ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 12, nº 5 (1995), p. 507-575
<http://www.numdam.org/item?id=AIHPC_1995__12_5_507_0>

Ann. Inst. Henri Poincaré,

Vol. 12, n° 5, 1995, p. 507-575 *Analyse non linéaire*

Nonlinear oblique boundary value problems for Hessian equations in two dimensions

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ABSTRACT. – We study nonlinear oblique boundary value problems for nonuniformly elliptic Hessian equations in two dimensions. These are equations whose principal part is given by a suitable symmetric function of the eigenvalues of the Hessian matrix D^2u of the solution u. An interesting feature of our second derivative estimates is the need for certain strong structural hypotheses on the boundary condition, which are not needed in the uniformly elliptic case. Restrictions of this type are natural in our context; we present examples showing that second derivative bounds may fail if we do not assume such conditions.

1. INTRODUCTION

In this paper we shall study nonlinear oblique boundary value problems of the form

$$F(D^2u) = g(x, u, Du) \quad \text{in} \quad \Omega, \tag{1.1}$$

$$b(x, u, Du) = 0 \quad \text{on} \quad \partial\Omega \tag{1.2}$$

A.M.S. Classification: Primary 35 J 25, 35 J 60, 35 J 65, Secondary 53 C 45

on bounded uniformly convex domains $\Omega \subset \mathbb{R}^2$. The function F is of a special type, as in [2]. It is given by

$$F(D^2u) = f(\lambda(D^2u)) \tag{1.3}$$

where $\lambda(D^2u)$ denotes the eigenvalues of the Hessian D^2u of u and f is a suitable symmetric function defined on an open, convex, symmetric (under interchange of λ_1 and λ_2) region $\Sigma \subset \mathbb{R}^2$. Σ is assumed to be closed under the addition of elements of the positive cone $\Gamma_+ = \{\lambda \in \mathbb{R}^2 : \lambda_1, \lambda_2 > 0\}$, *i. e.*, if $\lambda \in \Sigma, \tilde{\lambda} \in \Gamma_+$, then $\lambda + \tilde{\lambda} \in \Sigma$. Clearly then we are interested in solutions $u \in C^2(\Omega)$ of (1.1) such that at each point of Ω $\lambda(D^2u)$ belongs to Σ. We shall refer to such u as Σ-*admissible*, or briefly just *admissible*.

We assume that $f \in C^{0,1}(\Sigma) \cap C^0(\bar{\Sigma})$ is a positive function such that

$$f_i = \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{on} \quad \Sigma \quad \text{for} \quad i = 1, 2, \tag{1.4}$$

and for any compact set $K \subset \Sigma$ there is a positive constant $C(K)$ such that

$$\max\{f_1, f_2\} \leq C(K) \min\{f_1, f_2\} \quad \text{on} \quad K. \tag{1.5}$$

This follows automatically from (1.4) if $f \in C^1(\Sigma)$. In addition we assume

$$f \text{ is concave} \tag{1.6}$$

and

$$f \equiv 0 \quad \text{on} \quad \partial\Sigma. \tag{1.7}$$

Since f is generally only Lipschitz continuous, f_i exists only almost everywhere on Σ, and (1.4) is to be interpreted in this sense. We adopt this convention throughout the paper.

Conditions (1.4) and (1.5) imply that (1.1) is elliptic on admissible solutions, while (1.4) and (1.6) imply that F is a concave function on $M(\Sigma)$, the set of 2×2 real symmetric matrices whose eigenvalues belong to Σ (*see* [2]). It is easy to verify that $F_{ij} = \dfrac{\partial F}{\partial r_{ij}}(D^2u)$ is diagonal if D^2u is diagonal, and that the eigenvalues of $[F_{ij}]$ are f_1 and f_2.

We also assume that f satisfies the following structure conditions:

$$\sum f_i(\lambda)\,\lambda_i \geq 0 \quad \text{on} \quad \Sigma \tag{1.8}$$

and

$$\sum f_i(\lambda) \geq \sigma_0 \quad \text{on} \quad \{\lambda \in \Sigma : f(\lambda) \leq \mu\} \tag{1.9}$$

for any $\mu > 0$ and some positive constant $\sigma_0 = \sigma_0(f, \mu)$. It follows from (1.4) and (1.8) that Σ does not contain the origin and, also using (1.7), that $\partial\Sigma$ is asymptotic to $(\alpha, \alpha) + \partial\Gamma$ for some number $\alpha \geq 0$ and some open, convex, symmetric cone Γ with vertex at the origin and containing Γ_+. We shall assume without loss of generality that $\alpha = 0$. If $\alpha \neq 0$ we can replace Σ by $\tilde{\Sigma} = -(\alpha, \alpha) + \Sigma$ and $f(\lambda)$ by $\tilde{f}(\lambda) = f((\alpha, \alpha) + \lambda)$. If $\tilde{u}$ is a $\tilde{\Sigma}$-admissible solution of

$$\tilde{F}(D^2\tilde{u}) = \tilde{g}(x, \tilde{u}, D\tilde{u}) = g\left(x, \tilde{u} + \frac{1}{2}\alpha|x|^2, D\tilde{u} + \alpha x\right) \quad \text{in} \quad \Omega,$$

$$\tilde{b}(x, \tilde{u}, D\tilde{u}) = b\left(x, \tilde{u} + \frac{1}{2}\alpha|x|^2, D\tilde{u} + \alpha x\right) = 0 \quad \text{on} \quad \partial\Omega,$$

then $u = \tilde{u} + \frac{1}{2}\alpha|x|^2$ is clearly a Σ-admissible solution of (1.1), (1.2). It will be clear that $\tilde{g}, \tilde{b}$ satisfy similar regularity and structural hypotheses as g, b, with possibly new, but controlled, structural constants.

Since we are assuming $\alpha = 0$ and we are in two dimensions, only two types of Σ can arise: either

(i) $\quad \partial\Sigma$ is asymptotic to $\partial\Gamma_+$,

in which case we say Σ is of *type* 1, or

(ii) $\quad \partial\Sigma$ is asymptotic to $\partial\Gamma$ and $\Gamma \supsetneq \Gamma_+$,

in which case we say Σ is of *type* 2. We shall deal almost exclusively with type 1 regions in this paper, and henceforth we assume Σ is of type 1 unless otherwise stated. In case (ii) (1.1) is uniformly elliptic, so existence results for a large class of oblique boundary conditions follow from the work of Lieberman and Trudinger [8]. To prove uniform ellipticity for type 2 regions, let $\lambda \in \Sigma$ be a point at which $Df(\lambda)$ exists and let $w(\mu) = f(\lambda) + \sum f_i(\lambda)(\mu_i - \lambda_i)$ for any $\mu \in \mathbb{R}^2$. Then $w \geq f$ in Σ, so $L = \{\mu : w(\mu) = 0\}$ lies outside Σ. Consequently, L can be translated to give a parallel supporting line $\tilde{L}$ to $\partial\Sigma$. Since $Df(\lambda)$ is normal to $\tilde{L}$, we see that the ratio $f_1(\lambda)/f_2(\lambda)$ is bounded between two positive constants in case (ii). Thus (1.1) is uniformly elliptic in this case, and in addition, also strictly elliptic wherever $g(x, u, Du)$ is bounded, by virtue of (1.9).

In contrast, by similar reasoning we see that for type 1 regions, (1.1) is necessarily quite strongly nonuniformly elliptic. Consequently, the arguments used to obtain *a priori* estimates are somewhat different from those used in the uniformly elliptic case. Furthermore, the class of allowable boundary conditions is different in the two cases. For type 1 regions we shall allow the semilinear Neumann boundary condition

$$D_\nu u + \phi(x, u) = 0 \quad \text{on} \quad \partial\Omega \tag{1.10}$$

for suitable ϕ, where ν denotes the inner unit normal to $\partial\Omega$, but not the more general oblique boundary condition

$$(1.11) \qquad D_\beta u + \phi(x,u) = 0 \quad \text{on} \quad \partial\Omega\,,$$

unless the vector field β satisfies a certain structure condition (*see* (1.19)) which is not required in the uniformly elliptic case. Similarly, one of the structure conditions (*see* (1.27)) we shall require for fully nonlinear boundary conditions excludes the capillarity boundary condition

$$(1.12) \qquad D_\nu u + \theta(x,u)\sqrt{1+|Du|^2} = 0 \quad \text{on} \quad \partial\Omega\,,$$

unless θ is negative, a condition which is impossible to satisfy in our setting. The reasons for excluding (1.11) and (1.12) are not merely technical; for type 1 regions second derivative estimates may fail to hold if we do not make these strong assumptions on the boundary condition. We shall explain a little later why these hypotheses are natural for the problems we are considering.

Let us now proceed to our hypotheses on g and b. We assume that Ω is a $C^{2,1}$ uniformly convex domain in $\mathbb{R}^2$ and $g \in C^{1,1}(\bar\Omega \times \mathbb{R} \times \mathbb{R}^2)$ is a positive function satisfying

$$(1.13) \qquad g_z \geq 0 \quad \text{on} \quad \Omega \times \mathbb{R} \times \mathbb{R}^2\,.$$

We remark here that the positivity of f and g is convenient but inessential; it would be sufficient to assume

$$(1.14) \qquad f < \inf_{\Omega\times\mathbb{R}\times\mathbb{R}^2} g \quad \text{on} \quad \partial\Sigma\,.$$

It is convenient to consider the semilinear boundary condition (1.11) and the fully nonlinear case separately. For (1.11) we assume that $\phi \in C^{1,1}(\partial\Omega\times\mathbb{R})$ satisfies

$$(1.15) \qquad \phi_z < 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}\,,$$

$$(1.16) \qquad \phi(x,z) < 0 \quad \text{for all} \quad x \in \partial\Omega \quad \text{and all} \quad z \geq N$$

for some constant N, and

$$(1.17) \qquad \phi(x,z) \to \infty \quad \text{as} \quad z \to -\infty$$

uniformly for $x \in \partial\Omega$. We also assume that $\beta \in C^{1,1}(\partial\Omega; \mathbb{R}^2)$ is a unit vector field on $\partial\Omega$ with

$$\beta \cdot \nu > 0 \quad \text{on} \quad \partial\Omega \tag{1.18}$$

and

$$\left[-2\left(1+\left(\frac{\beta\cdot\tau}{\beta\cdot\nu}\right)^2\right)\delta_i\beta_j(x) - \phi_z(x,z)\,\delta_{ij}\right]\tau_i\tau_j > 0 \tag{1.19}$$

for all $(x,z) \in \partial\Omega \times \mathbb{R}$, where τ is a unit tangent vector to $\partial\Omega$ at x and $\delta = (\delta_1, \delta_2)$ denotes the tangential gradient operator relative to $\partial\Omega$ given by

$$\delta_i = (\delta_{ij} - \nu_i\nu_j)D_j\,.$$

Notice that (1.19) is automatically satisfied if $\beta \equiv \nu$, or more generally if β is a vector field with constant normal and tangential components; this follows easily from (1.15) and the uniform convexity of Ω.

For the case $g = g(x,u)$ we then have the following result.

THEOREM 1.1. – *Under the above hypotheses on $\Sigma, f, \Omega, g, \phi$ and β, the boundary value problem*

$$\begin{aligned} F(D^2u) &= g(x,u) \quad \textit{in} \quad \Omega\,, \\ D_\beta u + \phi(x,u) &= 0 \quad \textit{on} \quad \partial\Omega \end{aligned} \tag{1.20}$$

has a unique admissible solution u belonging to $C^{2,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$.

If g depends on Du we need to strengthen our hypotheses on f. We assume either

$$\mathcal{T}|\lambda|^{-1} \to \infty \quad \text{as} \quad |\lambda| \to \infty \quad \text{on} \quad \{\lambda \in \Sigma : f(\lambda) \le \mu\} \tag{1.21}$$

for any $\mu > 0$, or in the case that g is convex with respect to Du, the weaker condition

$$\mathcal{T} \to \infty \quad \text{as} \quad |\lambda| \to \infty \quad \text{on} \quad \{\lambda \in \Sigma : f(\lambda) \le \mu\} \tag{1.21$'$}$$

for any $\mu > 0$. Here and below $\mathcal{T} = f_1 + f_2$. We then have the following result for the case $g = g(x,u,Du)$.

THEOREM 1.2. – *Assume the above hypotheses on $\Sigma, f, \Omega, g, \phi$ and β, including* (1.21) *or* (1.21)$'$ *in the case that g is convex with respect to Du.*

Assume also that there is an admissible subsolution $\underline{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *of* (1.1). *Then the boundary value problem* (1.1), (1.11) *has a unique admissible solution* u *belonging to* $C^{2,\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0,1)$.

Analogues of Theorems 1.1 and 1.2 are valid in the uniformly elliptic case (ii) without a condition like (1.19) and we do not require Ω to be convex; however, if g depends on Du we need some restrictions on the growth of g with respect to Du (*see* [8]).

In our paper [16] on the oblique boundary condition (1.11) for two dimensional Monge-Ampère equations we assumed in place of (1.19) the condition

$$[-2\delta_i\beta_j(x) - \phi_z(x,z)\,\delta_{ij}]\,\tau_i\tau_j > 0 \tag{1.19'}$$

for all $(x,z) \in \partial\Omega \times \mathbb{R}$, where τ is as in (1.19). Unfortunately, the proof of the second derivative bound given in [16] is not completely correct; we need (1.19) rather than (1.19)′ to obtain the estimate (2.34) in [16]. Alternatively, (1.19)′ suffices if we assume in addition that $\|\beta - \nu\|_{C^1(\partial\Omega)}$ is sufficiently small.

We now consider fully nonlinear boundary conditions. We assume the strict obliqueness condition

$$\chi(x,z,p) = b_p(x,z,p) \cdot \nu(x) > 0 \tag{1.22}$$

for all $(x,z,p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^2$. It follows then that (1.2) can be written in the form

$$D_\nu u + \phi(x,u,\delta u) = 0 \quad \text{on} \quad \partial\Omega\,. \tag{1.23}$$

We assume furthermore that $\phi \in C^{1,1}(\partial\Omega \times \mathbb{R} \times \mathbb{R}^2)$ satisfies the conditions

$$\phi_z < 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R} \times \mathbb{R}^2\,, \tag{1.24}$$

$$\phi(x,z,0) < 0 \quad \text{for all} \quad x \in \partial\Omega \quad \text{and all} \quad z \geq N \tag{1.25}$$

for some constant N, and

$$\phi(x,z,p^T) \to \infty \quad \text{as} \quad z \to -\infty \tag{1.26}$$

uniformly for (x,p) lying in any compact subset of $\partial\Omega \times \mathbb{R}^2$, where $p^T = p - (p \cdot \nu(x))\,\nu(x)$. We also assume that ϕ satisfies the concavity condition

$$\phi_{p_ip_j}(x,z,p^T)\,\tau_i\tau_j < 0 \tag{1.27}$$

for all $(x,z,p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^2$ where τ is a unit tangent vector to $\partial\Omega$ at x.

We shall prove the following two results, for the cases $g = g(x, u)$ and $g = g(x, u, Du)$ respectively.

THEOREM 1.3. – *Let* Σ, f, Ω *and* g *satisfy the hypotheses of Theorem* 1.1 *and let* ϕ *satisfy the hypotheses above. Then the boundary value problem*

$$(1.28)\qquad \begin{aligned} F(D^2u) &= g(x,u) \quad \text{in } \Omega\,,\\ D_\nu u + \phi(x,u,\delta u) &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has a unique admissible solution u *belonging to* $C^{2,\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0,1)$.

THEOREM 1.4. – *Let* Σ, f, Ω *and* g *satisfy the hypotheses of Theorem* 1.2, *including either* (1.21), *or* (1.21)′ *in the case that* g *is convex with respect to* Du, *and let* ϕ *satisfy the hypotheses of Theorem* 1.3. *Assume also that there exists an admissible subsolution* $\underline{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *of* (1.1). *Then the boundary value problem* (1.1), (1.23) *has a unique admissible solution* u *belonging to* $C^{2,\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0,1)$.

Once the basic existence theorems above have been proved, it is also possible to obtain existence results in cases where the monotonicity assumptions (1.13) and (1.15) (respectively (1.24)) on g and ϕ are dropped. In these situations we do not have uniqueness in general. We now require the existence of an admissible subsolution $\underline{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ of the boundary value problem in question, not just of the differential equation as in Theorems 1.2 and 1.4. With this assumption condition (1.17) (respectively (1.26)) becomes redundant.

THEOREM 1.5. – *Suppose in each of Theorems* 1.1 *to* 1.4 *the hypotheses are modified as above and in addition to assuming* (1.21) *(respectively* (1.21)′*) if* g *depends on* Du *(respectively if* g *depends in a convex fashion on* Du*), we also assume* (1.21)′ if $g = g(x,u)$ *and* (1.13) *is not satisfied. Then each of the boundary value problems considered in Theorems* 1.1 *to* 1.4 *has an admissible solution* u *belonging to* $C^{2,\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0,1)$ *and satisfying* $u \geq \underline{u}$ *in* Ω.

In the semilinear case it suffices to assume $\partial\Omega \in C^{2,\alpha}$ for some $\alpha > 0$, but in the most important case where $\beta \equiv \nu$ we automatically have $\partial\Omega \in C^{2,1}$ since $\beta \in C^{1,1}(\partial\Omega; \mathbb{R}^2)$. Higher regularity of the solutions obtained in the above theorems follows from elliptic regularity theory [5], Theorem 6.30 and Lemma 17.16, in accordance with the regularity of the data. In particular, if f, g, β, ϕ and $\partial\Omega$ are C^∞, then the solution u belongs to $C^\infty(\bar{\Omega})$.

Conditions (1.19) and (1.27) are used in the second derivative estimation and, despite first appearances, seem to be natural for the problems we are considering. To see why, let us consider the semilinear boundary condition (1.11). As we shall see later, it is relatively simple to estimate $D_{\tau\beta}u$ and $D_{\beta\beta}u$ on $\partial\Omega$, where τ is a unit tangent vector field on $\partial\Omega$. In the uniformly elliptic case we can then solve (1.1) for $D_{\tau\tau}u$ to obtain a bound for $D_{\tau\tau}u$ on $\partial\Omega$, but this is not possible in our situation. We need to use the boundary condition to estimate $D_{\tau\tau}u$. After some computation we find that

$$D_{\tau\tau\beta}u - aD_{\tau\tau}u + b = 0 \quad \text{on} \quad \partial\Omega \tag{1.29}$$

where b is a bounded function and a is equal to the left-hand side of (1.19). If a is positive, which is equivalent to (1.19), an upper bound for $D_{\tau\tau}u$ follows at each boundary point where $D_{\tau\tau\beta}u \leq C$. This inequality can in fact be established at a suitable boundary point, leading to a full second derivative bound.

For the fully nonlinear boundary condition (1.23) we can show that

$$D_{\tau\tau\beta}u - a(D_{\tau\tau}u)^2 \geq -b(1 + |D^2u|) \quad \text{on} \quad \partial\Omega \tag{1.30}$$

for a suitable oblique vector field β, where b is a bounded function and now

$$a = -\phi_{p_ip_j}(x, u, \delta u)\, \tau_i\tau_j\,. \tag{1.31}$$

If $D_{\tau\tau\beta}u \leq C$ at some boundary point, we obtain a bound of the form

$$(D_{\tau\tau}u)^2 \leq C(1 + \sup_{\Omega} |D^2u|) \tag{1.32}$$

at that point provided a is positive, which is equivalent to (1.27).

In Section 6 we shall present examples showing that tangential second derivatives of the solution can become unbounded at points of $\partial\Omega$ where a goes to zero, both for semilinear and fully nonlinear boundary conditions. We do not have an example in two dimensions with D^2u unbounded and a negative everywhere on $\partial\Omega$. However, in higher dimensions we have an example involving a linear oblique boundary condition for which the quantity corresponding to a is negative. This suggests that $a > 0$, rather than $a \neq 0$, is the correct hypothesis on the boundary condition in the two dimensional case, at least for a semilinear boundary condition. It seems likely that this is also true for fully nonlinear boundary conditions of the form (1.23). It would be interesting to resolve this question in view of the fact that for the capillarity boundary condition (1.12) (rewritten in the form (1.23)) $a < 0$ if θ is positive, which is a natural assumption in our context.

It will be evident from the proofs that some slightly more general boundary conditions could be handled by the same arguments. For example, (1.11) could be replaced by

$$D_\beta u + \phi(x, u) + \epsilon\psi(x, u, Du) = 0 \quad \text{on} \quad \partial\Omega \tag{1.33}$$

for $\epsilon > 0$ sufficiently small, provided ϕ and β satisfy the hypotheses of Theorem 1.1 and $\psi \in C^{1,1}(\partial\Omega \times \mathbb{R} \times \mathbb{R}^2)$ satisfies

$$\psi_z(x, z, p) \leq 0, \tag{1.34}$$

$$\psi_p(x, z, p) \cdot \nu(x) \geq -C \tag{1.35}$$

and

$$\psi_{p_i p_j}(x, z, p)\, \xi_i \xi_j \leq 0 \tag{1.36}$$

for some positive constant C and all $(x, z, p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^2$ and $\xi \in \mathbb{R}^2$. We could also add $\psi(x, u, Du)$ to the boundary condition (1.23) if ψ satisfies (1.34), (1.35) and (1.36) with $C < 1$. We also note that conditions (1.21), (1.21)′ in Theorems 1.2, 1.4 and 1.5 could be weakened by imposing appropriate conditions on g instead.

For the special case of the Monge-Ampère equation

$$(\det D^2 u)^{1/2} = g(x, u, Du), \tag{1.37}$$

for which we take $f(\lambda) = (\lambda_1 \lambda_2)^{1/2}$, condition (1.21) is not satisfied but (1.21)′ is. However, the special structure of the determinant function allows us to prove the analogue of Theorems 1.2 and 1.4 for this case without any convexity conditions on g. Furthermore, in this case the existence of an admissible subsolution in Theorems 1.2 and 1.4 can be replaced by suitable structure conditions on g. This is also true for more general equations. We shall discuss these points further in Section 5.

