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On the regularity of edges in image segmentation *

by

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ABSTRACT. – We present a series of results on the regularity of edges in the Mumford-Shah minimization process. We prove that an isolated connected component of the edge set is a finite union of $C^{1,1}$ arcs. A fundamental step of the proof is the characterization of the four possible global-minimizers having a connected edge.

RÉSUMÉ. – Nous présentons une série de résultats sur la régularité des contours des minima de la fonctionnelle de Mumford-Shah. Sous certaines hypothèses de connexité on montre que le contours est une union finie d'arcs $C^{1,1}$. Une étape fondamentale de la démonstration est la caractérisation des quatre possibles minima globaux ayant un contour connexe.

CONTENTS

| | |
|--|-----|
| 1. Introduction | 486 |
| 2. Blow-up procedure | 490 |
| 2.1. Definition: blow-up, global minimizer | 490 |
| 2.2. Blow-up, limits are global-minimizers | 492 |

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| | |
|---|-----|
| 2.3. Sufficient condition for K_0 to be connected | 496 |
| 3. A monotonicity formula | 499 |
| 3.1. Global-minimizers of J^0 | 499 |
| 3.2. Minimizers of J | 502 |
| 4. Characterization of global-minimizers | 503 |
| 5. Finite number of pieces | 508 |
| 6. Regularity $C^{1,\alpha}$ | 510 |
| 7. Further regularity | 516 |
| 7.1. $C^{1,1}$ regularity | 516 |
| 7.2. $C^{k+2,\alpha}$ regularity when g is $C^{k,\alpha}$ | 518 |
| 7.3. Minimization with n connected components | 519 |
| 8. Regularity almost everywhere | 520 |
| 9. Color segmentation | 524 |
| 10. Appendix | 526 |

1. INTRODUCTION

In computer vision, an image is characterized by the collection of numbers giving the grey-level at each point of the screen. Mathematically, we will say that an image is given by a real L^∞ function g (the grey level) defined on a domain Ω (the screen) of \mathbf{R}^2 . At the end of this article we will consider color images. In this case, g will be a vector function (of the red, blue and green intensity) and will take its values in a cube of \mathbf{R}^3 .

In most images, human vision is able to detect structures, shapes of objects, etc. Those structures appear, when, for instance, the image is made of patches in which the grey level has but little variations. Sharp discontinuities are visible at the boundary of these patches. They are introduced by edges of objects, shadows, overlapping objects, etc.

The object of image segmentation is to extract, by a systematic algorithm, the meaningful discontinuities. The result of this operation will be a piecewise regular image u approximating the true image g and a set K of discontinuities of u . The set K is an estimation of the contours of the image. Although there is a number of different algorithms and softwares for image segmentation, it appears that all rely on the same principle [17]:

the minimization of an energy. Moreover, this energy is essentially made of the three terms of the Mumford-Shah functional.

In [19], Mumford and Shah proposed to define (u, K) , where $u \in C^1(\Omega \setminus K)$ and K is a closed set, as minimizers of the functional:

$$J(u, K) = \int_{\Omega \setminus K} \|\nabla u\|^2 + \mu \int_{\Omega} (u - g)^2 + \nu \mathcal{H}^1(K) \quad (1.1)$$

The first term of this energy penalizes the variations of u outside the edge set K , the second term is the distance to the true image and the third term is the 1-dimensional Hausdorff measure of the edge set K ([17] and [14]). In the energy J , μ and ν denote two positive constants. By a normalization we may assume without loss of generality that $\mu = \nu = 1$. For a simple geometric image one would expect that the set K is regular. However, for pictures of the real world it is not obvious that the edge set should be simple. More precisely, one cannot eliminate a priori the possibility of having a tree like edge set (Fig. 1) or lots of very small pieces (Fig. 2) or multibranch stars (Fig. 3). Mumford and Shah conjectured that there exists a minimizer of J such that the edge set K is the union of a finite set of C^1 -arcs and that each arc may end either as a crack-tip or in a triple junction (Fig. 4).

Existence of minimizers in the class of *SBV* functionals has been proved in [2], [11] and [6]. A function u is in $BV(\Omega)$ if the distributional gradient Du is a bounded Radon measure on Ω with values in \mathbf{R}^2 . If S_u denotes the singular set of $u \in BV(\Omega)$ then $\mathcal{H}^1(S_u) < \infty$ and for \mathcal{H}^1 -a.e. $x \in S_u$ one can define a normal vector ν_u and limit values u^+ and u^- of u on both sides of S_u . A function u is in $SBV(\Omega)$ (Special function of Bounded Variations) if $u \in BV(\Omega)$ and if its distributional gradient Du satisfies, for

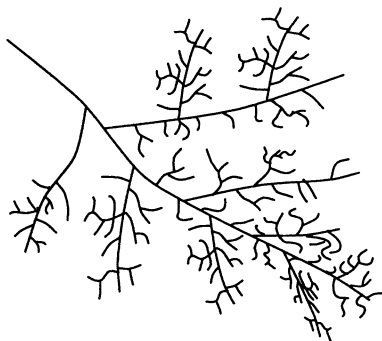


Fig. 1.

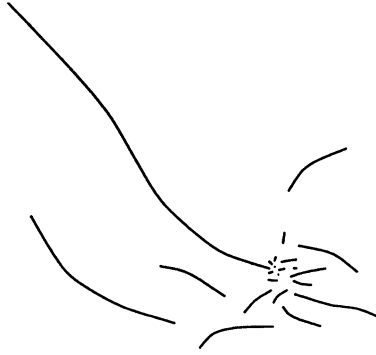


Fig. 2.

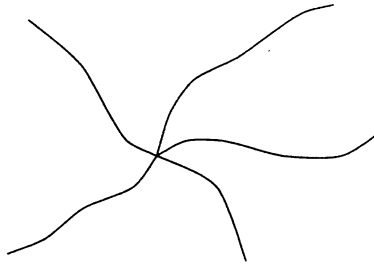


Fig. 3.

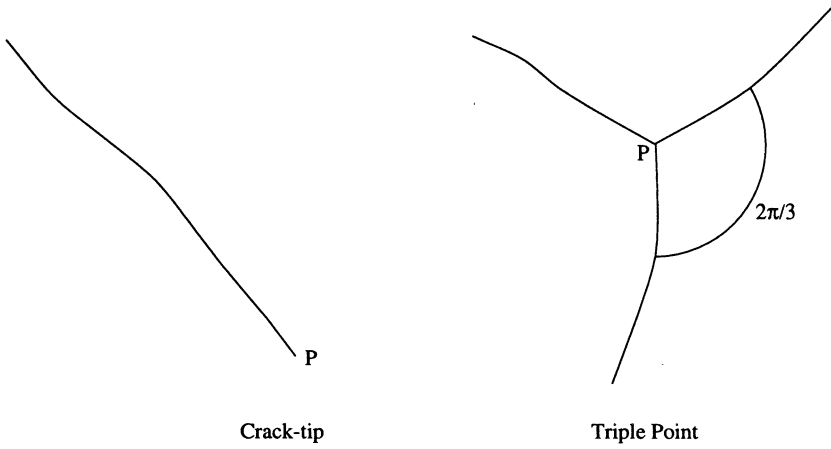


Fig. 4.

every C_0^1 vector field $\phi : \Omega \rightarrow \mathbf{R}^2$:

$$\int_{\Omega} \phi \cdot Du = \int_{\Omega} \phi \cdot \nabla u + \int_{S_u} (u^+ - u^-) \phi \cdot \nu_u \, d\mathcal{H}^1 \tag{1.2}$$

where ∇u is in $L^2(\Omega, \mathbf{R}^2)$.

It is proved in [6] that there is a correspondence between *SBV* minimizers and pairs (u, K) minimizing J . In the following, we choose to work with the initial formula of the function as depending upon (u, K) , $u \in C^1(\Omega \setminus K) \cap W^{1,2}(\Omega \setminus K)$ and K is a Hausdorff 1-dimensional closed set with finite Hausdorff measure.

Notice that we do not change $J(u, K)$ by adding a set of zero \mathcal{H}^1 measure to K . Therefore, we will assume in the following that K is a minimal closed set: there is no closed set $\tilde{K} \subset K$, $\tilde{K} \neq K$, such that u can be extended on $\Omega \setminus \tilde{K}$ with $J(u, \tilde{K}) \leq J(u, K)$. Under this assumption, K is an Ahlfors regular set: there are two positive constants c, C such that [6]:

$$\text{if } P \in K \quad \text{then } \forall r > 0, \quad cr \leq \mathcal{H}^1(K \cap B_r(P)) \leq Cr. \tag{1.3}$$

In [7], it is proved that K is contained in a single rectifiable Ahlfors regular curve whose length is proportional (with a universal ratio) to the length of K .

In this paper, we are interested in the local regularity of the edge set (see also [8] and [3]-[4]) and in the number of arcs. Our main ingredient is the characterization of all global-minimizers (*i.e.* pairs (u, K) defined on $\Omega = \mathbf{R}^2$ and such that for all balls B_R the energy $J_{B_R}^0(u, K) = \int_{B_R \setminus K} \|\nabla u\|^2 + \mathcal{H}^1(K \cap B_R)$ cannot be reduced by a compact perturbation of u and K inside the ball B_R .) Under the assumption that K is connected, it will be proved that there are only four types of global-minimizers. The characterization is carried out by a monotonicity property on the function $r \mapsto \frac{\int_{B_r \setminus K} \|\nabla u\|^2}{r}$. We will be able to prove that every blow-up limit is a global-minimizer. The characterization of all global-minimizers and therefore of the blow-up limits give strong information on the local behavior of K and u .

THEOREM 1.1. – *If G is an isolated connected component of K then, it is the union of a finite set of C^1 arcs. These arcs are $C^{1,1}$ away from crack-tips and can only merge through triple junctions.*

More precisely,

THEOREM 1.2. – *Let P be a point of the edge set K . If $B_R(P)$ intersects a single connected component of K , then there is a finite set of arcs l_j ,*

$j = 1, \dots, k$, with the same regularity property as above and such that

$$K \cap B_{R/2}(P) = \left(\bigcup_{j=1}^k l_j \right) \cap B_{R/2}(P). \quad (1.4)$$

Notice that this does not allow a minimizer to have an infinite number of arbitrary small pieces connected to each other. In July 1994, G. David announced a \mathcal{H}^1 -almost-everywhere regularity result, [8]. Actually, the method used to prove the theorems above allows one to get

THEOREM 1.3. – *For \mathcal{H}^1 almost every P in K there is a neighborhood $B_r(P)$ of P in which K is a $C^{1,1}$ arc.*

The above results were derived when g is L^∞ . If one assumes more regularity on g one may expect more regularity on K .

THEOREM 1.4. – *Assume that g is $C^{k,\alpha}$. If, in an open neighborhood U of $P \in K$, $K \cap U$ is a C^1 arc then it is $C^{k+2,\alpha}$ in U .*

Lastly, we would like to point out that it can be interesting to look for a minimizer (u^n, K^n) of J that has at most n connected components (see [17]). We have the complete result:

THEOREM 1.5. – *The edge set K^n is the union of a finite set of C^1 arcs, $C^{1,1}$ away from the crack-tips, and merging through triple junctions.*

To conclude this paper, we show, in section 9, that the analysis above holds for a model of color segmentation. A point of the image is characterized by its Red, Green and Blue intensity. In this case, $u(x)$ is a vector of $[0, 1]^3$ with a specific Riemannian metric which gives a slightly different energy functional [10].

Open questions. It is not known whether the edge set is regular up to the crack-tip.

The finite number of arcs is still an open question. Notice that if the characterization of global-minimizers was achieved without the assumption that K is connected, then the conjecture would be solved.

2. BLOW-UP PROCEDURE

2.1. Definition: blow-up, global minimizer

For a given open set ω and a piecewise continuous function ψ defined on $\partial\omega$ we may study the minimization of the functional

$$J_\omega(v, \tilde{K}) = \int_{\omega \setminus \tilde{K}} \|\nabla v\|^2 + \int_\omega (v - g)^2 + \mathcal{H}^1(\tilde{K} \cap \omega)$$

under the condition $v = \psi$ on $\partial\omega$.

Assume, as in the previous section, that (u, K) denotes a minimum segmentation. If $\omega \subset \Omega$ is such that $\partial\omega \cap K$ is finite, we define the trace ψ of u on $\partial\omega$. It is straightforward to see that (u, K) minimizes the functional J_ω with ψ as Dirichlet condition. This means that one cannot modify u and K inside ω and reduce the total energy $J = J_\omega + J_{\Omega \setminus \omega}$.

In the study of free-boundary problems, it is common to use blow-up procedures ([1], [16], etc.). Here, we introduce, for a given origin, the blow-up sequences $u_\varepsilon, g_\varepsilon, K_\varepsilon$:

$$\begin{aligned} u_\varepsilon(X) &= \frac{u(\varepsilon X) - c_\varepsilon(X)}{\sqrt{\varepsilon}}, \\ g_\varepsilon(X) &= \frac{g(\varepsilon X) - c_\varepsilon(X)}{\sqrt{\varepsilon}}, \\ K_\varepsilon &= \{X \in \mathbf{R}^2, \varepsilon X \in K\}. \end{aligned} \quad (2.1)$$

The real function $c_\varepsilon(X)$ is piecewise constant. It is introduced in order to be able to pass to the limit $\varepsilon \rightarrow 0$ in such a way that u_ε converges almost everywhere to some finite value. This will be detailed in the second part of this section. We first take $c \equiv 0$ and introduce:

$$\begin{aligned} U_\varepsilon(X) &= \frac{u(\varepsilon X)}{\sqrt{\varepsilon}}, \\ G_\varepsilon(X) &= \frac{g(\varepsilon X)}{\sqrt{\varepsilon}}. \end{aligned} \quad (2.2)$$

We notice that $\nabla U_\varepsilon(X) = \sqrt{\varepsilon} \nabla u(\varepsilon X)$ and that a ball B_ε with radius ε becomes B_1 by the blow-up procedure. Thus

$$J_{B_\varepsilon}(u, K) = \varepsilon \int_{B_1 \setminus K_\varepsilon} \|\nabla U_\varepsilon\|^2 + \varepsilon^3 \int_{B_1} (U_\varepsilon - G_\varepsilon)^2 + \varepsilon \mathcal{H}^1(K_\varepsilon \cap B_1). \quad (2.3)$$

We introduce the new functionals J_ω^ε :

$$J_\omega^\varepsilon(U_\varepsilon, K_\varepsilon) = \int_{\omega \setminus K_\varepsilon} \|\nabla U_\varepsilon\|^2 + \varepsilon^2 \int_\omega (U_\varepsilon - G_\varepsilon)^2 + \mathcal{H}^1(K_\varepsilon \cap \omega). \quad (2.4)$$

With this definition we have the straightforward statement on the minimization under Dirichlet conditions on ∂B_ε and ∂B_1 respectively:

$$(u, K) \text{ minimizes } J_{B_\varepsilon} \Leftrightarrow (U_\varepsilon, K_\varepsilon) \text{ minimizes } J_{B_1}^\varepsilon. \quad (2.5)$$

We remark that in the definition of J_ω^ε , the term $\varepsilon^2 \int_\omega (U_\varepsilon - G_\varepsilon)^2$ is of order ε . In the limit $\varepsilon = 0$ we define

$$J_\omega^0(u_0, K_0) = \int_{\omega \setminus K_0} \|\nabla u_0\|^2 + \mathcal{H}^1(K_0 \cap \omega). \quad (2.6)$$

In the following we will be interested in minimizers of the functional J^0 that are defined on the whole plane. Since there is no more image g in the functional J^0 and since the domain is now unbounded, we introduce a notion of compact-perturbation-global-minimizer of J^0 . For simplicity, this will be referred to in the following as “global-minimizer of J^0 ”.

