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Stabilization of second order evolution equations by unbounded nonlinear feedback

by

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ABSTRACT. – For an abstract evolution equation of the form $u_{tt} + Au + \partial\psi(u_t) \ni 0$, general conditions on the "unbounded" feedback are given, that ensure strong asymptotic stability. Essentially the directions determined by the convex of the minima of the functional ψ should not intersect the eigenspaces of A. Equivalently, the feedback on the velocity must dissipate enough energy, in the sense that the kernel of the form $\langle \partial\psi(\cdot), \cdot\rangle$ is not larger than the kernel of a "strategic" observation operator, for the uncontrolled system. The particular case where the control operator is the dual of the observation operator is specifically considered: the condition then corresponds to more classical rank conditions on the observation operator. The interest of the present framework is that it applies to boundary controls and to interior controls on thin sets (of zero measure but positive capacity). Several examples, including wave, beam and plate equations are considered.

Key words: Stabilization, Nonlinear feedback.

RÉSUMÉ. – Pour un problème d'évolution abstrait de la forme $u_{tt} + Au + \partial\psi(u_t) \ni 0$, on donne des conditions générales sur le feedback « non borné » pour assurer la stabilité asymptotique forte. Essentiellement les directions déterminées par le convexe des minima de ψ ne doivent pas être des directions propres de l'opérateur A. De façon équivalente, il faut que le bouclage sur la vitesse soit suffisamment dissipatif, en ce sens que le

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noyau de la forme $\langle \partial\psi(\cdot), \cdot\rangle$ ne doit pas être plus gros que le noyau d'un observateur « stratégique », pour le système non contrôlé. Le cas particulier où l'opérateur de contrôle est l'adjoint de l'opérateur d'observation est étudié; la condition se ramène alors à des conditions plus classiques de rang sur l'opérateur d'observation. Le cadre proposé ici englobe le cas de contrôles frontières ou intérieurs, distribués ou ponctuels, ainsi que des contrôles unilatéraux. Divers exemples concernant les équations des ondes, des poutres ou des plaques, éventuellement avec des contrôles sur des ensembles « fins », sont proposés.

0. INTRODUCTION

We consider here abstract evolution equations of the form

$$u_{tt} + Au + \partial\psi(u_t) \ni 0 \tag{0}$$

where A is linear self-adjoint and $\partial\psi(u_t)$ is a nonlinear dissipative mechanism built with the subdifferential $\partial\psi$ of a functional ψ. Our main result provides a necessary and sufficient condition on ψ to get global strong asymptotic stabilization in (0). This condition states essentially that the convex of the zeros of the nonnegative functional ψ should not intersect the eigenspaces of A. The approach adopted here to prove stabilization is rather classical: through LaSalle's invariance principle, it reduces to the analysis of a "uniqueness" or "unique continuation" property for the linear part of the equation. This is done by using spectral expansions. These techniques have been introduced and intensively used in [DA], [DA$_1$], [HA], [HA$_1$], [DA-SLE], [LA], [LAG$_1$], [SLE].

Our abstract framework makes a very systematic use of these ideas and provides a clean and general statement together with a very simple and elementary proof. It includes most of the stabilization characterizations for wave –and plate– like equations with distributed feedbacks, but also with boundary controls or controls on thin sets (of zero measure but of positive capacity). Applications to several (classical and less classical) examples are given in the second part of the paper.

1. ABSTRACT FRAMEWORK. WELLPOSEDNESS

We first describe an abstract framework for second order equations of the form

$$u_{tt} + Au + \partial\psi(u_t) \ni 0,$$

and we prove existence and uniqueneness of a solution and dissipativity of the energy along trajectories. This part is more or less classical.