In [4] Delanoë studied the boundary value problem

$$\begin{aligned} \det D^2 u &= g(x, u, Du) \quad \text{in} \quad \Omega, \\ Du(\Omega) &= \Omega^* \end{aligned} \tag{1.38}$$

where Ω^* is a uniformly convex domain in $\mathbb{R}^2$ (see also Pogorelov [11] for generalized and locally smooth solutions of this problem). The boundary condition can be reformulated in a more conventional way as

$$h(Du) = 0 \quad \text{on} \quad \partial\Omega \tag{1.39}$$

where h is a uniformly concave defining function for Ω^*, *i. e.*, $\Omega^* = \{p \in \mathbb{R}^2 : h(p) > 0\}$ and $Dh \neq 0$ on $\partial\Omega^*$. It is clear that if u is a convex solution of (1.38), then $H = h(Du)$ is positive in Ω and zero on $\partial\Omega$, and it follows that (1.39) is a degenerate oblique boundary condition on convex solutions. It is not immediately clear, however, that we have an *a priori* strict obliqueness estimate

$$(1.40) \qquad h_{p_i}(Du)\nu_i \geq c_0 > 0 \quad \text{on} \quad \partial\Omega\,,$$

so this type of boundary condition is not necessarily expressible in a form suitable for the application of Theorems 1.3 and 1.4. Nevertheless, we are able to treat this problem for a general class of Hessian equations. Our hypotheses on f and g are now somewhat different. We assume that f satisfies the conditions of Theorem 1.1 and in addition

$$(1.41) \qquad \lambda_1\lambda_2 \leq G(f(\lambda)) \quad \text{for} \quad \lambda \in \Sigma$$

for some real valued, continuous, increasing function G on $[0, \infty)$ with $G(0) = 0$. It follows that $\Sigma = \Gamma_+$. Further, we assume that $h \in C^{2,1}(\mathbb{R}^2)$ is a uniformly concave defining function for some $C^{2,1}$ uniformly convex domain $\Omega^* \subset \mathbb{R}^2$. Concerning g we assume that $g \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^2)$ is a positive function satisfying

$$(1.42) \qquad g(x,z,p) \to \infty \quad \text{as} \quad z \to \infty,$$

$$(1.43) \qquad g(x,z,p) \to 0 \quad \text{as} \quad z \to -\infty,$$

uniformly for all $(x,p) \in \bar{\Omega} \times \bar{\Omega}^*$. We then have the following result.

THEOREM 1.6. – *Let Σ, f, g, h, Ω and Ω^* satisfy the above hypotheses and in addition* (1.21) *in the case that g depends on Du ((1.21)′ suffices if g is convex with respect to Du). Then the boundary value problem*

$$(1.44) \qquad \begin{aligned} F(D^2u) &= g(x,u,Du) \quad \textit{in } \Omega\,, \\ h(Du) &= 0 \quad \textit{on} \quad \partial\Omega \end{aligned}$$

has a convex solution u belonging to $C^{2,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$. If in addition

$$(1.45) \qquad g_z > 0 \quad \text{on} \quad \Omega \times \mathbb{R} \times \Omega^*\,,$$

the solution is unique.

Our proof of Theorem 1.6 differs from that of Delanoë [4] for the Monge-Ampère case in that we establish directly the strict obliqueness estimate (1.40) and thereby reduce the second derivative estimation to the technique used in Theorems 1.3 and 1.4. Delanoë's method exploits the special structure of the determinant function, and it does not appear to extend to more general Hessian equations without making strong structural hypotheses on f; essentially one requires conditions of the type satisfied by the Monge-Ampère equation in two dimensions. As before, no convexity of g with respect to Du is required for the Monge-Ampère equation, even though in this case (1.21) is not satisfied. Higher regularity of the solution u follows as before from elliptic regularity theory if we have more regular data.

Apart from the papers [4], [10], [16] which deal exclusively with Monge-Ampère equations, the only result of which we are aware for oblique boundary value problems for nonuniformly elliptic Hessian equations is the paper [13] of Trudinger. The results there are very restrictive in that only the linear Neumann problem on balls is considered, although this is done in all dimensions. In future work we hope to extend our results here to Hessian equations in higher dimensions and to curvature equations. So far we have only partial results in these directions.

The rest of the paper is set out as follows. In Section 2 we prove some technical inequalities which we need to prove Theorems 1.1 to 1.6. In Section 3 we explain how to prove existence for (1.1), (1.2) using the continuity method, and we prove solution and gradient estimates. We also give some sufficient conditions for the existence of admissible subsolutions, as required in Theorems 1.2, 1.4 and 1.5. In Section 4 we prove second derivative estimates and complete the proofs of Theorems 1.1 to 1.6. This is the central part of the paper. The main ideas for second derivative bounds come from our earlier work [16] on Monge-Ampère equations, but even for this special case the results on fully nonlinear boundary conditions of the form (1.23) are new. In Section 5 we discuss extensions of our results to the degenerate situation where conditions (1.4), (1.5) and the positivity of g are weakened, and also the case that condition (1.8) is dropped and $\partial\Sigma$ is no longer assumed to be asymptotic to a cone. Finally, in Section 6 we discuss the necessity of conditions such as (1.19) and (1.27), and of conditions (1.9), (1.21) and (1.21)$'$.

2. PRELIMINARY LEMMAS

In this section we shall prove a number of technical inequalities which will be used in later sections of the paper. Unless otherwise stated, we assume f satisfies conditions (1.4) to (1.9) and Σ is of type 1, but it will be clear from the proofs that we do not need (1.4) and (1.5), and we could assume instead the weaker condition

$$(2.1)\qquad f_i \ge 0 \quad \text{on} \quad \Sigma \quad \text{for} \quad i = 1, 2\,.$$

In fact, this follows automatically from the concavity and positivity of f, and the fact that Σ is of type 1. We write f_i in place of $f_i(\lambda)$ and denote $f_1 + f_2$ by $\mathcal{T}$. For convenience we also assume $f \in C^1(\Sigma)$ in the following three lemmas, although this is not really necessary. In any case, for a number of technical reasons we will need to approximate (1.1), (1.2) by problems with more regular data than we have assumed in Section 1.

LEMMA 2.1. – *The following are true.*

(i) $\sum f_i\lambda_i \le f(\lambda) + a\mathcal{T}$ *where* (a, a) *is the point where the line* $\lambda_1 = \lambda_2$ *intersects* $\partial\Sigma$.

(ii) *For any* $\mu > 0$ *and any* $\lambda \in \Sigma_\mu = \{\lambda \in \Sigma : f(\lambda) \le \mu\}$ *with* $\lambda_1 \le \lambda_2$ *we have* $f_1 \ge \frac{1}{2}\sigma_0$ *where* σ_0 *is the constant from* (1.9).

(iii) $\lim_{t\to\infty} f(t, t) = \infty$.

(iv) $f_2/f_1 \to 0$ *as* $\lambda_2 \to \infty$, $\lambda \in \Sigma_\mu$.

If in addition f *satisfies* (1.21)′, *then*

(v) $f_1 \to \infty$ *as* $\lambda_2 \to \infty$, $\lambda \in \Sigma_\mu$, *and*

(vi) $\lim_{t\to\infty} f(s, t) = \infty$ *for any* $s > 0$.

Proof. – (i) (1.6) and (1.7) imply

$$0 = f(a, a) \le f(\lambda) + \sum f_i(a - \lambda_i)$$

which is (i).

(ii) The concavity and symmetry of f evidently imply $f_1 \ge f_2$ if $\lambda_1 \le \lambda_2$. (ii) follows from this and (1.9).

(iii) This is an immediate consequence of (1.9) and the symmetry of f.

(iv) First observe that if $\lambda \in \Sigma_\mu$ with $\lambda_1 \le \lambda_2$, then either $\lambda_1 \le a$ or

$$f(\lambda_1, \lambda_1) \le f(\lambda_1, \lambda_2) \le \mu\,,$$

so by (iii) there is a number $\delta_0 = \delta_0(f, \mu) > 0$ such that $\lambda_1 \le \delta_0$. For each $\eta \in (0, \mu]$ define

$$(2.2)\qquad L(\eta) = \{\lambda \in \Sigma : f(\lambda) = \eta\}\,.$$

Then $L(\eta)$ is a convex curve in Σ passing between $(0, 0)$ and (δ_0, δ_0) and asymptotic to a translate of $\partial\Gamma_+$. Clearly (f_1, f_2) is normal to $L(\eta)$, so $f_2/f_1 \to 0$ as $\lambda_2 \to \infty$ along any $L(\eta)$, uniformly for all $\eta \in (0, \mu]$.

(v) For $\lambda \in \Sigma_\mu$ with $\lambda_1 \leq \lambda_2$ we have $\mathcal{T} \leq 2f_1$, so (1.21)$'$ clearly implies (v).

(vi) Let $(\omega(t), t) \in \partial\Sigma$, so $\omega(t) \to 0$ as $t \to \infty$. Then, using (1.6) and (1.7), we obtain

$$\frac{1}{2}\mathcal{T}(s,t) \leq f_1(s,t) \leq \frac{f(s,t)}{s-\omega(t)}$$

for all $t \geq s$ so large that $(s,t) \in \Sigma$. (vi) now follows from this and (1.21)$'$.

LEMMA 2.2. – *Assume f satisfies* (1.21)$'$ *and in addition there exist positive constants K, L and positive functions $\psi_1, \psi_2 : [1,\infty) \to \mathbb{R}$ such that*

$$1 \leq \frac{\psi_2(t)}{\psi_1(t)} \leq K \tag{2.3}$$

and

$$\psi_1(t)\, f(\lambda) \leq f(t\lambda) \leq \psi_2(t)\, f(\lambda) \tag{2.4}$$

for all $t \geq 1$ and all $\lambda \in \Sigma$ with $\min\{\lambda_1, \lambda_2\} \geq L$. Then for any number $R \geq 1$ there is a number $\alpha = \alpha(f, K, L, R) > 0$ such that

$$f(t, (1+\alpha)t) \geq f(Rt, Rt) \tag{2.5}$$

for all $t \geq L$.

Proof. – For any $t \geq L$ and any $\alpha > 0$ we have

$$f(t, (1+\alpha)t) \geq \psi_1\left(\frac{t}{L}\right) f(L, (1+\alpha)L)$$

and

$$f(Rt, Rt) \leq \psi_2\left(\frac{t}{L}\right) f(RL, RL),$$

so it clearly suffices to prove there exists an $\alpha > 0$ such that

$$f(L, (1+\alpha)L) \geq K f(RL, RL).$$

But this follows immediately from Lemma 2.1 (vi).

Remark. – The technical conditions (2.3) and (2.4) in Lemma 2.2 are automatically satisfied (with $K = L = 1$ and $\psi_1(t) = \psi_2(t) = t^d$) if f is homogeneous of degree $d \in (0,1]$.

LEMMA 2.3. – *For any $\mu > 0$ and any $\epsilon > 0$ there is a positive constant $C(\epsilon)$, depending only on Σ, f, μ and ϵ, such that*

$$\sum f_i \lambda_i^2 \leq (C(\epsilon) + \epsilon|\lambda|)\mathcal{T} \quad on \quad \Sigma_\mu. \tag{2.6}$$

Proof. – Let $\lambda \in \Sigma_\mu$ with $\lambda_1 \leq \lambda_2$. From the proof of Lemma 2.1 (iv) there is a constant $\delta_0 > 0$ such that $\lambda_1 \leq \delta_0$. Thus $f_1\lambda_1^2 \leq \delta_0^2\mathcal{T}$, so only the term $f_2\lambda_2^2$ needs to be estimated. For $\eta \in (0, \mu]$ define $L(\eta)$ by (2.2). We consider two cases.

(i) *$L(\eta)$ is asymptotic to $\partial\Gamma_+$ for all $\eta \in (0, \mu]$.* This is the simpler case. Let $\epsilon > 0$. Then, since $L(\mu)$ is asymptotic to $\partial\Gamma_+$, there is a number $N = N(f, \epsilon) > 0$ such that if $\lambda \in L(\mu)$ with $\lambda_2 \geq N$, then $\lambda_1 < \epsilon$. But then, since f is increasing with respect to λ_1, for any $\lambda \in \Sigma_\mu = \cup_{\eta \in (0,\mu]} L(\eta)$ with $\lambda_2 \geq N$ we have $\lambda_1 < \epsilon$. Since the point (ϵ, N) lies inside all the curves $L(\eta)$ (*i.e.*, in the convex one of the two regions into which $L(\eta)$ divides $\mathbb{R}^2$), it follows that for any $\lambda \in L(\eta)$ we have

$$f_1(\lambda_1 - \epsilon) + f_2(\lambda_2 - N) \leq 0.$$

For $\lambda_2 \geq 2N$ we therefore have

$$\frac{1}{2} f_2\lambda_2 \leq f_2(\lambda_2 - N) \leq f_1(\epsilon - \lambda_1) \leq \epsilon f_1.$$

The estimate (2.6) now follows easily after replacing ϵ by $\epsilon/2$.

(ii) *Not all $L(\eta), \eta \in (0, \mu]$, are asymptotic to $\partial\Gamma_+$.* Let $\epsilon > 0$ and consider any $\eta \in (0, \mu]$ for which $L(\eta)$ is asymptotic to $(\sigma, \sigma) + \partial\Gamma_+$ for some $\sigma \in [0, \epsilon]$. Clearly, if $0 < \eta' \leq \eta$, then $L(\eta')$ also has this property, so we can assume η is the largest number in $(0, \mu]$ with this property. Then there is a number $N_0 = N_0(f, \epsilon) > 0$ such that $\lambda \in L(\eta)$ and $\lambda_2 \geq N_0$ imply $\lambda_1 \leq 2\epsilon$. But then $\lambda \in L(\eta')$ and $\lambda_2 \geq N_0$ imply $\lambda_1 \leq 2\epsilon$ for all $\eta' \in (0, \eta]$. By the argument used in case (i) we obtain

$$f_2\lambda_2^2 \leq 4\epsilon\lambda_2 f_1 \tag{2.7}$$

for all $\lambda \in \Sigma_\eta$ with $\lambda_2 \geq 2N_0$. If $\eta = \mu$ we are finished; if not, we still need to consider the cases $\eta' \in [\eta, \mu]$.

To do this we first observe that case (ii) occurs only if f is bounded on the set $\Sigma \cap [0, \delta_0] \times [0, \infty)$. In fact, if case (ii) occurs, then $L(\mu)$ is asymptotic to $(\sigma, \sigma) + \partial\Gamma_+$ for some $\sigma > 0$. Since $\partial\Sigma$ is asymptotic to $\partial\Gamma_+$, there is a number $N_1 = N_1(f, \sigma)$ such that $\lambda \in \partial\Sigma$ and $\lambda_2 \geq N_1$ imply $\lambda_1 < \sigma/2$. Thus by the concavity of f and the fact that $f \equiv 0$ on $\partial\Sigma$,

$$f_1(\sigma, \lambda_2) \leq 2f(\sigma, \lambda_2)\sigma^{-1} \leq 2\mu\sigma^{-1}$$

for $\lambda_2 \geq N_1$. But then, again using the concavity of f,

$$\begin{aligned} f(\delta_0, \lambda_2) &\leq f(\sigma, \lambda_2) + f_1(\sigma, \lambda_2)(\delta_0 - \sigma) \\ &\leq \mu + 2\mu(\delta_0 - \sigma)\sigma^{-1} \end{aligned}$$

for $\lambda_2 \geq N_1$. Using (2.1) we conclude that

$$0 \leq f(\lambda) \leq \mu + 2\mu(\delta_0 - \sigma)\sigma^{-1} \quad \text{on} \quad \Sigma \cap [0, \delta_0] \times [0, \infty)\,. \tag{2.8}$$

From this, the concavity of f and the fact that $\partial\Sigma$ is asymptotic to $\partial\Gamma_+$, it follows that there is a number $N_2 = N_2(\Sigma, \epsilon)$ such that

$$0 \leq f_1 \leq C\epsilon^{-1} \quad \text{on} \quad \Sigma \cap [\epsilon, \delta_0] \times [N_2, \infty)\,, \tag{2.9}$$

where C depends on μ, σ and δ_0, but not on ϵ.

Since f is bounded on $\Sigma \cap [0, \delta_0] \times [0, \infty)$ and increasing with respect to λ_2, $\hat{f}(\lambda_1) = \lim_{t\to\infty} f(\lambda_1, t)$ exists for all $\lambda_1 \in (0, \delta_0]$. Moreover, (2.9) implies that the convergence to the limit is uniform for all $\lambda_1 \in [\epsilon, \delta_0]$. Thus for any $\epsilon' > 0$ there is a number $t_0 = t_0(\Sigma, f, \epsilon, \epsilon') \geq N_2$ such that

$$\hat{f}(\lambda_1) - f(\lambda_1, t) \leq \epsilon'$$

for all $t \geq t_0$ and all $\lambda_1 \in [\epsilon, \delta_0]$. Since f is concave,

$$f_2(\lambda_1, t)(t - t_0) \leq \int_{t_0}^{t} f_2(\lambda_1, s)\, ds \leq \epsilon'$$

for all $t \geq t_0$ and all $\lambda_1 \in [\epsilon, \delta_0]$. Consequently,

$$f_2(\lambda_1, \lambda_2) \leq \frac{2\epsilon'}{\lambda_2} \tag{2.10}$$

for all $\lambda \in \Sigma \cap [\epsilon, \delta_0] \times [2t_0, \infty)$. Using (1.9) and (2.10) we then obtain

$$f_2\lambda_2^2 \leq 2\epsilon'\lambda_2 \leq 2\epsilon'\sigma_0^{-1}\lambda_2\mathcal{T} \tag{2.11}$$

on $\Sigma_\mu \cap [\epsilon, \delta_0] \times [2t_0, \infty)$. Recalling now that $L(\eta)$ is asymptotic to $(\epsilon, \epsilon) + \partial\Gamma_+$, we see, after setting $\epsilon' = \epsilon$, that (2.7) and (2.11) imply

$$f_2\lambda_2^2 \leq (4 + 2\sigma_0^{-1})\epsilon\lambda_2\mathcal{T} \quad \text{on} \quad \Sigma_\mu \cap \{\lambda : \lambda_2 \geq 2t_0\}\,. \tag{2.12}$$

The estimate (2.6) follows from this after replacing ϵ by $(4 + 2\sigma_0^{-1})^{-1}\epsilon$.

Remarks. – (i) Lemma 2.3 is much easier to prove if $\Sigma = \Gamma_+$ and f satisfies (1.21)′. For then, by Lemma 2.1 (i) with $a = 0$,

$$\sum f_i \lambda_i \leq f(\lambda) \leq \mu,$$

and hence, since $\lambda_i > 0$,

$$\sum f_i \lambda_i^2 \leq \mu |\lambda|. \tag{2.13}$$

If f satisfies (1.21)′, then for any $\epsilon > 0$ there is a number $C_1(\mu, \epsilon) > 0$ such that $\lambda \in \Sigma_\mu$ and $|\lambda| \geq C_1(\mu, \epsilon)$ imply $\mathcal{T} \geq \mu\epsilon^{-1}$. Thus (1.9) and (2.13) imply

$$\sum f_i \lambda_i^2 \leq \epsilon |\lambda| \mathcal{T} + C_2(\mu, \epsilon)\mathcal{T}.$$

(ii) If we also assume the structure condition (appearing in [2]) that for any compact set $K \subset \Sigma$ and any number $C > 0$ there is a number $R = R(\Sigma, f, K, C) > 0$ such that

$$f(\lambda_1, \lambda_2 + R) \geq C \quad \text{for all} \quad \lambda \in K, \tag{2.14}$$

then we do not need to consider the second case in the proof of Lemma 2.3. It is not difficult to verify that (2.14) is equivalent to (1.21)′.

(iii) The estimate (2.6) cannot be significantly improved without assuming further conditions on f. Suppose for example that the level line $\{f = 1\}$ is given by

$$\lambda_1 = (\log \lambda_2)^{-1} \quad \text{for} \quad \lambda_2 \gg \lambda_1.$$

Let $\Phi(\lambda) = \lambda_1 - (\log \lambda_2)^{-1}$. Then along $\{\Phi = 0\}$ we find that

$$\begin{aligned} D_1\Phi\lambda_1^2 + D_2\Phi\lambda_2^2 &= \lambda_1^2 + (\log \lambda_2)^{-2}\lambda_2 \\ &= O\big(\lambda_2(\log \lambda_2)^{-2}\big) \quad \text{as} \quad \lambda_2 \to \infty. \end{aligned}$$

Since $\dfrac{D\Phi}{|D\Phi|} = \dfrac{Df}{|Df|}$ along $\{\Phi = 0\}$, we conclude that

$$\sum f_i \lambda_i^2 = O\big(\lambda_2(\log \lambda_2)^{-2}\big)\mathcal{T}$$

as $\lambda_2 \to \infty$ along $\{f = 1\}$.

To conclude this section we make a few remarks about our hypotheses on Σ and f. If Σ satisfies either condition (i) or (ii) of Section 1 and f

satisfies (2.1), (1.6) and (1.7), then (1.8) and (1.9) can be derived from the alternative condition

$$\lim_{t\to\infty} f(t,t) = \infty\,. \tag{2.15}$$

To see this, first note that (2.15) implies that for any compact set $K \subset \Sigma$ and any number $C > 0$ there is a number $R = R(\Sigma, f, K, C) \geq 1$ such that

$$f(R\lambda) \geq C \quad \text{for all} \quad \lambda \in K. \tag{2.16}$$

In particular

$$\lim_{t\to\infty} f(t\lambda) = \infty \quad \text{for all} \quad \lambda \in \Sigma\,. \tag{2.17}$$

This is clear if Σ satisfies condition (i); if Σ satisfies condition (ii) it is only a little more difficult and is left to the reader to verify. To prove (1.8) (which is trivial for Σ of type 1) we use the positivity and concavity of f to obtain

$$0 \leq f(t\lambda) \leq f(\lambda) + (t-1)\sum f_i\lambda_i \tag{2.18}$$

for all $\lambda \in \Sigma$ and all $t \geq 1$. (1.8) follows from this by dividing by $t-1$ and letting $t \to \infty$. To prove (1.9) we use the concavity of f and (1.8) to obtain

$$\sum f_i \geq \frac{f(T,T) - f(\lambda)}{T} \tag{2.19}$$

for any $T > 0$ such that $(T,T) \in \Sigma$. (1.9) follows from this and (2.15).