DEFINITION 2.1. – A segmentation (u, K) is a global-minimizer of J^0 if it is defined on the whole space and if for any bounded open sub-domains U, V of \mathbf{R}^2 with $U \subset \subset V$ there are no function \tilde{u} and closed edge set \tilde{K} defined on V such that:

- (i) \tilde{u} and u (resp. \tilde{K} and K) coincide in $V \setminus \bar{U}$,
- (ii) for every pair of points X, Y in $V \setminus (\bar{U} \cup K)$, if X and Y are in two distinct connected components of $V \setminus K$ so they are in $V \setminus \tilde{K}$,
- (iii) $J_V^0(\tilde{u}, \tilde{K}) < J_V^0(u, K)$.

Notice that we take $V \neq U$ in order to avoid problems arising from $K \cap \partial U \neq \emptyset$. Condition (ii) is coherent with the introduction of the piecewise constant function $c_\varepsilon(X)$ in the definition of the blow-up sequences. Indeed, if P is a regular point of K , then the blow-up limit at P , (u_0, K_0) , is the segmentation of the plane such that K_0 is a straight line $\{y = 0\}$ and u is constant on both sides of K ($u_0(x, y) = c^+$ if $y > 0$, $u_0(x, y) = c^-$ if $y < 0$). With condition (ii), (u_0, K_0) is a global-minimizer. However, whatever the value of c^+ and c^- , the value of $J_{B_{2R}}^0$ can be reduced for large enough R by excising a large segment out of K (this excision violates (ii)).

2.2. Blow-up limits are global-minimizers

The fundamental result in the blow-up procedure is the following theorem which says that blow-up sequences converge to global-minimizers.

THEOREM 2.2. – Let (u, K) be a minimal segmentation of J . The blow-up sequence $(u_\varepsilon, K_\varepsilon)$ is such that there is a subsequence converging to (u_0, K_0) in the following sense:

$$\begin{aligned}
 &K_{\varepsilon_q} \rightarrow K_0 \text{ locally in the Hausdorff metric,} \\
 &u_{\varepsilon_q} \rightarrow u_0 \text{ strongly in } H_{loc}^1(\mathbf{R}^2 \setminus K_0), \tag{2.7} \\
 &J_\omega^0(u_0, K_0) \leq \liminf J_\omega^{\varepsilon_q}(U_{\varepsilon_q}, K_{\varepsilon_q}) \text{ for any bounded open set } \omega, \\
 &J_{B_r}^0(u_0, K_0) = \lim J_{B_r}^{\varepsilon_q}(U_{\varepsilon_q}, K_{\varepsilon_q}) \text{ for almost every } r > 0. \tag{2.8}
 \end{aligned}$$

Moreover (u_0, K_0) is a global-minimizer.

THEOREM 2.3. – If (v_n, K_n) minimizes J for an image g^n such that $\|g^n\|_{L^\infty} \leq 1$ (or other uniform L^p bound with $p \geq 2$) the blow-up sequence

$(v_{\varepsilon_n}^n, K_{\varepsilon_n}^n)$ with $\varepsilon_n \rightarrow 0$ has a convergent subsequence to a global-minimizer (v_0, K_0) of J^0 in the same sense as in theorem 2.2.

Remark 2.4. – Given a blow-up sequence $(u_\varepsilon, K_\varepsilon)$ there may exist several blow-up limits. The object of section 4 will be to characterize all the possible blow-up limits.

Proof. – Theorem 2.3 is a straightforward generalization of theorem 2.2. Therefore we will concentrate on the proof of theorem 2.2. The proof of convergence relies as in [6] and [11] on the L^2 bound on ∇u and on the compactness of Hausdorff metric. However, because of the introduction of the functions c_ε in the definition of u_ε and since u_ε is not obviously bounded, we will detail the proof hereafter.

(i) **Convergence of K_ε to K_0**

First we recall the definition of the Hausdorff distance \mathbf{d} of two sets A and B (for two points x and y , $d(x, y)$ denotes the usual euclidean distance):

$$\mathbf{d}(A, B) = \max(\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)) \tag{2.9}$$

$$d(x, B) = \inf_{y \in B} d(x, y). \tag{2.10}$$

The compactness of the Hausdorff metric \mathbf{d} for subsets of a compact domain gives the existence of a subsequence K_{ε_q} converging locally to a set K_0 .

(ii) **Convergence of u_ε to u_0**

In each connected component of $\mathbf{R}^2 \setminus K_0$ we choose a point Y_i . Let c_ε be the piecewise constant function defined on $\mathbf{R}^2 \setminus K_0$ by

$$c_\varepsilon(X) = u_\varepsilon(Y_i) \text{ if } X \text{ and } Y_i \text{ are in the same} \\ \text{connected component of } \mathbf{R}^2 \setminus K_0.$$

We study now the convergence of u_ε . We notice that the gradients ∇U_ε are uniformly locally bounded in the L^2 norm (by an excision argument one gets $J_{BR}^\varepsilon(U_\varepsilon, K_\varepsilon) \leq 2\pi R$, see [6] for the details). Consequently $\int_{B_R \setminus (K_0 \cup K_\varepsilon)} \|\nabla u_\varepsilon\|^2 \leq 2\pi R$. Let F be a bounded open set, $F \subset\subset \mathbf{R}^2 \setminus K_0$. For ε small, $F \subset \Omega_\varepsilon \setminus K_\varepsilon$. Since $|\Delta u_\varepsilon| = \varepsilon^2 |U_\varepsilon - G_\varepsilon| \leq \varepsilon^{3/2}$, Δu_ε is

uniformly bounded for $\varepsilon \leq 1$ in F . The L^2 bound on ∇u_ε and the L^∞ bound on Δu_ε give a bound on $\|\nabla u_\varepsilon\|_{L^p(F)}$, for all $p \geq 1$, uniform in ε small. By Sobolev embedding, we get a uniform bound on $\|u_\varepsilon\|_{C^{0,\alpha}(F)}$. Consequently, $\exists M, \forall \varepsilon < \varepsilon_0$

$$\|u_\varepsilon\|_{W^{2,p}(F)} < M \tag{2.11}$$

and a subsequence of u_ε converges to u_0 in $W_{loc}^{1,p}(\mathbf{R}^2 \setminus K_0)$.

(iii) The blow-up limit (u_0, K_0) is a global-minimizer

Let us finally check that (u_0, K_0) is a global-minimizer. We notice that by construction,

$$\int_{B_R \setminus K_0} \|\nabla u_0\|^2 \leq \liminf \int_{B_R \setminus K_\varepsilon} \|\nabla u_\varepsilon\|^2. \tag{2.12}$$

Inequality (2.12) and a lower semicontinuity lemma for the Hausdorff measure [6] are the two main ingredients of the proof of the claim that (u_0, K_0) is a global-minimizer.

We recall definition 0.9, lemma 0.10 and theorem 3.38 of [6]:

DEFINITION 2.5. – [6] *Let A be a Borel subset of $\bar{\Omega}$. We say that A satisfies the concentration property in Ω if for every $\varepsilon > 0$ there exists $\alpha = \alpha(\varepsilon) > 0$ such that, if $B_R = B(x_0, R)$ is any disk contained in Ω with $x_0 \in A$ and $0 < R < 1$, then there exists a disk $B(x, r)$ contained in B_R such that*

$$\begin{aligned} \text{diam}(B(x, r)) &\geq \alpha \text{diam}(B_R), \\ \mathcal{H}^1(B(x, r) \cap A) &\geq (1 - \varepsilon) \text{diam}(B(x, r)). \end{aligned}$$

LEMMA 2.6. – [6] *Let (K_k) be a sequence of closed subsets of Ω which converges in the Hausdorff metric to a closed subset K of $\bar{\Omega}$. Assume that the sets K_k satisfy the concentration property (definition 2.5) in Ω uniformly with respect to k (i.e. with $\alpha(\varepsilon)$ independent of k). Then*

$$\mathcal{H}^1(K \cap \Omega) \leq \liminf \mathcal{H}^1(K_k \cap \Omega). \tag{2.13}$$

THEOREM 2.7. – [6] *Let u, K be a minimum segmentation of J . Then K satisfies the concentration property in Ω with a function $\varepsilon \mapsto \alpha(\varepsilon)$ which does not depend on the data Ω, g, u ($\|g\|_{L^\infty} \leq 1$).*

It is straightforward to see that if K satisfies the concentration property, then the blow-up K_ε for $\varepsilon \leq 1$ satisfies the same property with the same function $\varepsilon \mapsto \alpha(\varepsilon)$. Consequently, for the converging subsequence K_{ε_q} of the blow-up procedure we have the following lower semicontinuity property on any open domain ω of \mathbf{R}^2 ,

$$\mathcal{H}^1(K_0 \cap \omega) \leq \liminf \mathcal{H}^1(K_{\varepsilon_q} \cap \omega). \tag{2.14}$$

This, together with (2.12), gives the lower semicontinuity result on the blow-up sequence,

$$J_\omega^0(u_0, K_0) \leq \liminf J_\omega^{\varepsilon_q}(U_{\varepsilon_q}, K_{\varepsilon_q}). \tag{2.15}$$

We assume by contradiction that (u_0, K_0) is not a global-minimizer of J^0 . Then there is $U = B_R$ and $V = B_{2R}$ and a couple $(\tilde{u}_0, \tilde{K}_0)$ satisfying conditions (i), (ii) and (iii) of definition 2.1. We construct hereafter a compact perturbation of $(U_{\varepsilon_q}, K_{\varepsilon_q})$ and prove that for large q , $(U_{\varepsilon_q}, K_{\varepsilon_q})$ cannot be a minimizer of J^{ε_q} .

Since $\mathcal{H}^1(K_0 \cap B_r) \leq 2\pi r$, the intersection $K_0 \cap \partial B_r$ is a finite set of points for almost every r . We choose $r_0, r_0 > R$ such that $K_0 \cap \partial B_{r_0}$ is finite. We denote $K_0 \cap \partial B_{r_0} = \{y_1, \dots, y_n\}$. Choose $\zeta > 0$ such that

$$3n\zeta < J_{B_{2r_0}}^0(u_0, K_0) - J_{B_{2r_0}}^0(\tilde{u}_0, \tilde{K}_0). \tag{2.16}$$

We introduce two subsets of ∂B_{r_0} :

$$L = \partial B_{r_0} \cap (\cup_{i=1}^n B(y_i, \zeta)), \tag{2.17}$$

$$\Gamma = \partial B_{r_0} \setminus (\cup_{i=1}^n B(y_i, \zeta)). \tag{2.18}$$

The set Γ is compact in $\mathbf{R}^2 \setminus K_0$. For a subset A of \mathbf{R}^2 we define $A^\eta = \{X \in \mathbf{R}^2, d(X, A) \leq \eta\}$. There is a $\eta > 0$ such that $\Gamma^{3\eta} \subset \mathbf{R}^2 \setminus K_0$. Then for q large, the convergence of K_{ε_q} to K_0 in the Hausdorff metric gives $\Gamma^{2\eta} \subset \mathbf{R}^2 \setminus K_{\varepsilon_q}$. Then for $p > 2$, the estimates (2.11) and the compact embedding of $W^{2,p}$ in $C^{1,\alpha}$ ($\alpha = 1 - \frac{2}{p}$) gives the convergence of a subsequence u_{ε_q} to u_0 in Γ_η for the $C^{1,\alpha}$ norm. We introduce the continuous function w_q in ∂B_{r_0} such that: $w_q = u_{\varepsilon_q} - u_0$ on Γ and w_q is linear in the polar angle θ on every connected component of $\partial B_{r_0} \setminus \Gamma$ (i.e. $w_q(r_0, \theta) = \alpha + \beta\theta$ where α and β are chosen such that w_q is continuous on ∂B_{r_0}). Then, for $\|x\| < r_0$ we define

$$w_q(x) = \psi\left(\frac{\|x\|}{r_0}\right) w_q\left(r_0 \frac{x}{\|x\|}\right)$$

where ψ is a C^∞ nonnegative function in $[0, 1]$ such that $\psi = 1$ in a neighborhood of 1 and $\psi = 0$ in a neighborhood of 0. With this construction $w_q \in W^{1,\infty}(B_{r_0})$ and there is a constant $C = C(r_0, K_0, \eta)$ such that almost everywhere:

$$\begin{aligned} \|\nabla w_q\| &\leq C \left(\left\| \frac{\partial}{\partial \tau} (u_\varepsilon - u_0) \right\|_{L^\infty(\Gamma)} + \|u_\varepsilon - u_0\|_{L^\infty(\Gamma)} \right) \\ &\leq C \|u_\varepsilon - u_0\|_{C^{1,\alpha}(\Gamma)}. \end{aligned}$$

We define $\tilde{U}_{\varepsilon_q}$ by

$$\begin{aligned} \tilde{U}_{\varepsilon_q} &= \tilde{u}_0 + w_q + c_\varepsilon \quad \text{in } B_{r_0} \setminus K_0, \\ \tilde{U}_{\varepsilon_q} &= U_{\varepsilon_q} \quad \text{in } \mathbf{R}^2 \setminus (B_{r_0} \cup K_0). \end{aligned} \tag{2.19}$$

and construct $\tilde{K}_{\varepsilon_q}$ from K_{ε_q} by adding the piece L and replacing K_{ε_q} by \tilde{K}_0 inside the ball B_{r_0} :

$$\tilde{K}_{\varepsilon_q} = (K_{\varepsilon_q} \setminus \bar{B}_{r_0}) \cup L \cup (\tilde{K}_0 \cap B_{r_0}). \tag{2.20}$$

Then $\tilde{U}_{\varepsilon_q}$ is such that

$$\int_{B_{2r_0} \setminus \tilde{K}_{\varepsilon_q}} \|\nabla \tilde{U}_{\varepsilon_q}\|^2 \rightarrow \int_{B_{2r_0} \setminus \tilde{K}_0} \|\nabla \tilde{u}_0\|^2.$$

This, together with (2.16) and (2.14), gives for q large and ζ small:

$$J_{B_{2r_0}}^{\varepsilon_q}(\tilde{U}_{\varepsilon_q}, \tilde{K}_{\varepsilon_q}) < J_{B_{2r_0}}^{\varepsilon_q}(U_{\varepsilon_q}, K_{\varepsilon_q}) \tag{2.21}$$

which contradicts the minimality of U_ε .

As a consequence of this argument we have proved that for almost every r

$$J_{B_r}^0(u_0, K_0) = \lim J_{B_r}^{\varepsilon_q}(U_{\varepsilon_q}, K_{\varepsilon_q}) \tag{2.22}$$

which completes the proof of theorem 2.2.

2.3. Sufficient condition for K_0 to be connected

In section 4 we will be interested in global-minimizers such that the edge set K is connected. The following lemma will ensure that the edge set K_0 of the blow-up limit is connected.