Let H be a Hilbert space, and let A be a linear operator with dense domain $D(A)$. We assume A is self-adjoint, coercive on H, and we define $V = D(A^{1/2})$, equipped with the scalar product

$$\langle u, v\rangle_{V\times V} = (A^{1/2} u, A^{1/2} v)_{H\times H} = \langle \tilde{A} u, v\rangle_{V'\times V} \tag{1}$$

where $\tilde{A} \in \mathcal{L}(V, V')$ is defined by the bilinear form $\langle , \rangle_{V\times V}$ and extends A. As usual, we identified H with its dual. Then $V \subset H \subset V'$, with the following consistency relation

$$\forall h \in H, \quad \forall v \in V, \quad \langle h, v\rangle_{V'\times V} = (h, v)_{H\times H}. \tag{2}$$

Let be given a proper, convex, lower semi-continuous (l.s.c.) function

$$\psi : \quad V \to]-\infty, \infty], \qquad \psi \not\equiv +\infty \tag{3}$$

with effective domain $D(\psi) = \{v \in V;\ \psi(v) < \infty\}$. We consider the sub-differential $\partial\psi$ of ψ defined by

$$\begin{cases} \forall u \in V, \\ \partial\psi(u) = \{f \in V';\ \psi(u+v) - \psi(u) \geqq \langle f, v\rangle_{V'\times V},\ \forall v \in V\}, \end{cases} \tag{4}$$

with $D(\partial\psi) = \{u \in V;\ \partial\psi(u) \neq \emptyset\}$.

It is known that $\partial\psi$ is a maximal monotone graph from V to V', and that $D(\partial\psi) \subset D(\psi)$, with a dense inclusion [BA-PRE].

Next, on the space $V \times H$ equipped with the natural product Hilbert structure, we define the nonlinear operator B by

$$\begin{aligned} D(B) = \{(v, h) \in V \times H;\ h \in D(\partial\psi);& \\ \exists f \in \partial\psi(h) \text{ such that } \tilde{A} v + f \in H\}& \end{aligned} \tag{5}$$

and if $(v, h) \in D(B)$

(6) $B(v, h) = \{(-h, \tilde{A}\, v + f);\ \forall f \in \partial\psi(h) \text{ such that } \tilde{A}\, v + f \in H\}$.

Equivalently, in terms of graphs considered as subsets of $V \times H$

$$(7) \quad B = \{(v, h) \times (-h, [\tilde{A}\, v + \partial\psi(h)] \cap H);\ (v, h) \in V \times H\}.$$

We now prove that we have an adequate general framework to define a second order evolution equation.

PROPOSITION 1. – *B is maximal monotone on $V \times H$.*

Proof. – (i) The monotonicity is easy to establish. Let $(v, h), (\tilde{v}, \tilde{h}) \in D(B)$, $f \in \partial\psi(h)$, $\tilde{f} \in \partial\psi(\tilde{h})$, such that $\tilde{A}\, v + f$ and $\tilde{A}\, \tilde{v} + \tilde{f} \in H$. Then [*see* (1) and (2)]

$$\begin{aligned}(8) \quad & \langle -h + \tilde{h},\, v - \tilde{v}\rangle_{V \times V} + (\tilde{A}\, v + f - \tilde{A}\, \tilde{v} - \tilde{f},\, h - \tilde{h})_{H \times H} \\ &= \langle \tilde{A}\,(v - \tilde{v}),\, \tilde{h} - \tilde{h}\rangle_{V' \times V} + \langle \tilde{A}\, v + f - \tilde{A}\, \tilde{v} - \tilde{f},\, h - \tilde{h}\rangle_{V' \times V} \\ &= \langle f - \tilde{f},\, h - h\rangle_{V' \times V} \geqq 0, \text{ since } \partial\psi \text{ is monotone on } \mathrm{V} \times \mathrm{V}'.\end{aligned}$$

(ii) We prove that $I + B$ is onto on $V \times H$: for $(F, G) \in V \times H$, we have to solve the system

$$(9) \quad \begin{cases} (v, h) \in D(B) \\ v - h = F \\ h + \tilde{A}\, v + f = G, \quad f \in \partial\psi(h) \qquad (\text{thus } \tilde{A}\, v + f \in H) \end{cases}$$

which is equivalent to the system

$$(10) \quad \begin{cases} v \in V, \quad h \in D(\partial\psi) \subset V & (a) \\ v = F + h & (b) \\ h + \tilde{A}\, h + f = G - \tilde{A}\, F, \quad f \in \partial\psi(h) & (c) \end{cases}$$

[observe that (9) $\Rightarrow$ (10) obviously and that, if (10) holds, then $\tilde{A}\, v + f = \tilde{A}\, h + f + \tilde{A}\, F = G - h \in H$].