To summarize, in place of (1.8) and (1.9) we could have assumed (2.15), but then we would have had to impose (i) or (ii) as additional hypotheses on Σ in order to derive (1.8) and (1.9). Some results for the case that (1.8) is dropped and Σ does not satisfy either (i) or (ii) of Section 1 will be given in Section 5.

3. SOLUTION AND GRADIENT BOUNDS

In this section we use the method of continuity, which is discussed in [5], Sections 17.2 and 17.9, to reduce the proofs of Theorems 1.1 to 1.4, and of Theorem 1.6 in the case that (1.45) holds, to the derivation of suitable *a priori* estimates. We also prove estimates for u and Du, and give some simple sufficient conditions for the existence of subsolutions, as required in Theorems 1.2, 1.4 and 1.5.

For functional analytic reasons the method of continuity requires more regular data than we have assumed in Section 1, and it requires the monotonicity hypotheses (1.13) and (1.15) (respectively (1.24)), or (1.45) in the case of Theorem 1.6. In addition, the second derivative estimates of the following section will be proved for solutions u belonging to $C^4(\Omega) \cap C^3(\bar{\Omega})$ (the proof in fact requires only $u \in W^{4,2}_{loc}(\Omega) \cap C^3(\bar{\Omega})$), a regularity hypothesis which is not guaranteed by our regularity assumptions on the data. For these reasons we shall assume initially that all the data are C^∞. Theorems 1.1 to 1.4 and Theorem 1.6 in the case that (1.45) holds then follow by standard approximation arguments coupled with uniqueness results for admissible solutions of the appropriate boundary value problem. In each case, if u and v are two admissible solutions, then by the mean value theorem $w = u - v$ satisfies a linear elliptic oblique problem

$$
\begin{aligned}
a^{ij} D_{ij} w + b^i D_i w + cw &= 0 \quad \text{in} \quad \Omega\,, \\
D_\beta w + \gamma w &= 0 \quad \text{on} \quad \partial\Omega
\end{aligned}
\tag{3.1}
$$

with $c \leq 0$ ($c < 0$ in Theorem 1.6 if (1.45) holds) and $\gamma < 0$ ($\gamma \equiv 0$ in Theorem 1.6). Uniqueness is then an immediate consequence of the maximum principle and the Hopf boundary point lemma.

When the monotonicity conditions (1.13) and (1.15) (respectively (1.24)) or (1.45) do not hold, as in Theorems 1.5 and 1.6, the continuity method alone is not sufficient to prove existence. We also need a suitable fixed point theorem. This procedure and the proofs of Theorems 1.5 and 1.6 will be given in Section 4.

Since $g_z \geq 0$, in the case $g = g(x, u)$ (and more generally if $g = g(x, u, Du)$ is bounded with respect to Du), for suitable positive constants A and B

$$
\underline{u}(x) = A(|x|^2 - B) \tag{3.2}
$$

is an admissible subsolution of (1.1). If g depends on Du, we take $\underline{u}$ to be the subsolution whose existence is assumed in Theorems 1.2 and 1.4. Since g is positive and $f \equiv 0$ on $\partial\Sigma$, it is clear that there is a number $\delta > 0$ such that $u_0(x) = \delta|x|^2$ is admissible and

$$
F(D^2 u_0) \leq g(x, \underline{u}, D\underline{u})\,. \tag{3.3}
$$

For $t \in [0, 1]$ consider the family of boundary value problems

$$
\begin{aligned}
F(D^2 u) &= tg(x, u, Du) + (1 - t)\, F(D^2 u_0) \quad \text{in} \quad \Omega\,, \\
b(x, u, Du) &= (1 - t)\, b(x, u_0, Du_0) \quad \text{on} \quad \partial\Omega\,.
\end{aligned}
\tag{3.4}
$$

For $t = 1$ this is our original problem (1.1), (1.2), while u_0 is an admissible solution of (3.4) when $t = 0$. By our structural hypotheses in Section 1, standard linear elliptic theory ([5], Chapter 6) and the implicit function theorem ([5], Theorem 17.6), the set $\mathcal{S}$ of $t \in [0,1]$ for which (3.4) is solvable with admissible solution $u_t \in C^{2,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$ is relatively open. If we can also prove the *a priori* estimates

$$\|u_t\|_{C^{2,\alpha}(\bar{\Omega})} \leq C \tag{3.5}$$

for some $\alpha \in (0,1)$ with C independent of t, then $\mathcal{S}$ is also closed, and hence equal to $[0,1]$. Thus our original problem (1.1), (1.2) is solvable. Clearly, g_t and b_t given by

$$g_t(x,z,p) = tg(x,z,p) + (1-t)\,F\big(D^2 u_0(x)\big)$$

and

$$b_t(x,z,p) = b(x,z,p) - (1-t)\,b\big(x, u_0(x), Du_0(x)\big)$$

satisfy similar structure conditions as g and b, with structure constants controlled independently of t, and by (3.3), (3.4), $\underline{u}$ is a subsolution of

$$F(D^2 u) = g_t(x,u,Du) \tag{3.6}$$

for each $t \in [0,1]$. In proving our *a priori* estimates we shall therefore ignore any dependence on t.

We begin with the estimate for u. This part of the argument does not require b to have a special form such as (1.11) or (1.23). All we require is that b satisfy

$$b(x,z,0) < 0 \quad \text{for all} \quad x \in \partial\Omega \quad \text{and all} \quad z \geq N_1 \tag{3.7}$$

for some constant N_1,

$$b(x,z,p) > 0 \quad \text{for all} \quad (x,p) \in \partial\Omega \times K \quad \text{and all} \quad z \leq N_2 \tag{3.8}$$

for any compact set $K \subset \mathbb{R}^2$ and some number $N_2 = N_2(K)$, together with the degenerate obliqueness condition

$$b_p(x,z,p) \cdot \nu(x) \geq 0 \quad \text{for all} \quad (x,z,p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^2 . \tag{3.9}$$

It is easily checked that (3.7) and (3.8) hold for oblique boundary conditions of the form (1.11) (respectively (1.23)) by virtue of our hypotheses (1.16) and (1.17) (respectively (1.25) and (1.26)).

Since u is convex, u attains its maximum on $\partial\Omega$, say at a point $x_0 \in \partial\Omega$, and hence

$$D_\nu u(x_0) \leq 0, \qquad \delta u(x_0) = 0\,.$$

Using (3.9) we obtain

$$0 \leq b\big(x_0, u(x_0), 0\big)\,,$$

which implies, with the aid of (3.7),

$$\sup_\Omega u \leq N_1\,. \tag{3.10}$$

To obtain a lower bound we observe that by the mean value theorem $\underline{u} - u$ satisfies a linear elliptic differential inequality

$$a^{ij} D_{ij}(\underline{u} - u) + b^i D_i(\underline{u} - u) + c(\underline{u} - u) \geq 0 \quad \text{in} \quad \Omega$$

with $c \leq 0$ by virtue of (1.13), and hence by the maximum principle $\underline{u} - u$ attains its maximum on $\partial\Omega$, say at a point $y_0 \in \partial\Omega$. Thus

$$D_\nu(\underline{u} - u)(y_0) \leq 0\,, \qquad \delta(\underline{u} - u)(y_0) = 0\,,$$

and hence by (3.9),

$$0 \geq b\big(y_0, u(y_0), D\underline{u}(y_0)\big)\,,$$

which implies, by (3.8),

$$u(y_0) \geq N_2 = N_2(|D\underline{u}|_{0;\Omega})\,.$$

It follows then that

$$\inf_\Omega u \geq -C(\underline{u})\,, \tag{3.11}$$

so the estimate for u is proved.

We now come to the gradient bound. Here we take advantage of the convexity of u and use the following result [10], Theorem 2.2.

LEMMA 3.1. – *Let Ω be a $C^{1,1}$ bounded domain in $\mathbb{R}^n$ and $u \in C^1(\bar{\Omega})$ a convex function satisfying*

$$D_\beta u \geq -M \quad on \quad \partial\Omega \tag{3.12}$$

for some constant M, where β is a unit vector field on $\partial\Omega$ with

$$\beta\cdot\nu\geq\beta_0 \quad on \quad \partial\Omega \tag{3.13}$$

for some positive constant β_0. Then

$$\sup_{\Omega}|Du|\leq C \tag{3.14}$$

where C depends only on $n, \beta_0, |u|_{0;\Omega}, M$ and Ω.

Lemma 3.1 immediately implies a gradient bound in the semilinear case (1.11) if β satisfies (1.18). In the case of a boundary condition of the form (1.23) with ϕ satisfying (1.27), we immediately obtain

$$0\leq D_\nu u+\phi(x,u,0)+\phi_{p_i}(x,u,0)\delta_i u \quad \text{on} \quad \partial\Omega\,,$$

and a gradient bound follows from Lemma 3.1.

Remark. – Gradient estimates can also be obtained for boundary conditions of the form (1.23) under more general structure conditions than the concavity condition (1.27). However, we cannot obtain second derivative bounds under such general conditions, and for this reason we shall not pursue this further here.

We now consider the boundary value problem (1.44). If u, h and Ω^* are as in Theorem 1.6, then $H = h(Du)$ is positive in Ω and zero on $\partial\Omega$, so $D_\nu H\geq 0$ and $\delta H = 0$ on $\partial\Omega$. Thus

$$D_i H = D_\nu H\nu_i = h_{p_k}(Du)D_{ik}u \quad \text{on} \quad \partial\Omega\,. \tag{3.15}$$

Since D^2u is invertible, we see that

$$\chi = h_{p_k}(Du)\nu_k = D_\nu H u^{\nu\nu}\geq 0\,, \tag{3.16}$$

where $u^{\nu\nu} = u^{ij}\nu_i\nu_j$ and $[u^{ij}] = [D^2u]^{-1}$. Thus (1.44) is degenerate oblique. However, from (3.15) we also see that

$$D_\nu H\chi = D_{ik}uh_{p_i}h_{p_k}\,. \tag{3.17}$$

Combining this with (3.16) we obtain

$$\chi = \left[u^{\nu\nu}\,D_{ij}uh_{p_i}\,h_{p_j}\right]^{1/2}, \tag{3.18}$$

which is positive provided $u\in C^2(\bar\Omega)$ with $D^2u > 0$. Thus (1.44) is in fact strictly oblique on convex $C^2(\bar\Omega)$ solutions if g is positive, and so the method of continuity can be used. Later we shall prove the strict obliqueness estimate (1.40).

To set up a suitable family of problems for (1.44) we assume for convenience that Ω and Ω^* both contain the origin. For $t \in [0,1]$ let $\{\Omega_t\}$ be a family of smooth, uniformly convex domains in $\mathbb{R}^2$, uniformly bounded in the C^3 sense and with curvatures bounded from below by a positive constant independent of t, and with $\Omega_0 = B_1(0)$ and $\Omega_t \to \Omega_1 = \Omega$ as $t \to 1$. Let $u_0(x) = \dfrac{1}{2}|x|^2$, $h_0(p) = \dfrac{1}{2}(1 - |p|^2)$, and set

$$h_t = th_1 + (1-t)\,h_0\,,$$

where $h_1 = h$ is a smooth, uniformly concave defining function for Ω^*. Then h_t is a smooth, uniformly concave defining function for $\Omega_t^* = \{p \in \mathbb{R}^2 : h_t(p) > 0\}$, and since $h_0, h_1 > 0$ in $B_r(0)$ for some $r > 0$, we see that $\Omega_t^* \supset B_r(0)$ for each $t \in [0,1]$. Evidently $\|h_t\|_{C^3(\overline{\Omega_t^*})}$ is bounded independently of t, and we can also assume without loss of generality that

$$(3.19) \qquad |Dh_t| \geq 1 \quad \text{on} \quad \partial\Omega_t^*$$

for each $t \in [0,1]$.

Consider the family of boundary value problems

$$(3.20) \qquad \begin{aligned} &F(D^2u) = tg(x,u,Du) + (1-t)\,e^{u-u_0}\,F(D^2u_0) \quad \text{in} \quad \Omega_t\,,\\ &h_t(Du) = 0 \quad \text{on} \quad \partial\Omega_t\,. \end{aligned}$$

By construction u_0 is a convex solution of (3.20) for $t = 0$. As before, we see (using our hypotheses in Section 1, strict obliqueness, linear elliptic theory and the implicit function theorem) that the boundary value problem (1.44), which coincides with (3.20) when $t = 1$, is solvable with convex solution $u \in C^{2,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$, provided we can prove the estimates (3.5). We also note that the hypotheses (1.42) and (1.43) are satisfied uniformly for $t \in [0,1]$ for the function g_t given by

$$g_t(x,z,p) = tg(x,z,p) + (1-t)\,e^{z-u_0(x)}\,F\big(D^2u_0(x)\big)\,.$$

It suffices therefore to prove the estimates (3.5) for $t = 1$. We also note that the solution of (1.44) which we obtain by this procedure satisfies $Du(\Omega) = \Omega^*$, since $Du_t(\Omega_t) = \Omega_t^*$ for $t = 0$ and this is preserved by the homotopy.

We now proceed to the estimates for (1.44). The gradient estimate is trivial since $Du(\Omega) = \Omega^*$. To prove an upper bound for u we observe that the concavity of f implies

$$f(\lambda) \leq f(b,b) + \sum f_i(b,b)(\lambda_i - b)$$

for all $b > 0$ such that $(b,b) \in \Sigma$, and hence

$$F(D^2u) \leq C_1(f,b) + \sigma\Delta u$$

where $\sigma = f_1(b,b) > 0$. Fixing a suitable b and integrating this over Ω, we obtain

$$\begin{aligned}\int_\Omega g(x,u,Du) &\leq C_2 + \sigma\int_\Omega \Delta u\\ &= C_2 - \sigma\int_{\partial\Omega} D_\nu u\\ &\leq C_3\,.\end{aligned}$$

The hypothesis (1.42) now implies that u is bounded above somewhere in Ω, and hence also

$$\sup_\Omega u \leq C_4\,, \tag{3.21}$$

since Du is bounded.

To obtain a lower bound, we use (1.41) to obtain

$$\begin{aligned}\int_\Omega G(g(x,u,Du)) &= \int_\Omega G(F(D^2u))\\ &\geq \int_\Omega \det D^2u\\ &= |\Omega^*|\,.\end{aligned}$$

The hypothesis (1.43) then implies that u is bounded from below at some point of $\bar{\Omega}$, and hence also

$$\inf_\Omega u \geq -C_5\,. \tag{3.22}$$

We conclude this section by giving some simple sufficient conditions for the existence of admissible subsolutions. We assume that f satisfies conditions (1.4) to (1.9) and (1.21)′ and Σ is of type 1, although not all of these conditions are needed for the construction of admissible subsolutions.

In addition, we assume that f satisfies the technical condition of Lemma 2.2. Namely, there exist positive constants K, L and positive functions $\psi_1, \psi_2 : [1, \infty) \to \mathbb{R}$ such that

$$1 \leq \frac{\psi_2(t)}{\psi_1(t)} \leq K \tag{3.23}$$

and

$$\psi_1(t) f(\lambda) \leq f(t\lambda) \leq \psi_2(t) f(\lambda) \tag{3.24}$$

for all $t \geq 1$ and all $\lambda \in \Sigma$ with $\min\{\lambda_1, \lambda_2\} \geq L$. Recall that this condition is automatically satisfied if f is homogeneous of some degree $d \in (0, 1]$. We also assume that for some positive constant C such that $(C, C) \in \Sigma$ we have

$$g(x, z, p) \leq f(C(1 + |p|), C(1 + |p|)) \tag{3.25}$$

for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$ with $z \leq 0$. Under these conditions we can construct an admissible subsolution $\underline{u} \in C^2(\bar{\Omega})$ of (1.1). Our construction is similar to one of Lions [9], Section IV (see also [1], Section 7) for the special case of the Monge-Ampère equation.

Let ψ be a C^2 uniformly convex defining function for Ω with $D^2\psi \geq I$ in Ω and let

$$\underline{u} = \mu\,(e^{\alpha\psi} - 1) \tag{3.26}$$

for some positive constants μ and α to be determined. We shall first fix α, and then fix μ, depending on α, but for now all we need is that μ and α are such that

$$\mu\alpha\, e^{\alpha\psi} \geq L\,, \tag{3.27}$$

so that $(\mu\alpha\, e^{\alpha\psi}, \mu\alpha\, e^{\alpha\psi}) \in \Sigma$. Differentiating $\underline{u}$ we obtain

$$\begin{aligned} D_i\underline{u} &= \mu\alpha\, e^{\alpha\psi}\, D_i\psi\,, \\ D_{ij}\underline{u} &= \mu\alpha\, e^{\alpha\psi}(D_{ij}\psi + \alpha D_i\psi D_j\psi) \\ &\geq \mu\alpha\, e^{\alpha\psi}(\delta_{ij} + \alpha D_i\psi D_j\psi) \end{aligned}$$

since $D^2\psi \geq I$ in Ω. The eigenvalues of $[\delta_{ij} + \alpha D_i\psi D_j\psi]$ are 1 and $1 + \alpha|D\psi|^2$, so

$$F(D^2\underline{u}) \geq f(\mu\alpha\, e^{\alpha\psi}, \mu\alpha\, e^{\alpha\psi}(1 + \alpha|D\psi|^2))\,. \tag{3.28}$$

Let $\delta = (2C)^{-1}$ where C is the constant in (3.25). We consider two cases.

(i) $|D\psi| \geq \delta$.

In this case, (3.28) and Lemma 2.2 imply that for α fixed sufficiently large we have

$$\begin{aligned} F(D^2\underline{u}) &\geq \hat{f}(\mu\alpha\, e^{\alpha\psi}(1 + C|D\psi|_{0;\Omega})) \\ &\geq \hat{f}(C(1 + |D\underline{u}|)), \end{aligned}$$

provided μ is so large that

$$\mu\alpha\, e^{\alpha\psi} \geq C\,. \tag{3.29}$$

For convenience here and below we denote $f(t, t)$ by $\hat{f}(t)$.

(ii) $|D\psi| \leq \delta$.

In this case (3.28) implies

$$\begin{aligned} F(D^2\underline{u}) &\geq \hat{f}(\mu\alpha\, e^{\alpha\psi}) \\ &\geq \hat{f}\left(\frac{1}{2}\,\mu\alpha\, e^{\alpha\psi} + \frac{1}{2}\,\mu\alpha e^{\alpha\psi}\,\delta^{-1}\,|D\psi|\right) \\ &\geq \hat{f}(C(1 + |D\underline{u}|)), \end{aligned}$$

provided

$$\frac{1}{2}\,\mu\alpha e^{\alpha\psi} \geq C\,. \tag{3.30}$$

Consequently, for α fixed as above and μ fixed so large that (3.27) and (3.30) are satisfied, we see that $\underline{u}$ is an admissible subsolution of (1.1).

It is worth observing that the growth condition on g given by (3.25) is generally optimal. This is shown by the Monge- Ampère equation

$$\det D^2 u = (1 + |Du|^2)^{\gamma/2}\,. \tag{3.31}$$

Since $\det D^2 u$ is the Jacobian of the gradient map $Du : \Omega \to \mathbb{R}^2$, by integrating we obtain

$$|\Omega| = \int_{Du(\Omega)} \frac{dp}{(1 + |p|^2)^{\gamma/2}} \leq \int_{\mathbb{R}^2} \frac{dp}{(1 + |p|^2)^{\gamma/2}}\,.$$

The last integral is infinite if $\gamma \leq 2$, but is finite if $\gamma > 2$. So for $\gamma > 2$ we cannot construct a convex subsolution of (3.31) on domains of arbitrary measure. But if diam Ω is small enough, we can always construct subsolutions, not just for (3.31), but also for (1.1). Furthermore, for this we do not need to assume (1.21)′ or the technical conditions (3.23) and (3.24).

To prove this, let ψ be as above and set

$$\underline{u} = A\psi \tag{3.32}$$

for some positive constant A to be chosen. Since $D^2\psi \geq I$, we have

$$F(D^2\underline{u}) \geq f(A, A)$$

for A so large that $(A, A) \in \Sigma$. We now fix A so large that

$$f(A, A) \geq g(x, z, p)$$

for all $x \in \Omega$, $|z| \leq 1$ and $|p| \leq 1$. Since $\psi \in C^2(\bar{\Omega})$ and $\psi \equiv 0$ on $\partial\Omega$, we have

$$|\psi| \leq Cd^2\,, \quad |D\psi| \leq Cd \quad \text{on} \quad \bar{\Omega}$$

for some positive constant C depending on $\sup_\Omega |D^2\psi|$, where $d = \operatorname{diam}\Omega$. Thus for d so small that

$$ACd^2, ACd \leq 1\,,$$

we see that $\underline{u}$ is an admissible subsolution of (1.1).

A straightforward argument also shows that if

$$g(x, z, p) \leq o(f(|p|, |p|)) \quad \text{as} \quad |p| \to \infty\,, \tag{3.33}$$

uniformly for $x \in \Omega$ and $z \in \mathbb{R}$, then $\underline{u}$ given by (3.32) is an admissible subsolution of (1.1) for A fixed sufficiently large.

