

$k = 1, 2, \ldots$; then, by Theorem 2.1, $I(B)$ is well defined w.r.t. $\Gamma \vee \tilde{\Gamma}$. This proves that the index w.r.t. $\tilde{\Gamma}$ is the same with Γ.

Now, we give

DEFINITION 2.3. – *For a given compact linear symmetric operator B, let P_B be the orthogonal projection onto* $\ker(A+B)$, *we define*

$$N(B) = \dim\ker(A+B),$$
$$I_-(B) = I(B + P_B);$$

and call the pair $(I_-(B), N(B))$ the abstract Maslov index of B w.r.t. A.

By definition, we immediately have

$$I_-(B) + N(B) = I(B - P_B). \tag{2.8}$$

The following theorem is a generalization of Theorem 2.8 in [CL].

THEOREM 2.4. – *Assume that the functional f defined in* (2.1), *satisfies the assumptions* (*A*) *and* (*G*). *Then f has a critical point. Moreover, if θ is a critical point, and if G is C^2 in a neighbourhood of θ and one of the following conditions hold*:

(1) $f(\theta) = 0$,

(2) $f(\theta) < 0$ *and* $I_-(B_\infty) \notin [I_-(B_0), I_-(B_0) + N(B_0)]$,

(3) $f(\theta) > 0$ *and* $I_-(B_\infty) + N(B_\infty) \notin [I_-(B_0), I_-(B_0) + N(B_0)]$,

where $B_0 = d^2 G(\theta)$;

then f possesses at least a critical point other than θ.

Proof. – We take a sequence of orthogonal projections P_n such that. $H_n = P_n H$ is invariant under $A + B_\infty$. Since B_∞ is compact, by Lemma 2.1, $\Gamma = \{P_n\}$ is an approximation scheme w.r.t. A. According to lemma 3.1 in [CL], f satisfies $(PS)_c^*$ condition for $c \neq 0$, *i.e.*, any sequence

$x_n \in \mathcal{H}_n$ satisfying $f_n(x_n) \to c \neq 0$ and $f'_n(x_n) \to \theta$ possesses a strong convergent subsequence, where $f_n = f|_{\mathcal{H}_n}$, the restriction of f on $\mathcal{H}_n$.

Next define

$$F(u, s) = \begin{cases} f(u+s) & (u, s) \in \mathcal{H}_\infty^\perp \times \mathcal{H}_\infty, \\ \frac{1}{2}((A+B_\infty)u, u) & (u, s) \in \mathcal{H}_\infty^\perp \times \{\infty\}, \end{cases}$$

and let $F^n = F|_{\mathcal{H}_n^\perp \times \Sigma}$, where $\sum = \mathcal{H}_\infty \cup \{\infty\} \simeq S^{m_0}$, $m_0 = \dim \mathcal{H}_\infty$, $\mathcal{H}_n^\perp = \mathcal{H}_\infty^\perp \cap \mathcal{H}_n$, and $\mathcal{H}_\infty = \ker(A + B_\infty)$.

We apply lemma 2.6 in [CL] to F^n, and obtain two subordinate classes $\alpha_n^* \prec \alpha_n$ in the relative homology groups $H_*((F^n)_d, (F^n)_a)$ for large d and $-a$, where $(F^n)_c$ is the level set of F^n; and $a, d > 0$. We know from the same lemma,

$$q_n = \dim \alpha_n = m(P_n(A+B_\infty)P_n) + N(B_\infty), \quad \text{and}$$
$$q_n^* = \dim \alpha_n^* = m(P_n(A+B_\infty)P_n).$$

Let

$$c_n = \inf_{z \in \alpha_n} \sup_{x \in |z|} F^n(x)$$
$$c_n^* = \inf_{z \in \alpha_n^*} \sup_{x \in |z|} F^n(x)$$

These are critical values of F^n, if they are not zero. Since c_n, c_n^* are bounded, we have convergent subsequences such that

$$c = \lim_{n \to \infty} c_n, \qquad c^* = \lim_{n \to \infty} c_n^*.$$

It is easily seen:

(1) $c^* \leq c$.