Essentially, we have to prove that (10 *c*) holds *i. e.* that the operator $I + \tilde{A} + \partial\psi$ from V to V' is onto. But this is easy to do, and more or less standard. We recall briefly the procedure. Let

$$J(\theta) = \frac{1}{2}(\theta, \theta)_{H \times H} + \frac{1}{2}\langle \theta, \theta\rangle_{V \times V} + \psi(\theta) - \langle G - \tilde{A}F, \theta\rangle_{V' \times V}$$

which is a convex, l.s.c. functional on V. Since ψ is convex, l.s.c. and proper, it is bounded below by an affine function, and hence, $\lim J(\theta) = +\infty$, as $|\theta|_V \to +\infty$. Therefore, J admits at least a minimum at $h \in V$. Then

$$J(h) \leqq J(h + t(\theta - h)), \qquad \forall \theta \in V, \quad \forall t \in [0, 1].$$

Writing this inequality, using the convexity of ψ, then dividing by t and finally letting $t \searrow 0_+$, we get

$$\begin{aligned}\langle G - \tilde{A}F, \theta - h\rangle_{V' \times V} \leqq (h, \theta - h)_{H \times H} \\ + \langle h, \theta - h\rangle_{V \times V} + \psi(\theta) - \psi(h), \qquad \forall \theta \in V.\end{aligned}$$

Since $\langle h, \theta - h\rangle_{V \times V} = \langle \tilde{A} h, \theta - h\rangle_{V' \times V}$ and $(h, \theta - h)_{H \times H} = \langle h, \theta - h\rangle_{V' \times V}$, we deduce that $G - \tilde{A} F - h - \tilde{A} h \in \partial \psi(h)$, which is just (10 c).

We denote by $S^B(t)$ the nonlinear semi-group of contractions on $\overline{D(B)}$ generated by B. As a consequence of the general theory [BRE], we have the following properties, where $(u(t), v(t)) = S^B(t)(u_0, u_1)$:

(11) $\quad \forall (u_0, u_1) \in \overline{D(B)}, \quad t \to (u(t), v(t)) \in \mathcal{C}([0, \infty[; V \times H)$

and, for any $(u_0, u_1) \in D(B)$

(12) $$\forall t \geqq 0, \quad (u(t), v(t)) \in D(B)$$

(13) $t \to (u(t), v(t)) \in V \times H$ is Lipschitz continuous and a.e. differentiable on $[0, \infty[$

(14) $$\begin{cases} v(t) = u_t(t), \; u_{tt}(t) + \tilde{A} u(t) + f(t) = 0, \\ \qquad f(t) \in \partial \psi(v(t)), \; \text{a.e. } t \geqq 0. \end{cases}$$

Moreover, for any $(u_0, u_1) \in D(B)$

(15) $t \to (-v(t), [\tilde{A} u(t) + \partial \psi(v(t))]^0) \in V \times H$ is right-continuous everywhere, where $[\,]^0$ denotes the minimal section.

(16) $$\begin{aligned}&\|(-v(t), [\tilde{A} u(t) + \partial \psi(v(t))]^0)\|_{V \times H} \\ &\quad \leqq \|(-u_1, [\tilde{A} u_0 + \partial \psi(u_1)]^0)\|_{V \times H}, \qquad \forall t \geqq 0.\end{aligned}$$

Note that in particular, if $(u_0, u_1) \in D(B)$ then $u \in W^{2,\infty}(0, \infty; H) \cap W^{1,\infty}(0, \infty; V)$.

For any $(u_0, u_1) \in \overline{D(B)}$, we define

$$E(t, u_0, u_1) = \frac{1}{2}|u(t)|_V^2 + \frac{1}{2}|v(t)|_H^2. \tag{17}$$

PROPOSITION 2. – (i) *Assume* $(u_0, u_1) \in D(B)$, *then* $\forall 0 \leqq s \leqq t$

$$E(t, u_0, u_1) - E(s, u_0, u_1) = -\int_s^t \langle f(\sigma), \nu(\sigma)\rangle_{V' \times V}\, d\sigma, \tag{18}$$

where $f(\sigma) \in \partial\psi(\nu(\sigma))$ *a.e.* σ *satisfies* (14).