Finally, if we strengthen (3.25) by requiring it to hold for all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$, not just for $z \leq 0$, we see that for $\underline{u}$ given by (3.26), $\underline{u} + N$ is an admissible subsolution for any constant N. Likewise, if we assume

$$g(x, z, p) \leq C \tag{3.34}$$

for some constant C and all $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$ with $|p| \leq 1$, then $A\psi + N$ is also an admissible subsolution of (1.1) for any constant N, provided A is sufficiently large and $\operatorname{diam}\Omega$ is sufficiently small. For any oblique unit vector field β on $\partial\Omega$ and $\underline{U} = \underline{u} + N$ we have

$$D_\beta \underline{U} + \phi(x, \underline{U}) \geq -\sup_{\partial\Omega} |D\underline{u}| + \phi(x, N)\,.$$

Consequently, if there exists a number N such that

$$\phi(x, N) \geq \sup_{\partial\Omega} |D\underline{u}| \tag{3.35}$$

for all $x \in \partial\Omega$, then $\underline{U}$ is an admissible subsolution of (1.1), (1.11). The argument for the boundary condition (1.23) is virtually identical. Since $\delta\underline{u} \equiv 0$ on $\partial\Omega$, we now require the existence of an N such that

$$\phi(x, N, 0) \geq \sup_{\partial\Omega} |D\underline{u}| \tag{3.36}$$

for all $x \in \partial\Omega$.

Finally, we remark that if f satisfies

$$\lambda_1\lambda_2 \leq G(f(\lambda)) \quad \text{for} \quad \lambda \in \Sigma \tag{3.37}$$

for some strictly increasing function G, criteria for the existence of admissible subsolutions can be obtained from various existence results for the Dirichlet problem for Monge-Ampère equations (*see* [1], [3], [6], [9], [11], [15]).

4. SECOND DERIVATIVE BOUNDS

In this section we shall prove second derivative bounds. This is the most difficult of the *a priori* estimates, and it is here that the two dimensionality enters in a crucial way.

Since we have already bounded u and Du, we have

$$g(x, u, Du) + |Dg(x, u, Du)| + |D^2 g(x, u, Du)| \leq \mu \tag{4.1}$$

for some positive constant μ. Similarly, for the function ϕ in (1.23) we have

$$|\phi(x, u, \delta u)| + |D\phi(x, u, \delta u)| + |D^2\phi(x, u, \delta u)| \leq \tilde{\mu} \tag{4.2}$$

for some positive constant $\tilde{\mu}$. A similar inequality of course also holds for the function ϕ in (1.11). Furthermore, for (1.23) we have

$$b_{p_k}(x, u, Du) = \nu_k + \phi_{p_l}(x, u, \delta u)(\delta_{kl} - \nu_k\nu_l)\,. \tag{4.3}$$

Since u and Du have already been bounded, the vector field $\beta = b_p(x, u, Du)/\ |b_p(x, u, Du)|$ satisfies

$$\beta \cdot \nu \geq \beta_0 \quad \text{on} \quad \partial\Omega \tag{4.4}$$

for some positive constant β_0. The use of β to denote the vector field $b_p/|b_p|$ is convenient and should cause no confusion with the β appearing in (1.11). The concavity condition (1.27) can now also be rewritten in the form

$$\phi_{p_i p_j}(x, u, \delta u)\,\tau_i\tau_j \leq -\theta \quad \text{on} \quad \partial\Omega \tag{4.5}$$

for all directions τ tangential to $\partial\Omega$ at x and some positive constant θ.

We now differentiate equation (1.1) twice in a direction ξ to obtain

$$F_{ij}\, D_{ij\xi}u = g_\xi + g_z D_\xi u + g_{p_k}\, D_{k\xi}u \tag{4.6}$$

and

$$\begin{aligned} F_{ij}\, D_{ij\xi\xi}u &+ F_{ij,rs}\, D_{ij\xi}u D_{rs\xi}u \\ &= g_{\xi\xi} + 2g_{\xi z}\, D_\xi u + 2g_{\xi p_k} D_{k\xi}u + g_{zz}(D_\xi u)^2 \\ &\quad + 2g_{zp_k}\, D_\xi u\, D_{k\xi}u + g_{p_k p_l}\, D_{k\xi}u\, D_{l\xi}u \\ &\quad + g_z\, D_{\xi\xi}u + g_{p_i}\, D_{i\xi\xi}u\,. \end{aligned} \tag{4.7}$$

Using the concavity of F and (4.1) we obtain

$$F_{ij}\, D_{ij\xi\xi}u - g_{p_i}\, D_{i\xi\xi}u \geq -C(1 + |D^2u|^2)\,. \tag{4.8}$$

If g is convex with respect to Du, or g is independent of Du, we instead obtain

$$F_{ij}\, D_{ij\xi\xi}u - g_{p_i}\, D_{i\xi\xi}u \geq -C(1 + |D^2u|) \tag{4.8$'$}$$

or

$$F_{ij}\, D_{ij\xi\xi}u \geq -C \tag{4.8$''$}$$

respectively. In obtaining (4.8)$''$ (but not (4.8) or (4.8)$'$) we have also used (1.13); if we do not assume (1.13), we get

$$F_{ij}\, D_{ij\xi\xi}u \geq -C(1 + |D^2u|)\,. \tag{4.8$'''$}$$

Let $\epsilon > 0$, to be fixed later. We denote by $C(\epsilon), C_0(\epsilon), C_1(\epsilon), \ldots$ various positive constants depending on ϵ as well as on other parameters. Any constants not depending on ϵ will be denoted by $C, C_0, C_1, \ldots$. As usual, different constants will often be denoted by the same symbol.

Let

$$w = w(x, \gamma) = D_{\gamma\gamma}u + K(1 + \epsilon M)|x|^2 ,$$

where $M = \sup_\Omega |D^2u|$ and K is a positive constant to be chosen, and assume that w attains its maximum over all points of $\bar{\Omega}$ and all directions $\gamma \in S^1$ at $(x_0, \xi) \in \bar{\Omega} \times S^1$. We will show that if $x_0 \in \Omega$, then $D_{\xi\xi}u(x_0) \leq C(\epsilon)$. In case (4.8) we have

$$\begin{aligned} Lw &= F_{ij}\, D_{ij}w - g_{p_i}\, D_i w \\ &\geq -C_1(1 + |D^2u|^2) + 2K(1 + \epsilon M)\mathcal{T} - C_2 K(1 + \epsilon M) . \end{aligned}$$

Using (1.21) we see there is a constant $C_0(\epsilon) \geq 1$ such that if $|D^2u| \geq C_0(\epsilon)$, then $\mathcal{T} \geq \epsilon^{-1}|D^2u|$, and consequently $|D^2u|^2 \leq \epsilon M\mathcal{T}$. So whenever $|D^2u| \geq C_0(\epsilon)$ and hence also $\mathcal{T} \geq \epsilon^{-1}$, we find that $Lw > 0$ for $K \geq 2C_1$ and $\epsilon \leq C_2^{-1}$. Fixing $K = 2C_1$ we conclude that if $x_0 \in \Omega$ and

$$\epsilon \leq \min\left\{C_2^{-1}, \frac{1}{2}K^{-1}(\text{diam}\Omega)^{-2}\right\}$$

(assuming the origin is contained in Ω), then

$$\sup_\Omega |D^2u| \leq C(\epsilon) . \tag{4.9}$$

In cases (4.8)′ and (4.8)″ the argument is almost identical, except that for (4.8)′ it is sufficient to use (1.21)′ in place of (1.21), while for (4.8)″ it suffices to consider in place of w the function

$$\widetilde{w} = D_{\gamma\gamma}u + K|x|^2 ,$$

and it is not necessary to assume (1.21)′ and to introduce M and ϵ. In case (4.8)‴, which occurs only if (1.13) is not satisfied, we assume (1.21)′ and argue as for (4.8)′.

In all cases then, we need to consider the possibility that w attains its maximum at $x_0 \in \partial\Omega$ and a direction $\xi \in S^1$. We shall carry out the argument for the general case $g = g(x, u, Du)$, and only indicate where a simplification occurs if g is independent of Du.

We make a rotation of coordinates so that at x_0 $\nu_1 = \nu \cdot e_1$ and $\nu_2 = \nu \cdot e_2$ are nonnegative, and so that ξ is a coordinate direction, say $\xi = e_2$. Then we have

$$D_\beta w(x_0) \leq 0 , \qquad D_2 w(x_0) \leq 0 , \tag{4.10}$$

which can be rewritten as

$$(4.11)\qquad D_{22\beta}u(x_0) \le C(1+\epsilon M)\,, \qquad D_{222}u(x_0) \le C(1+\epsilon M)\,.$$

To proceed further we need to prove some preliminary second derivative bounds. This part of the argument does not require the boundary condition to be written in a particular form, or indeed, does not even require the boundary condition to be oblique, although to obtain useful estimates we should assume $b_p \neq 0$. We therefore carry out this part of the proof for the general boundary condition (1.2). We assume that the function b has been extended in a $C^{1,1}$ fashion to $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^2$. Differentiating (1.2) tangentially we obtain

$$0 = \delta_i b = (b_{x_j} + b_z D_j u + b_{p_k} D_{jk} u)(\delta_{ij} - \nu_i \nu_j) \quad \text{on} \quad \partial\Omega\,,$$

so

$$(4.12)\qquad |D_{\tau\beta} u(x_0)| \le C$$

where τ is any unit tangent vector to $\partial\Omega$ at x_0 and $\beta = b_p/|b_p|$.

Next, differentiating the function $B = b(x, u, Du)$ we obtain

$$\begin{aligned} D_i B =& b_{x_i} + b_z D_i u + b_{p_k} D_{ik} u\,, \\ D_{ij} B =& b_{x_i x_j} + b_{x_i z} D_j u + b_{x_i p_k} D_{jk} u \\ &+ b_{x_j z} D_i u + b_{zz} D_i u D_j u + b_{z p_k} D_i u D_{jk} u \\ &+ b_{x_j p_k} D_{ik} u + b_{z p_k} D_j u D_{ik} u + b_{p_k p_l} D_{ik} u D_{jl} u \\ &+ b_z D_{ij} u + b_{p_k} D_{ijk} u\,. \end{aligned}$$

Using (4.6) we obtain

$$(4.13)\qquad \begin{aligned} F_{ij} D_{ij} B =& F_{ij}(b_{x_i x_j} + 2b_{x_i z} D_j u + b_{zz} D_i u D_j u) \\ &+ 2F_{ij}(b_{x_i p_k} D_{jk} u + b_{z p_k} D_i u D_{jk} u) \\ &+ b_{p_k p_l} F_{ij} D_{ik} u D_{jl} u + b_z F_{ij} D_{ij} u \\ &+ b_{p_k}(g_{x_k} + g_z D_k u + g_{p_l} D_{kl} u)\,. \end{aligned}$$

We now need to estimate the various terms in (4.13). Most of these can be estimated in an obvious way, but those involving products of F_{ij} and $D_{jk}u$ require a little explanation. For example, the term $F_{ij} b_{x_i p_k} D_{jk} u$ is the trace of the product XYZ of the three matrices $X = [F_{ij}]$, $Y = [b_{x_i p_k}]$

and $Z = [D_{jk}u]$, so if Q is an orthogonal matrix which diagonalizes D^2u (and hence also $[F_{ij}]$) at some point $x \in \Omega$, we have

$$\mathrm{trace(XYZ)} = \mathrm{trace(QXQ^T\,QYQ^T\,QZQ^T)}\,.$$

It is sufficient therefore to estimate $F_{ij}b_{x_ip_k}D_{jk}u$ under the assumption that D^2u and $[F_{ij}]$ are diagonal. Thus if λ_1 and λ_2 are the eigenvalues of D^2u and f_1 and f_2 are the eigenvalues of $[F_{ij}]$ we have

$$|F_{ij}b_{x_ip_k}D_{jk}u| \le C_1 \sum f_i\lambda_i \le C_2\mathcal{T}\,.$$

Here we have used Lemma 2.1 (i) and (1.9), together with the fact that λ_1 and λ_2 are nonnegative, to obtain the last inequality. The terms $F_{ij}b_{zp_k}D_iuD_{jk}u$ and $b_zF_{ij}D_{ij}u$ can be estimated similarly. We proceed in the same way to estimate the quadratic term $b_{p_kp_l}F_{ij}D_{ik}uD_{jl}u$. With the aid of Lemma 2.3 we find that for any $\epsilon > 0$ we have

$$|b_{p_kp_l}F_{ij}D_{ik}uD_{jl}u| \le C\big(C(\epsilon) + \epsilon M\big)\,\mathcal{T}\,.$$

Finally, if g depends on Du, so $g_p \not\equiv 0$, we can use (1.21)′ and (1.9) to estimate

$$|b_{p_k}g_{p_l}D_{kl}u| \le C|D^2u| \le \epsilon M\mathcal{T} + C_1(\epsilon) \le \big(\epsilon M + C_2(\epsilon)\big)\,\mathcal{T}$$

for any $\epsilon > 0$. Inserting the above estimates into (4.13) and replacing ϵ by ϵC^{-1} for a suitable constant C, we arrive at the inequality

$$(4.14)\qquad |F_{ij}D_{ij}B| \le \big(C(\epsilon) + \epsilon M\big)\,\mathcal{T} \quad \text{in} \quad \Omega$$

for any $\epsilon > 0$. If ψ is a C^2 uniformly convex defining function for Ω with $D^2\psi \ge I$, we see that for $A \ge C(\epsilon)$

$$F_{ij}D_{ij}\big((A + \epsilon M)\,\psi \pm B\big) \ge 0 \quad \text{in} \quad \Omega\,.$$

Since $(A + \epsilon M)\,\psi \pm B = 0$ on $\partial\Omega$ we conclude that

$$D_\nu\big((A + \epsilon M)\,\psi \pm B\big) \le 0 \quad \text{on} \quad \partial\Omega$$

and hence

$$(4.15)\qquad |D_{\nu\beta}u| \le C_1\big(C(\epsilon) + \epsilon M\big) \quad \text{on} \quad \partial\Omega\,.$$

Combining this with (4.12) we obtain

$$(4.16)\qquad |D_{\gamma\beta}u(x_0)| \le C_2\big(C(\epsilon) + \epsilon M\big)$$

for any direction $\gamma \in S^1$.

Remarks. – (i) If b is a semilinear boundary condition, the term $b_{p_k p_l} F_{ij} D_{ik}u D_{jl}u$ does not appear in (4.13) and it is not necessary to use Lemma 2.3. If g is independent of Du, the term $b_{p_k} g_{p_l} D_{kl}u$ does not appear and it is not necessary to assume (1.21)'.

(ii) In the uniformly elliptic case in two dimensions all the second derivatives at x_0 can be estimated from (4.16) and equation (1.1) itself, but this is not possible in our situation. We need to use the inequalities (4.11) and the boundary condition.

We now express various vectors at x_0 in the basis e_1, e_2. Recall that we have assumed that $\nu \cdot e_1$ and $\nu \cdot e_2$ are nonnegative at x_0 and $\xi = e_2$. At x_0 we have

$$\nu = ae_1 + be_2 \tag{4.17}$$

for some nonnegative constants a, b satisfying $a^2 + b^2 = 1$. Let

$$\tau = -be_1 + ae_2\,, \tag{4.18}$$

so that τ is tangential to $\partial\Omega$ at x_0. Let c, d be constants satisfying $c^2+d^2 = 1$ and such that at x_0 we have

$$\beta = ce_1 + de_2\,. \tag{4.19}$$

Since $D_{12}u(x_0) = 0$ we have

$$\begin{aligned} D_{\beta\beta}u(x_0) &= c^2\, D_{11}u(x_0) + d^2\, D_{22}u(x_0) \\ &\geq d^2\, D_{22}u(x_0)\,, \end{aligned}$$

so by (4.16),

$$D_{22}u(x_0) \leq C_2\big(C(\epsilon) + \epsilon M\big)\, d^{-2}\,. \tag{4.20}$$

We therefore need to consider only the case that $|d|$ is small, say $|d| \leq d_0$ for a suitably chosen positive constant d_0.

The main step now is to show that the two inequalities (4.11) imply

$$D_{\tau\tau\beta}u(x_0) \leq C_3\big(C(\epsilon) + \epsilon M\big)\,, \tag{4.21}$$

provided $|D^2u(x_0)|$ is sufficiently large and $|d|$ is sufficiently small. This part of the argument depends strongly on the two dimensionality. Once we have this, the required bound for $D^2u(x_0)$ will follow with the aid of our hypotheses on the boundary condition.

To prove (4.21) we first differentiate (1.1) in the directions e_1 and e_2. Noting that $D_{12}u(x_0) = 0$, we obtain at x_0

$$D_{111}u = \frac{g_{x_1} + g_z D_1 u + g_{p_1} D_{11}u}{F_{11}} - \frac{F_{22}}{F_{11}} D_{122}u \tag{4.22}$$

and

$$D_{112}u = \frac{g_{x_2} + g_z D_2 u + g_{p_2} D_{22}u}{F_{11}} - \frac{F_{22}}{F_{11}} D_{222}u . \tag{4.23}$$

Using (4.17), (4.18), (4.19), (4.22) and (4.23) we then obtain at x_0

$$\begin{aligned} D_{\tau\tau\beta}u &= b^2 D_{11\beta}u - 2ab\, D_{12\beta}u + a^2 D_{22\beta}u \\ &= b^2 c D_{111}u + b^2 d D_{112}u - 2abc\, D_{112}u \\ &\quad - 2abd\, D_{122}u + a^2 D_{22\beta}u \\ &= b^2 c\left(\frac{g_{x_1} + g_z D_1u + g_{p_1} D_{11}u}{F_{11}} - \frac{F_{22}}{F_{11}} D_{122}u\right) \\ &\quad + (b^2 d - 2abc)\left(\frac{g_{x_2} + g_z D_2u + g_{p_2} D_{22}u}{F_{11}} - \frac{F_{22}}{F_{11}} D_{222}u\right) \\ &\quad - 2abd\, D_{122}u + a^2 D_{22\beta}u \\ &= \left(a^2 - b^2 \frac{F_{22}}{F_{11}} - \frac{2abd}{c}\right) D_{22\beta}u \\ &\quad + \left(\frac{2abd^2}{c} + 2abc \frac{F_{22}}{F_{11}}\right) D_{222}u \\ &\quad + \frac{1}{F_{11}}\Big\{b^2 c\big(g_{x_1} + g_z D_1u + g_{p_1} D_{11}u\big) \\ &\quad + (b^2 d - 2abc)\big(g_{x_2} + g_z D_2u + g_{p_2} D_{22}u\big)\Big\} . \end{aligned} \tag{4.24}$$

Next, using the obliqueness condition

$$\beta \cdot \nu = ac + bd \geq \beta_0$$

for some positive constant β_0, we see that

$$ac \geq \beta_0 - bd \geq \beta_0 - |d| \geq \frac{1}{2}\beta_0 ,$$

provided

$$|d| \leq \frac{1}{2}\beta_0 . \tag{4.25}$$

Assuming henceforth that (4.25) is satisfied, we obtain, since $a^2, c^2 \leq 1$ and $a \geq 0$,

$$\frac{1}{2}\beta_0 \leq a, c \leq 1\,, \tag{4.26}$$

and hence also

$$0 \leq \frac{abd^2}{c} \leq \frac{1}{2}\beta_0 \tag{4.27}$$

and

$$0 \leq abc \leq 1\,. \tag{4.28}$$

Using Lemma 2.1 (iv) we see there is a constant $N_0 = N_0(f, \mu, \beta_0)$ such that if

$$D_{22}u(x_0) \geq N_0\,, \tag{4.29}$$

then

$$\frac{F_{22}}{F_{11}} \leq \frac{1}{8}\beta_0^2\,. \tag{4.30}$$

If we assume in addition that

$$|d| \leq d_0 = \frac{1}{32}\beta_0^3\,, \tag{4.31}$$

we obtain

$$0 \leq a^2 - b\frac{F_{22}}{F_{11}} - \frac{2abd}{c} \leq 1 + \frac{1}{8}\beta_0^2\,. \tag{4.32}$$

Next, from (4.27), (4.28), (4.29) and (4.30) we obtain

$$0 \leq \frac{2abd^2}{c} + 2abc\frac{F_{22}}{F_{11}} \leq \beta_0 + \frac{1}{4}\beta_0^2\,. \tag{4.33}$$

Finally, to control the last term of (4.24), we observe that if g is independent of Du, this term is bounded, by Lemma 2.1 (ii), while if g depends on Du and we assume (1.21)$'$, then by Lemma 2.1 (v), for any $\epsilon > 0$ the last term of (4.24) is bounded by $C\epsilon(1 + M)$ provided

$$D_{22}u(x_0) \geq N_1 \tag{4.34}$$

for some positive constant $N_1 = N_1(f, \mu, \epsilon)$. Using the above estimates and (4.11) in (4.24) we finally arrive at (4.21), provided (4.25), (4.26), (4.29), (4.31) and (4.34) are satisfied.