LEMMA 2.8. – Assume that $\omega \subset \Omega$ intersects only one connected component of K . Let P_n be a sequence of points of an open set $U \subset \subset \omega$. For

every blow-up sequence $u_n(X) = \frac{u(P_n + \varepsilon_n X) - c_n(X)}{\sqrt{\varepsilon_n}}$, $K_n = \{X, P_n + \varepsilon_n X \in K\}$ converging to a blow-up limit (u_0, K_0) , the limit edge set K_0 is connected.

Proof. – The proof will proceed by contradiction. We assume that K_0 is not connected. Notice that since K_n is a 1-dimensional Hausdorff connected set, it is arc-wise connected: for any two points X and Y in K_n there is a continuous function $\psi : [0, 1] \rightarrow K_n$ such that $\psi(0) = X$ and $\psi(1) = Y$ (see [13]).

The edge set K_0 is the limit in the Hausdorff distance of the connected edge sets K_n . Therefore, K_0 cannot have a bounded isolated connected component and has at least two unbounded connected components. One may choose a point M in one component and a point N in another. Since K_n converges to K_0 in the Hausdorff distance there are two sequences of points $M_n \in K_n$ and $N_n \in K_n$ converging to M and N respectively.

NOTATION. – Let A, B be two points of K . We denote by $d_K(A, B)$ the minimal length of an arc connecting A to B in K . ($d_K(A, B) = +\infty$ if there is no arc connecting A to B in K).

Since K_n is closed there is an arc of minimum length ψ_n , parameterized by arc-length, connecting the points M_n and N_n in K_n :

$$\begin{aligned} \psi_n : [0, l_n] &\rightarrow K_n, & \psi_n(0) &= M_n, & \psi_n(l_n) &= N_n, \\ l_n &= d_{K_n}(M_n, N_n). \end{aligned}$$

Obviously, this is a Jordan arc (ψ_n is one-to-one) and ψ_n is 1-Lipschitz continuous. We notice that $l_n \rightarrow +\infty$. Indeed, if l_n does not go to $+\infty$ then a subsequence of ψ_n will converge to a ψ connecting M to N in K_0 . This contradicts the choice of M and N in two different connected components of K_0 . We choose a new blow-up sequence such that the image by this blow-up of the arc connecting M_n to N_n is with length 1:

$$v_n(X) = \frac{u(P_n + \varepsilon_n l_n X)}{\sqrt{\varepsilon_n l_n}}, \quad G_n = \{X \in \mathbf{R}^2, P_n + \varepsilon_n l_n X \in K\}. \quad (2.23)$$

To have a blow-up sequence, we have to check that $\varepsilon_n l_n$ goes to 0. We argue by contradiction and assume that $\varepsilon_n l_n \rightarrow k > 0$. We introduce

$$\xi_n : [0, 1] \rightarrow \mathbf{R}^2, \quad \xi_n(t) = \varepsilon_n \psi_n(l_n t) + P_n.$$

We notice that, for all $t \in [0, 1]$, $\psi_n(l_n t) \in K_n = \{X \in \mathbf{R}^2, P_n + \varepsilon_n X \in K\}$. This implies: $\xi_n(t) \in K$. Moreover, since $\varepsilon_n l_n \rightarrow k$, ξ_n is $2k$ -Lipschitz

for n large and a subsequence converges uniformly to $\xi : [0, 1] \rightarrow K$. We may assume that $P_n \rightarrow P \in \omega$. Since ξ_n connects the points $\hat{M}_n = \varepsilon_n M_n + P_n$ and $\hat{N}_n = \varepsilon_n N_n + P_n$ in K we deduce that $\xi(0) = \xi(1) = P$. From the construction of ξ we get:

$$\mathcal{H}^1(\xi([0, 1])) \leq \liminf \mathcal{H}^1(\xi_n([0, 1])) = k. \tag{2.24}$$

Since $\mathcal{H}^1(K) < \infty$ and since the arcs ξ_n converge to ξ in K we shall have

$$\mathcal{H}^1(\xi_n([0, 1]) \setminus \xi([0, 1])) \rightarrow 0.$$

Consequently, for large n there are two points μ_n and ν_n in $\xi_n([0, 1]) \cap \xi([0, 1])$ such that $d_K(\hat{M}_n, \mu_n) < k/8$ and $d_K(\hat{N}_n, \nu_n) < k/8$. Since, μ_n and ν_n are two points of the closed curve with length at most k defined by ξ , we have $d_K(\mu_n, \nu_n) < k/2$. Consequently

$$d_K(\hat{M}_n, \hat{N}_n) < 3k/4$$

which contradicts the relation

$$d_K(\hat{M}_n, \hat{N}_n) = \varepsilon_n d_{K_n}(M_n, N_n) = \varepsilon_n l_n \rightarrow k \quad \text{as } n \rightarrow +\infty.$$

This contradiction shows that $\varepsilon_n l_n \rightarrow 0$ and we may apply theorem 2.3 and get the convergence of a blow-up subsequence (v_n, G_n) to some (v_0, G_0) . We consider arcs

$$\zeta_n : [0, 1] \rightarrow \mathbf{R}^2, \quad \zeta_n(t) = \frac{1}{l_n} \psi_n(l_n t).$$

For a subsequence, the arcs ζ_n with length 1 converge uniformly to an arc ζ with length at most 1 and such that $\zeta(0) = \zeta(1) = 0$. Thus, ζ defines a closed curve in G_0 . Since (v_0, G_0) is a global-minimizer, the interior of the domain defined by ζ must be empty (see lemma 4.8). Therefore every point of $\zeta([0, 1])$ is reached twice by ζ . Since ζ_n is a Jordan arc with length 1 we get

$$\mathcal{H}^1(\zeta([0, 1])) \leq \frac{1}{2}.$$

This can be written as $\mathcal{H}^1(\zeta([0, 1])) \leq \mathcal{H}^1(\zeta_n([0, 1])) - 1/2$. The argument of the proof of theorem 2.2 gives then: for almost all $r > 1/2$:

$$J_{B_r}^0(v_0, K_{v_0}) \leq \liminf J_{B_r}^{\varepsilon_n l_n}(v_n, \tilde{K}_n) - \frac{1}{2} \tag{2.25}$$

which contradicts the equality (2.8). This completes the proof by contradiction of the lemma.

Notice that the proof of lemma 2.8 gives immediately that any connected component of K is a chord-arc set:

LEMMA 2.9. – *There is a constant C such that if M and N are in the same connected component of K then*

$$d_K(M, N) \leq C d(M, N)$$

where $d(M, N)$ denotes the euclidean distance.

3. A MONOTONICITY FORMULA

3.1. Global-minimizers of J^0

To characterize the global-minimizers of J^0 , introduced in the previous section, we prove a monotonicity formula

THEOREM 3.1. – *Let (u, K) be a global-minimizer and let $\phi(r) = \int_{B_r \setminus K} \|\nabla u\|^2$. If K is connected then $r \mapsto \frac{\phi(r)}{r}$ is a non-decreasing function of r . If $\frac{\phi(r)}{r}$ is constant, then in some polar coordinates (r, θ) we have:*

$$u(r, \theta) = C\sqrt{r} \cos \frac{\theta}{2} \quad \text{for } \theta \in [0, 2\pi]$$

and K is the half axis $\{\theta = 0\}$.

Remark 3.2. – If (u, K) is defined in a ball B_R , K is a closed connected set and u minimizes $\int_{B_R \setminus K} \|\nabla u\|^2$ under Dirichlet conditions $u = u_0$ on ∂B_R then the conclusion of theorem 3.1 holds for $r \in (0, R)$.

Monotonicity formulas of another type have been used in the study of two-phase free-boundary problems (see for instance [1]). The proof of theorem 3.1 follows a series of lemmas and propositions. We want to compare the two integrals $\int_{B_r} \|\nabla u\|^2$ and $r \int_{\partial B_r \setminus K} \|\nabla u\|^2$ which appears in the derivative of $\frac{\phi(r)}{r}$. We first state a Green formula

LEMMA 3.3. – *Let (u, K) be a global-minimizer, then for almost all r :*

$$\int_{B_r} \|\nabla u\|^2 = \int_{\partial B_r \setminus K} u \frac{\partial u}{\partial \nu}.$$

This lemma is analogous to lemma 3.1 in [6]. The reader is referred to [6] for the detailed proof.

PROPOSITION 3.4. – *Let (u, K) be a global-minimizer. If K is connected then for almost every r*

$$r \int_{\partial B_r \setminus K} \|\nabla u\|^2 \geq \int_{\partial B_r \setminus K} u \frac{\partial u}{\partial \nu}.$$

Proof. – The proof of this proposition is based on an estimate of $\int_{\partial B_r \setminus K} |u_\tau|^2$ where u_τ is the tangential derivative on the sphere ∂B_r . We know that for almost every r , $\partial B_r \cap K$ is a finite set and $\int_{\partial B_r \setminus S} \|\nabla u\|^2 < +\infty$ [6]. Therefore, we will prove the inequality under the assumption that $\partial B_r \cap K$ is a finite set $\{y_1, \dots, y_k\}$. In polar coordinates, y_i has coordinates (r, θ_i) and we may assume that $0 \leq \theta_1 < \theta_2 < \dots < \theta_k < 2\pi$. We introduce $\theta_{k+1} = \theta_1 + 2\pi$ (the point y_{k+1} with coordinates (r, θ_{k+1}) coincides with y_1). We now evaluate the integral of $\|\nabla u\|^2$ on each arc $y_i \widehat{y}_{i+1}$:

$$\begin{aligned} \int_{y_i \widehat{y}_{i+1}} \|\nabla u\|^2 &= r \int_{\theta_i}^{\theta_{i+1}} \|\nabla u\|^2 d\theta \\ &= r \int_{\theta_i}^{\theta_{i+1}} \left(|u_r|^2 + \frac{1}{r^2} |u_\theta|^2 \right) d\theta \\ &\geq 2r \left(\int_{\theta_i}^{\theta_{i+1}} |u_r|^2 \right)^{1/2} \left(\frac{1}{r^2} \int_{\theta_i}^{\theta_{i+1}} |u_\theta|^2 \right)^{1/2}. \end{aligned} \tag{3.1}$$

Here, u_r and u_θ denotes the radial and azimuthal partial derivatives. Introducing the average value $\bar{u}_i = \frac{1}{\theta_{i+1} - \theta_i} \int_{\theta_i}^{\theta_{i+1}} u$ of u on the arc $y_i \widehat{y}_{i+1}$ we may use the following lemma which is a straightforward consequence of a theorem of Wirtinger (see [15] theorem 258):

LEMMA 3.5. – *Let $f \in H^1([-\pi, \pi], \mathbf{R})$ and let $\bar{f} = \frac{1}{2\pi} \int_{-\pi}^\pi f$ then*

$$\frac{1}{4} \int_{-\pi}^\pi (f - \bar{f})^2 dx < \int_{-\pi}^\pi (f')^2 dx$$

unless $f(x) = a + b \sin \frac{x}{2}$.

By a straightforward change of variables, we get

$$\int_{\theta_i}^{\theta_{i+1}} |u_\theta|^2 \geq \left(\frac{2\pi}{\theta_{i+1} - \theta_i} \right)^2 \frac{1}{4} \int_{\theta_i}^{\theta_{i+1}} (u - \bar{u}_i)^2. \tag{3.2}$$

Using this together with (3.1) we deduce:

$$\begin{aligned} \int_{y_i \widehat{y}_{i+1}} \|\nabla u\|^2 &\geq 2r \left(\int_{\theta_i}^{\theta_{i+1}} |u_r|^2 \right)^{1/2} \left(\frac{1}{4r^2} \int_{\theta_i}^{\theta_{i+1}} (u - \bar{u}_i)^2 \right)^{1/2} \\ &\geq \int_{\theta_i}^{\theta_{i+1}} u_r (u - \bar{u}_i), \end{aligned}$$

that is

$$\int_{y_i \widehat{y}_{i+1}} \|\nabla u\|^2 \geq \frac{1}{r} \int_{y_i \widehat{y}_{i+1}} u_\nu (u - \bar{u}_i). \tag{3.3}$$

This inequality was derived without any particular assumptions on u and K .

Now, if K is connected then the set $y_i \widehat{y}_{i+1}$ defines a bounded domain \mathcal{D} of boundary $\partial \mathcal{D} = F \cup y_i \widehat{y}_{i+1}$ where $F \subset K$. Then, by a similar argument as in lemma 3.1 [6] we observe that since \bar{u}_i is a constant, $\int_{\partial \mathcal{D}} u_\nu \bar{u}_i = \int_{\mathcal{D}} \nabla u \nabla \bar{u}_i = 0$. Moreover, u satisfies Neumann boundary conditions on F and then $\int_{\partial \mathcal{D}} u_\nu \bar{u}_i = \int_{y_i \widehat{y}_{i+1}} u_\nu \bar{u}_i$. Consequently if K is connected (or more generally if y_1, \dots, y_k belongs to the same connected component of K) we have $\int_{y_i \widehat{y}_{i+1}} u_\nu \bar{u}_i = 0$ which gives the inequality:

$$\int_{y_i \widehat{y}_{i+1}} \|\nabla u\|^2 \geq \frac{1}{r} \int_{y_i \widehat{y}_{i+1}} u_\nu u. \tag{3.4}$$

Summing over all $i = 1, \dots, k$ this reads

$$\int_{\partial B_r \setminus K} \|\nabla u\|^2 \geq \frac{1}{r} \int_{\partial B_r \setminus K} u_\nu u \tag{3.5}$$

which is the conclusion of proposition 3.4.

Proof of theorem 3.1. –

Lemma 3.3 and proposition 3.4 implies

$$r\phi'(r) \geq \phi(r) \text{ a.e.}$$

which is $\left(\frac{\phi(r)}{r}\right)' \geq 0$.

The second part of the theorem is concerned with the situations where $\left(\frac{\phi(r)}{r}\right)' \equiv 0$. If this occurs then the inequality in (3.5) is in fact an equality for almost every r . Consequently, the inequality given in lemma 3.5 is an equality and there are three functions $\alpha(r)$, $\beta(r)$ and $\theta(r)$ such that for a.e. r and for $\theta \in [\theta(r), \theta(r) + 2\pi)$ u is given by $u(r, \theta) = \beta(r) + \alpha(r) \cos\left(\frac{\theta - \theta(r)}{2}\right)$. Since u is harmonic in $\mathbf{R}^2 \setminus K$ we get $\alpha(r) = \alpha_0 \sqrt{r}$, $\beta(r) = \beta_0$ and $\theta(r) = \theta_0$. This completes the proof of theorem 3.1.