(ii) *Assume* $0 \in \partial\psi(0)$; *then* $\forall (u_0, u_1) \in \overline{D(B)}$, $t \to E(t, u_0, u_1)$ *is nonincreasing.*

Proof. – (i) Let $(u_0, u_1) \in D(B)$; by the regularity property (13), we have, a.e. t:

$$\begin{aligned}\frac{d}{dt} E(t, u_0, u_1) &= \langle u_t(t), u(t)\rangle_{V \times V} + (v_t(t), v(t))_{H \times H}\\ &= \langle \tilde{A} u(t), u_t(t)\rangle_{V' \times V} + (-\tilde{A} u(t) - f(t), v(t))_{H \times H},\end{aligned}$$

where $f(t) \in \partial\psi(v(t))$ satisfies (14)

$$\begin{aligned}&= \langle \tilde{A} u(t), u_t(t)\rangle_{V' \times V} - \langle \tilde{A} u(t) + f(t), v(t)\rangle_{V' \times V} \quad \text{by (2)}\\ &= -\langle f(t), v(t)\rangle_{V' \times V}.\end{aligned}$$

Then (18) follows by integration.

(ii) Since $\partial\psi$ is monotone and $0 \in \partial\psi(0)$, we have

$$\langle f(\sigma), v(\sigma)\rangle_{V' \times V} \geqq 0.$$

Thus $t \to E(t, u_0, u_1)$ is nonincreasing for $(u_0, u_1) \in D(B)$. By density, and continuity of $E(t, .,.)$ on $V \times H$, the property is true for $(u_0, u_1) \in \overline{D(B)}$.

Remark 1. – A frequent situation is when ψ is given by $\psi = \phi \circ C$ where $C \in \mathcal{L}(V, U)$, U is another Hilbert space and $\phi : U \to]-\infty, \infty]$ is proper convex and l.s.c. It fits in our framework if $R(C) \cap D(\phi) \neq \emptyset$. It has been proved in [LA] (*see* also [CO-PI$_1$], Appendix 1) that if C is surjective, then $\partial\psi = C^* \partial\phi C$ and that the equation

$$u_{tt} + \tilde{A} u + C^* \partial\phi C u_t \ni 0$$

defines a nonlinear semi-group of contractions on a closed subset of $V \times H$. This situation corresponds to the case where the control operator C^* is the adjoint of the observation operator C, with a nonlinear interaction between observation and control described by $\partial\psi$. Therefore, we have proved a result which includes a familiar framework in control theory.

Remark 2. – The fact that $\partial\psi$ acts from V to V' and not necessarily from H to H, is essential to describe boundary or pointwise feedback between observation and control in P.D.E's (when $\psi = \phi \circ C$).

Remark 3. – Even in the restricted case where $\psi = \phi \circ C$ and C is surjective, it may be as convenient to keep the formulation with ψ, and to compute $\partial\psi$ directly, without using $C^* \, \partial\varphi \, C$. This will be observed later on, in the applications (Section 3).

2. STRONG ASYMPTOTIC STABILITY

We now study the asymptotic behaviour of the semi-group $S^B(t)$. The main tool will be the invariance principle of LASALLE. As usual, some compactness has to be assumed.

We suppose that $0 \in \partial\psi(0)$, thus after a normalization

$$\min_{v \in V} \psi(v) = \psi(0) = 0 \tag{19}$$

We denote by K_ψ the closed convex set where ψ attains its minimum

$$K_\psi = \{\psi \in V;\ \psi(v) = 0\}. \tag{20}$$

Moreover, we assume B has compact resolvent (for conditions ensuring the compactness, *see* [CO-PI$_1$], Appendix 2). It follows from this fact and from (19) that the trajectories of $S^B(t)$ are relatively compact in $V \times H$, and that for any $(u_0, u_1) \in \overline{D(B)}$, the ω-limit set $\omega(u_0, u_1)$ is a non-empty closed set [DA-SLE]. Moreover, we have the following:

PROPOSITION 3. – *Assume* (19) *and* B *has compact resolvent. Then*

(21) • *If* $(u_0, u_1) \in D(B)$, *then* $\omega(u_0, u_1) \in D(B)$.