(2) If c or c^* is not zero, then it is a critical value of f.

(3) If $c = c^* = 0$, then f has a noncompact critical set.

It remains to show: if either $c^* < 0$ or $c > 0$, then we have a nontrivial critical point with the critical value c^* or c.

If not, the only critical point of F is θ and $F(\theta) = c^*$ (or c).

We only consider the case $F(\theta) = c^*$, the other case is similar.

On one hand, we have $n_0 > 0$ such that $\forall\, n \geq n_0$,

$$\begin{aligned} q_n^* &= m\left(P_n\left(A+B_\infty\right)P_n\right) \\ &= m\left(P_n\left(A+B_\infty+P_\infty\right)P_n\right) \\ &= I_-\left(B_\infty\right)+m\left(P_n\left(A+P\right)P_n\right) \\ &\notin \left[I_-\left(B_0\right),\, I_-\left(B_0\right)+N\left(B_0\right)\right]+m\left(P_n\left(A+P\right)P_n\right) \\ &= \left[m\left(P_n\left(A+B_0+P_0\right)P_n\right),\, m\left(P_n\left(A+B_0-P_0\right)P_n\right)\right], \end{aligned}$$

where P_0 is the orthogonal projection onto $\ker(A+B_0)$.

One may find $\delta > 0$ such that

$$m\left(P_n\, d^2 f\left(x\right) P_n\right) \in \left[m\left(P_n\left(A+B_0+P_0\right)P_n\right),\, m\left(P_n\left(A+B_0-P_0\right)P_n\right)\right]$$

$\forall x \in B(\theta, \delta)$, the δ ball centered at θ, $\forall n \geq n_0$, by Theorem 2.2.

According to the $(PS)_d^*$, $d < 0$, for n_0 large, we have

$$df^n(x) \neq \theta \quad \text{as } \|x\| > \delta/2 \qquad \text{and} \qquad n \geq n_0.$$

$\forall n$, by Marino Prodi Theorem [MP], one constructs a functional $\tilde{F}^n$ on $\mathcal{H}_n$, which satisfies:

(1) $(PS)_d \qquad \forall d < 0$,

(2) $(\tilde{F}^n)_{c_n^* \pm \varepsilon} = (F^n)_{c_n^* \pm \varepsilon}$ for some ε,

(3) $\tilde{F}^n = F^n$ in $\mathcal{H}_n \backslash B(\theta, \varepsilon)$,

(4) $\tilde{F}^n$ has only nondegenerate critical points $y_1, \ldots, y_l$ all concentrated in $B(\theta, \delta/2) \cap \mathcal{H}_n$.

Thus,

$$m\left(\left(d^2 \tilde{F}^n\right)\left(y_j\right)\right) \in \left[m\left(P_n\left(A+B_0+P_0\right)P_n\right),\, m\left(P_n\left(A+B_0-P_0\right)P_n\right)\right],$$

$j = 1, 2, \ldots, l$.

On the other hand, by the definition of c_n^*, we have

$$H_{q_n^*}\left((\tilde{F}^n)_{c_n^*+\varepsilon},\, (\tilde{F}^n)_{c_n^*-\varepsilon}\right) = H_{q_n^*}\left(F^n_{c_n^*+\varepsilon},\, F^n_{c_n^*-\varepsilon}\right) \neq 0$$

This contradicts with the Morse inequalities, if we choose $c^* + \varepsilon < 0$.

THEOREM 2.5. – *Assume* (A) *and*

$(G')\, G : \mathcal{H} \to \mathbb{R}^1$ *is a* C^1 *functional with compact differential* G'. *There is a linear compact symmetric operator* B_∞ *such that*

$$\|G'(x) - B_\infty x\| = 0(1) \qquad \textit{and} \qquad \|x\| \to \infty \tag{2.9}$$

and

$$G(x) - \frac{1}{2}(B_\infty x, x) \to -\infty \,(\text{or } +\infty) \tag{2.10}$$

as $\|P_\infty x\| \to \infty$, *where* P_∞ *is the orthogonal projection onto* $\ker(A+B_\infty)$.