We now study global-minimizers defined in a half-space $\mathbf{R}^+ \times \mathbf{R}$. In this case, $\theta_{i+1} - \theta_i \leq \pi$ in (3.2) and it is straightforward to prove the

COROLLARY 3.6. – *If (u, K) is a global-minimizer in $\mathbf{R}^+ \times \mathbf{R}$ such that $K \cup \partial(\mathbf{R}^+ \times \mathbf{R})$ is connected then $r \mapsto \frac{\phi(r)}{r^2}$ is a nondecreasing function of r .*

Remark 3.7. – *If $U = \{y > f(x)\}$ is a Lipschitz domain ($f(0) = 0$) and if (u, K) is a global-minimizer in U such that $K \cup \partial U$ is connected then there is a $\alpha \in (1, 2]$ such that $r \mapsto \frac{\phi(r)}{r^\alpha}$ is a nondecreasing function of r .*

3.2. Minimizers of J

We consider now a minimizer (u, K) of the Mumford-Shah functional

$$J(u, K) = \int_{\Omega \setminus K} \|\nabla u\|^2 + \int_{\Omega} (u - g)^2 + \mathcal{H}^1(K).$$

LEMMA 3.8. – *Assume that $\omega \subset \Omega$ intersects only one connected component of K . Let U be an open set $U \subset\subset \omega$. There is a constant $C > 0$ and a real r_0 such that for all $P \in U$ we have:*

$$\text{for a.e. } r \in (0, r_0), \int_{\partial B_r(P) \setminus K} \|\nabla u\|^2 \geq \frac{1}{r} \int_{B_r(P) \setminus K} \|\nabla u\|^2 - Cr \tag{3.6}$$

Proof. – The lemma will be derived from inequality (3.3) by an estimation of $\int_{y_i y_{i+1}} u_\nu \bar{u}_i$. Since y_i and y_{i+1} are in the same connected component there is an arc of minimum length ψ_i connecting y_i to y_{i+1} . Lemma 2.9 implies that $\mathcal{H}^1(\psi_i([0, 1])) \leq Cd(y_i, y_{i+1}) \leq C'r$. Therefore, integrating in the domain ω_i defined by $\psi_i([0, 1])$ and the arc $y_i y_{i+1}$ we get

$$\begin{aligned} \int_{\widehat{y_i y_{i+1}}} u_\nu \bar{u}_i &= \int_{\omega_i \setminus K} \nabla u \nabla \bar{u}_i + \int_{\omega_i \setminus K} \Delta u \bar{u}_i \\ &= \int_{\omega_i} (u - g) \bar{u}_i \end{aligned}$$

Since $\mathcal{H}^1(\partial\omega_i) < (C' + 2\pi)r$ we get, by isoperimetric inequality, $|\omega_i| < C''r^2$ and then $|\int_{\omega_i} (u - g) \bar{u}_i| \leq C''r^2$. Inequality (3.3) gives then the conclusion of the lemma.

4. CHARACTERIZATION OF GLOBAL-MINIMIZERS

The main theorem of this section is:

THEOREM 4.1. – *If (u, K) is a global-minimizer such that K is connected then (u, K) is one of the following:*

- (i) *K is empty and u is constant.*
- (ii) *K is a straight line defining two half-planes and u is constant on each half-plane.*
- (iii) *K is the union of three half lines with angle $2\pi/3$ and u is constant on each sector.*
- (iv) *in a polar set of coordinates $u(r, \theta) = C \pm \sqrt{\frac{2}{\pi}} \sqrt{r} \cos \frac{\theta}{2}$ for $\theta \in [0, 2\pi)$ and K is the half axis $\theta = 0$.*

Proof. – The proof relies on a careful use of the monotonicity formula given in the previous section.

LEMMA 4.2. – *If (u, K) is a global-minimizer such that $\frac{\phi(r)}{r}$ is constant then either*

- (i) *$\frac{\phi(r)}{r} \equiv 1$ and in a polar set of coordinates $u(r, \theta) = C \pm \sqrt{\frac{2}{\pi}} \sqrt{r} \cos \frac{\theta}{2}$ for $\theta \in [0, 2\pi)$.*
- (ii) *or $\frac{\phi(r)}{r} \equiv 0$.*

Proof. – The determination of the constant $\sqrt{\frac{2}{\pi}}$ can be found in [19]. The lemma is then a straightforward consequence of theorem 3.1.

To prove theorem 4.1 we will show that $\frac{\phi(r)}{r}$ is a constant for a properly chosen origin. For this purpose we study the limits $\lim_{+\infty} \frac{\phi(r)}{r}$ and $\lim_0 \frac{\phi(r)}{r}$ using a blow-up and blow-down technique. The blow-up is defined as in section 2, that is,

$$U_\varepsilon(X) = \frac{u(\varepsilon X)}{\sqrt{\varepsilon}} \quad \text{and} \quad u_\varepsilon(X) = \frac{u(\varepsilon X) - c_\varepsilon(X)}{\sqrt{\varepsilon}}, \quad (4.1)$$

$$K_\varepsilon = \{X \in \mathbf{R}^2, \varepsilon X \in K\} \quad (4.2)$$

and the blow-down is given by

$$u_l(X) = \frac{u(lX)}{\sqrt{l}}, \quad (4.3)$$

$$K_l = \{X \in \mathbf{R}^2, lX \in K\} \quad (4.4)$$

where $\varepsilon \rightarrow 0$ and $l \rightarrow +\infty$.

From theorem 2.2 we deduce the,

COROLLARY 4.3. – *Let (u, K) be a global-minimizer then*

(*) *a blow-up subsequence $u_{\varepsilon_q}, K_{\varepsilon_q}$ (i.e. $\varepsilon_q \rightarrow 0$ as $q \in \mathbf{N}$ goes to ∞) converges to a global-minimizer (u_0, K_0) ,*

(**) *a blow-down subsequence u_{l_p}, K_{l_p} (i.e. $l_p \rightarrow \infty$ as $p \in \mathbf{N}$ goes to ∞) converges to a global-minimizer (u_∞, K_∞) .*

The convergences are to be understood in the same sense as stated in theorem 2.2. We can deduce then

LEMMA 4.4. – *Let $\phi_0(r) = \int_{B_r} \|\nabla u_0\|^2$ and $\phi_\infty(r) = \int_{B_r} \|\nabla u_\infty\|^2$. Then*

$$\frac{\phi_0(r)}{r} \equiv \lim_0 \frac{\phi(\rho)}{\rho} \quad \text{and} \quad \frac{\phi_\infty(r)}{r} \equiv \lim_\infty \frac{\phi(\rho)}{\rho}. \tag{4.5}$$

With lemma 4.2 this implies,

LEMMA 4.5. – *If (u, K) is a global-minimizer such that K is connected then $\lim_0 \frac{\phi(r)}{r}$ and $\lim_\infty \frac{\phi(r)}{r}$ can only take the value 0 or 1.*

Consequently, if $\lim_\infty \frac{\phi(r)}{r} = 0$ then the monotonicity formula gives $\phi(r) \equiv 0$. Thus, u is a constant in any connected component of $\mathbf{R}^2 \setminus K_0$ and we are reduced to minimizing $\mathcal{H}^1(K_0)$ under the condition (ii) of definition 2.1 which gives situations (i), (ii) or (iii) of theorem 4.1.

Let (u, K) be a minimizer of the Mumford-Shah functional $J(u, K)$ or $J^0(u, K)$. Assume that K has a finite number of connected components. We consider the blow-up at a point P : $u_\varepsilon(X) = \frac{u(P+\varepsilon X) - c_\varepsilon(X)}{\sqrt{\varepsilon}}$. The next lemma gives additional information about the way blow-ups converge in case (i), (ii), (iii).

LEMMA 4.6. – *If a blow-up limit at a point P (i.e. the limit of a subsequence $(u_{\varepsilon_n}, K_{\varepsilon_n})$ is of type (i), (ii) or (iii) then for any other subsequence the blow-up limit at P is of the same type.*

Proof. – We study the function

$$\xi : r \mapsto \frac{\int_{B_r(P) \setminus K} \|\nabla u\|^2}{r} \tag{4.6}$$

Under the hypotheses of lemma 4.6, lemma 3.8 implies that $\xi'(r) \geq -C$ for r small. Then, ξ has a limit l as $r \rightarrow 0$.

We claim that for any converging subsequence, the blow-up limit (u_0, K_0) is such that the ratio $\frac{\int_{B_r(0) \setminus K} \|\nabla u_0\|^2}{r}$ is constant and equal to l . This is a straightforward consequence of equations (2.12), (2.14) and (2.8). Under the

assumptions of lemma 4.6, this implies that $l = 0$ and that every blow-up limit is such that $\frac{\int_{B_r(0) \setminus K} \|\nabla u_0\|^2}{r}$ is zero. Therefore, every blow-up limit cannot be of type (iv), it is of type (i), (ii) or (iii). This is a very strong constraint on the solutions.

Assume first that a blow-up limit is of type (i). We recall [6] that there is a constant C such that if $\mathcal{H}^1(K \cap B_r) < Cr$ then $K \cap B_{\frac{r}{2}} = \emptyset$. If the blow-up limit is of type (i), then (2.14) implies that $\liminf \mathcal{H}^1(K_\varepsilon \cap B_1) = 0$. Therefore there is ε such that $\mathcal{H}^1(K_\varepsilon \cap B_1) < C$, that is, $\mathcal{H}^1(K \cap B_\varepsilon) < C\varepsilon$. This implies that $K \cap B_{\frac{\varepsilon}{2}} = \emptyset$ and then every blow-up limit is of type (i).

It remains to understand what happens when the blow-up limit is of type (ii) or (iii). We argue by contradiction and assume that there are two blow-up sequences ε_p and η_p such that $(u_{\varepsilon_p}, K_{\varepsilon_p})$ converges to a type (ii) and (u_{η_p}, K_{η_p}) converges to a type (iii). We may assume that $\varepsilon_p \geq \eta_p > \varepsilon_{p+1}$. Notice that this implies the convergence $\frac{\eta_p}{\varepsilon_p} \rightarrow 0$ (if not, the blow-up limit should be at the same time a type (ii) and a type (iii)). We will now prove that we can construct a blow-up sequence inbetween ε_p and η_p which converges to a global-minimizer which cannot be of one of the types of theorem 4.1.

Let us denote by k^2 the limit edge set of type (ii). It is a line passing through the origin 0. Let M be the triple point of the limit edge set of type (iii). We denote this set by $M + k^3$ where k^3 is the union of three half lines meeting at the origin and making $2\pi/3$ angles. For any R we have the convergence: $K_{\eta_p} \cap B_R \rightarrow (M + k^3) \cap B_R$ in the Hausdorff metric. We may choose a sequence l_p going slowly to $+\infty$ such that $l_p \eta_p < \varepsilon_p$ and for all R , $K_{l_p \eta_p} \cap B_R \rightarrow k^3 \cap B_R$. For simplicity, we will assume in the following that the sequence η_p is chosen such that $K_{\eta_p} \cap B_R \rightarrow k^3 \cap B_R$. Theorem 2.2 implies that $\mathcal{H}^1(B_R \cap K_{\eta_p}) \rightarrow 3R$ uniformly in any compact interval. For the same reasons $\mathcal{H}^1(B_R \cap K_{\varepsilon_p}) \rightarrow 2R$ uniformly in any compact interval. We introduce $R_p = \inf\{R \geq 1, \mathcal{H}^1(B_R \cap K_{\eta_p}) \leq 2.5R\}$. Since $\varepsilon_p > \eta_p$, we have $R_p < \frac{\varepsilon_p}{\eta_p}$ and $R_p \rightarrow +\infty$. We are interested in the blow-up sequence $(u_{\zeta_p}, K_{\zeta_p})$ for $\zeta_p = R_p \varepsilon_p$. Since $\mathcal{H}^1(B_R \cap K)$ is an increasing function of R , the definition of R_p and ζ_p gives

$$\forall R \in (1/R_p, 1), \quad 2.5R \leq \mathcal{H}^1(B_R \cap K_{\zeta_p}) \leq 2.5. \tag{4.7}$$

A subsequence of $(u_{\zeta_p}, K_{\zeta_p})$ converges to some (u_0, K_0) such that

$$\forall R \in (0, 1), \quad 2.5R \leq \mathcal{H}^1(B_R \cap K_0) \leq 2.5 \tag{4.8}$$

None of the global-minimizers of theorem 4.1 satisfies this relation. This contradiction completes the proof of lemma 4.6.

So far, we have proved that either we are in case (i), (ii), (iii) of theorem 4.1, or we have $\lim_{\infty} \frac{\phi(r)}{r} = 1$.

PROPOSITION 4.7. – *If $\lim_{\infty} \frac{\phi(r)}{r} = 1$ then there is a point P such that if we denote $\phi_P(r) = \int_{B_r(P)} \|\nabla u\|^2$ then*

$$\lim_0 \frac{\phi_P(r)}{r} = 1. \quad (4.9)$$

We will say that P is a crack-tip for the edge set K .

Proof. – We argue by contradiction. If the conclusion of the proposition is not true then for every point P the blow-up limit at P gives situations (i), (ii) or (iii) of theorem 4.1. We consider a point $P \in K$ and Jordan arcs in K parameterized by arc length $\psi : (a, b) \rightarrow K$ such that $\psi(0) = P$, (ψ is one to one). We consider maximal Jordan arcs passing through P : ψ is such that if $\tilde{\psi} : (\tilde{a}, \tilde{b}) \rightarrow K$ is a Jordan arc in K parameterized by arc length and $(a, b) \subset (\tilde{a}, \tilde{b})$ and $\tilde{\psi}|_{(a, b)} = \psi$ then $(a, b) = (\tilde{a}, \tilde{b})$ and $\tilde{\psi} = \psi$. Since K is a 1-dimensional Hausdorff connected set, it is arcwise connected. By assumption, K is unbounded and then, there is a maximal Jordan arc $\psi : (a, b) \rightarrow K$ such that $\psi(0) = P$ and such that $a - b$ is arbitrary large. If $(a, b) = \mathbf{R}$ then $\forall r > 0, \mathcal{H}^1(K \cap B_r(P)) \geq 2r$ which contradicts the fact that the blow-down is of type (iv). Therefore we may assume that $b < +\infty$. and we may introduce $N = \psi(b)$.

* If $N \in \psi([a, b))$ ($N = \psi(c)$) then $\psi : [c, b] \rightarrow K$ defines a closed Jordan curve defining a bounded connected component of $\mathbf{R}^2 \setminus K$. This contradicts lemma 4.8 below.

* If $\psi(b) \notin \psi([a, b))$ then the blow-up limit at N is either a type (ii) or a type (iii).

Assume that one blow-up limit is of type (ii). Lemma 4.6 says that every blow-up limit is of type (ii). This reads: $\forall \varepsilon > 0 \exists r_0, \forall r < r_0, \exists L^r, L^r$ is a straight line passing through N such that $K \cap B_r(N) \subset \{X \in \mathbf{R}^2, d(X, L^r) < \varepsilon r\}$. Let $\varepsilon = 1/2$. Then, $\partial B_r(N) \cap \{X \in \mathbf{R}^2, d(X, L^r) < r/2\}$ is made of two components C_1^r and C_2^r . There exists r_1 such that for $r < r_1, \psi((a, b)) \cap C_1^r \neq \emptyset$ implies $\psi((a, b)) \cap C_2^r = \emptyset$. Indeed, otherwise this would imply (for r arbitrary small) that $\mathcal{H}^1(K \cap B_r(N)) \geq (1 + \sqrt{3})r$. However, since the

blow-up limit is of type (ii), we should have $\lim_{r \rightarrow 0} \frac{\mathcal{H}^1(K \cap B_r(N))}{r} = 2$. Therefore we will assume that $\psi((a, b))$ intersects only C_1^r for $r < r_1$. Since the blow-up limit is a line, there is a sequence $N_n \in K \cap C_2^{r_n}$, ($r_n = \|N_n - N\| > 0$) converging to N . Let $\xi_n : (b, c) \rightarrow K$ be an arc of minimal length connecting N to N_n in K . This is a Jordan arc and we may assume that it is parameterized by arc length. Lemma 2.9 implies that its length goes to 0. Then for n large enough $\xi_n(b, c) \subset B_{r_1}(N)$ and we can prove easily by connexity that $\xi_n(t) \in C_{r_n(t)}^2$ (where $r_n(t) = \|\xi_n(t) - N\|$) for all $t \in (b, c)$. Consequently $\xi_n(b, c) \cap \psi(a, b) = \emptyset$. We may then extend ψ on (a, c) by ξ_n and contradict the maximality of ψ .