• $\omega(u_0, u_1)$ *is invariant under* $S^B(t)$, *and the restriction of* $S^B(t)$ *on* $\omega(u_0, u_1)$ *is an isometry for the* $V \times H$*-norm.*

• *Let* $(u_0, u_1) \in D(B)$, $(w_0, w_1) \in \omega(u_0, u_1)$ *and* $(w(t), w_t(t)) = S^B(t)(w_0, w_1)$.

Then

$$0 = \langle f(t), w_t(t)\rangle_{V' \times V}, \qquad f(t) \in \partial\psi(w_t(t)) \text{ a.e. } t, \tag{22}$$

where $f(t)$ *is defined as in* (14).

• *If moreover, ψ satisfies the following property*

$$\partial\psi(0) = \{0\} \tag{23}$$

then

$$\begin{cases} (w_0, w_1) \in D(B_0), \\ \textit{and} \\ (w(t), w_t(t)) = S^{B_0}(t)(w_0, w_1), \end{cases} \tag{24}$$

where B_0 is the operator B corresponding to $\partial\psi \equiv 0$, i.e.

$$\begin{cases} D(B_0) = D(A) \times V, \\ \textit{and} \\ \forall (v, h) \in D(B_0), \quad B_0(v, h) = (-h, Av). \end{cases} \tag{25}$$

Proof. – The claim (21) follows directly from the maximality of B and the fact that $t \to \|B^0(u(t), u_t(t))\|_{V\times H}$ is uniformly bounded, and in fact decreasing along the trajectories [*see* (16)].

The fact that $S^B(t)$ is an isometry on $\omega(u_0, u_1)$ is a consequence of the invariance principle of LASALLE, since

$$t \to E(t, u_0, u_1) = \frac{1}{2}\|(u(t), v(t))\|^2_{V\times H}$$

is a Lyapunov function [*See* Proposition 2 (ii)].

Property (22) follows from formula (18) applied to (w_0, w_1) instead of (u_0, u_1) : $t \to E(t, w_0, w_1)$ is constant, and $\langle f(\sigma), w_t(\sigma)\rangle_{V'\times V}$ is nonnegative a.e. For properties (24) and (25), we first observe that since $f(t) \in \partial\psi(w_t(t))$ a.e., we get

$$\psi(0) \geqq \psi(w_t(t)) - \langle f(t), w_t(t)\rangle_{V'\times V}$$

and thus, from (22) we deduce

$$\psi(w_t(t)) \leqq \psi(0), \text{ thus } \psi(w_t(t)) = 0 = \psi(0).$$

Let $v \in V$ be arbitrary. Then

$$\begin{aligned} \psi(v) - \psi(0) &= \psi(v) - \psi(w_t(t)) \\ &\geqq \langle f(t), v - w_t(t)\rangle_{V'\times V} = \langle f(t), v\rangle_{V'\times V} \quad \text{[using (22)]}. \end{aligned}$$

This implies $f(t) \in \partial\psi(0)$, and by assumption (23) we get $f(t) = 0$ a.e. t. Therefore, w satisfies the equation derived from (14)

$$w_{tt} + \tilde{A}w(t) = 0 \quad \text{a.e. } t \geqq 0,$$

thus $\tilde{A}w(t) \in H$ a.e. $t \geqq 0$. This implies

$$w(t) \in D(A) \text{ a.e. } t \geqq 0, \qquad \text{and} \qquad \tilde{A}w(t) = Aw(t),$$

and also

$$B\,w(t), w_t(t))^0 = B_0(w(t)) \text{ a.e. } t.$$

But by (16)

$$\begin{aligned}\|B_0(w(t), w_t(t))\|_{V\times H} &= \|B(w(t), w_t(t))^0\|_{V\times H}\\ &\leqq B(w_0, w_1)\|_{V\times H} \quad \text{a.e. } t.\end{aligned}$$

Letting t tend to zero, we get $(w_0, w_1) \in D(B_0)$, by maximality of the operator B_0. Clearly, then $(w(t), w_t(t)) = S^{B_0}(t)(w_0, w_1)$.