Then f *has a critical point. Moreover,* f *possesses a nontrivial critical point, if* θ *is a critical point and if* G *is* C^2 *in a neighbourhood of* θ *and*

$$I_-(B_\infty) \notin [I_-(B_0),\, I_-(B_0) + N(B_0)]$$
$$(\text{or } I_-(B_\infty) + N(B_\infty) \notin [I_-(B_0),\, I_-(B_0) + N(B_0)] \text{ resp.}),$$

where $B_0 = d^2 G(\theta)$.

The proof is similar to the previous one, but simpler. Because the Landesman Lazer Condition (2.10) implies (PS), there is no need to be concerned with the critical point at infinity. Only q_n^* (or q_n) is used in the same argument to show the existence of a nontrivial critical point.

Remark 2.6. – In Theorem 2.5, if further, $N(A + B_\infty) = 0$, then (2.9) can be replaced by $\|G'(x) - B_\infty x\| = 0(\|x\|)$ as $\|x\| \to \infty$ and (2.10) is not needed.

Remark 2.7. – In both theorems 2.4 and 2.5, if we are only concerned with the existence of a solution, then the local C^2 condition of G at θ can be dropped out.

3. HAMILTONIAN SYSTEMS

Now, we return to the problem (1.1). Let $\mathcal{H}$ be the fractional Sobolev space $H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n})$, where S^1 is the unit circle, which is diffeomorphic to $[0, 1]/\{0, 1\}$. Define a bounded self adjoint operator A on $\mathcal{H}$ by the bilinear form:

$$(Ax, x) = \int_0^1 \left\langle -J\frac{dx}{dt}, x \right\rangle dt \tag{3.1}$$

$\forall x \in C^1(S^1, \mathbb{R}^{2n})$. The functional G

$$G(x) = -\int_0^1 H(t, x)\, dt \tag{3.2}$$

is C^1 on $\mathcal{H}$, if we assume that H_x is of polynomial growth in x.

The critical points of the functional

$$f(x) = \frac{1}{2}\int_0^1 \left\langle -J\frac{dx}{dt}, x \right\rangle dt - \int_0^1 H(t, x)\, dt \tag{3.3}$$

are solutions of (1.1).

Let $B(t)$ be a continuous 1-periodic symmetric matrix function, then the multiplication $x(t) \mapsto B(t)\,x(t)$ defines a compact linear self adjoint operator on $H^{\frac{1}{2}}(S^1, \mathbb{R}^{2n})$, and then $A + B$ is again self adjoint. Let $\Gamma_B = \{P_n | n = 1, 2...\}$, where P_n is finite-dimensional projection, strongly converges to I the identity, and commutes with $A + B$. Then Γ_B is an approximation scheme w.r.t. A. Indeed, only (3) in Definition 2.1 is needed to verify. Noticing

$$[P_n, A] = [P_n, A + B] - [P_n, B] = -[P_n, B],$$

and that the right hand side converges to zero in the operator norm provided by lemma 2.1, our verification is complete.

We turn out to study the relationship between abstract and concrete Maslov indices.