If the blow-up limit at N is of type (iii) (union of 3 half lines D_i) the same argument can be carried out with two connected components of $\partial B_r \cap \{X \in \mathbf{R}^2, \exists i, d(X, D_i) < \varepsilon r\}$ instead of C_1^r and C_2^r and give again a contradiction. This completes the proof of proposition 4.7.

End of the proof of theorem 4.1. – We have proved that $\lim_{\infty} \frac{\phi(r)}{r}$ can only take the value 0 or 1 (lemma 4.5). If $\lim_{\infty} \frac{\phi(r)}{r} = 0$ then we get (i), (ii) or (iii) in theorem 4.1. If $\lim_{\infty} \frac{\phi(r)}{r} = 1$ then there is a P such that $\frac{\phi_P(r)}{r}$ is constant and equal to 1. This gives the situation (iv) of theorem 4.1 whose proof is completed.

We have been using in the previous proof the fact that $K \setminus \mathbf{R}^2$ cannot have a bounded connected component.

LEMMA 4.8. – *If (u, K) is a global-minimizer, then $\mathbf{R}^2 \setminus K$ cannot have a bounded connected component.*

Proof. – Notice that by an obvious excision argument, any bounded connected component of $\mathbf{R}^2 \setminus K$ is convex. By contradiction, assume that $\mathbf{R}^2 \setminus K$ has a bounded convex connected component \mathcal{O} . Now, u is necessary a constant in the bounded connected component \mathcal{O} . Let A be a regular point of $\partial \mathcal{O}$. The blow-up at A is a line (situation (ii)). There is a limit value of u on both side of K at A . We may choose a compact perturbation of u inside \mathcal{O} such that those two values are identical. Then the blow-up at A can be proceeded with a function c constant in the whole plane. Therefore, the blow-up limit has to be a global-minimizer with respect to any compact perturbation (the assumption (ii) of definition 2.1 is not required). Here, the blow-up limit (u_0, K_0) is such that K_0 is a line and $u_0 \equiv 0$ in $\mathbf{R}^2 \setminus K_0$. Consequently for any ball B_R one can take $\tilde{K}_0 = K_0 \setminus B_R$ and $\tilde{u} \equiv 0$ in $\mathbf{R}^2 \setminus \tilde{K}_0$ and reduce the J^0 energy. This contradicts the minimality of (u_0, K_0) .

5. FINITE NUMBER OF PIECES

In this section we will prove that any isolated connected component of K is made of a finite number of Jordan arcs.

THEOREM 5.1. – *Let Ω be a Lipschitz domain and G be an isolated connected component of K . Then,*

$$G = \bigcup_{i=1}^k \psi_i([0, 1]) \quad (5.1)$$

where ψ_i , $i = 1, \dots, k$, are one to one Lipschitz continuous functions and $\psi_i((0, 1)) \cap \psi_j((0, 1)) = \emptyset$ if $i \neq j$.

Let P be a point in a neighborhood of G . Theorem 2.2 and lemma 2.8 ensure that a subsequence of the blow-up sequence of (u, K) at P converges to a global-minimizer which satisfies the hypotheses of theorem 4.1. As noted in remark 2.4 there may be several blow-up limits depending of the chosen subsequence. However we have

PROPOSITION 5.2. – *All blow-up limits at P are of the same type (that is, (i), (ii), (iii) and (iv) of theorem 4.1). Type (i) is obtained when $P \notin K$, for type (ii) P will be called a flat point, for type (iii) P is a triple point and for type (iv) P is a crack-tip.*

Proof. – We first notice that lemma 2.8 implies that any blow-up limit is such that the edge set is connected. Consequently, theorem 4.1 implies that the blow-up limit has to be of type (i), (ii), (iii) or (iv). Lemma 4.6 gives then the conclusion.

Proof of theorem 5.1. – Let us first assume that $G \subset\subset \Omega$. We know (proposition 5.2) that any point of K is a flat point or a triple point or a crack-tip. Let us prove that there is a finite number of triple points. By contradiction, assume that there is a sequence T_n of distinct triple points in G . By compactness we may assume that T_n converges to a point P of G . We introduce the blow-up sequence $u_{\varepsilon_n}(X) = \frac{u(T_n + \varepsilon_n r(X)) - c_{\varepsilon_n}(X)}{\sqrt{\varepsilon_n}}$, $K_{\varepsilon_n} = \{X \in \mathbf{R}^2, T_n + \varepsilon_n r(X) \in K\}$ where $\varepsilon_n = \|T_{n+1} - T_n\|$ and r is the rotation which sends e_x on $\frac{T_{n+1} - T_n}{\varepsilon_n}$. For this transformation, K_{ε_n} has a triple point at the origin and at the point $A = (1, 0)$. A subsequence of $(u_{\varepsilon_n}, K_{\varepsilon_n})$ converges to a global-minimizer (u_0, K_0) of J^0 . The origin and A are in the set K_0 and at least one of the two is not a triple point (theorem 4.1). Assume that the origin is not a triple point. It is then a crack-tip or a flat point. Therefore,

$$\mathcal{H}^1(K_{\varepsilon_n} \cap B_r(0)) \xrightarrow{n \rightarrow +\infty} r \text{ (resp. } 2r) \text{ uniformly in } r \in (0, 1). \quad (5.2)$$

Since T_n is a triple point we have

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^1(K_{\varepsilon_n} \cap B_r)}{r} = 3. \tag{5.3}$$

As in the proof of lemma 4.6 we introduce $R_n = \inf\{r > 0, \mathcal{H}^1(K_{\varepsilon_n} \cap B_r) \leq 2.5r\}$. We notice that $R_n \rightarrow 0$ and for $\eta_n = R_n \varepsilon_n$ we consider the blow-up $u_{\eta_n}(X) = \frac{u(T_n + \eta_n r(X)) - c_{\eta_n}(X)}{\sqrt{\eta_n}}$, $K_{\eta_n} = \{X \in \mathbf{R}^2, T_n + \eta_n r(X) \in K\}$. Then,

$$\forall R \in (0, 1), \quad 2.5R \leq \mathcal{H}^1(K_{\varepsilon_n} \cap B_r) \leq 2.5 \tag{5.4}$$

As in lemma 4.6 the blow-up limit satisfies the above inequality and this contradicts the result of theorem 4.1. Consequently, when $G \subset\subset \Omega$, there is a finite number of triple points.

Let us now consider the case where G meets $\partial\Omega$. Assume first that Ω is regular. We study then a blow-up at a boundary point. This gives a global-minimizer defined in a half-space $\mathbf{R}^+ \times \mathbf{R}$. From corollary 3.6 and the fact that $\phi(r) \leq Cr$ it is obvious that the blow-up limit satisfies: $\phi \equiv 0$ and K is the half line $\mathbf{R}^+ \times \{0\}$. A similar argument as before shows that there cannot be a sequence of triple points converging to the boundary. If Ω is only Lipschitz then remark 3.7 gives a similar characterization of blow-up limits ($\phi \equiv 0$, K is a half-line ending at the origin, etc.) The same argument as before gives the finite number of triple points.

Similarly, it is even easier to prove that there is at most a finite number of crack-tips.

To conclude the proof of theorem 5.1 we have to prove that we have a finite number of curves. We consider a flat point P of G . As in the proof of lemma 4.7, we can prove that there is a Jordan arc parameterized by $\psi : [a, b] \rightarrow K$ of maximal length, passing trough P and such that every point of $\psi((a, b))$ is flat. Then, the endpoints $\psi(a)$ and $\psi(b)$ are either triple points or crack-tips. We note also that for any Jordan arc $\psi' : [a', b'] \rightarrow K$, if every point of $\psi'((a', b'))$ is flat then either

$$\psi'((a', b')) \cap \psi((a, b)) = \emptyset \quad \text{or} \quad \psi'((a', b')) \subset \psi((a, b)). \tag{5.5}$$

Indeed, consider $(t_1, t_2) \subset [a', b']$ such that $\psi'((t_1, t_2)) \cap \psi((a, b)) = \emptyset$ and $\psi'(t_1)$ (or $\psi'(t_2)$) is in $\psi([a, b])$. Either $\psi'(t_1)$ is an endpoint of $\psi([a, b])$ (i.e. coincide with $\psi(a)$ or $\psi(b)$) or $\psi'(t_1)$ is a triple point. If $\psi'(t_1)$ is a triple point then, by construction of $\psi([a, b])$, it is an endpoint: $\psi(a)$ or $\psi(b)$.

Consequently, G is a union of disjoint Jordan arcs connecting a finite family of triple points and crack-tips with each other. The family of

arcs is then obviously finite ($\mathcal{H}^1(K) < \infty$). This completes the proof of theorem 5.1.

PROPOSITION 5.3. – *Let P be a point of K . Assume that $B_r(P)$ intersects only one connected component of K . Then*

$$K \cap B_{\frac{r}{2}}(P) = \left(\bigcup_{i=1}^k \psi_i([0, 1]) \right) \cap B_{\frac{r}{2}}(P) \tag{5.6}$$

where ψ_i are one to one Lipschitz continuous functions and $\psi_i((0, 1)) \cap \psi_j((0, 1)) = \emptyset$ if $i \neq j$.

Proof. – As in the proof of theorem 5.1, there is a finite number of triple points and crack-tips in $B_{\frac{3r}{4}}(P)$. For each $M \in K \cap B_{\frac{r}{2}}(P)$ there is a Jordan arc $\psi : [a, b] \rightarrow K$ of maximal length, passing through M , such that every point of $\psi((a, b))$ is flat and such that $\psi((a, b)) \subset B_{\frac{3r}{4}}(P)$. As before those arcs are pairwise disjoint. The endpoints $\psi(a)$ and $\psi(b)$ of this arc are either triple points or crack-tips or points of the boundary $\partial B_{\frac{3r}{4}}(P)$. If $\psi(a)$ or $\psi(b)$ is in $\partial B_{\frac{3r}{4}}(P)$ then $\mathcal{H}^1(\psi((a, b))) \geq r/4$. Since $\mathcal{H}^1(K \cap B_r) \leq Cr$ there is a finite number of those Jordan arcs intersecting $B_{\frac{r}{2}}(P)$ and ending on $\partial B_{\frac{3r}{4}}(P)$. We already know (as in theorem 5.1) that there is a finite number of those arcs ending at a triple-point or at a crack-tip. Consequently the family of maximal Jordan arcs made of flat points of K and intersecting $B_{\frac{r}{2}}(P)$ is finite and satisfies the conclusion of the proposition.

6. REGULARITY $C^{1,\alpha}$

Let U be a domain, we recall the definition of $C^{1,\alpha}(\bar{U})$.

$$C^{0,\alpha}(\bar{U}) = \left\{ f \in C(\bar{U}), \sup_{\substack{x,y \in \bar{U} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < +\infty \right\}$$

$$C^{m,\alpha}(\bar{U}) = \{ f \in C^m(\bar{U}), D^\beta f \in C^{0,\alpha}(\bar{U}) \ \forall \beta \text{ with } |\beta| = m \}$$

A curve c is $C^{1,\alpha}$ if locally one can find local coordinates (x, y) such that $c \cap B(0, r)$ is a graph $\{y = f(x)\}$ with $f \in C^{1,\alpha}$.

THEOREM 6.1. – *Let (u, K) be a minimizer of the Mumford-Shah functional. Let P be a flat point of K . Assume that there is an open neighborhood ω of P such that ω intersects only one connected component*

of K . Then, there is a neighborhood of P in which K is a $C^{1,\alpha}$ arc for all $\alpha < 1/2$.

Let $\phi_P : r \mapsto \int_{B_r(P) \setminus K} \|\nabla u\|^2$. We will prove a weak monotonicity formula

LEMMA 6.2. – Assume that for any $R < R_0$ there is a straight line L_R passing through P such that $K \cap B_R(P) \subset \{X \in B_R(P), d(X, L_R) < lR\}$ where $0 < l < 1$ is a constant. Assume that only one component of K intersects B_{R_0} and that for all $R < R_0$, K intersects both connected components of $\partial B_R(P) \cap \{X \in B_R(P), d(X, L_R) < lR\}$, then there is $\beta = \beta(l)$ and a constant $C(R_0, \beta)$ such that

$$\phi_P(R) \leq CR^\beta, \quad \forall R < R_0.$$

The constant β is such that $\beta(l) \rightarrow 2$ as $l \rightarrow 0$.

Proof. – Let $\eta = \sin^{-1} l$. Inequality (3.2) with $|\theta_{i+1} - \theta_i| \leq \pi + 2\eta$ allows us to adapt the proof of lemma 3.8 and get for $r < R_0$,

$$\int_{\partial B_r(P) \setminus K} \|\nabla u\|^2 \geq \frac{1}{r} \frac{2\pi}{\pi + 2\eta} \int_{B_r(P) \setminus K} \|\nabla u\|^2 - Cr. \tag{6.1}$$

We define $\beta = \frac{2\pi}{\pi + 2\eta}$, $1 < \beta < 2$. Inequality (6.1) reads then

$$\frac{d}{dr} \left(\frac{\phi_P(r)}{r^\beta} \right) \geq -Cr^{1-\beta}. \tag{6.2}$$

A direct integration between $R < R_0$ and R_0 gives

$$\phi_P(R) \leq R^\beta \left(\frac{\phi_P(R_0)}{R_0^\beta} + \frac{C}{2 - \beta} R_0^{2-\beta} \right),$$

which is exactly the conclusion of the lemma with $\beta(l) = \frac{2\pi}{\pi + 2\sin^{-1} l}$. Obviously, $1 < \beta(l) < 2$ and $\beta(l) \rightarrow 2$ as $l \rightarrow 0$.

LEMMA 6.3. – Under the assumptions of lemma 6.2 and for $l < \frac{\sqrt{3}}{2}$ there is a constant $C(R_0, \beta)$ such that for $R < R_0$,

$$\mathcal{H}^1(K \cap B_R(P)) \leq 2R + CR^\beta. \tag{6.3}$$

Proof. – We remark that under the hypotheses of lemma 6.2 and for $l < \frac{\sqrt{3}}{2}$ any blow-up sequence at P will yields a type (ii) blow-up limit

(theorem 4.1). This is indeed the only type compatible with the hypotheses. Therefore we know that

$$\lim_{R \rightarrow 0} \frac{\mathcal{H}^1(K \cap B_R(P))}{R} = 2. \tag{6.4}$$

Let $n_0 \in \mathbb{N}$ be chosen such that $R_0 \in [2^{-n_0-1}, 2^{-n_0}]$. The limit (6.4) and inequality (6.1) imply then the existence of a constant $C_1 > 0$ such that for all $n \geq n_0$,

$$\exists r \in [2^{-n}, 2^{-n+1})$$

such that

$$\int_{\partial B_r(P)} \|\nabla u\|^2 \leq C_1 r^{\beta-1} \quad \text{and} \quad \#(K \cap B_r(P)) = 2. \tag{6.5}$$

Let $K \cap \partial B_r(P) = \{y_1, y_2\}$. We introduce a compact perturbation (\tilde{u}, \tilde{K}) of (u, K) . In $B_r(P)$ we replace K by the union of the two segments: $\tilde{K} \cap B_r(P) = [P, y_1] \cup [P, y_2]$. Outside $B_r(P)$ we define $\tilde{K} \setminus B_r(P) = K \setminus B_r(P)$. The function \tilde{u} is introduced as: $\tilde{u} = u$ in $\Omega \setminus B_r(P)$ and \tilde{u} minimizes the energy $\int_{B_r(P) \setminus \tilde{K}} \|\nabla v\|^2$ under the Dirichlet condition $v = u$ on $\partial B_r(P)$.