Next, we come to our main results of the paper, namely a characterization of asymptotic stabilization in terms of ψ (more precisely of K_ψ).

We will assume that the resolvent of A is compact and denote by F_i the associated eigenspaces. They are of finite dimension and

$$V = \bigoplus_{i\geqq 1} F_i.$$

THEOREM 4. – *Assume A and B have compact resolvent and* (19), (23). *Then*

$$\text{(26)}\qquad \forall (u_0, u_1) \in \overline{D(B)}, \qquad \lim_{t\nearrow\infty} S^B(t)(u_0, u_1) = 0 \quad \textit{in } V\times H$$

if and only if

$$\text{(27)}\qquad \forall i \geqq 1, \quad K_\psi \cap (-K_\psi) \cap F_i = \{0\}.$$

Proof. – Proposition 3 shows that, on the ω-limit set $\omega(u_0, u_1)$, the nonlinear semi-group $S^B(t)$ coincides with the linear semigroup $S^{B_0}(t)$ associated with the uncontrolled operator B_0. Therefore, our first step will consist in representing the solution of the uncontrolled linear equation in terms of the spectrum of A.

We denote by ζ_n, $n \geqq 0$ the eigenfunctions of A, with associated eigenvalues $\mu_n^2 > 0$ (not necessarily distinct), so that $\{\zeta_n\}_n$ is an orthonormal basis of H (and an orthogonal basis of V) and

$$|\zeta_n|_H = 1, \qquad |\zeta_n|_V = |A^{1/2}\zeta_n|_H = \mu_n > 0.$$

It is straightforward that $\frac{1}{\sqrt{2}}(\zeta_n/\mu_n, \mp i\,\zeta_n)$ is a Hilbert basis of eigenfunctions of B_0 in $V\times H$ associated with eigenvalues $\pm i\,\mu_n$.

Let $(w_0, w_1) \in \overline{D(B_0)} = V \times H$, $(w(t), w_t(t)) = S^{B_0}(t)(w_0, w_1)$. By a straightforward computation we get

(28) $$w(t) = \sum_{\mathbb{N}} [a_n \cos \mu_n t + b_n \sin \mu_n t] \zeta_n / \mu_n$$

(29) $$w_t(t) = \sum_{\mathbb{N}} [b_n \cos \mu_n t - a_n \sin \mu_n t] \zeta_n$$

where (28) [resp. (29)] converges in V, (resp. H). Moreover

$$a_n = \langle w_0, \zeta_n/\mu_n \rangle_{V \times V} \in l^2(\mathbb{N}), \text{ since } w_0 \in V \text{ and } |\zeta_n/\mu_n|_V = 1,$$

$$b_n = (w_1, \zeta_n)_{H \times H} \in l^2(\mathbb{N}), \text{ since } w_1 \in H \text{ and } |\zeta_n|_H = 1.$$

Now, if we assume that $(w_0, w_1) \in D(B_0)$, the convergences in (28) and (29) can be improved.

First, $w_0 \in D(A)$, thus

$$a_n = \frac{1}{\mu_n} (A^{1/2} w_0, A^{1/2} \zeta_n)_{H \times H} = \frac{1}{\mu_n} (A w_0, \zeta_n)_{H \times H}$$

which implies

$$\mu_n a_n = (A w_0, \zeta_n)_{H \times H} \in l^2(\mathbb{N}), \text{ since } A w_0 \in H, \text{ and } |\zeta_n|_H = 1.$$

Next $w_1 \in V$, thus

$$b_n = (w_1, \zeta_n)_{H \times H} = \frac{1}{\mu_n^2} (w_1, A \zeta_n)_{H \times H}$$

$$= \frac{1}{\mu_n^2} (A^{1/2} w_1, A^{1/2} \zeta_n)_{H \times H} = \frac{1}{\mu_n} \left\langle w_1, \frac{\zeta_n}{\mu_n} \right\rangle_{V \times V}$$

which implies

$$\mu_n b_n = \left\langle w_1, \frac{\zeta_n}{\mu_n} \right\rangle_{V \times V} \in l^2(\mathbb{N}), \text{ since } w_1 \in V, \text{ and } |\zeta_n/\mu_n|_V = 1.$$

Since $V = D(A^{1/2}) = \{\sum_{\mathbb{N}} \alpha_n \zeta_n; \sum_{\mathbb{N}} \alpha_n^2 \zeta_n^2 < \infty\}$, we deduce that if $(w_0, w_1) \in D(B_0)$, then, in particular, the series (29) converges in V, uniformly with respect to t.