We now need to show that (4.21) implies a bound for $D^2u(x_0)$. We first consider the semilinear case (1.11). Computing the second tangential derivatives of (1.11) on $\partial\Omega$ we get

$$(4.35)\qquad \begin{aligned} &D_k u \delta_i\delta_j\beta_k + \delta_i\beta_k\delta_j D_k u + \delta_j\beta_k\delta_i D_k u \\ &\qquad + \beta_k\delta_i\delta_j D_k u + \delta_i\delta_j\phi \quad = 0 \quad \text{on} \quad \partial\Omega\,, \end{aligned}$$

and hence at x_0,

$$(4.36)\qquad \begin{aligned} D_{\tau\tau\beta}u \geq &- 2(\delta_i\beta_k)\, D_{jk}u\tau_i\tau_j - \phi_z\, D_{ij}u\tau_i\tau_j \\ &+ \ (\delta_i\nu_j)\,\tau_i\tau_j\, D_{\nu\beta}u - C\,. \end{aligned}$$

Using (4.16) and (4.21) we obtain at x_0

$$(4.37)\qquad -2(\delta_i\beta_k)\, D_{jk}u\tau_i\tau_j - \phi_z\, D_{\tau\tau}u \leq C_4\big(C(\epsilon) + \epsilon M\big)\,.$$

To proceed further it is convenient to choose coordinates so that $\tau = e_1$ and $\nu = e_2$ at x_0. Then (4.37) becomes

$$(4.38)\quad -2(\delta_1\beta_1)\, D_{11}u - 2(\delta_1\beta_2)\, D_{12}u - \phi_z D_{11}u \leq C_4\big(C(\epsilon) + \epsilon M\big)\,.$$

To control the second term on the left, we observe that since β is a unit vector field, $\delta|\beta|^2 = 0$ on $\partial\Omega$, and so

$$\beta_1\delta_1\beta_1 + \beta_2\delta_1\beta_2 = 0 \quad \text{at} \quad x_0\,.$$

If we now write $e_2 = \tilde{a}e_1 + \tilde{b}\beta$ at x_0, we find that $\tilde{a} = -\beta_1/\beta_2$ and $\tilde{b} = 1/\beta_2$. It follows then that at x_0

$$\begin{aligned} -2(\delta_1\beta_2)\, D_{12}u &= 2\,\frac{\beta_1}{\beta_2}\,(\delta_1\beta_1)\, D_{12}u \\ &= 2\,\frac{\beta_1}{\beta_2}\,(\delta_1\beta_1)\Big(-\frac{\beta_1}{\beta_2}\, D_{11}u + \frac{1}{\beta_2}\, D_{1\beta}u\Big) \\ &\geq -2\Big(\frac{\beta_1}{\beta_2}\Big)^2\,(\delta_1\beta_1)\, D_{11}u - C_5\big(C(\epsilon) + \epsilon M\big)\,, \end{aligned}$$

by virtue of (4.16). Inserting this into (4.38) we obtain

$$(4.39)\qquad \Big\{-2\Big(1 + \Big(\frac{\beta_1}{\beta_2}\Big)^2\Big)(\delta_1\beta_1) - \phi_z\Big\}\, D_{11}u \leq C_6\big(C(\epsilon) + \epsilon M\big)$$

at x_0, and this in turn gives us a bound

$$D_{\tau\tau}u(x_0) \leq C_7\big(C(\epsilon) + \epsilon M\big)\,, \tag{4.40}$$

by virtue of (1.19).

We now write

$$\xi = \bar{a}\tau + \bar{b}\beta(x_0)\,.$$

Since β satisfies the strict obliqueness condition (1.18) we have

$$|\bar{a}|,\, |\bar{b}| \leq C\,,$$

and hence, using (4.16) and (4.40),

$$\begin{aligned} D_{\xi\xi}u &= \bar{a}^2 D_{\tau\tau}u + 2\bar{a}\bar{b}D_{\tau\beta}u + \bar{b}^2 D_{\beta\beta}u \\ &\leq C_8\big(C(\epsilon) + \epsilon M\big) \end{aligned}$$

at x_0. For the function $w = D_{\xi\xi}u + K(1+\epsilon M)\,|x|^2$, which attains its maximum at (x_0, ξ), we then have

$$w \leq C_9\big(C(\epsilon) + \epsilon M\big)\,.$$

At the point $y_0 \in \bar{\Omega}$ at which $|D^2u(y_0)| = M = \sup_\Omega |D^2u|$ we therefore have

$$M \leq C_{10}\big(C(\epsilon) + \epsilon M\big)\,.$$

For $\epsilon > 0$ finally fixed so small that $C_{10}\epsilon \leq \dfrac{1}{2}$ and so that any other smallness requirements on ϵ are satisfied, this gives

$$\sup_\Omega |D^2u| \leq C_{11}\,. \tag{4.41}$$

This completes the proof of the second derivative bound for the semilinear boundary condition (1.11).

We now complete the proof in the fully nonlinear case (1.23). After some computation we see with the aid of (4.2) that at x_0 we have

$$\begin{aligned} D_{\tau\tau k}u\big(\nu_k + \phi_{p_l}(\delta_{kl} - \nu_k\nu_l)\big) + \phi_{p_kp_l}(\delta_i\delta_k u)(\delta_j\delta_l u)\tau_i\tau_j \\ \geq -C(1+|D^2u|)\,. \end{aligned} \tag{4.42}$$

Using (4.2) once again, the concavity condition (4.5) and the estimate (4.21) we obtain

$$(4.43) \qquad \big(D_{\tau\tau}u(x_0)\big)^2 \le C(1+M).$$

The remainder of the proof proceeds as before, so the second derivative bound is proved for (1.23) as well.

Remark. – It is evident from the proof above that somewhat weaker inequalities than (4.11) would suffice. In the case of a semilinear boundary condition we still require the estimate (4.11) for $D_{22\beta}u(x_0)$ but for $D_{222}u(x_0)$ it suffices to show

$$(4.11') \qquad D_{222}u(x_0) \le C(1+M).$$

This is clear since the coefficient of $D_{222}u$ in (4.24), namely $\dfrac{2abd^2}{c} + 2abc\,\dfrac{F_{22}}{F_{11}}$, is small if $|d|$ and $\dfrac{F_{22}}{F_{11}}$ are small. The smallness of $\dfrac{F_{22}}{F_{11}}$ follows from Lemma 2.1 (iv) if $D_{22}u(x_0)$ is large. In the case of a fully nonlinear boundary condition (1.23) with ϕ satisfying (1.27), the estimates

$$(4.11'') \qquad D_{22\beta}u(x_0) \le \epsilon M^2 + C(\epsilon)$$

for $\epsilon > 0$ sufficiently small and

$$(4.11''') \qquad D_{222}u(x_0) \le C(1+M^2)$$

are sufficient.

The hypotheses (1.7) and $g > 0$ and the second derivative bound imply that the eigenvalues of D^2u lie in a fixed compact subset of Σ. Thus (1.4) and (1.5) imply that equation (1.1) is uniformly elliptic. Consequently, we can apply the theory developed in [7] and [8] to deduce a second derivative Hölder estimate

$$(4.44) \qquad [D^2u]_{\alpha;\Omega} \le C$$

for some $\alpha > 0$. As noted in [8] (see also [7], Theorem 3), the estimate (4.44) can be proved much more easily in two dimensions than in higher dimensions, and in the two variable case (4.44) is in fact valid under our somewhat weaker regularity hypotheses. In particular, at this stage we require only $\partial\Omega \in C^{2,\gamma}$ for some $\gamma > 0$. The proofs of Theorems 1.1 to

1.4 for the case of C^∞ data are now complete. However, once we have this result, the above estimates and a standard approximation argument (approximating f, g, β, ϕ and Ω by smooth functions and domains satisfying the appropriate structure conditions of Section 1 and having uniform bounds in suitable norms) lead to Theorems 1.1 to 1.4 under our regularity hypotheses. We note that it is clearly possible to carry out this procedure in such a way that the subsolution $\underline{u}$ in Theorems 1.2 and 1.4 is also a subsolution of the approximating equations. With these observations our proofs of Theorems 1.1 to 1.4 are complete.

Remark. – An examination of the proof above shows that it suffices to assume $\partial\Omega \in C^{2,\gamma}$ for some $\gamma > 0$ in Theorems 1.1 and 1.2. Of course, in the important special case $\beta \equiv \nu$, the regularity requirement on β implies $\partial\Omega \in C^{2,1}$. A similar weakening of the regularity hypothesis on $\partial\Omega$ is not evident for the fully nonlinear boundary condition (1.23), since the form of the boundary condition and the second derivative estimation seem to require $\nu \in C^{1,1}$.

As mentioned in the introduction, for the Monge-Ampère case $f(\lambda) = (\lambda_1\lambda_2)^{1/2}$ (1.21)′ is satisfied but (1.21) is not, so the above proof of the second derivative bounds does not include the Monge-Ampère equation (1.37) in the case that g depends in a nonconvex fashion on Du. We now show how the proof needs to be modified to work for this case as well. It suffices to consider only the semilinear boundary condition (1.11); the fully nonlinear boundary condition (1.23) can be handled more simply.

For this it is convenient to write the Monge-Ampère equation in the form

$$F(D^2u) = \det D^2u = \tilde{g}(x, u, Du) \tag{4.45}$$

where $\tilde{g} = g^2$. Then

$$F_{ij} = (\det D^2u)\, u^{ij} = \tilde{g}u^{ij} \tag{4.46}$$

and

$$F_{ij,rs} = -\tilde{g}^{-1} F_{ir}F_{js} + \tilde{g}^{-1} F_{ij}F_{rs}\,. \tag{4.47}$$

Since u and Du are already bounded and g is positive, there is a number $\delta_0 > 0$ such that

$$\tilde{g}(x, u, Du) \geq \delta_0\,. \tag{4.48}$$

By Theorem 1.1, for each $\rho \in (0, 1)$ there is a unique convex function $v = v_\rho \in C^2(\bar{\Omega})$ solving the boundary value problem

$$\begin{aligned} \det D^2v &= \frac{1}{2}\,\delta_0 \quad \text{in} \quad \Omega, \\ D_\beta(v + \rho\psi) + \varphi(x, v + \rho\psi) &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{4.49}$$

where $\psi \in C^{2,1}(\bar{\Omega})$ is any uniformly convex function with $D_\beta \psi < 0$ on $\partial\Omega$. In particular, if $\partial\Omega \in C^{2,1}$, ψ can be chosen to be a defining function for Ω. By the estimates proved above

$$\sup_{\rho\in(0,1)} |v_\rho|_{2;\Omega} \leq \Lambda \tag{4.50}$$

for some positive constant Λ, independent of $\rho \in (0,1)$. From (4.50) and the fact that $\delta_0 > 0$, we also obtain

$$D^2 v_\rho \geq \lambda_0 I \tag{4.51}$$

for some positive constant λ_0, independent of $\rho \in (0,1)$. Setting $\bar{v} = v + \rho\psi$ we see that

$$\det D^2\bar{v} \leq \det D^2 v + C(\rho\Lambda + \rho^2),$$

so fixing $\rho > 0$ small enough we have

$$\det D^2 v \leq \delta_0 \quad \text{in} \quad \Omega. \tag{4.52}$$

By the mean value theorem $u - \bar{v}$ satisfies an elliptic differential inequality

$$a^{ij} D_{ij}(u - \bar{v}) \geq 0 \quad \text{in} \quad \Omega,$$

together with the boundary condition

$$D_\beta(u - \bar{v}) + \gamma(u - \bar{v}) = 0 \quad \text{on} \quad \partial\Omega$$

for some negative function γ. From the maximum principle we deduce that $u - \bar{v} \leq 0$ in Ω, and hence that $D_\beta(u - \bar{v}) \leq 0$ on $\partial\Omega$. Thus

$$D_\beta(v - u) \geq -\rho D_\beta \psi \geq \sigma\rho \quad \text{on} \quad \partial\Omega \tag{4.53}$$

for some positive constant σ.

We now consider on $\bar{\Omega} \times S^1$ the function

$$w = w(x, \gamma) = e^{\alpha(v-u)} D_{\gamma\gamma} u$$

where $v \in C^2(\bar{\Omega})$ is the unique convex solution of (4.49) with $\rho \in (0,1)$ fixed as above, and α is a positive constant to be chosen. As before, we need to bound w from above. Assume first that w attains its maximum at a point $x_0 \in \Omega$ and a direction $\xi \in S^1$. Differentiating $w = w(\cdot, \xi)$ we obtain

$$\frac{D_i w}{w} = \frac{D_{i\xi\xi}u}{D_{\xi\xi}u} + \alpha D_i(v-u)\,, \tag{4.54}$$

$$\frac{D_{ij}w}{w} - \frac{D_i w D_j w}{w^2} = \frac{D_{ij\xi\xi}u}{D_{\xi\xi}u} - \frac{D_{i\xi\xi}uD_{j\xi\xi}u}{(D_{\xi\xi}u)^2} + \alpha D_{ij}(v-u)\,. \tag{4.55}$$

Differentiating (4.45) twice in the direction ξ we get

$$F_{ij}D_{ij\xi\xi}u + F_{ij,rs}D_{ij\xi}uD_{rs\xi}u = D_{\xi\xi}\tilde{g}\,. \tag{4.56}$$

Using these inequalities and (4.51) we see that at x_0 we have

$$\begin{aligned} 0 \geq & F_{ij}\, D_{ij\xi\xi}u - \frac{1}{D_{\xi\xi}u} F_{ij}\, D_{i\xi\xi}u\, D_{j\xi\xi}u \\ & + \alpha D_{\xi\xi}u\, F_{ij}\, D_{ij}(v-u) \\ \geq & - F_{ij,rs}\, D_{ij\xi}uD_{rs\xi}u - \frac{1}{D_{\xi\xi}u} F_{ij}\, D_{i\xi\xi}u\, D_{j\xi\xi}u \\ & + \alpha\lambda_0 \mathcal{T}\, D_{\xi\xi}u - 2\alpha\tilde{g}D_{\xi\xi}u + D_{\xi\xi}\tilde{g}\,. \end{aligned} \tag{4.57}$$

Since $Dw(x_0) = 0$, (4.54) implies that at x_0 we have

$$\begin{aligned} D_{\xi\xi}\tilde{g} &\geq -C_1(1+|D^2u|^2) + \tilde{g}_{p_i}\, D_{i\xi\xi}u \\ &= -C_1(1+|D^2u|^2) - \alpha\tilde{g}_{p_i}\, D_i(v-u)\, D_{\xi\xi}u \\ &\geq -C_1(1+|D^2u|^2) - C_2\alpha D_{\xi\xi}u\,. \end{aligned} \tag{4.58}$$

To handle the third derivative terms we make a rotation of coordinates so that $D^2u(x_0)$ is diagonal with ξ a coordinate direction, say $\xi = e_2$. Using (4.46) and (4.47) we see that at x_0

$$\begin{aligned} & - F_{ij,rs}\, D_{ij\xi}uD_{rs\xi}u - \frac{1}{D_{\xi\xi}u} F_{ij}u\, D_{i\xi\xi}uD_{j\xi\xi}u \\ & = \tilde{g}^{-1}\{F_{ii}F_{jj}D_{ij2}uD_{ij2}u - (F_{ii}D_{ii2}u)^2 - F_{22}F_{ii}D_{i22}uD_{i22}u\} \\ & = \tilde{g}^{-1}\{F_{11}F_{jj}D_{j12}uD_{j12}u - |D_2\tilde{g}|^2\} \\ & \geq -\tilde{g}^{-1}\,|D_2\tilde{g}|^2\,. \end{aligned} \tag{4.59}$$

Since $\tilde{g} = g^2$ and g satisfies (4.1), we have

$$\tilde{g}^{-1}\,|D_2\tilde{g}|^2 = 4|D_2 g|^2 \le C_3(1+|D^2u|^2)\,. \tag{4.60}$$

Inserting (4.58), (4.59) and (4.60) into (4.57), and also observing that $\mathcal{T} = \Delta u$, we obtain at x_0

$$0 \ge \alpha\lambda_0(D_{\xi\xi}u)^2 - C_4\alpha D_{\xi\xi}u - C_5\big(1+(D_{\xi\xi}u)^2\big)\,. \tag{4.61}$$

A bound for $D_{\xi\xi}(x_0)$, and hence for $w(x_0,\xi)$, now follows by fixing α sufficiently large.

We now consider the case that w attains its maximum at a point $x_0 \in \partial\Omega$ and a direction $\xi \in S^1$. Assuming as before that $\xi = e_2$ and $\nu\cdot e_1$, $\nu\cdot e_2$ are nonnegative, we have

$$D_\beta w(x_0),\; D_2 w(x_0) \le 0\,, \tag{4.62}$$

which can be rewritten in the form

$$D_{22\beta}u \le -\alpha D_\beta(v-u)\,D_{22}u \le -\alpha\sigma\rho D_{22}u \tag{4.63}$$

and

$$D_{222}u \le -\alpha D_2(v-u)\,D_{22}u \le CD_{22}u \tag{4.64}$$

at x_0. The remainder of the proof of the second derivative bound (including the proof of (4.16)) now proceeds as before with (4.63) and (4.64) being used in place of (4.11). The only observation that needs to be made is that the coefficient of $D_{222}u$ in (4.24), namely $\dfrac{2abd^2}{c} + 2abc\dfrac{F_{22}}{F_{11}}$, is small if $|d|$ and $\dfrac{F_{22}}{F_{11}} = \dfrac{D_{11}u}{D_{22}u}$ are small enough.

Remark. – As noted in the remark following (4.43), weaker conditions than (4.63) and (4.64) suffice for the proof. In particular, in the case of a fully nonlinear boundary condition (4.11)$''$ and (4.11)$'''$ are sufficient. To obtain these it suffices to consider the function

$$\widetilde{w}(x,\gamma) = e^{\alpha|x|^2}\,D_{\gamma\gamma}u$$

in place of w and there is no need to introduce v. Consequently, the lower bound δ_0 for $g(x,u,Du)$ is not needed in this case.

We now complete the proof of the second derivative bound for the problem (1.44). To do this it suffices to establish the strict obliqueness estimate (1.40); since h is uniformly concave, the argument used above to bound D^2u can then be applied with only very minor modifications, while the second derivative Hölder estimate follows from uniformly elliptic theory as before.

To prove (1.40) we consider the function

$$w = \chi + AH$$

where $\chi = h_{p_k}(Du)\nu_k$, $H = h(Du)$ and A is a positive constant to be chosen. Here ν is assumed to have been extended in a $C^{1,1}$ fashion to $\bar{\Omega}$. We have

$$\begin{aligned} D_i\chi =& h_{p_kp_l}\, D_{il}u\nu_k + h_{p_k}D_i\nu_k\,, \\ D_{ij}\chi =& h_{p_kp_lp_m}\, D_{il}uD_{jm}u\nu_k + h_{p_kp_l}\, D_{ijl}u\nu_k \\ &+\ h_{p_kp_l}\, D_{il}uD_j\nu_k + h_{p_kp_l}\, D_{jl}uD_i\nu_k + h_{p_k}\, D_{ij}\nu_k \end{aligned}$$

and

$$\begin{aligned} D_iH &= h_{p_k}D_{ik}u\,, \\ D_{ij}H &= h_{p_kp_l}D_{ik}uD_{jl}u + h_{p_k}D_{ijk}u\,. \end{aligned}$$

Consequently,

$$\begin{aligned} Lw =& F_{ij}\, D_{ij}w - g_{p_i}D_iw \\ =& F_{ij}\, D_{il}uD_{jm}u\big(h_{p_kp_lp_m}\nu_k + Ah_{p_lp_m}\big) \\ &+\ h_{p_kp_l}\big(g_{x_l} + g_zD_lu\big)\,\nu_k + 2F_{ij}\, D_{il}uD_j\nu_kh_{p_kp_l} \\ &+\ F_{ij}\, D_{ij}\nu_kh_{p_k} + Ah_{p_k}\big(g_{x_k} + g_zD_ku\big) - D_i\nu_kh_{p_k}g_{p_i}\,. \end{aligned}$$

As in the second derivative estimation, the term $2F_{ij}D_{il}uD_j\nu_kh_{p_kp_l}$ can be estimated under the assumption that $[F_{ij}]$ and D^2u are diagonal at the point at which we are computing. Thus

$$2F_{ij}D_{il}uD_j\nu_kh_{p_kp_l} \leq C\mathcal{T}\,.$$

To handle the term which is quadratic in D^2u we use the uniform concavity of h and fix A so large that this term is negative. The remaining terms are easily estimated, and we arrive at

$$Lw \leq C_1(1+\mathcal{T}) \leq C_2\mathcal{T} \quad \text{in} \quad \Omega\,. \tag{4.65}$$

Now let x_0 be the point on $\partial\Omega$ at which $w\big|_{\partial\Omega}$ attains its minimum. We want to prove

$$w(x_0) = \chi(x_0) \geq c_0 \tag{4.66}$$

for some positive constant c_0. To do this we first show that

$$D_\nu w(x_0) \geq -C\,. \tag{4.67}$$

This is clear in the case that g is independent of Du, for then $\tilde{\psi} = w(x_0) + B\psi$, where ψ is a C^2 uniformly convex defining function for Ω and B is a large positive constant, is a lower barrier for w at x_0. But if g depends on Du, we need a more careful barrier argument. We adapt the one used in [1], Section 7, to our situation.

Suppose for convenience that x_0 is the origin with the positive x_2 axis in the direction of the inner normal to $\partial\Omega$ at 0. Near 0 we can represent $\partial\Omega$ as

$$x_2 = \omega(x_1) = \frac{1}{2}\,\kappa x_1^2 + O(|x_1|^3)\,, \tag{4.68}$$

where $\kappa > 0$ is the curvature of $\partial\Omega$ at 0. In $\Omega_\epsilon = \Omega \cap B_\epsilon(0)$ we shall use the barrier function

$$v = \frac{1}{2}\,(\kappa - \sigma)\,x_1^2 + \frac{1}{2}\,N x_2^2 - x_2$$

with $\sigma > 0$ fixed so small that $\kappa - \sigma > 0$. First we fix $N \geq 2$ so that

$$Lv \geq c\mathcal{T} \quad \text{in} \quad \Omega_\epsilon \tag{4.69}$$

for $\epsilon > 0$ small enough, where c is a positive constant. To do this, first observe that

$$\begin{aligned} F_{ij}\,D_{ij}v &= (\kappa - \sigma)\,F_{11} + NF_{22} \\ &\geq c_1(\mathcal{T} + NF_{22}) \end{aligned}$$

for some positive constant c_1. Let $\tilde{N}$ be the matrix $\tilde{N} = diag(1, N)$. We assume N is so large that $(1, N) \in \Sigma$. Then, using the concavity of F, we see that

$$F(\tilde{N}) \leq F(D^2u) + F_{ij}(D^2u)(\tilde{N}_{ij} - D_{ij}u)$$

and consequently

$$\begin{aligned} F_{11} + NF_{22} &\geq F(\tilde{N}) - F(D^2u) + F_{ij}(D^2u)\,D_{ij}u \\ &\geq F(\tilde{N}) - C_1\,. \end{aligned}$$

Since $|g_p| \leq C$, we see that in Ω_ϵ

$$Lv \geq \frac{1}{2}\,c_1\mathcal{T} + \frac{1}{2}\,c_1(F(\tilde{N}) - C_1) - CN\epsilon - C\,.$$

Assuming now that (1.21)′ holds, we see from Lemma 2.1 (vi) that we can fix N so large that

$$\frac{1}{2}c_1(F(\tilde{N}) - C_1) \geq 2C. \tag{4.70}$$

If we now require ϵ to be so small that

$$N\epsilon \leq 1, \tag{4.71}$$

we obtain (4.69) with $c = \frac{1}{2}c_0$.