We claim that $\int_{B_r(P) \setminus \tilde{K}} \|\nabla \tilde{u}\|^2$ is less than $C_2 r^\beta$. Indeed $B_r(P) \setminus \tilde{K}$ has two sectors σ_i of angles $\gamma_i \pi$ ($i = 1, 2$ and $\gamma_1 + \gamma_2 = 2$) which can be mapped by a conformal mapping (we identify \mathbb{R}^2 and \mathbb{C}) $z \in \mathbb{C} \mapsto \lambda(z - z_P)^{\frac{1}{\gamma_i}}$ (where $\lambda \in \mathbb{C}$, $|\lambda| = 1$) into the half ball $B_{\frac{1}{r^{\gamma_i}}} \cap (\mathbb{R} + i\mathbb{R}^+)$. If v is the image of \tilde{u} by this conformal mapping then we have $\frac{\partial v}{\partial \nu} = 0$ on the diameter of the half ball. By a reflection we extend v to an harmonic function w on the ball $B_{\frac{1}{r^{\gamma_i}}}$. We have then the identity [6]:

$$\int_{\partial B_{\frac{1}{r^{\gamma_i}}}} |w_\nu|^2 = \int_{\partial B_{\frac{1}{r^{\gamma_i}}}} |w_\tau|^2. \tag{6.6}$$

The monotonicity formula (theorem 3.1) gives:

$$\int_{\partial B_{\frac{1}{r^{\gamma_i}}}} \|\nabla w\|^2 \geq \frac{2}{r^{\frac{1}{\gamma_i}}} \int_{B_{\frac{1}{r^{\gamma_i}}}} \|\nabla w\|^2. \tag{6.7}$$

(The factor 2 comes from (3.2) with $\theta_{i+1} - \theta_i = \pi$.) We notice that by the construction of w we have

$$\int_{B_{\frac{1}{r^{\gamma_i}}}} \|\nabla w\|^2 = 2 \int_{\sigma_i} \|\nabla \tilde{u}\|^2 \tag{6.8}$$

and

$$\int_{\partial B_{\frac{1}{r^{\gamma_i}}}} |w_\tau|^2 = 2\gamma_i r^{1-\frac{1}{\gamma_i}} \int_{y_i \widehat{y_{i+1}}} |\tilde{u}_\tau|^2. \tag{6.9}$$

where w_τ (resp. \tilde{u}_τ) is the tangential partial derivative on $\partial B_{\frac{1}{r^{\gamma_i}}}$ (resp. on the associated arc $y_i \widehat{y_{i+1}}$ of $\partial B_r(P)$). From the construction of \tilde{u} we know that $\tilde{u}_\tau = u_\tau$ on $\partial B_r(P)$. Identities (6.6), (6.7) and (6.8) gives then:

$$\int_{\sigma_i} \|\nabla \tilde{u}\|^2 \leq \gamma_i r \int_{y_i \widehat{y_{i+1}}} |u_\tau|^2. \tag{6.10}$$

Using (6.5) we deduce

$$\int_{B_r(P) \setminus \tilde{K}} \|\nabla \tilde{u}\|^2 \leq 2\pi C_1 r^\beta. \tag{6.11}$$

Since (u, K) is a minimizer of the Mumford-Shah functional we shall have

$$J(u, K) \leq J(\tilde{u}, \tilde{K}) \tag{6.12}$$

which implies

$$\begin{aligned} \mathcal{H}^1(K \cap B_r(P)) &\leq \mathcal{H}^1(\tilde{K} \cap B_r(P)) + \int_{B_r(P) \setminus \tilde{K}} \|\nabla \tilde{u}\|^2 + \int_{B_r(P)} (u - g)^2 \\ &\leq 2r + 2\pi C_1 r^\beta + \pi r^2. \end{aligned} \tag{6.13}$$

For any $R < R_0$, $\exists n \geq n_0$ such that $R \in [2^{-n-1}, 2^{-n})$. We have proved that there is a $r \in [2^{-n}, 2^{-n+1})$ such that (6.13) holds. The hypotheses of the lemma implies that for $R < r$,

$$\mathcal{H}^1(K \cap B_R(P)) + 2(r - R) \leq \mathcal{H}^1(K \cap B_r(P)).$$

Therefore we have

$$\mathcal{H}^1(K \cap B_R(P)) \leq 2R + 2\pi C_1 r^\beta + \pi r^2. \tag{6.14}$$

Let $C = 2^{1+2\beta} \pi C_1 + 16\pi R_0^{2-\beta}$, since $r/4 < R < R_0$ and $\beta < 2$ we get

$$\forall R < R_0, \quad \mathcal{H}^1(K \cap B_R(P)) \leq 2R + CR^\beta. \tag{6.15}$$

PROPOSITION 6.4. – *Under the assumptions of lemma 6.2 there is a constant $C(R_0, \beta)$ and a straight line L passing through P such that for all $R < R_0/2$:*

$$K \cap B_R(P) \subset \{X, d(X, L) \leq CR^{\frac{\beta+1}{2}}\} \quad (6.16)$$

Proof. – Let R be given, $R < R_0/2$. Let M and N be two points of $K \cap \partial B_R(P)$ chosen in each connected component of $\partial B_R(P) \cap \{X \in B_R(P), d(X, L_R) < lR\}$ (see lemma 6.2). If there is a point $X \in K \cap B_R(P)$ and a real $m \leq 2l$ such that $d(X, [P, M]) \geq mR > 0$ and $d(X, [P, N]) \geq mR > 0$ then a straightforward estimation gives a constant C such that $\mathcal{H}^1(K \cap B_{2R}(P)) \geq 2R + CRm^2$. We deduce from (6.3) that there is a constant C such that $m \leq CR^{\frac{\beta-1}{2}}$. This is:

$$K \cap B_R(P) \subset (\{X, d(X, [P, M]) \leq CR^{\frac{\beta+1}{2}}\} \cup \{X, d(X, [P, N]) \leq CR^{\frac{\beta+1}{2}}\}). \quad (6.17)$$

We can choose two sequences M_k and N_k of points in each connected component of $K \cap \partial B_{R/2^k}(P)$. Identity (6.17) gives the estimates:

$$d(M_{k+1}, [P, M_k]) \leq C (R/2^k)^{\frac{\beta+1}{2}} \text{ and } d(N_{k+1}, [P, N_k]) \leq C (R/2^k)^{\frac{\beta+1}{2}}. \quad (6.18)$$

Let θ_k (resp. Θ_k) denotes a determination of the angle that the vector PM_k (resp. PN_k) makes with a fixed vector e . Inequalities (6.18) implies that the determination of θ_{k+1} and Θ_{k+1} can be chosen such that:

$$|\theta_{k+1} - \theta_k| \leq 2C (R/2^k)^{\frac{\beta-1}{2}} \text{ and } |\Theta_{k+1} - \Theta_k| \leq 2C (R/2^k)^{\frac{\beta-1}{2}}. \quad (6.19)$$

This gives the convergence of the sequences θ_k and Θ_k to real numbers θ_∞ and Θ_∞ . We notice that since the blow-up limit at P is a straight line we shall have $\theta_\infty = \Theta_\infty + (2n+1)\pi$ where n is an integer. This defines a line L passing through P . Inequalities (6.19) give the existence of a constant C : $|\theta_k - \theta_\infty| \leq C (R/2^k)^{\frac{\beta-1}{2}}$ and $|\Theta_k - \Theta_\infty| \leq C (R/2^k)^{\frac{\beta-1}{2}}$. Together with (6.17) this gives the conclusion of the proposition.

Let P be a flat point of K , we assume that there is an open neighborhood ω of P such that ω intersects only one connected component of K . We will prove then that K is $C^{1,\alpha}$ in a neighborhood of P .

LEMMA 6.5. – *There is a neighborhood U of P such that $K \cap U$ is a Jordan arc and such that for all l , $0 < l < 1$ there is a $R_0 > 0$ such that*

for all $M \in K \cap U$, M is a flat point and for all $R < R_0$ there is a straight line L_R^M passing through M such that

$$\left. \begin{aligned} K \cap B_R(M) &\subset \{X \in B_R(M), d(X, L_R^M) < lR\}, \\ K &\text{ intersects each connected component} \\ &\text{of } \partial B_R(M) \cap \{X \in B_R(M), d(X, L_R^M) < lR\}. \end{aligned} \right\} \quad (6.20)$$

Proof. – We first notice that proposition 5.3 gives the existence of a neighborhood of P in which K is a Jordan arc. In this neighborhood, any point of K is a flat point. By a contradiction argument, if the conclusion of the lemma is not true then one can construct a sequence of flat points M_n converging to a point P and a sequence of positive real numbers R_n converging to 0 such that (6.20) does not hold. By construction, the associated blow-up sequence $(u_n(X) = u(M_n + R_n X) / \sqrt{R_n}, K_n)$ cannot converge to a type (ii) global-minimizer. Since K is an arc in a neighborhood of P , $\mathcal{H}^1(K \cap B_{R_n}(M_n)) \geq 2$ and the blow-up limit cannot be a type (iv) (neither (i)). Finally, a similar argument as in lemma 4.6 proves that the blow-up limit cannot be of type (iii) (triple-point). This contradicts theorems 2.2 and 4.1.

Proof of theorem 6.1. – From lemma 6.5 we know that the hypotheses of lemma 6.2 and 6.3 are satisfied uniformly in a neighborhood of P . Then the conclusion of proposition 6.4 holds uniformly in this neighborhood. This gives exactly the $C^{1,\alpha}$ regularity for $\alpha = \frac{\beta-1}{2}$. Since this is true for any l , $0 < l < \frac{\sqrt{3}}{2}$ and since $\beta(l) \rightarrow 2$ when $l \rightarrow 0$ we have proved the $C^{1,\alpha}$ regularity for any $\alpha < 1/2$.

Notice that the same arguments as in lemma 6.2 in the case of a triple point P give the

LEMMA 6.6. – *Let P be a triple point of K , such that there is an open neighborhood ω of P such that ω intersects only one connected component of K . Then for any $\beta < 3$ there is a constant C such that*

$$\phi_P(R) \leq CR^\beta.$$

Then, the procedure of lemma 6.3 and proposition 6.4 yields

PROPOSITION 6.7. – *Under the conditions of lemma 6.6, there are three half lines D_i meeting at P with angle $2\pi/3$ such that for any $\beta < 3$ there is a constant C such that*

$$K \cap B_R \subset \{X, d(X, D_1 \cup D_2 \cup D_3) \leq CR^{\frac{\beta+1}{2}}\}. \quad (6.21)$$

Taking this into account in the proof of theorem 6.1 we write the:

THEOREM 6.8. – *Let P be a triple point of K , such that there is an open neighborhood ω of P such that ω intersects only one connected component of K . Then there is a neighborhood U of P in which K is the union of three $C^{1,\alpha}$ arcs ($\forall \alpha < 1/2$) making a $2\pi/3$ angle at P .*

COROLLARY 6.9. – *If G is an isolated connected component of K then it is the union of a finite number of arcs. Those arcs are $C^{1,\alpha}$ away from the crack-tips and can only merge through a finite number of triple points.*

7. FURTHER REGULARITY

7.1. $C^{1,1}$ regularity

In this section we will prove in theorem 7.2 that if K is, locally, a $C^{1,\alpha}$ curve then it is $C^{1,1}$. This result together with theorem 5.1 and 6.1 completes the proof of theorem 1.1 and 1.2. We first start with a proposition which gives the $C^{1,\alpha}$ regularity of u up to the edge set K . In the following, (u, K) is a minimizer of the Mumford-Shah functional.

PROPOSITION 7.1. – *Let P be a point of K such that in a neighborhood $B_{4R}(P)$ of P the edge set K is a $C^{1,\alpha}$ curve: $\{(x, y), y = f(x), x \in I\}$. K defines two sets $B_R^+ = \{(x, y) \in B_R(P), y \geq f(x)\}$ and $B_R^- = \{(x, y) \in B_R(P), y \leq f(x)\}$. The minimizer u can be extended on each side of K such that*

$$u \in C^{1,\alpha}(B_R^+) \quad \text{and} \quad u \in C^{1,\alpha}(B_R^-). \quad (7.1)$$

Proof. – We will prove that u is $C^{1,\alpha}$ in $B_R^+ = \{(x, y) \in B_R(P), y \geq f(x)\}$. The minimizer u satisfies $-\Delta u + u - g = 0$ in the interior of B_{4R}^+ . In the sense of lemma 3.1 in [6] we can say that u satisfies Neumann boundary conditions $\frac{\partial u}{\partial \nu} = 0$ on K . We want to prove that a solution of $-\Delta u + u - g = 0$ in B_{4R}^+ with $\frac{\partial u}{\partial \nu} = 0$ on K is $C^{1,\alpha}$ in B_R^+ . Since g is L^∞ , the $C^{1,\alpha}$ regularity in the case of Dirichlet conditions is given by theorem 5.5.5' of [18]. Following [18], it is possible in fact to prove the $C^{1,\alpha}$ regularity in the case of Neumann boundary conditions. For sake of completeness the proof is given in the appendix.