If we now denote by $0 < \omega_1^2 < \omega_2^2 < \cdots < \omega_i^2 < \cdots$ the **distinct** eigenvalues of A, with associated eigenspaces F_i of dimension p_i, and $\{\zeta_i^j\}_{j=1, \cdots, p_i}$ a basis of eigenfunctions of F_i such that the whole system corresponds to $\{\zeta_n\}_n$, then (28) and (29) can be rewritten (with obvious notations)

$$w(t) = \sum_{i=1}^{\infty} \sum_{j=1}^{p_i} [a_i^j \cos \omega_i t - b_i^j \sin \omega_i t] \frac{\zeta_i^j}{\omega_i}, \tag{30}$$

$$w_t(t) = \sum_{i=1}^{\infty} \sum_{j=1}^{p_i} [b_i^j \cos \omega_i t - a_i^j \sin \omega_i t] \zeta_i^j, \tag{31}$$

where the series converges in V, uniformly in t. This finishes our first step and we will now use the above representation to prove the sufficient and necessary conditions announced in the theorem.

Sufficiency. – Assume (26). By the contraction property of the semi-group $S^B(t)$, it is enough to prove (26) for $(u_0, u_1) \in D(B)$.

Let $(u_0, u_1) \in D(B)$, and $(w_0, w_1) \in \omega(u_0, u_1)$ It is enough to prove that $(w_0, w_1) = 0$.

Let $(w(t), w_t(t)) = S^B(t)(w_0, w_1)$. By Proposition 3, (23) and (24) imply that $(w_0, w_1) \in D(B_0)$, $(w(t), w_t(t)) = S^{B_0}(t)(w_0, w_1)$, thus formula (31) is valid, and can be more simply written as

$$w_t(t) = \sum_{i=1}^{\infty} \cos \omega_i t \, \varphi_i + \sin \omega_i t \, \psi_i, \qquad \text{with} \quad \varphi_i, \psi_i \in F_i \tag{32}$$

where the series converges in V, uniformly in t. Moreover (*see* proof of Proposition 3)

$$w_t(t) \in K_\psi, \quad \text{a.e. } t \geqq 0. \tag{33}$$

From the uniform convergence in (32) we deduce that

$$\forall \varepsilon > 0, \quad \exists N_0 \in \mathbb{N} \text{ such that for all } N \geqq N_0$$

$$\left| w_t(t) - \sum_{i=1}^{\infty} [\cos \omega_i t \, \varphi_i + \sin \omega_i t \, \psi_i \right|_V \leqq \varepsilon, \qquad \forall t \geqq 0$$

and thus, with $\eta = \pm 1$, we deduce

$$\begin{aligned} &\left| \frac{1}{T} \int_0^T \frac{1 + \eta \cos \omega_p t}{2} w_t(t) \right. \\ &\left. - \sum_{i=1}^{N} \frac{1}{T} \int_0^T \frac{1 + \eta \cos \omega_p t}{2} (\cos \omega_i t \, \varphi_i + \sin \omega_i t \, \psi_i) \right|_V \\ &\leqq \left| \frac{1 + \eta \cos \omega_p t}{2} \right|_{L^\infty} \varepsilon = \varepsilon. \end{aligned} \tag{34}$$