Next we examine v on $\partial\Omega_\epsilon$. On $\partial\Omega \cap B_\epsilon$ we have, by (4.68)

$$v \leq -\frac{1}{2}\sigma x_1^2 + C|x_1|^3,$$

and hence

$$v \leq -\frac{1}{4}\sigma x_1^2 \quad \text{on} \quad \partial\Omega \cap B_\epsilon \tag{4.72}$$

for ϵ small enough. On the remaining part of $\partial\Omega_\epsilon$, where $|x| = \epsilon$, we consider two cases.

(i) $\frac{1}{2}\sigma x_1^2 > N x_2^2 = N(\epsilon^2 - x_1^2)$.

In this case, since $x_2 > \omega(x_1)$, we have on $\Omega \cap \partial B_\epsilon$

$$\begin{aligned} v &\leq \frac{1}{2}(\kappa - \sigma)x_1^2 + \frac{1}{2}Nx_2^2 - \omega(x_1) \\ &\leq -\frac{1}{2}\sigma x_1^2 + C|x_1|^3 + \frac{1}{2}Nx_2^2 \\ &\leq -\frac{1}{4}\sigma x_1^2 + C|x_1|^3 \\ &\leq -c_2\epsilon^2 \end{aligned} \tag{4.73}$$

for some positive constant c_2 and for ϵ small enough, depending only on σ and N, which have already been fixed.

(ii) $\frac{1}{2}\sigma x_1^2 \leq N x_2^2$.

On this portion of $\Omega \cap \partial B_\epsilon$ we have

$$x_2 \geq c_3\epsilon$$

for some positive constant c_3, again depending only on σ and N. Consequently,

$$
\begin{aligned}
v &\leq Cx_1^2 + \frac{1}{2} Nx_2^2 - x_2 \\
&\leq (C+N)\,\epsilon^2 - c_3\epsilon \\
&\leq -\frac{1}{2}\, c_3\epsilon
\end{aligned}
\tag{4.74}
$$

for ϵ small enough, depending only on σ and N. Finally, fixing $\epsilon > 0$ so small that (4.71), (4.72), (4.73) and (4.74) are satisfied, we conclude that $v(0) = 0$ and $v < 0$ on $\partial\Omega_\epsilon - \{0\}$.

Returning now to (4.65), we see that for $B > 0$ sufficiently large, $Bv + w(x_0)$ is a lower barrier for w at x_0, and (4.67) follows.

We now show that (4.67) implies a positive lower bound for χ at x_0. Recalling the definition of w we see that (4.67) can be rewritten as

$$
h_{p_k p_l} D_{il}u\nu_i\nu_k + h_{p_k}(D_i\nu_k)\,\nu_i + Ah_{p_k} D_{ik}u\nu_i \geq -C \quad \text{at} \quad x_0\,. \tag{4.75}
$$

We now make a rotation of coordinates so that the positive x_2 axis is in the direction of $\nu(x_0)$. Then (4.75) implies at x_0

$$
\begin{aligned}
-h_{p_2p_2}D_{22}u &\leq C_1 + Ah_{p_1}D_{12}u + Ah_{p_2}D_{22}u + h_{p_1p_2}D_{12}u \\
&\leq C_1 + C_2|D_{12}u| + Ah_{p_2}D_{22}u\,.
\end{aligned}
\tag{4.76}
$$

Since $H = h(Du) = 0$ on $\partial\Omega$, we also have

$$
h_{p_1}D_{11}u + h_{p_2}D_{12}u = 0 \quad \text{at} \quad x_0\,. \tag{4.77}
$$

Assume now that $\chi(x_0) = h_{p_2}(Du(x_0))$ is small, say $0 \leq \chi(x_0) \leq \epsilon$. Since

$$
|Dh| \geq 1 \quad \text{on} \quad \partial\Omega^* \tag{4.78}
$$

(*see* (3.19)), we see from (4.77) that

$$
D_{11}u \leq C\epsilon|D_{12}u| \quad \text{at} \quad x_0\,. \tag{4.79}
$$

Together with the convexity of u, this implies

$$
|D_{12}u|^2 \leq D_{11}uD_{22}u \leq C\epsilon|D_{12}u|\,D_{22}u \quad \text{at} \quad x_0\,, \tag{4.80}
$$

which simplifies to

$$(4.81)\qquad |D_{12}u| \le C\epsilon D_{22}u \quad \text{at} \quad x_0\,.$$

Inserting (4.81) into (4.76), using the fact that $-h_{p_2p_2} \ge \theta_0$ for some positive constant θ_0, and fixing $\epsilon > 0$ sufficiently small, we deduce

$$(4.82)\qquad D_{22}u(x_0) \le C\,.$$

But then, the estimates (4.79) and (4.81) imply

$$(4.83)\qquad D^2u(x_0) \le CI\,,$$

and this in turn implies

$$(4.84)\qquad D^2u(x_0) \ge \sigma I$$

for some positive constant σ, since $g(x, u, Du)$ has a positive lower bound. If we now use (4.83) and (4.84) in (3.18), we obtain the strict obliqueness estimate (1.40). The proof of Theorem 1.6 is now complete, in the case that g satisfies (1.45).

To conclude this section we show how to prove the existence of solutions under the modified hypotheses of Theorem 1.5. We shall use the Leray-Schauder fixed point theorem [5], Theorem 11.6, which was used for a similar purpose in [8]. Our procedure here is a little more complicated because we wish to find a solution $u \ge \underline{u}$.

We consider the semilinear case first. We assume initially that f, g, β, ϕ and $\partial\Omega$ are C^∞. These assumptions will be removed later. We fix a constant $K > 0$ so that

$$(4.85)\qquad \left[-2\left(1 + \left(\frac{\beta\cdot\tau}{\beta\cdot\nu}\right)^2\right)\delta_i\beta_j(x) + K\delta_{ij}\right]\tau_i\tau_j > 0$$

for all $x \in \partial\Omega$ and any direction τ tangential to $\partial\Omega$ at x. We also fix constants $\alpha_0 > 0$ and $\alpha_1 \in \mathbb{R}$ so that

$$(4.86)\qquad g(x, \underline{u}, D\underline{u}) > \alpha_0 \quad \text{in} \quad \Omega$$

and

$$(4.87)\qquad D_\beta\underline{u} - K\underline{u} > \alpha_1 \quad \text{on} \quad \partial\Omega\,.$$

Recall that $\underline{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is assumed to be an admissible subsolution of the problem

$$
\begin{aligned}
F(D^2 u) &= g(x, u, Du) \quad \text{in} \quad \Omega\,, \\
D_\beta u + \phi(x, u) &= 0 \quad \text{on} \quad \partial\Omega\,,
\end{aligned}
\tag{4.88}
$$

i. e., $\underline{u}$ satisfies

$$
\begin{aligned}
F(D^2 \underline{u}) &\geq g(x, \underline{u}, D\underline{u}) \quad \text{in} \quad \Omega\,, \\
D_\beta \underline{u} + \phi(x, \underline{u}) &\geq 0 \quad \text{on} \quad \partial\Omega\,.
\end{aligned}
\tag{4.89}
$$

Let u_0 be the unique admissible solution of

$$
\begin{aligned}
F(D^2 u_0) &= \alpha_0 \quad \text{in} \quad \Omega\,, \\
D_\beta u_0 - K u_0 &= \alpha_1 \quad \text{on} \quad \partial\Omega\,.
\end{aligned}
\tag{4.90}
$$

Such a u_0 exists by (4.85) and Theorem 1.1, and moreover, by elliptic regularity theory [5], Theorem 6.3 and Lemma 17.16, we have $u_0 \in C^\infty(\bar{\Omega})$. For $t \in [0, 1]$ and $v \in C^3(\bar{\Omega})$ consider the family of problems

$$
\begin{aligned}
F(D^2 u_t) &= t g(x, u_0 + v, Du_0 + Dv) + (1 - t)\,\alpha_0 \quad \text{in} \quad \Omega\,, \\
D_\beta u_t &+ t\{\phi(x, u_0 + v) - K(u_t - u_0 - v)\} \\
&- K(1 - t)\,u_t - (1 - t)\,\alpha_1 = 0 \quad \text{on} \quad \partial\Omega\,.
\end{aligned}
\tag{4.91}
$$

By (4.85), Theorem 1.1 and elliptic regularity theory, for each such v and t (4.91) has a unique admissible solution $u_t \in C^{3,\alpha}(\bar{\Omega})$ for any $\alpha < 1$. Consequently, the map $T : C^3(\bar{\Omega}) \times [0, 1] \to C^3(\bar{\Omega})$ given by $T(v, t) = u_t - u_0$ is continuous and compact, and $T(v, 0) = 0$ for all $v \in C^3(\bar{\Omega})$. If we can also show that for all the fixed points of $T(\cdot, t)$, $t \in [0, 1]$, *i. e.*, for any admissible solution u_t of

$$
\begin{aligned}
&F(D^2 u_t) = t g(x, u_t, Du_t) + (1 - t)\,\alpha_0 \quad \text{in} \quad \Omega, \\
&\quad D_\beta u_t + t\phi(x, u_t) - K(1 - t)\,u_t - (1 - t)\,\alpha_1 = 0 \quad \text{on} \quad \partial\Omega\,,
\end{aligned}
\tag{4.92}
$$

we have

$$
\|u_t\|_{C^3(\bar{\Omega})} \leq C
\tag{4.93}
$$

with C independent of t, then by [5], Theorem 11.6, $T(\cdot, 1)$ has a fixed point, or equivalently, (4.88) has an admissible solution u belonging to $C^3(\bar{\Omega})$, and hence also to $C^\infty(\bar{\Omega})$ by elliptic regularity theory. We shall prove (4.93) and in addition

$$
u_t \geq \underline{u} \quad \text{in} \quad \Omega\,.
\tag{4.94}
$$

First, by arguing exactly as in Section 3, using (1.16) and the boundary condition in (4.92), we see that

$$\sup_{\Omega} u_t \leq \max\{N, -\alpha_1 K^{-1}\}. \tag{4.95}$$

To prove the lower bound (4.94) we first observe that by our choice of α_0 and α_1, $\underline{u}$ is a subsolution of (4.92) for each $t \in [0, 1]$, and in fact, a strict subsolution if $t < 1$; consequently $u_0 > \underline{u}$ in $\bar{\Omega}$. Since T is continuous, we then also have $u_t > \underline{u}$ in $\bar{\Omega}$ for t sufficiently small. Let $t \in [0, 1]$ be the smallest number for which $u_t \geq \underline{u}$ in $\bar{\Omega}$ with equality somewhere. If there is no such t, or if $t = 1$, then (4.94) is trivially true. If $t < 1$, then equality cannot occur at an interior point, since for $t < 1$ $\underline{u}$ is a strict subsolution of (4.92). Thus equality can occur only on $\partial\Omega$, say at a point $x_0 \in \partial\Omega$, in which case we have

$$D_\beta \underline{u} < D_\beta u_t \quad \text{at} \quad x_0, \tag{4.96}$$

by the Hopf boundary point lemma. However, since $\underline{u}(x_0) = u_t(x_0)$, we see easily from the boundary conditions in (4.89) and (4.92) and our choice of α_1 that

$$D_\beta \underline{u} \geq D_\beta u_t \quad \text{at} \quad x_0, \tag{4.97}$$

which contradicts (4.96). So equality cannot occur on $\partial\Omega$ either, and (4.94) is proved.

A gradient bound for u_t now follows directly from Lemma 3.1, while the second derivative bound is proved exactly as before. We need only observe that (4.85) implies that the analogue of the structure condition (1.19) for the problem (4.92) is satisfied. Higher order estimates, and in particular (4.93), follow as before from the uniformly elliptic theory, so Theorem 1.5 is proved for the semilinear case with smooth data.

To obtain the result under the stated regularity hypotheses we approximate f, g, β, ϕ and Ω by smooth functions and domains in the usual way, as explained previously. The only point that needs to be noted is that since $\underline{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$, the approximation procedure can be carried out in such a way that $\underline{u}$ is a subsolution of the approximating problems. This is clear, since we can find a family of smooth, uniformly convex subdomains $\Omega_\epsilon \subset\subset \Omega$ uniformly bounded in the $C^{2,1}$ sense and with $\Omega_\epsilon \to \Omega$ in the $C^{2,\alpha}$ sense for any $\alpha < 1$ as $\epsilon \to 0$, such that

$$\begin{aligned} F(D^2\underline{u}) &\geq g(x, \underline{u}, D\underline{u}) \quad \text{in} \quad \Omega_\epsilon, \\ D_\beta \underline{u} + \phi(x, \underline{u}) + \epsilon &\geq 0 \quad \text{on} \quad \partial\Omega_\epsilon. \end{aligned} \tag{4.98}$$

The remaining details of the approximation procedure are straightforward.

The fully nonlinear case can be handled in a similar way. We fix $\alpha_0 > 0$ as before, but now fix $\alpha_1 \in \mathbb{R}$ so that

$$D_\nu \underline{u} - |\delta \underline{u}|^2 - \underline{u} > \alpha_1 \quad \text{on} \quad \partial\Omega, \tag{4.99}$$

where now $\underline{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is an admissible subsolution of

$$\begin{aligned} F(D^2 u) &= g(x, u, Du) \quad \text{in} \quad \Omega, \\ D_\nu u + \phi(x, u, \delta u) &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.100}$$

We now take u_0 to be the unique admissible solution of

$$\begin{aligned} F(D^2 u_0) &= \alpha_0 \quad \text{in} \quad \Omega, \\ D_\nu u_0 - |\delta u_0|^2 - u_0 &= \alpha_1 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{4.101}$$

and in place of (4.91) we consider the family of problems

$$\begin{aligned} F(D^2 u_t) &= t g(x, u_0 + v, Du_0 + Dv) + (1-t)\,\alpha_0 \quad \text{in} \quad \Omega, \\ D_\nu u_t &+ t\{\phi(x, u_0 + v, \delta u_t) - (u_t - u_0 - v)\} \\ &- (1-t)(|\delta u_t|^2 + u_t) - (1-t)\,\alpha_1 = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.102}$$

The solutions u_t corresponding to fixed points of T are then admissible solutions of

$$\begin{aligned} F(D^2 u_t) &= t g(x, u_t, Du_t) + (1-t)\,\alpha_0 \quad \text{in} \quad \Omega, \\ D_\nu u_t &+ t\phi(x, u_t, \delta u_t) - (1-t)(|\delta u_t|^2 + u_t) \\ &- (1-t)\,\alpha_1 = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.103}$$

The remainder of the proof is as before, except of course that now we use the strict concavity condition (1.27) on ϕ for the second derivative estimation in place of condition (1.19) for β. This completes the proof of Theorem 1.5.

Finally, to prove Theorem 1.6 without the monotonicity condition (1.45), we proceed in a similar fashion. We consider the family of problems

$$\begin{aligned} F(D^2 u_t) &= t\{g(x, u_0 + v, Du_t) + e^{u_t - u_0 - v}\} \\ &\quad + (1-t)\, e^{u_t} \quad \text{in} \quad \Omega, \\ h(Du_t) &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{4.104}$$

where u_0 is the unique convex solution of

$$\begin{aligned} F(D^2 u_0) &= e^{u_0} \quad \text{in} \quad \Omega, \\ h(Du_0) &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{4.105}$$

5. EXTENSIONS

In this section we present some extensions of the results proved in the previous sections.

(i) Degenerate Equations

The arguments we have used in the previous sections allow us to prove existence theorems for oblique boundary value problems for certain types of degenerate equations. In our situation the equation can degenerate in two ways. First, if g is merely nonnegative rather than positive, the eigenvalues of D^2u may not lie in a compact subset of Σ even if D^2u is bounded, and so in general we cannot deduce uniform ellipticity in the way we did before. Second, in place of (1.4) and (1.5) we may assume the degenerate ellipticity condition

$$f_i \geq 0 \quad \text{in} \quad \Sigma, \qquad i = 1, 2. \tag{5.1}$$

In fact, this follows automatically from the positivity and concavity of f, if Σ satisfies either condition (i) or (ii) of Section 1. An example of such a degenerate function f is

$$f(\lambda) = \min\{\lambda_1, \lambda_2\}. \tag{5.2}$$

Using a suitable approximation argument, which we shall describe below, we can obtain $C^{1,1}$ admissible solutions of (1.1), (1.2). Admissibility should now be interpreted as meaning that the eigenvalues of D^2u belong to $\bar{\Sigma}$ wherever D^2u exists.

To find suitable approximations to f we consider a number of cases. First, if $\partial\Sigma \subset \Gamma_+$, we define $\tilde{f}$ to be 1 on $\partial\Sigma$ and extend $\tilde{f}$ to be homogeneous of degree one. For $\epsilon > 0$ we set

$$f_\epsilon = f + \epsilon \hat{f} \tag{5.3}$$

where $\hat{f} = \tilde{f} - 1$. In the second case, if $\partial\Sigma \cap \partial\Gamma_+ \neq \emptyset$, we have $(\partial\Sigma - \partial\Gamma_+) - B_R(0) = \emptyset$ for some sufficiently large R. We may then set $\hat{f}(\lambda) = (\lambda_1\lambda_2)^{1/2}$ and define f_ϵ by (5.3). In either case, we easily verify that if $f \in C^{0,1}(\Sigma) \cap C^0(\bar{\Sigma})$ satisfies (5.1) and (1.6) to (1.9), then f_ϵ satisfies (1.4) to (1.9) (with $C(K)$ in (1.5) now depending on ϵ and K), with the possible exception of (1.7), which is satisfied in the first case ($\partial\Sigma \subset \Gamma_+$). In the second case we instead have

$$f_\epsilon \leq \epsilon R \quad \text{on} \quad \partial\Sigma. \tag{5.4}$$

Consider the family of problems

$$(5.5)\qquad \begin{aligned} F_\epsilon(D^2u) &= g(x,u) + \epsilon(R+1) \quad \text{in} \quad \Omega\,, \\ b(x,u,Du) &= 0 \quad \text{on} \quad \partial\Omega\,, \end{aligned}$$

where F_ϵ corresponds to f_ϵ in the usual way. In the case $\partial\Sigma \subset \Gamma_+$ we take R = 0. Under the hypotheses on g and b assumed in Theorem 1.1 (respectively Theorem 1.3), with the exception that g is now merely nonnegative rather than positive, we see that for $\epsilon > 0$ problem (5.5) has a unique admissible solution $u_\epsilon \in C^{2,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$, depending on ϵ. This follows directly from Theorem 1.1 (respectively Theorem 1.3); in the case that f_ϵ satisfies (5.4) rather than (1.7), we need only replace Σ by $\tilde{\Sigma} = \{\lambda \in \Sigma : f(\lambda) > \epsilon R\}$. Furthermore, the bounds of Sections 3 and 4 imply that $\|u_\epsilon\|_{C^2(\bar{\Omega})}$ is bounded independently of ϵ for $\epsilon \in (0,1)$. Here we use the observation that these bounds do not depend on a positive lower bound for $g(x,u)$ or on positive upper and lower bounds for f_1 and f_2. Consequently, as $\epsilon \to 0^+$ u_ϵ converges in any $C^{1,\alpha}(\bar{\Omega})$ norm, $\alpha < 1$, to a $C^{1,1}(\bar{\Omega})$ admissible solution u of the limit problem

$$(5.6)\qquad \begin{aligned} F(D^2u) &= g(x,u) \quad \text{in} \quad \Omega\,, \\ b(x,u,Du) &= 0 \quad \text{on} \quad \partial\Omega\,. \end{aligned}$$

We have therefore proved the following extension of Theorems 1.1 and 1.3.

THEOREM 5.1. – *Suppose all the hypotheses of Theorem* 1.1 *(respectively Theorem* 1.3*) are satisfied, except that* g, f_1 *and* f_2 *are merely nonnegative rather than positive. Then the boundary value problem* (1.20) *(respectively* (1.28)*) has a unique admissible solution* u *belonging to* $C^{1,1}(\bar{\Omega})$.

Exactly the same procedure works in the case that g depends on Du, provided we also assume (1.21), or (1.21)$'$ if g is convex with respect to Du, and we can find an *a priori* bound

$$(5.7)\qquad \sup_{\Omega} |u_\epsilon| \le C\,,$$

with C independent of ϵ for $\epsilon > 0$ sufficiently small, for admissible solutions u_ϵ of the approximating problems

$$(5.8)\qquad \begin{aligned} F_\epsilon(D^2u) &= g(x,u,Du) + \epsilon(R+1) \quad \text{in} \quad \Omega\,, \\ b(x,u,Du) &= 0 \quad \text{on} \quad \partial\Omega\,. \end{aligned}$$

This follows as in Section 3 provided we can find a strict admissible subsolution $\underline{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ of (1.1), *i. e.*, $\underline{u}$ satisfies

$$(5.9)\qquad F(D^2\underline{u}) > g(x,\underline{u},D\underline{u}) \quad \text{in} \quad \Omega\,.$$

We can then conclude the following degenerate analogue of Theorems 1.2 and 1.4.