THEOREM 7.2. – *Let P be a point of K such that in a neighborhood U of P the edge set K is a $C^{1,\alpha}$ curve, then K is $C^{1,1}$ in U .*

Proof. – For every point M of $K \cap U$ we may take a ball $B_r(M) \subset U$ and a coordinate system centered at M in which $K \cap B_r(M)$ is the graph

of a $C^{1,\alpha}$ function $f : I \rightarrow \mathbf{R}$ such that $f(0) = 0$ and $f'(0) = 0$:

$$K \cap B_r(M) = \{(x, y) \in U, x \in I \text{ and } y = f(x)\}. \tag{7.2}$$

We would like to estimate if we can reduce the Mumford-Shah energy by moving the edge set near M . This can be achieved for instance by adding a $C_0^\infty(I)$ function ζ to f . Assume that $\zeta \geq 0$, let us replace K in $B_r(M)$ by the set $\{(x, y) \in B_r(M), y = f(x) + \zeta(x)\}$ and get a new edge set \tilde{K} . We define now a new function \tilde{u} from u in $B_r(M) \setminus \tilde{K}$:

$$\begin{aligned} \forall (x, y) \in B_r(M), \text{ if } f(x) < y < f(x) + \zeta(x) \\ \text{then } \tilde{u}(x, y) &= u(x, 2f(x) - y), \\ \text{if not, then } \tilde{u}(x, y) &= u(x, y). \end{aligned}$$

It is easy to compute:

$$\begin{aligned} J(\tilde{u}, \tilde{K}) - J(u, K) &= \int_I \left(\sqrt{1 + (f' + \zeta')^2} - \sqrt{1 + (f')^2} \right) dx \\ &\quad + \int_{\{(x,y), f(x) < y < f(x) + \zeta(x)\}} \left((\tilde{u} - g)^2 - (u - g)^2 \right. \\ &\quad \left. + \|\nabla \tilde{u}\|^2 - \|\nabla u\|^2 \right) dx dy. \end{aligned}$$

Since (u, K) is a minimizer we must have $J(\tilde{u}, \tilde{K}) - J(u, K) \geq 0$ that is:

$$\begin{aligned} &\int_I \left(\sqrt{1 + (f' + \zeta')^2} - \sqrt{1 + (f')^2} \right) dx \\ &\quad + \int_{\{(x,y), f(x) < y < f(x) + \zeta(x)\}} \left((\tilde{u} - g)^2 - (u - g)^2 \right. \\ &\quad \left. + \|\nabla \tilde{u}\|^2 - \|\nabla u\|^2 \right) dx dy \geq 0. \end{aligned} \tag{7.3}$$

We may replace ζ by $t\zeta$ and let $t > 0$ go to 0. We get then:

$$\frac{1}{t} \int_I \left(\sqrt{1 + (f' + \zeta')^2} - \sqrt{1 + (f')^2} \right) \rightarrow \int_I \frac{f'\zeta'}{\sqrt{1 + (f')^2}}. \tag{7.4}$$

We denote by κ the distributional first order derivative of $\frac{f'}{\sqrt{1+(f')^2}}$. It is the curvature of K and we can write:

$$\int_I \frac{f'\zeta'}{(1 + (f')^2)^{1/2}} = - \int_I \kappa \zeta. \tag{7.5}$$

Since u and f are $C^{1,\alpha}$ and since g is bounded we deduce from (7.3) that there is a constant C such that:

$$\int_I \kappa \zeta \leq C \int_I \zeta \tag{7.6}$$

A similar construction with $-\zeta$ gives the existence of a constant C such that for all $C_0^\infty(I)$ function θ :

$$\left| \int_I \kappa \theta \right| \leq C \int_I |\theta| \tag{7.7}$$

This means that κ (and therefore f'') is an L^∞ function and that $K \cap B_r(M)$ is a $C^{1,1}$ curve. This is true for every point $M \in K \cap U$. The proof of theorem 7.2 is completed.

7.2. $C^{k+2,\alpha}$ regularity when g is $C^{k,\alpha}$

In this section we prove theorem 1.4. From theorem 6.1, we know that if K is a C^1 curve then it is $C^{1,\alpha}$. We follow then the same construction as in the proof of theorem 7.2. With the same notations, we start over from inequality (7.3). We denote by $u^+(x)$, $\nabla u^+(x)$, $\tilde{u}^+(x)$ and $\nabla \tilde{u}^+(x)$ the limit value of u , ∇u , \tilde{u} and $\nabla \tilde{u}$ on K from above at the point $(x, f(x))$. We write (7.3) with $t\zeta$ in place of ζ . We notice that the continuity of g allows to rewrite (7.3) in the limit $t \rightarrow 0$ as:

$$\int_I \zeta \left(- \frac{f''}{(1 + (f')^2)^{3/2}} + (\tilde{u}^+ - g(x, f(x)))^2 - (u^+ - g(x, f(x)))^2 + \|\nabla \tilde{u}^+\|^2 - \|\nabla u^+\|^2 \right) dx \geq 0 \tag{7.8}$$

We denote by $u^-(x)$ and $\nabla u^-(x)$ the limit value of u and ∇u on K from below at the point $(x, f(x))$. Inequality (7.8) gives at almost every $x \in I$:

$$- \frac{f''(x)}{(1 + (f'(x))^2)^{3/2}} + (u^-(x) - g(x, f(x)))^2 - (u^+(x) - g(x, f(x)))^2 + \|\nabla u^-(x)\|^2 - \|\nabla u^+(x)\|^2 \geq -4\|\nabla u^-(x)\|^2 (|f'(x)| + (f'(x))^2) \tag{7.9}$$

With $-\zeta$ we derive at almost every $x \in I$:

$$- \frac{f''(x)}{(1 + (f'(x))^2)^{3/2}} + (u^-(x) - g(x, f(x)))^2 - (u^+(x) - g(x, f(x)))^2 + \|\nabla u^-(x)\|^2 - \|\nabla u^+(x)\|^2 \leq 4\|\nabla u^+(x)\|^2 (|f'(x)| + (f'(x))^2) \tag{7.10}$$

Since $f'(0) = 0$, there exists a function ε_0 with $\lim_0 \varepsilon_0(x) = 0$ such that for almost every $x \in I$:

$$\left| -\frac{f''(x)}{(1+(f'(x))^2)^{3/2}} + (u^-(x) - g(x, f(x)))^2 - (u^+(x) - g(x, f(x)))^2 + \|\nabla u^-(x)\|^2 - \|\nabla u^+(x)\|^2 \right| \leq \varepsilon_0(x)$$

This inequality has been derived by an argument centered at M of coordinates $(0,0)$. The same argument at a point $N(x_0, f(x_0))$ would give the existence of a function ε_{x_0} with $\lim_{x_0} \varepsilon_{x_0}(x) = 0$ such that for almost every $x \in I$:

$$\left| -\frac{f''(x)}{(1+(f'(x))^2)^{3/2}} + (u^-(x) - g(x, f(x)))^2 - (u^+(x) - g(x, f(x)))^2 \right| \leq \varepsilon_{x_0}(x) \tag{7.12}$$

$$+ \|\nabla u^-(x)\|^2 - \|\nabla u^+(x)\|^2 \Big| \leq \varepsilon_{x_0}(x) \tag{7.13}$$

Consequently f'' is continuous and at every point $N(x, f(x))$ of $K \cap B_r(M)$ we have the expression of the curvature of K at the point N :

$$\frac{f''(x)}{(1+(f'(x))^2)^{3/2}} = (u^-(x) - g(N))^2 - (u^+(x) - g(N))^2 + \|\nabla u^-(x)\|^2 - \|\nabla u^+(x)\|^2. \tag{7.14}$$

From this equation we are able to start a classical bootstrap argument. If g is $C^{0,\alpha}$ then f is $C^{2,\alpha}$ and then (elliptic regularity) u is $C^{2,\alpha}$ up to K .

By induction, if g is $C^{k,\alpha}$ and u is $C^{k+1,\alpha}$ up to K then f is $C^{k+2,\alpha}$ and u is $C^{k+2,\alpha}$ up to K . This completes the proof of theorem 1.4.

Remark 7.3. – Notice (7.14) makes it easy to construct an example of an L^∞ function g such that K is $C^{1,1}$ and not C^2 .

7.3. Minimization with n connected components

One can study the problem: minimize the Mumford-Shah functional under the condition that K is made at most of n connected components

(see proposition 15.44 of [17]). The previous analysis gives immediately the proof of theorem 1.5.

Remark 7.4. – The proof through blow-up procedure (theorem 2.3) gives that the maximum number of arcs and triple points do not depend on the image g , $0 \leq g \leq 1$.

8. REGULARITY ALMOST EVERYWHERE

In this section we will not make any assumption on the connected components of K and prove theorem 1.3. This result is a direct consequence of the following proposition and of theorem 6.1 and 7.2. The proof of proposition 8.1 involves an argument of coarea formula as in [7]. The coarea formula is used, here, together with blow-up techniques and precise estimates on the length of the level set of a distance function.

PROPOSITION 8.1. – *For \mathcal{H}^1 almost every point P of K there is a ball $B_\rho(P)$ and a one-to-one Lipschitz map $\psi : [0, 1] \rightarrow \Omega$ such that*

$$P \in \psi((0, 1)) \quad \text{and} \quad B_\rho(P) \cap K = B_\rho(P) \cap \psi([0, 1]). \tag{8.1}$$

Proof. – \mathcal{H}^1 almost every point of K has a tangent and has density one. Therefore by a blow-up at such points one gets a global minimizer (u_0, K_0) where K_0 is included in a line (the tangent) and where $\mathcal{H}^1(K_0 \cap B_r) = 2r$ for all r . Then, K_0 is a line and u_0 is a constant on both sides. Let P be such a point of K and $(u_\varepsilon, K_\varepsilon)$ be a blow-up sequence at P .

Step 1. – There are no holes in K_ε near P .

Assume that K_0 coincides with the x axis. For ε small, $(u_\varepsilon, K_\varepsilon)$ is arbitrary close to (u_0, K_0) and for any $A > 0$ there is $\varepsilon > 0$ such that $(U_\varepsilon(X) = u(P + \varepsilon X)/\sqrt{\varepsilon} - Const, K_\varepsilon)$ satisfies in the rectangle $R_2 = [-2, 2] \times [-1, 1]$:

$$\left. \begin{aligned} U_\varepsilon(x, y) &\geq 2A, & \text{if } x \geq 1/2, & (x, y) \in R_2, \\ U_\varepsilon(x, y) &\leq -2A, & \text{if } x \leq -1/2, & (x, y) \in R_2, \\ K_\varepsilon \cap R_2 &\subset \{-1/2 < x < 1/2\}, \\ |\mathcal{H}^1(K_\varepsilon) - 2| &\leq 1/A, \quad \mathbf{d}(K_\varepsilon, K_0) \leq 1/A, \\ \int_{R_2 \setminus K_\varepsilon} \|\nabla U_\varepsilon\|^2 &\leq 1/A. \end{aligned} \right\} \tag{8.2}$$

Let $x_0 \in [-2, 2]$ be such that K_ε intersects the vertical segment $\{x_0\} \times [-1, 1]$ in a single point (x_0, y_0) . Assume that

$$\exists y \in (y_0, 1/2), \quad U_\varepsilon(x_0, y_0) < A \text{ or } \exists y \in (-1/2, y_0), \quad U_\varepsilon(x_0, y_0) > -A \tag{8.3}$$

then $\int_{-1}^1 \|\nabla U_\varepsilon\|^2 dy \geq A^2$. For large A , $\int_{R_2 \setminus K_\varepsilon} \|\nabla U_\varepsilon\|^2$ is small and there is at least a $x_0 \in [1, 2]$ such that K_ε intersects $\{x_0\} \times [-1, 1]$ and $\{-x_0\} \times [-1, 1]$ in a single point and such that (8.3) does not occur. Replace ε by εx_0 and let $R_1 = [-1, 1] \times [-1, 1]$. The set K_ε intersects ∂R_1 at two points $M_1 = (-1, y_1)$ and $M_2 = (1, y_2)$ and divides $\partial R_1 \setminus K_\varepsilon$ in two parts: the upper one where $U_\varepsilon > A$ and the lower one where $U_\varepsilon < -A$. Moreover, for ε small, properties (8.2) are still satisfied in R_1 .

We will now prove that for A large, there are no holes in K_ε , i.e. M_1 and M_2 are in the same connected component of $K_\varepsilon \cap R_1$. Consider a path $\psi : [0, l] \rightarrow R_2$ connecting M_1 to M_2 and such that $\|\nabla \psi\| = 1$ a.e. and such that $\gamma = \mathcal{H}^1(\psi[0, l] \setminus K_\varepsilon)$ is minimal. Assume by contradiction that $\gamma \neq 0$ (there are holes in K_ε). We want to prove that taking $\tilde{K} = K_\varepsilon \cup \psi([0, l])$ instead of K we reduce the energy $\int_{\{R_1 \setminus K_\varepsilon\}} \|\nabla U_\varepsilon\|^2$ by more than γ and contradict the minimality of $(U_\varepsilon, K_\varepsilon)$.

For this purpose we notice that if:

$$\begin{aligned} \phi : [0, a] \rightarrow R_1 \setminus K_\varepsilon \text{ is a one-to-one path such that} \\ \|\nabla \phi\| = 1 \text{ a.e., } \phi(0) \text{ and } \phi(a) \in \partial R_1, \\ U_\varepsilon(\phi(0)) \leq -A \text{ and } U_\varepsilon(\phi(a)) \geq A \end{aligned} \tag{8.4}$$

then $\int_0^a \|\nabla U_\varepsilon(\phi(t))\|^2 dt \geq 4A^2/a$. We will then construct a family of such curves as part of the level surfaces of a distance function.

Assume that $K_\varepsilon \cap R_1$ is made of a finite number of pieces and that the minimum path ψ touches successively the components $F_0 \ni M_1, F_1, \dots, F_n \ni M_2$. We construct the dilation of F_0 . For δ small, $R_1 \cap \partial\{N \in \mathbf{R}^2, d(N, F_0) \leq \delta\}$ does not intersect K_ε and one component of this boundary is a curve ϕ_δ satisfying (8.4), such that its length l_δ is less than $2\mathcal{H}^1(F_0) + \pi\delta$. We proceed until $\phi_\delta([0, l_\delta])$ touches K_ε for $\delta = \delta_0$, then we replace F_0 by the union G_1 of $\{N \in R_1, d(N, F_0) \leq \delta_0\}$ and the connected components of K_ε touched by $\phi_{\delta_0}([0, l_{\delta_0}])$. We proceed as before: for $\delta > \delta_0$, ϕ_δ is the component satisfying (8.4) of the set $R_1 \cap \partial\{N \in \mathbf{R}^2, d(N, G_1) \leq \delta - \delta_0\}$. We proceed until $\phi_\delta([0, l_\delta])$ touches K_ε , etc. We remark that by construction, ϕ_δ intersects $\psi([0, l])$ at a point $\psi(\tau)$ such that $\mathcal{H}^1(\psi([0, \tau]) \setminus K_\varepsilon) = \delta$. And if $\psi(\tau) \notin K_\varepsilon$ then, $\phi_\delta([0, l_\delta])$

cannot intersect any of the F_i , or this would contradict the minimality of ψ . Consequently, the construction can be done for any $\delta \in [0, \gamma]$. For all δ , except a finite number of them, ϕ_δ satisfies condition (8.4) with a length l_δ less than $2\mathcal{H}^1(K_\varepsilon \cap R_1) + \pi\delta$. Moreover, we have $d(\phi_\delta(t), \phi_{\delta'}(t')) \geq |\delta - \delta'|$ for all δ, δ' and $t \in [0, l_\delta], t' \in [0, l_{\delta'}]$. With this construction, the curves ϕ_δ can be viewed as part of the level surfaces of the distance function D where $D(X)$ is the infimum of $\mathcal{H}^1(\xi([0, 1]) \setminus K_\varepsilon)$ with $\xi(0) = M_1$ and $\xi(1) = X$. On each curve ϕ_δ , $\int_{\phi_\delta} \|\nabla U_\varepsilon\|^2$ is larger than $4A^2/(2\mathcal{H}^1(K_\varepsilon \cap R_1) + \pi\gamma)$. The coarea formula [12] gives

$$\int_{R_1 \setminus K_\varepsilon} \|\nabla U_\varepsilon\|^2 \geq \left(\frac{4A^2}{2\mathcal{H}^1(K_\varepsilon \cap R_1) + \pi\gamma} \right) \gamma. \quad (8.5)$$

In general, we take the sequence $(F_n)_{n \in \mathbb{N}}$ of components of $K_\varepsilon \cap R_1$ touched by ψ and $(G_n)_{n \in \mathbb{N}}$ the other components of $K_\varepsilon \cap R_1$. We first remove the G_n . Then we choose a finite number of holes of total length larger than $\gamma/2$ (i.e. a finite number of disjoint intervals (a_k, b_k) of total length larger than $\gamma/2$ in $\psi^{-1}(\psi([0, l] \setminus K_\varepsilon))$ and fill in the other holes with the appropriate pieces of $\psi([0, l])$ (i.e. we take $K' = \cup F_n \cup \psi([0, l] \setminus \cup (a_k, b_k))$). This way, we are reduced, with K' , to the previous case with a finite number of pieces and may construct the curves ϕ_δ for $\delta \in [0, \gamma/2]$. We introduce the one-to-one function $h_0(t, \delta) = \phi_\delta(t)$ for $t \in [0, l_\delta]$. We may introduce successively the G_n and construct a function h_n (h_n is a small perturbation of h_{n-1}). The sequence h_n converges to a function h such that for almost every δ , $t \mapsto h(t, \delta)$ satisfies (8.4), has a length less than $2\mathcal{H}^1(K_\varepsilon \cap R_1) + \gamma + \pi\delta$ and $d(h(t, \delta), h(t', \delta')) \geq |\delta - \delta'|$. Again, the coarea formula gives $\int_{R_1 \setminus K_\varepsilon} \|\nabla U_\varepsilon\|^2 \geq CA^2\gamma$ where $C = 2/(2\mathcal{H}^1(K_\varepsilon \cap R_1) + (\pi + 1)\gamma)$.