THEOREM 3.1. – *For any given continuous* 1-*periodic symmetric matrix function* $B(t)$, *we have*

$$N(B) = n(B) \tag{3.4}$$

$$I_-(B) = i_-(B) \tag{3.5}$$

Proof. – By definition

$$N(B) = \dim\ker(A + B) = \dim\ker(W(1) - I) = n(B).$$

In order to show (3.5), firstly, according to $[AZ\,2]$, for a nondegenerate constant matrix B_0, we have

$$I_-(B_0) = I(B_0) = i(B_0) = i_-(B_0),$$

provided by choosing a special $\Gamma = \Gamma_{B_0}$. Secondly, for general nondegenerate matrix function $B(t)$, according to [CZ], there exists a nondegenerate constant matrix B_0, homotopic to $B(t)$ in nondegenerate class. By homotopic invariance of the Morse indices, and the definition of Maslov index, we obtain

$$I_-(B) = I(B) = I(B_0) = i(B_0) = i(B) = i_-(B). \tag{3.6}$$

Finally, for degenerate $B(t)$, according to Long [Lo1], we have on one hand

$$i_-(B) = \varliminf_{C \to B} i(C) \qquad i_-(B) + n(B) = \varlimsup_{C \to B} i(C)$$

where $C(t)$ is nondegenerate. On the other hand, by the lower semi-continuity of the Morse index,

$$\varliminf_{C \to B} I(C) \geq I_-(B),$$

and

$$\varlimsup_{C \to B} I(C) \leq I_-(B) + N(B).$$

It follows from (3.4) and (3.6)

$$I_-(B) \leq i_-(B) \leq i_-(B) + n(B) \leq I_-(B) + n(B).$$

(3.5) is proved.

LEMMA 3.2. – *Assume* (1.5) *and* (1.6). *Then* (G) *is satisfied.*

Proof. – We want to prove the following conclusion:
For $\xi \in H^{\frac{1}{2}}(S^1, R^{2n})$,

$$\|P_\infty \xi\| \to \infty \Leftrightarrow |(P_\infty \xi)(t)| \to \infty \text{ uniformly in } [0, 1] \tag{3.7}$$

Indeed, let $\{e_1(t), \ldots, e_d(t)\}$ be a basis in $\ker(A + B_\infty)$, where $d = \dim \ker(A + B_\infty)$. On one hand, let $M = \operatorname{Max}\{|e_j(t)|_{\mathbb{R}^{2n}} | 1 \leq j \leq d,\, t \in [0, 1]\}$, we have

$$|z(t)|_{\mathbb{R}^{2n}} \leq M \sum_{j=1}^{d} |v_j|,$$

for every $z \in \ker(A + B_\infty)$, and $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$, with

$$z(t) = \sum_{j=1}^{d} v_j\, e_j(t).$$

On the other hand, since $\{e_j(t)\}$ is linearly independent $\forall t \in [0, 1]$, by compactness of $[0, 1]$, one finds $\varepsilon_0 > 0$ such that

$$|z(t)|_{\mathbb{R}^{2n}} \geq \varepsilon_0 \sum_{j=1}^{d} |v_j|$$

However, $\sum_{j=1}^{d} |v_j|$ is an equivalent norm of $\ker(A + B_\infty)$. This proves (3.7).

Let $z = P_\infty x$, $y = (I - P_\infty) x$. Suppose

$$\|y_n\| \leq M, \qquad \|z_n\| \to \infty,$$

then $\forall\, \varepsilon > 0$, $\exists$ measurable sets E_n satisfying

$$\begin{cases} |y_n(t)| \le M/\sqrt{\varepsilon} & \text{for } t \notin E_n \\ \text{and mes}\,(E_n) \le \varepsilon. \end{cases}$$

We have

$$\left| G(x_n) - \frac{1}{2}(B_\infty x_n, x_n) \right| \le \int_0^1 |h(t, y_n(t) + z_n(t))|\, dt$$
$$= \int_{E_n} + \int_{CE_n} \le (\mathrm{Max}\,|h| + 1)\,\varepsilon$$

as n large enough, because $|z_n(t)|_{\mathbb{R}^{2n}} \to +\infty$.

Similarly

$$(G'(x_n) - B_\infty x_n, v)_{L^2} = \int_0^1 h_x(t, y_n(t) + z_n(t)) \cdot v(t)\, dt \to 0,$$

$\forall\, v \in L^2(S^1, \mathbb{R}^{2n})$. Since $H^{\frac{1}{2}} \hookrightarrow L^2$ is compact, (G) is verified.