THEOREM 5.2. – *Suppose all the hypotheses of Theorem* 1.2 *(respectively Theorem* 1.4*) are satisfied, except that now* g, f_1, *and* f_2 *are merely nonnegative rather than positive, and* $\underline{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *is now a strict admissible subsolution of* (1.1). *Then the boundary value problem* (1.1), (1.11) *(respectively* (1.1), (1.23)) *has a unique admissible solution* u *belonging to* $C^{1,1}(\bar{\Omega})$.

We can also obtain a degenerate version of Theorem 1.5 by a similar procedure. We need to strengthen the hypothesis on $\underline{u}$ only slightly by requiring $\underline{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ to be an admissible function satisfying

$$(5.10)\qquad \begin{aligned} F(D^2\underline{u}) &> g(x, \underline{u}, D\underline{u}) \quad \text{in} \quad \Omega, \\ b(x, \underline{u}, D\underline{u}) &\geq 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

We cannot obtain $C^{1,1}(\bar{\Omega})$ solutions of (1.44) in the degenerate case $g(x, u, Du) \geq 0$, because a positive lower bound for $g(x, u, Du)$ enters into the obliqueness estimate (1.40), which in turn is used in the second derivative estimation. But we can allow a degeneracy of the type (5.1) and still obtain $C^{1,1}(\bar{\Omega})$ solutions. The structure condition (1.41) is clearly preserved by the regularization

$$(5.11)\qquad f_\epsilon(\lambda) = f(\lambda) + \epsilon(\lambda_1\lambda_2)^{1/2}.$$

Finally, we mention that if $\partial\Sigma \subset \Gamma_+$ and (1.4) is satisfied, then (1.5) holds for any compact set $K \subset \bar{\Sigma}$. In this case we obtain admissible solutions $u \in C^{2,\alpha}(\bar{\Omega})$ for some $\alpha > 0$ in Theorems 1.1 to 1.5 even if g is nonnegative rather than positive.

(ii) Monge-Ampère Equations

In the case of the Monge-Ampère equation (1.37), the second derivative estimate proved in Section 4 is independent of a positive lower bound for $g(x, u, Du)$ if g is convex with respect to Du. This is true even if g is not convex with respect to Du in the case of the fully nonlinear boundary condition (1.23) (see the remark following (4.64)), but it is not true for the semilinear boundary condition (1.11). Recall that in the semilinear case the dependence on $\inf g(x, u, Du) > 0$ enters through the construction of the function v in (4.49). However, it seems reasonable to expect that the convexity condition on g can be dropped in this case as well, although we shall not pursue this here.

As we have already mentioned in the introduction, the existence of a subsolution $\underline{u}$ in Theorems 1.2 and 1.4 for the Monge-Ampère case can be replaced by suitable structure conditions on g. The relevant estimate for $\sup_{\Omega} |u|$ is proved in [10] for a semilinear degenerate oblique boundary condition of the form (1.11), but the same argument is applicable to more general oblique, possibly degenerate, boundary conditions. The following theorem summarizes our results for the Monge-Ampère equation.

THEOREM 5.3. – *Let Ω be a $C^{2,1}$ uniformly convex domain in $\mathbb{R}^2$ and let $g \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^2)$ be a nonnegative function such that*

$$g_z \geq 0 \quad in \quad \Omega \times \mathbb{R} \times \mathbb{R}^2 \tag{5.12}$$

and

$$(g(x, N, p))^2 \leq \tilde{g}(x)/\tilde{h}(p) \tag{5.13}$$

for all $(x, p) \in \Omega \times \mathbb{R}^2$ and some constant N, where $\tilde{g}, \tilde{h}$ are nonnegative functions in $L^1(\Omega), L^1_{loc}(\mathbb{R}^2)$ respectively such that

$$\int_{\Omega} \tilde{g} < \int_{\mathbb{R}^2} \tilde{h}\,. \tag{5.14}$$

(i) *If ϕ and β satisfy the hypotheses assumed in Theorem* 1.1 *and g is convex with respect to Du, then the problem*

$$\begin{aligned} (\det D^2 u)^{1/2} &= g(x, u, Du) \quad in \quad \Omega\,, \\ D_{\beta} u + \phi(x, u) &= 0 \quad on \quad \partial\Omega \end{aligned} \tag{5.15}$$

has a unique convex solution u belonging to $C^{1,1}(\bar{\Omega})$.

(ii) *If ϕ satisfies the hypotheses assumed in Theorem* 1.3, *then the problem*

$$\begin{aligned} (\det D^2 u)^{1/2} &= g(x, u, Du) \quad in \quad \Omega\,, \\ D_{\nu} u + \phi(x, u, \delta u) &= 0 \quad on \quad \partial\Omega \end{aligned} \tag{5.16}$$

has a unique convex solution u belonging to $C^{1,1}(\bar{\Omega})$.

(iii) *If in addition g is positive in $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^2$, then the solutions in* (i) *and* (ii) *belong to $C^{2,\alpha}(\bar{\Omega})$ for any $\alpha < 1$, and the convexity condition on g in* (i) *can be dropped.*

Remarks. – (i) The hypothesis in Theorems 1.2 and 1.4 concerning the existence of an admissible subsolution can be dropped if f satisfies the structure condition (1.41) and g satisfies (5.12) and

$$G(g(x, N, p)) \leq \tilde{g}(x)/\tilde{h}(p) \tag{5.17}$$

for all $(x,p) \in \Omega \times \mathbb{R}^2$ and some constant N, where $\tilde{g}, \tilde{h}$ are nonnegative functions in $L^1\Omega), L^1_{loc}(\mathbb{R}^2)$ satisfying (5.14). This is also true in the case of Theorem 5.2, provided that for small enough $\epsilon > 0$ the approximating functions f_ϵ and $g_\epsilon = g + \epsilon(R+1)$ in (5.3) and (5.8) satisfy the above conditions for suitable $G, \tilde{g}$ and $\tilde{h}$.

(ii) For the special case of the equation of prescribed Gauss curvature

$$\det D^2 u = K(x)(1+|Du|^2)^2\,, \tag{5.18}$$

conditions (5.13) and (5.14) take the simple form

$$\int_\Omega K < \pi\,. \tag{5.19}$$

(iii) Type 3 Regions

It is also possible to obtain some existence results if we drop condition (1.8). In addition to the two possibilities (i) and (ii) in Section 1 (possibly modified by translation along the diagonal $\lambda_1 = \lambda_2$), we now also have a third:

(iii) $\partial\Sigma$ is not asymptotic to the boundary of any cone.

In this case we say Σ is a *type* 3 region.

If Σ satisfies either (i) or (ii) of Section 1 (modulo a translation along the diagonal) and f satisfies (1.4) to (1.7) and (1.9), then we can make a translation so that $\partial\Sigma$ is asymptotic to the boundary of a cone with vertex at the origin. This changes the equation and boundary condition as explained in Section 1. Now, (1.9) trivially implies

$$\lim_{t\to\infty} f(t,t) = \infty\,, \tag{5.20}$$

and as observed at the end of Section 2, (5.20) implies (1.8) if Σ is of type 1 or 2. Thus for type 1 or 2 regions we reduce either to the case considered above, or to the uniformly elliptic case treated in [8].

For type 3 regions, however, some additional complications arise and we cannot obtain results of the generality of Theorems 1.1 to 1.4. If we make a translation along the diagonal so that the origin belongs to $\partial\Sigma$, we have

$$\sum f_i\lambda_i \le f(\lambda) \tag{5.21}$$

by (1.7) and the concavity of f, but in place of (1.8) we have the weaker condition

$$\sum f_i\lambda_i \ge \min\{\lambda_1,\lambda_2\}\,\mathcal{T}\,. \tag{5.22}$$

Now consider the problem

$$F(D^2u) = g(x) \quad \text{in} \quad \Omega, \tag{5.23}$$

$$D_\beta u + \gamma(x)u + \phi(x) = 0 \quad \text{on} \quad \partial\Omega, \tag{5.24}$$

where Ω is a $C^{2,1}$ uniformly convex domain in $\mathbb{R}^2$, $g \in C^{1,1}(\bar{\Omega})$ is positive, $\beta \in C^{1,1}(\partial\Omega; \mathbb{R}^2)$ is a unit vector field satisfying (1.18) and (1.19), and $\phi, \gamma \in C^{1,1}(\partial\Omega)$ with $\gamma \leq -\gamma_0$ for some positive constant γ_0. As in Section 3, we easily get a bound

$$\sup_\Omega |u| \leq C, \tag{5.25}$$

while from Lemma 3.1, or more precisely, from an examination of its proof in [10] to see that the bound depends linearly on $\sup_\Omega |u|$, we obtain

$$\sup_\Omega |Du| \leq C(1 + \tilde{M}), \tag{5.26}$$

where

$$\tilde{M} = -\min\left\{0, \inf_{\substack{x\in\Omega \\ \xi\in S^1}} D_{\xi\xi}u(x)\right\}.$$

To estimate the second derivatives, we observe that as before $w = D_{\gamma\gamma}u + K|x|^2$ attains its maximum at a point $x_0 \in \partial\Omega$ and a direction $\xi \in S^1$, provided K is sufficiently large. Thus if we choose coordinates so that $\nu \cdot e_1$ and $\nu \cdot e_2$ are nonnegative at x_0 and $\xi = e_2$, we have

$$D_{22\beta}u(x_0),\ D_{222}u(x_0) \leq C. \tag{5.27}$$

To estimate the second derivatives at x_0 we proceed as before. By tangentially differentiating the boundary condition (5.24) and using (5.25) and (5.26) we obtain

$$|D_{\tau\beta}u(x_0)| \leq C(1 + \tilde{M}). \tag{5.28}$$

To estimate $D_{\nu\beta}u(x_0)$ we extend β, ϕ and γ in a $C^{1,1}$ fashion to $\bar{\Omega}$ and set $h = D_\beta u + \gamma u + \phi$. We find that

$$\begin{aligned} F_{ij}D_{ij}h =& F_{ij}D_{ij}\beta_k D_k u + 2F_{ij}D_i\beta_k\, D_{jk}u + g_k\beta_k \\ &+ F_{ij}D_{ij}\gamma u + 2F_{ij}D_i\gamma D_j u + \gamma F_{ij}D_{ij}u + F_{ij}D_{ij}\phi\,. \end{aligned} \tag{5.29}$$

As before, the term $F_{ij}(D_i\beta_k)\, D_{jk}u$ can be estimated by first making a rotation of coordinates so that D^2u and $[F_{ij}]$ are diagonal at the point at which we are computing. Thus if $\lambda_1 \leq \lambda_2$ are the eigenvalues of D^2u and f_1, f_2 are the eigenvalues of $[F_{ij}]$ at that point, we obtain

$$\begin{aligned} |F_{ij}D_i\beta_k D_{jk}u| &\leq C \sum f_i|\lambda_i| \\ &\leq C\Big(\sup_{\Omega} g + 2f_1|\lambda_1|\Big) \\ &\leq C(1+\tilde{M})\mathcal{T}. \end{aligned}$$

Here we have used (5.21) and (5.23) to obtain the second inequality. Estimating the remaining terms in (5.29) with the aid of (5.21), (5.22), (5.25) and (5.26) we arrive at

$$(5.30) \qquad |F_{ij}\, D_{ij}h| \leq C(1+\tilde{M})\mathcal{T} \quad \text{in} \quad \Omega .$$

If now ψ is a C^2 uniformly convex defining function for Ω with $D^2\psi \geq I$, we see that

$$F_{ij}\, D_{ij}\big(A(1+\tilde{M})\,\psi \pm h\big) \geq 0 \quad \text{in} \quad \Omega$$

for $A \geq C$, and consequently

$$(5.31) \qquad |D_{\nu\beta}u(x_0)| \leq C(1+\tilde{M}) .$$

We now proceed as before using (5.27). We see that if β satisfies (1.18) and (1.19), then

$$(5.32) \qquad D_{\xi\xi}u(x_0) \leq C(1+\tilde{M}) .$$

However, since Σ is of type 3, $\tilde{M} = o(M)$ where $M = \sup_{\Omega} |D^2u|$, and consequently, we finally obtain

$$(5.33) \qquad \sup_{\Omega} |D^2u| \leq C .$$

Second derivative Hölder estimates now follow from the uniformly elliptic theory of [8], as explained before. Thus for Σ of type 3 we have the following result.

THEOREM 5.4. – *Under the hypotheses made above on* $\Sigma, f, g, \beta, \gamma$ *and* ϕ, *the boundary value problem* (5.23), (5.24) *has a unique admissible solution* u *belonging to* $C^{2,\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0,1)$.

Remarks. – (i) Because of the way in which $\tilde{M}$ enters into the estimates (5.26) and (5.30), the second derivative estimation above depends strongly on the linearity of the boundary condition and the fact that g is independent of u and Du. If we allow a nonlinear boundary condition or more general g, the argument breaks down in general.

(ii) The use of Lemma 3.1 to obtain the gradient bound can be avoided by carrying $\sup_\Omega |Du|$ through the computations, and finally obtaining in place of (5.33) the estimate

$$\sup_\Omega |D^2u| \leq C(1 + \sup_\Omega |Du|)\,. \tag{5.34}$$

Bounds for Du and D^2u then follow by interpolation. But this does not avoid the introduction of $\tilde{M}$, since it also enters in estimating the terms $2F_{ij}D_i\beta_k D_{jk}u$ and $\gamma F_{ij}D_{ij}u$ in (5.29). It is not clear how this can be avoided.

(iii) If we could prove in place of (5.28) and (5.31) the stronger estimate

$$|D_{\tau\beta}u(x_0)| + |D_{\nu\beta}u(x_0)| \leq C\,, \tag{5.35}$$

we could conclude a full second derivative bound without having to assume the structure condition (1.19) on β. For at x_0 we can write

$$D_{\xi\xi}u = a^2 D_{\tau\tau}u + 2abD_{\tau\beta}u + b^2 D_{\beta\beta}u$$

for some controlled (since β is strictly oblique) constants a and b, and so, by (5.35), $D_{\xi\xi}u(x_0)$ is bounded from above unless $D_{\tau\tau}u(x_0)$ is positive. But if $D_{\tau\tau}u(x_0)$ is positive, then for $D_{\gamma\gamma}u$ equal to the minimum eigenvalue of D^2u, we have for suitable controlled constants $\tilde{a}$ and $\tilde{b}$,

$$\begin{aligned} D_{\gamma\gamma}u &= \tilde{a}^2\, D_{\tau\tau}u + 2\tilde{a}\tilde{b}\, D_{\tau\beta}u + \tilde{b}^2\, D_{\beta\beta}u \\ &\geq -C \end{aligned}$$

at x_0, by (5.35). However, this also implies an upper bound for $D_{\xi\xi}u(x_0)$, since

$$\lim_{t\to\infty} f(s,t) = \infty \tag{5.36}$$

for any $s \in \mathbb{R}$. The proof of this is almost identical to the proof of Lemma 2.1(vi). The only difference is that if Σ is of type 3, the function ω satisfies $\omega(t) \to -\infty$ as $t \to \infty$, and the proof is valid for any $s \in \mathbb{R}$, not just for $s > 0$.

(iv) Using (5.36) we easily see that if Σ is of type 3 and f satisfies (1.4) to (1.7) and (1.9), then for any $\mu > 0$ $\{\lambda \in \Sigma : f(\lambda) > \mu\}$ is also a type 3 region. This implies that f_1/f_2 grows relatively slowly as $\lambda_2 \to \infty$

along any level line of f, so in this respect type 3 regions are intermediate between type 1 and type 2. The mild nonuniform ellipticity for type 3 regions suggests that it may be possible to drop the structure condition (1.19) on β and to allow Ω to be an arbitrary $C^{2,1}$ bounded domain in $\mathbb{R}^2$, at least for a certain class of type 3 regions. We may investigate these questions further in the future.

(iv) Other Hessian Equations

Relatively minor modifications of our arguments yield estimates and corresponding existence theorems for equations of the form

$$F\big(D^2u - \sigma(x,u)\big) = g(x,u,Du) \quad \text{in} \quad \Omega \tag{5.37}$$

subject to boundary conditions of the types (1.11) or (1.23), where g, β and ϕ satisfy the appropriate hypotheses from Section 1 and $\sigma \in C^{1,1}(\bar{\Omega} \times \mathbb{R})$ is a 2×2 symmetric matrix valued function. We should also assume, at least initially, that $\sigma_z \geq 0$. This requirement can then be dropped, along with the monotonicity conditions on g and ϕ, provided we have an admissible subsolution $\underline{u} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ of the boundary value problem in question. Admissibility is now taken to mean that at each point of Ω the eigenvalues of $D^2\underline{u} - \sigma(x,\underline{u})$ belong to Σ.

6. EXAMPLES

In this section we present a number of examples showing the necessity of some conditions such as (1.19) and (1.27). We shall first give examples in higher dimensions and later we shall construct examples in two dimensions. However, our examples in two dimensions have the disadvantage of being degenerate. We shall also give some examples showing the necessity of (1.9), and of (1.21)$'$ (at least) if g depends on Du.

(i) Higher Dimensions

We recall an example of Pogorelov [12]. He showed that for any $n \geq 3$ the function

$$u(x) = \big(1 + x_n^2\big)\left(\sum_{k<n} x_k^2\right)^{1-1/n} \tag{6.1}$$

is a convex generalized solution of the equation

$$\det D^2u = g(x) = (2\beta)^{n-1}\big(1 + x_n^2\big)^{n-2}\big(\beta - 1 - (\beta + 1)\, x_n^2\big) \tag{6.2}$$

in a small ball $B = B_\rho(0) \subset \mathbf{R}^n$, where $\beta = 2 - 2/n$. Evidently, for ρ sufficiently small g is C^∞ and positive, while $u \in C^{1,1-2/n}(\bar{B})$ and is C^∞ except along the x_n axis.

Near the point $(0, \rho) \in \partial B$ consider the vector field

$$\beta = \left(\theta x', -\sqrt{1 - \theta^2 |x'|^2}\right) \tag{6.3}$$

where θ is a positive constant and $x' = (x_1, \dots, x_{n-1})$. After some computation we find that

$$\begin{aligned} D_\beta u &= |x'|^{2-2/n} \left[2\theta \left(1 - \frac{1}{n}\right)(1 + x_n^2) - 2x_n (1 - \theta^2 |x'|^2)^{1/2} \right] \\ &= \left[2\theta \left(1 - \frac{1}{n}\right) - \frac{2x_n}{1 + x_n^2} (1 - \theta^2 |x'|^2)^{1/2} \right] u \,. \end{aligned} \tag{6.4}$$

If we set

$$\sigma(x) = -2\theta \left(1 - \frac{1}{n}\right) + \frac{2x_n}{1 + x_n^2} (1 - \theta^2 |x'|^2)^{1/2} \,, \tag{6.5}$$

we see that $D_\beta u + \sigma u = 0$ on ∂B near $(0, \rho)$ and furthermore, β is smooth and strictly oblique on ∂B near $(0, \rho)$ and $\sigma < 0$ for θ sufficiently large, depending only on n and ρ. By reflection we can define β and σ near $(0, -\rho)$ to have the same properties. On ∂B away from $(0, \pm\rho)$ u is smooth, so there we can define β and σ so that β, σ and $D_\beta u + \sigma u$ are C^∞ with β strictly oblique and σ negative. We see then that u given by (6.1) is a convex generalized solution of

$$\begin{aligned} \det D^2 u &= g(x) \quad \text{in} \quad B \,, \\ D_\beta u + \sigma u &= \phi \quad \text{on} \quad \partial B \end{aligned} \tag{6.6}$$

where g, β, σ and ϕ are C^∞, g is positive in $\bar{B}$, β is strictly oblique and σ is negative. From the construction of β it is clear that we can make β close to ν in the C^0 norm, but not in the C^1 norm.

It is interesting to examine (6.6) a little further. From [10] we know that (6.6) has a unique convex solution $u \in C^\infty(\bar{B})$ provided g, σ and ϕ are C^∞ with $g > 0$ and $\sigma < 0$, and $\beta \equiv \nu$. In fact, since B is a ball, we can say a little more. Let us assume for convenience that $B = B_1(0)$; this can be achieved by a suitable rescaling of the coordinates. Following the argument of [10], Section 4, we see that for K fixed sufficiently large,

$$w = (\delta_{kl} - x_k x_l) \, D_{kl} u + K|x|^2$$

attains its maximum on ∂B, say at a point $x_0 \in \partial B$ which we may take to be $(0,\dots,0,-1)$ by a suitable rotation of coordinates. Thus

$$D_\beta w(x_0) \le 0\,, \tag{6.7}$$

which implies

$$\sum_{\alpha<n} D_{\alpha\alpha\beta}u(x_0) \le C\,. \tag{6.8}$$

Here we have used the fact that

$$\begin{aligned} -D_\beta(x_k x_l)\, D_{kl}u &= -2(D_\beta x_k)\, x_l\, D_{kl}u \\ &= -2\beta_k\, x_l\, D_{kl}u \end{aligned}$$

is bounded on ∂B, by virtue of a standard barrier argument, for example the one used in Section 4 (see also [10], Section 3).

To proceed further, we tangentially differentiate the boundary condition twice, as we did in Section 4, to obtain

$$[-2(\delta_i\beta_k)\, D_{jk}u - \sigma\,\delta_{ij}]\,\tau_i\,\tau_j \le C + D_{\tau\tau\beta}u \tag{6.9}$$

at any point $x \in \partial B$ and any direction τ tangential to ∂B at x. In particular, summing over $\tau = e_1,\dots,\tau = e_{n-1}$ at x_0 and using (6.8), we obtain, at x_0,

$$-2\sum_{\alpha<n}\sum_{k=1}^{n}(\delta_\alpha\beta_k)\, D_{\alpha k}u - \sigma\sum_{\alpha<n} D_{\alpha\alpha}u \le C\,. \tag{6.10}$$

If we rotate the $e_1,\dots,e_{n-1}$ directions so that $[D_{ij}u(x_0)]_{i,j<n}$ is diagonal, then (6.10) becomes

$$\begin{aligned} -2\sum_{\alpha<n}(\delta_\alpha\beta_\alpha)\, D_{\alpha\alpha}u - \sigma\sum_{\alpha<n} D_{\alpha\alpha}u & \\ \le C + 2\sum_{\alpha<n}(\delta_\alpha\beta_n)\, D_{\alpha n}\,u & \end{aligned} \tag{6.11}$$

at x_0.