Now, if we replace K_ε by $\tilde{K} = K_\varepsilon \cup \psi([0, 1])$, then $\mathcal{H}^1(\tilde{K}) = \mathcal{H}^1(K) + \gamma$ and \tilde{K} cut R_1 in two parts. In the upper part, take $\tilde{U} = \sup(A, U_\varepsilon)$ and in the lower part take $\tilde{U} = \inf(-A, U_\varepsilon)$. For almost every curve $t \mapsto h(t, \delta)$ there is a minimum segment $[\alpha, \beta]$ in which $t \mapsto U_\varepsilon(h(t, \delta))$ goes from $-A$ to A and one gets as above $\int_{\{-A \leq U_\varepsilon \leq A\} \setminus K_\varepsilon} \|\nabla U_\varepsilon\|^2 \geq CA^2\gamma$ which is larger than γ for large A . Therefore, (\tilde{U}, \tilde{K}) is a compact perturbation of $(U_\varepsilon, K_\varepsilon)$ which reduces the energy. This contradicts the minimality of $(U_\varepsilon, K_\varepsilon)$.

Consequently, for ε small, M_1 and M_2 are in the same connected component of K_ε

Step 2. – The blow-up limit at any point near P is a line.

Let P be a point where K has a tangent and has density one. Then, for any $\beta, \beta', \exists r_0$ such that $\forall r \leq r_0$

(a) the set $G_r = \{\rho \in [r/2, r), \#(K \cap \partial B_\rho(P)) = 1\}$ satisfies $meas(G_r) > (1 - \beta)r/2$,

(b) $\forall \rho \in G_r; K \cap \partial B_\rho(P)$ is made of two points of the same connected component of K (consequence of step 1) and of polar coordinates centered at $P, (\rho, \theta_1), (\rho, \theta_2)$ such that $\pi\beta' > |\theta_2 - \theta_1 - \pi|$.

Those two properties hold then for the blow-up $K_\varepsilon, \forall r \leq r_0/\varepsilon$. The convergence of K_ε to the line K_0 gives the existence of r_1 and ε_1 such that $\forall \varepsilon < \varepsilon_1, \forall Q \in K_\varepsilon \cap B_1(O), \forall r, 1 \leq r \leq r_1/\varepsilon_1$ properties (a) and (b) above holds at Q . Assume that there are sequences $\varepsilon_n \rightarrow 0, Q_n \in K_{\varepsilon_n} \cap B_1$ and $r_n, 0 < r_n < 1$ such that (a) and (b) are satisfied for $r, r_n < r < r_1/\varepsilon_n$ but not at $r = r_n$. Consider the blow-up sequence $V_n(X) = \frac{U_{\varepsilon_n}(Q_n + r_n X)}{\sqrt{r_n}}, v_n(X) = V_n(X) - c_n(X)$ (where c_n is piecewise constant as in section 2), $K_n = \{X, Q_n + r_n X \in K_{\varepsilon_n}\}$. A subsequence will converge to a global minimizer (v, K_v) such that $O \in K_v$, identities (a) and (b) (with \geq instead of $>$) are satisfied for $r > 1$ and

$$meas(G_1) \leq (1 - \beta)1/2 \text{ or } \exists \rho \in G_1 \text{ such that } \pi\beta' \leq |\theta_2 - \theta_1 - \pi|. \tag{8.6}$$

For $r > 2$, the computation for the monotonicity formula gives for all $\rho \in G_r$

$$\int_{\partial B_\rho \setminus K_v} \|\nabla v\|^2 \geq \frac{1}{\rho} \frac{2}{(1 - \beta)} \int_{B_\rho \setminus K_v} \|\nabla v\|^2 \tag{8.7}$$

and then

$$\int_{B_{r/2} \setminus K_v} \|\nabla v\|^2 \leq 2^{-2+\tau} \int_{B_r \setminus K_v} \|\nabla v\|^2, \tag{8.8}$$

where $\tau(\beta, \beta') \rightarrow 0$ when β, β' goes to 0. Assume that β, β' were chosen such that $\tau < 1$. Then, since $\int_{B_r \setminus K_v} \|\nabla v\|^2 \leq 2\pi r$ one gets for all $r \geq 1$, and $n \geq 1, \int_{B_r \setminus K_v} \|\nabla v\|^2 \leq 2^{n(-2+\tau)} 2\pi r 2^n = 2\pi r 2^{n(-1+\tau)}$. Thus, $\int_{B_r \setminus K_v} \|\nabla v\|^2 = 0$ for all r and the global minimizer (v, K_v) is of type (i), (ii) or (iii) of theorem 4.1. Only (ii) satisfies (a) and (b) above. Therefore K_v is a line, which contradicts (8.6). Consequently, (a) and (b) holds for all $r < r_1$ at any point $Q \in K \cap B_{\varepsilon_1}$. The blow-up at Q is then a line.

Step 3. – Conclusion.

We have proved that there is a ball B_σ such that $\partial B_\sigma \cap K = \{N_0, N_1\}$ and there is a Jordan arc $\psi : [0, l] \rightarrow \bar{B}_\sigma \cap K$ with endpoints N_0 and N_1 . We know also that for any $Q \in B_\sigma \cap K$ the blow-up limit is a line. As in section 5 we conclude then that $\bar{B}_\sigma \cap K = \psi[0, l]$ which completes the proof of the proposition.

9. COLOR SEGMENTATION

In the previous sections we have concentrate on black-and-white images. Mumford-Shah segmentation procedure can also be applied to color images. Following [10] we say that each point of a color image is characterized by its red, green and blue intensity. The true image is then the vector function $g : \Omega \rightarrow \mathbf{R}^3$ of red, green and blue intensity. We may assume that g (and therefore the regularized image u) takes its values in the cube $[0, 1]^3$ on which we define a Riemannian metric:

$$(ds)^2 = \sum_{i,j} \phi_{ij}(u) du^i du^j \quad (9.1)$$

where ϕ_{ij} are continuous functions. We assume that there are two positive constants, λ and μ such that

$$\forall u_0 \in [0, 1]^3, \quad \forall x \in \mathbf{R}^3, \quad \lambda \|x\|^2 \leq \sum_{i,j} \phi_{ij}(u_0) x^i x^j \leq \mu \|x\|^2. \quad (9.2)$$

Mumford-Shah functional is then replaced by

$$E(u, K) = \int_{\Omega \setminus K} d(u(x), g(x)) + \int_{\Omega \setminus K} \sum_{i,j} \phi_{ij}(u) \nabla u^i \cdot \nabla u^j + \mathcal{H}^1(K). \quad (9.3)$$

The distance $d(u(x), g(x))$ is classically defined as the infimum

$$d(u(x), g(x)) = \inf_v \left(\int_0^1 \sqrt{\sum_{i,j} \phi_{ij} \frac{dv^i}{dt} \frac{dv^j}{dt}} dt \right) \quad (9.4)$$

over paths v connecting $u(x)$ to $g(x)$ (a path is a smooth function $v : [0, 1] \rightarrow [0, 1]^3$ such that $v(0) = u(x)$ and $v(1) = g(x)$).

We may study, as in section 2, blow-up sequences $(u_\varepsilon, K_\varepsilon)$. Condition (9.2) ensures the convergence of a blow-up sequence to a blow-up limit (u_0, K_0) . Since ϕ_{ij} is continuous, we may choose a subsequence such that in each connected component of $\mathbf{R}^2 \setminus K_0$, $\phi_{ij}(u_\varepsilon)$ converges locally uniformly to a constant. Since K_0 is a closed set there is a countable family of connected component C^n . For each C^n we denote by a_{ij}^n the limit value of $\phi_{ij}(u_\varepsilon)$ in C^n . The argument of the proof of theorem 2.2 gives then

PROPOSITION 9.1. – *For any bounded open sub-domains U, V of \mathbf{R}^2 with $U \subset\subset V$ there is no function \tilde{u}_0 and closed edge set \tilde{K}_0 defined on V*

such that

- (i) \tilde{u}_0 and u_0 (resp. \tilde{K}_0 and K_0) coincide in $V \setminus \bar{U}$,
- (ii) for every pair of points X, Y in $V \setminus (\bar{U} \cup K_0)$, if X and Y are in two distinct connected components of $V \setminus K_0$ so they are in $V \setminus \tilde{K}$,
- (iii) $(\tilde{u}_0, \tilde{K}_0)$ lowers the energy

$$\begin{aligned} & \sum_m \int_{\tilde{C}^m} \sum_{i,j} \tilde{a}_{ij}^m \nabla \tilde{u}_0^i \cdot \nabla \tilde{u}_0^j + \mathcal{H}^1(\tilde{K} \cap V) \\ & \leq \sum_n \int_{C^n \cap V} \sum_{i,j} a_{ij}^n \nabla u_0^i \cdot \nabla u_0^j + \mathcal{H}^1(K \cap V), \end{aligned} \tag{9.5}$$

where \tilde{C}^m denotes the connected component of $V \setminus \tilde{K}_0$ \tilde{a}_{ij}^m is defined as

$\tilde{a}_{ij}^m = \tilde{a}_{ij}^n$ if $\tilde{C}^m \cap C^n \cap (V \setminus U) \neq \emptyset$ (there is at most one n satisfying this condition),

$\tilde{a}_{ij}^m = \delta_{ij}$ if $\forall n, \tilde{C}^m \cap C^n \cap (V \setminus U) = \emptyset$.

Remark 9.2. – If $\forall n, \tilde{C}^m \cap C^n \cap (V \setminus U) = \emptyset$ then $\tilde{C}^m \subset U$ is bounded and one can lower the energy by taking \tilde{u} constant in \tilde{C}^m whatever the definition of the definite positive matrix \tilde{a}_{ij}^m .

We will now concentrate on each connected component \tilde{C}^n . There is an orthonormal basis of \mathbf{R}^3 in which matrix a_{ij}^n is diagonal. We denote by b_i^n the element of the diagonal in this basis and by (U_0^1, U_0^2, U_0^3) the components of u_0 in the same basis. The energy of (u, K) in a ball B_R can be written as

$$\int_{C^n \cap B_R} \sum_{i,j} a_{ij}^n \nabla u_0^i \cdot \nabla u_0^j = \int_{C^n \cap B_R} \sum_i b_i^n \|\nabla U^i\|^2. \tag{9.6}$$

Obviously, since (u_0, K_0) is a global-minimizer, U_0^i is harmonic and satisfies Neumann boundary conditions on ∂C^n . Therefore, if ∂C^n is connected, theorem 3.1 and remark 3.2 implies that the functions

$$r \mapsto \frac{\int_{C^n \cap B_r} \|\nabla U^i\|^2}{r} \tag{9.7}$$

are nondecreasing functions of r . Therefore the analysis of section 4 gives the

THEOREM 9.3. – *If (u_0, K_0) is a global-minimizer of E such that K_0 is connected then (u_0, K_0) is one of the following:*

- (i) K_0 is empty and u_0 is constant.

(ii) K_0 is a straight line defining two half-planes and u_0 is constant on each half-plane.

(iii) K_0 is the union of three half lines with angle $2\pi/3$ and u_0 is constant on each sector.

(iv) in a polar set of coordinates and in the appropriate basis of \mathbb{R}^3

$$U^i(r, \theta) = \alpha_i \sqrt{\frac{2}{\pi}} \sqrt{r} \cos \frac{\theta}{2} \quad \text{for } \theta \in [0, 2\pi)$$

and K is the half axis $\theta = 0$. The constants α_i are such that $\sum_i b_i(\alpha_i)^2 = 1$.

Consequently, it is easy to check that the proofs presented in sections 5 to 8 for a black-and-white image holds for the case of a color image. Theorems 1.1 to 1.5 are true for (u, K) minimizer of the energy E defined in equation (9.3)-(9.1) and under condition (9.2).

10. APPENDIX

Proof of proposition. 7.1. – Let w be the solution of $\Delta w = u - g$ in B_{4R}^+ with Dirichlet conditions $w = 0$ on ∂B_{4R}^+ . Since u and g are in L^∞ and K is $C^{1,\alpha}$ then theorem 5.5.5' of [18] gives that w is in $C^{1,\alpha}(B_{2R}^+)$.

We would like now to prove that $v = u - w$ is also in $C^{1,\alpha}(B_{2R}^+)$. By construction v is harmonic, it has an adjoint V such that $v + iV$ is holomorphic. We may assume that $V(P) = 0$. If γ is an oriented regular curve connecting two points A and B in the interior of B_{2R}^+ and if ν denotes the normal to γ then

$$V(B) - V(A) = \int_\gamma \frac{\partial v}{\partial \nu} = \int_\gamma \frac{\partial u}{\partial \nu} - \frac{\partial w}{\partial \nu} \tag{10.1}$$

For any points M of $K \cap B_{2R}$ let \mathcal{C}_M be the portion of K connecting P and M . Since w is in $C^{1,\alpha}(B_{2R}^+)$ we may introduce a function:

$$\chi(M) = \int_{\mathcal{C}_M} \frac{\partial w}{\partial \nu} \tag{10.2}$$

For any point (x, y) of $B_{2R}(P)$ we may define χ by

$$\chi(x, y) = \chi(x, f(x)). \tag{10.3}$$

The function χ is then $C^{1,\alpha}$ in $B_{2R}(P)$.

Assume that A and B are two points of K . The curve γ connecting A to B can be chosen arbitrary close to K . Lemma 3.1 of [6] and equation (10.1) give then

$$V(B) - V(A) = \chi(B) - \chi(A). \tag{10.4}$$

Since we assumed $V(P) = 0$, we have for all $M \in K \cap B_{2R}$,

$$V(M) = \chi(M) \tag{10.5}$$

Since χ is $C^{1,\alpha}$, theorem 5.5.5' of [18] give now that $V \in C^{1,\alpha}(B_R^+)$. The adjoint v of V is in the same space and we have proved that $u \in C^{1,\alpha}(B_R^+)$ which completes the proof of proposition 7.1.

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