Proof of Theorem 1.1 *(or* 1.2). – We are going to show that Theorem 1.1 (or 1.2) is a special case of Theorem 2.4 (or 2.5 resp.), if we choose A, G and f as in (3.1), (3.2) and (3.3) respectively. Obviously, (A) is satisfied, and (G) follows from Lemma 3.2. By the Sobolev embedding theorem and the Holder inequality, we have

$$\int H_{xx}(t, x(t))\, y(t)\, z(t) \le C\, \|x\|^s\, \|y\|\, \|z\|.$$

This implies that $G \in C^2$. Theorem 3.1 identifies the concrete Maslov indices with the abstract. Theorem 1.1 (or 1.2) now follows directly from Theorem 2.4 (or 2.5 resp.) directly.

REFERENCES

[AZ1] H. Amann and E. Zehnder, Nontrivial Solutions for a class of nonresonance problems and applications to nonlinear differential equations, *Ann. Sc. Super. Pisa, CC. Sci.*, IV, Ser. 7, 1980, pp. 539-603.

[AZ2] H. Amann and E. Zehnder, Periodic solutions of asymptotically Hamiltonian systems, *Manuscr. Math.*, Vol. **32**, 1980, pp. 149-189.

[Ch1] K. C. Chang, Solutions of asymptotically linear operator equations via Morse theory, *Comm. Pure Appl. Math.*, Vol. **34**, 1981, pp. 693-712.

[Ch2] K. C. Chang, Applications of homology theory to some problems in differential equations, *Nonlinear Func. Anal.*, (F.E. Browder Ed.), *Proc. Symp. Pure Math.*, AMS, 1986, pp. 253-261.

[Ch3] K. C. CHANG, On the homology method in the critical point theory, PDE and related subjects, (M. MIRANDA Ed.) Pitman, Vol. **269**, 1992, pp. 59-77.

[CL] K. C. CHANG and J. Q. LIU, A strong resonance problem, *Chinese Ann. of Math.*, Vol. **11B**, 1990, pp. 191-210.

[CZ] C. CONLEY and E. ZEHNDER, Morse type index theory for flows and periodic solutions for Hamiltonian equations, *Comm. Pure Appl. Math.*, Vol. **37**, 1984, pp. 207-253.

[DL] Y. H. DING and J. Q. LIU, Periodic solutions of asymptotically linear Hamiltonian systems, *J. Sys. Sci. Math. Sci.*, Vol. **9**, 1990, pp. 30-39.

[LL] S. J. LI and J. Q. LIU, Morse theory and asymptotically linear Hamiltonian systems, JDE 78, 1989, pp. 53-73.

[Li] M. J. LIU, Morse theory for strong indefinite functional and applications, Thesis, Institute of System Science, Beijing, 1990.

[Lo] Y. M. LONG, Maslov index, degenerate critical points and asymptotically linear Hamiltonian systems, *Science in China*, Ser. A, Vol. **33**, 1990, pp. 1409-1419.

[LZ] Y. M. LONG and E. ZEHNDER, Morse theory for forced oscillations of asymptotically linear Hamiltonian systems, Stochastic Processes, Physics and Geometry, *World Sci. Press,* 1990, pp. 528-563.

[MP] A. MARINO and G. PRODI, Metodi perturbativi nella teoria di Morse, Boll. Un. Mate. Italia, *Suppl. Fasc.* 3, 1975, pp. 1-32.

[Sa] A. SALVATORE, Periodic solutions of asymptotically linear systems without symmetry, *Rend. Sem. Mat. Univ.*, Padova, Vol. **74**, 1985, pp. 147-161.

[Sz] A. SZULKIN, Cohomology and Morse theory for strong indefinite functionals, MZ 209, 1992, pp. 375-418.

(Manuscript received October 24, 1994;
version revised August 29, 1995.)