As in the two dimensional case, at x_0 we write

$$e_n = a_1 e_1 + \cdots + a_{n-1}\, e_{n-1} + b\beta\,,$$

so that $a_\alpha = -\dfrac{\beta_\alpha}{\beta_n}$ and $b = \dfrac{1}{\beta_n}$. We then find that at x_0

$$\begin{aligned} 2\sum_{\alpha<n}(\delta_\alpha\beta_n)\, D_{\alpha n}u &= 2\sum_{\alpha<n}(\delta_\alpha\beta_n)\left(\sum_{\gamma<n} a_\gamma D_{\alpha\gamma}u + bD_{\alpha\beta}u\right) \\ &\le -2\sum_{\alpha<n}\frac{\beta_\alpha}{\beta_n}\,(\delta_\alpha\beta_n)\, D_{\alpha\alpha}u + C\,, \end{aligned}$$

since $D_{\tau\beta}u(x_0)$ is bounded for any tangential direction τ at x_0; this follows as in the two dimensional case by tangentially differentiating the boundary condition in the direction τ. Combining this with (6.11) we obtain, at x_0,

$$-2\sum_{\alpha<n}(\delta_\alpha\beta_\alpha)\,D_{\alpha\alpha}u + 2\sum_{\alpha<n}\frac{\beta_\alpha}{\beta_n}(\delta_\alpha\beta_n)\,D_{\alpha\alpha}u - \sigma\sum_{\alpha<n}D_{\alpha\alpha}u \le C. \tag{6.12}$$

Since $D_{\alpha\alpha}u \geq 0$, an upper bound

$$D_{\alpha\alpha}u(x_0) \le C \tag{6.13}$$

for each $\alpha = 1,\dots,n-1$ follows whenever the coefficient of each $D_{\alpha\alpha}u$ in (6.12) is positive, that is, when

$$-2(\delta_\alpha\beta_\alpha) + \frac{2\beta_\alpha}{\beta_n}(\delta_\alpha\beta_n) - \sigma \ge \sigma_0 \tag{6.14}$$

at x_0 for each $\alpha = 1,\dots,n-1$ and some positive constant σ_0. A full second derivative bound

$$\sup_B |D^2u| \le C \tag{6.15}$$

then follows. Coupling this with the solution and gradient estimates proved in [10] and the second derivative Hölder estimates proved in [8], we conclude that (6.6) has a unique convex solution $u \in C^\infty(\bar{B})$ whenever $g \in C^\infty(\bar{B})$ is positive, $\sigma \in C^\infty(\partial B)$ is negative and $\beta \in C^\infty(\partial B;\mathbb{R}^n)$ is strictly oblique and satisfies (6.14) at each point $x_0 \in \partial B$ for each $\alpha = 1,\dots,n-1$, in any coordinate system such that $x_0 = (0,\dots,0,-1)$.

We have not assumed $|\beta| \equiv 1$ on ∂B to arrive at (6.14), but if we assume this, in the two dimensional case (6.14) can be put in the form (1.19), as in Section 4. Thus (6.14) appears to be the analogue of (1.19) in higher dimensions.

We also note that a similar argument works for the fully nonlinear boundary condition (1.23), provided ϕ satisfies conditions (1.24) to (1.27).

For the vector field β given by (6.3), the left hand side of (6.14) at $(0,\rho)$ for $\alpha = 1$ is

$$-2(\delta_1\beta_1) - \sigma = -\frac{2\theta}{n} - \frac{2\rho}{1+\rho^2},$$

which can be made less than any preassigned negative number by choosing θ large enough. So (6.14) fails for this vector field. This example also suggests that in two dimensions we need (1.19) to derive second derivative bounds, and that it is not sufficient to assume merely that the left hand side of (1.19) is nonzero. However, we do not know of a two dimensional example showing that this is indeed the case.

We also observe that u given by (6.1) is the unique convex generalized solution in $C^1(\bar{B})$ of (6.6). To prove this, let v be another such solution and suppose $w = u - v > 0$ somewhere in B. We show this leads to a contradiction. If $w \leq 0$ on ∂B, then by the comparison principle for generalized solutions of Monge-Ampère equations (*see* [3] or [11]) $w \leq 0$ in B. If $w > 0$ somewhere on ∂B, let x_0 be the point where $w/_{\partial B}$ attains its maximum. Then $\tilde{u} = u - u(x_0)$ and $\tilde{v} = v - v(x_0)$ satisfy the same equation as u, v and $\tilde{u} \leq \tilde{v}$ on ∂B. By the comparison principle for generalized solutions $\tilde{u} \leq \tilde{v}$ in B. Consequently $\tilde{u} - \tilde{v}$ attains its maximum at x_0 and

$$D_\beta(\tilde{u} - \tilde{v}) = D_\beta w \leq 0 \quad \text{at} \quad x_0 .$$

But from the boundary condition in (6.6) we have

$$D_\beta w = -\sigma w > 0 \quad \text{at} \quad x_0 ,$$

contradicting the above inequality.

One can also verify that u given by (6.1) is a convex generalized solution of

$$\begin{aligned} \det D^2 u &= g(x) \quad \text{in} \quad B, \\ D_\nu u + \gamma(x) u &= 0 \quad \text{on} \quad \partial B , \end{aligned} \tag{6.16}$$

where

$$\gamma(x) = \frac{2}{\rho}\left(1 - \frac{1}{n} + \frac{x_n^2}{1 + x_n^2}\right) .$$

For this boundary condition we see that

$$[-2(\delta_i \nu_j)(x) - \gamma(x)\,\delta_{ij}]\,\tau_i \tau_j = \frac{2}{\rho}\left(\frac{1}{n} - \frac{x_n^2}{1 + x_n^2}\right) > 0$$

on ∂B for ρ small enough, where as usual τ is any direction tangential to ∂B at x. Thus for ρ small enough, the data in (6.16) satisfy the regularity and structure conditions necessary to derive a second derivative bound by the method used above. But the solution u does not satisfy the necessary regularity hypotheses; the argument above requires $u \in W^{4,n}_{loc}(B) \cap C^3(\bar{B})$.

It is also interesting to observe that (6.16) has a globally smooth convex solution in addition to the nonsmooth one given by (6.1). This follows from the *a priori* estimate (6.15) proved above (or the results in [10], [13]) and an argument very similar to those used at the end of Section 4 to prove Theorem 1.5. All we need to do is find a convex subsolution $\underline{u} \in C^2(B) \cap C^1(\bar{B})$, and this is easily done. Since $\gamma \geq \gamma_0$ for some positive constant γ_0 if ρ is small enough, we see that

$$\underline{u}(x) = A(|x|^2 - \rho^2) + B$$

satisfies

$$\det D^2\underline{u} = (2A)^n \geq \sup_B g \quad \text{in} \quad B,$$
$$D_\nu\underline{u} + \gamma\underline{u} = -2A\rho + B\gamma > 0 \quad \text{in} \quad \partial B$$

if A and B are fixed large enough.

Next we demonstrate the necessity of (1.27). Straightforward computations show that on ∂B we have

$$|\delta u|^2 = A(x_n)|x'|^{2-4/n} + B(x_n)|x'|^{4-4/n}$$

for some smooth functions A and B. Thus on ∂B

$$\begin{aligned}|x'|^2|\delta u|^2 &= (A(x_n) + B(x_n)|x'|^2)|x'|^{4-4/n}\\ &= \frac{A(x_n) + B(x_n)|x'|^2}{(1 + x_n^2)^2} u^2\\ &= G(x)u^2\,.\end{aligned}$$

If β and σ are now as in (6.6), we see that u satisfies the strictly oblique boundary condition

$$D_\beta u + \sigma u - |x'|^2|\delta u|^2 - \phi(x) + G(x)u^2 = 0 \quad \text{on} \quad \partial B\,. \tag{6.17}$$

Here the concavity condition (1.27) fails at $(0, \pm\rho)$.

Now let ψ be a smooth bounded function on $\mathbb{R}$ with $|\psi'| \leq C$ and $\psi(t) = t^2$ for $|t| \leq 1$. Then, since $\sigma < 0$, we see that for ϵ and ρ small enough, u also satisfies the boundary condition

$$D_\beta u + \sigma u - \epsilon|x'|^2|\delta u|^2 - \phi(x) + \epsilon G(x)\psi(u) = 0 \quad \text{on} \quad \partial B\,, \tag{6.18}$$

and the left hand side is strictly decreasing with respect to u. An argument very similar to other one given above shows that u is the unique convex generalized solution of (6.2), (6.18) belonging to $C^1(\bar{B})$.

The above example shows that second derivative bounds may fail if we do not assume strict inequality in (1.27). Since u given by (6.1) also satisfies

$$D_\beta u + \sigma u + |x'|^2|\delta u|^2 - \phi(x) + G(x)u^2 = 0 \quad \text{on} \quad \partial B\,, \tag{6.19}$$

they can also fail if we have the weak convexity condition

$$\phi_{p_i p_j}(x, z, p^T)\, \tau_i \tau_j \geq 0 \tag{6.20}$$

for all $(x, z, p) \in \partial B \times \mathbb{R} \times \mathbb{R}^n$ and all directions τ tangential to ∂B at x. But we do not know of an example showing that second derivative bounds fail under the stronger condition

$$\phi_{p_i p_j}(x, z, p^T)\, \tau_i \tau_j > 0 \tag{6.21}$$

for all $(x, z, p) \in \partial B \times \mathbb{R} \times \mathbb{R}^n$ and all τ as above. As mentioned in the introduction, it would be interesting to resolve this question, since the capillarity boundary condition (1.12) satisfies (6.21) under the natural (in our context) condition $\theta(x, u) > 0$.

Although we have used the Monge-Ampère equation to construct our examples, we can easily obtain examples involving more general equations. We illustrate the technique here. Near $(0, \rho) \in \partial B$ let β be given by (6.3) with θ a positive constant. For σ given by (6.5) we then have $D_\beta u + \sigma u = 0$ in a neighbourhood of $\xi_+ = (0, \rho)$, say in $\bar{B} \cap B_r(\xi_+)$ for some $r = r(\theta) > 0$. By reflection we obtain a vector field β with similar properties in $\bar{B} \cap B_r(\xi_-)$ where $\xi_- = (0, -\rho)$. Using the fact that u is smooth everywhere except along the x_n axis, we can define β on all of $\bar{B} - B_{\rho - r}(0)$ so that β and $D_\beta u + \sigma u$ are smooth, and $\beta \cdot x < 0$ in this region. Now let η be a smooth, nonnegative function with $\eta \leq 1$ such that $\eta \equiv 1$ on ∂B and $\eta \equiv 0$ in $B_{\rho - r/2}(0)$. Define a vector field $\tilde{\beta}$ on $\partial \tilde{B}$, where $\tilde{B} = B^N_\rho(0)$, $N > n$, is the open ball in $\mathbb{R}^N$ of radius ρ centred at the origin, by

$$\tilde{\beta}(X) = \eta(x)\beta(x) - (1 - \eta(x)) \sum_{j=n+1}^{N} x_j e_j \tag{6.22}$$

where $X = (x_1, \ldots, x_N) \in \partial B$, $x = (x_1, \ldots, x_n)$ and e_j are the unit coordinate vectors in $\mathbb{R}^N$. Clearly $\tilde{\beta}$ is smooth on ∂B, and it is strictly oblique since

$$\tilde{\beta}(X) \cdot X = \eta(x)\, \beta(x) \cdot x - (1 - \eta(x)) \sum_{j=n+1}^{N} x_j^2,$$

which is negative by our construction of β and our definition of η. If we now extend u given by (6.1) to be constant in the $x_{n+1}, \ldots, x_N$ directions, and similarly extend σ and define $\tilde{\sigma} = \eta\sigma$, we find that $D_{\tilde{\beta}}u + \tilde{\sigma}u$ is smooth on ∂B and u solves

$$F_n(D^2u) = g(x_1, \ldots, x_n) \quad \text{in} \quad \tilde{B} \tag{6.23}$$

in the generalized sense of [14] or [17]. Here F_n is given by

$$F_n(D^2u) = S_n(\lambda) = \sum_{1 \le i_1 < \ldots < i_n \le N} \lambda_{i_1} \ldots \lambda_{i_n} \tag{6.24}$$

where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of D^2u. Recall that (6.23) is one of the main examples of Hessian equations considered in [2] and [13].

We can follow a similar procedure and show that u also satisfies a boundary condition similar to (6.17) on ∂B. We leave this to the reader.

(ii) Two Dimensions

In the two dimensional case it appears to be more difficult to construct examples which are nondegenerate because we do not know of an explicit nonregular solution of a suitable nondegenerate equation as we do in higher dimensions. But degenerate examples are easy to construct and we shall be content with this. Recall however, that the second derivative bounds proved in Section 4 for the boundary conditions (1.11) and (1.23) are independent of any positive lower bound for g, and are valid even if (1.4) and (1.5) are replaced by the degenerate condition (5.1). The only exception to this is the special case of the Monge-Ampère equation (1.37) with g depending on Du in a non-convex fashion, and even this only for the semilinear boundary condition (1.11). So despite being degenerate, our two dimensional examples show that some conditions such as (1.19) or (1.27) are necessary for second derivative estimates, and we cannot dispense with these without somehow using the positivity of g or imposing some additional conditions on f.

For $\epsilon > 0$ let

$$u(x, y) = (\epsilon^2 + x^2)^{1/2}, \tag{6.25}$$

so that

$$\det D^2u = 0. \tag{6.26}$$

Let B be the open unit ball in $\mathbb{R}^2$ and let $\beta = (0, 1)$ near $(0, -1)$. Then β is strictly oblique and $D_\beta u = 0$ on ∂B near $(0, -1)$. As before, we can extend β to all of ∂B and obtain a smooth, strictly oblique vector field with

$$D_\beta u = \phi(x, y) \quad \text{on} \quad \partial B \tag{6.27}$$

for some smooth function ϕ. However, u and Du are bounded independently of ϵ for $\epsilon \in (0,1]$, while $D_{xx}u = \epsilon^{-1}$ along $x = 0$. In this example we can also make β close to ν in the C^0 norm, but not in the C^1 norm.

Next we have

$$|\delta u|^2 = \frac{x^2 - x^4}{\epsilon^2 + x^2} \quad \text{on} \quad \partial B,$$

so u also satisfies the boundary condition

$$D_\beta u - (\epsilon^2 + x^2)|\delta u|^2 = \phi(x,y) - (x^2 - x^4) \quad \text{on} \quad \partial B. \tag{6.28}$$

Again we see the necessity of assuming (1.27), at least in the degenerate case $g \geq 0$.

By arguing as in the higher dimensional case, we can also obtain counterexamples for the equation

$$F_2(D^2 u) = 0 \quad \text{in} \quad B_1^N(0), \tag{6.29}$$

where $F_2(D^2u)$ is defined by (6.24) with $n = 2$.

Next we give an example where g is positive, but the equation has a degeneracy of the form (5.1). Let

$$u(x,y) = (\epsilon^2 + x^2)^{1/2} + \frac{1}{2}(x^2 + y^2). \tag{6.30}$$

Then if f is given by

$$f(\lambda) = \min\{\lambda_1, \lambda_2\}, \tag{6.31}$$

we see that u solves

$$F(D^2 u) = 1 \quad \text{in} \quad B. \tag{6.32}$$

But for the same vector field as above we find that

$$D_\beta u = \tilde{\phi}(x,y) \quad \text{on} \quad \partial B \tag{6.33}$$

for some smooth function $\tilde{\phi}$. In addition, a computation shows that on ∂B we have

$$|\delta u|^2 = x^2 y^2 (\epsilon^2 + x^2)^{-1},$$

so for any positive integer m u satisfies

$$D_\beta u - x^{2m}|\delta u|^2 = \tilde{\phi}(x,y) - x^{2m+2} y^2 (\epsilon^2 + x^2)^{-1} \quad \text{on} \quad \partial B. \tag{6.34}$$

For any positive integer k we can fix m so large that the right hand side of (6.34) is bounded in the $C^k(\bar{B})$ norm independently of $\epsilon \in (0,1]$. This shows the necessity of some conditions such as (1.19) or (1.27) even if g is positive.

(iii) Conditions (1.9) and (1.21)′

We now give some examples showing the necessity of (1.9), and of (1.21)′ (or a stronger condition such as (1.21)) if g depends on Du.

An example of [2] shows the necessity of (1.9). For any positive integer k the function f given by

$$f(\lambda) = 2 - (\lambda_1\lambda_2)^{-2k} \tag{6.35}$$

satisfies (1.4) to (1.8) on $\Sigma = \{\lambda \in \Gamma_+ : f(\lambda) > 0\}$, but not (1.9). In the unit ball $B \subset \mathbb{R}^2$ the function

$$\tilde{u}(x,y) = u(r) = r^2 - 1 + \frac{1}{3}(1-r^2)^{3/2}, \quad r^2 = x^2 + y^2, \tag{6.36}$$

is a convex solution of

$$\begin{aligned} F(D^2\tilde{u}) &= g(x,y) \quad \text{in} \quad B, \\ D_\nu \tilde{u} &= 2 \quad \text{on} \quad \partial B, \end{aligned} \tag{6.37}$$

where $g \geq 1$ belongs to $C^k(\bar{B})$, but not to $C^{k+1}(\bar{B})$. However $u \notin C^2(\bar{B})$. Clearly $\tilde{u}$ also satisfies the boundary condition

$$D_\nu \tilde{u} - |\delta \tilde{u}|^2 = 2 \quad \text{on} \quad \partial B. \tag{6.38}$$

Next, consider the same function $\tilde{u}$, but now take

$$f(\lambda) = \frac{\lambda_1\lambda_2}{\lambda_1 + \lambda_2}. \tag{6.39}$$

Then f satisfies (1.4) to (1.9) on $\Sigma = \Gamma_+$, but not (1.21)′, since

$$\mathcal{T} = \frac{\lambda_1^2 + \lambda_2^2}{(\lambda_1 + \lambda_2)^2}. \tag{6.40}$$

For the function $\tilde{u}$ the eigenvalues of $D^2\tilde{u}$ are given by

$$\begin{aligned} \lambda_1 &= \frac{\dot{u}}{r} = 2 - (1-r^2)^{1/2} \geq 1, \\ \lambda_2 &= \ddot{u} = 2 - (1-r^2)^{1/2} + r^2(1-r^2)^{-1/2} \geq \lambda_1. \end{aligned}$$

We find that

$$F(D^2\tilde{u}) = \frac{(2-A)(2-A+r^2A^{-1})}{2(2-A)+r^2A^{-1}} \tag{6.41}$$

where $A = A(r) = (1 - r^2)^{1/2}$. Evidently the right hand side of (6.41) is smooth for $0 \leq r < 1$, but not at $r = 1$. Let η be a smooth nonnegative function such that $\eta \equiv 1$ in $B_{1/4}$, $\eta \equiv 0$ in $B_1 - B_{3/4}$, and for $\epsilon > 0$ small enough, let $\tilde{\eta}$ be a smooth positive function on $\mathbb{R}^2$ such that $\tilde{\eta}(x) = |x|$ outside B_ϵ. Then since $A(r) = 2 - |D\tilde{u}|/r$, we see that for $\epsilon > 0$ small enough

$$A(r) = \eta A(r) + (1 - \eta)\left(2 - \frac{\tilde{\eta}(D\tilde{u})}{\tilde{\eta}(x)}\right),$$

and the right-hand side is a smooth function of x and $D\tilde{u}$. Thus $\tilde{u}$ solves

$$F(D^2\tilde{u}) = g(x, D\tilde{u}) \quad \text{in} \quad B \tag{6.42}$$

for some smooth positive function g.

This shows the necessity of some condition such as (1.21)′ if g depends on Du, but we do not know whether (1.21)′ needs to be strengthened to (1.21) if g depends on Du in a nonconvex fashion, or whether some weaker condition suffices. As we have already observed, for the Monge-Ampère case $f(\lambda) = (\lambda_1\lambda_2)^{1/2}$ (1.21) just fails, but we do not need a convexity condition on g (except in the degenerate case $g \geq 0$ with a semilinear boundary condition). Presumably (1.21), and possibly even (1.21)′, can be weakened in certain other cases by somehow exploiting the finer structure of f. From the proof of the second derivative bounds it is clear that (1.21) (respectively (1.21)′) can be weakened slightly by requiring $\mathcal{T}|\lambda|^{-1}$ (respectively $\mathcal{T}$) to be sufficiently large (depending on g, β, ϕ and Ω) as $|\lambda| \to \infty$ on $\Sigma_\mu = \{\lambda \in \Sigma : f(\lambda) \leq \mu\}$ for any $\mu > 0$. But this is no longer a condition just on f.

We also assumed (1.21)′ in the case $g = g(x, u)$ and g does not satisfy

$$g_z \geq 0 \quad \text{in} \quad \Omega \times \mathbb{R}. \tag{6.43}$$

We do not know whether this is necessary in general. It is evident from the proof of the second derivative bounds that (1.21)′ can be dropped if

$$g_z \geq -\epsilon \quad \text{in} \quad \Omega \times \mathbb{R} \tag{6.44}$$

for $\epsilon > 0$ small enough, depending on g, β, ϕ and Ω. The necessity of (1.21)′ if (6.44) fails, and of (1.21) in the case that g depends in a nonconvex fashion on Du, is evidently closely related to the question of whether a global second derivative estimate follows from a boundary second derivative estimate under these conditions.

ACKNOWLEDGEMENT

Some of the research for this paper was carried out while the author was visiting the University of Bonn from May to July 1991. The author wishes to thank the Sonderforschungsbereich 256 for its support during this time.

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(Manuscript received February 16, 1994;
accepted June 21, 1994.)