On the other hand, a straightforward computation gives

$$\int_0^T (1+\eta\cos\omega_p t)\ \cos\omega_i t\,dt = \frac{1}{\omega_i}\sin\omega_i T + \frac{\eta}{2}\begin{cases}\dfrac{1}{\omega_i+\omega_p}\sin(\omega_i+\omega_p)T \\ \quad +\dfrac{1}{\omega_i-\omega_p}\sin(\omega_i-\omega_p)T, & \text{if } i\neq p \\ T+\dfrac{1}{2\,\omega_p}\sin 2\,\omega_p T, & \text{if } i=p\end{cases} \tag{35}$$

$$-\int_0^T (1+\eta\cos\omega_p t)\ \sin\omega_i t\,dt = \frac{1}{\omega_i}\cos w_i T + \frac{\eta}{2}\begin{cases}\dfrac{1}{\omega_i+\omega_p}[\cos(\omega_i+\omega_p)T-1] \\ \quad +\dfrac{1}{\omega_i-\omega_p}[\cos(\omega_i-\omega_p)T-1], & \text{if } i\neq p \\ \dfrac{1}{2\,\omega_p}\cos 2\,\omega_p T, & \text{if } i=p\end{cases} \tag{36}$$

From (34), (35) and (36) one obtains for $N > p$

$$\left|\lim_{T\nearrow\infty}\frac{1}{T}\int_0^T \frac{1+\eta\cos\omega_p t}{2}\,w_t(t)\,dt-\frac{\eta}{4}\varphi_p\right|_V \leqq \varepsilon. \tag{37}$$

By (33) $w_t(t)\in K_\psi$ a.e. t, and $\dfrac{1+\eta\cos\omega_p t}{2}\in[0,\,1]$ by the choice of η. Since K_ψ is a closed convex set containing the origin, we get

$$\frac{1+\eta\cos\omega_p t}{2}\,w_t(t)\in K_\psi \quad \text{a.e. } t,$$

and consequently, with the same argument,

$$\lim_{T\nearrow\infty}\frac{1}{T}\int_0^T \frac{1+\eta\cos\omega_p t}{2}\,w_t(t)\in K_\psi. \tag{38}$$

From (37) and (38) one deduces

$$\pm\frac{1}{4}\varphi_p\in K_\psi. \tag{39}$$

But, by definition, $\varphi_p \in F_p$. So (27) implies $\varphi_p = 0$. Similarly, multiplying (32) by $\dfrac{1+\eta\,\sin\,\omega_p\,t}{2}$ instead of $\dfrac{1+\eta\,\cos\,\omega_p\,t}{2}$ and integrating over $(0,\,T)$, one gets $\psi_p = 0$, $\forall\,p$. Hence $w_t\,(t) = 0$ for all t, and by (32), (31), (30), $w_t\,(t) = 0$ for all t also. This proves (26).

Necessity. – We argue by contradiction. Suppose that, for some i, there exists

$$v \in K_\psi \cap (-K_\psi) \cap F_i, \qquad v \neq 0.$$

Then consider $u\,(t) = -\dfrac{1}{\omega_i}\,\cos\,\omega_i\,t\,v$, $u_t\,(t) = \sin\,\omega_i\,t\,v$.

It is clear that $u_{tt}\,(t) = -\omega_i^2\,u\,(t)$ and thus

$$u_{tt} + A\,u = 0. \tag{40}$$

So $(u\,(t),\,u_t\,(t)) = S^{B_0}\,(t)\left(-\dfrac{1}{\omega_i}\,v,\,0\right)$.

Since $v \in K_\psi \cap (-K_\psi)$, for any $\lambda \in [-1,\,1]$, $\lambda\,v \in K_\psi$, and thus $\psi\,(\lambda\,v) = 0$. Deriving this identity w.r.t. λ yields

$$\langle f,\,v\rangle = 0, \qquad \forall\,f \in \partial\psi\,(\lambda\,v), \quad \text{or else}$$

$$\langle f,\,\lambda\,v\rangle = 0, \qquad \forall\,f \in \partial\psi\,(\lambda\,v).$$

As in the proof of Proposition 3, one deduces $f \in \partial\psi\,(0)$, and from condition (23), one obtains

$$\partial\psi\,(\lambda\,v) = \{0\}, \qquad \forall\,\lambda \in [-1,\,1].$$

Applying the result to $\lambda = \sin\,\omega_i\,t$ one gets

$$\partial\psi\,(u_t) = \{0\}.$$

Thus u, which is a solution of (40) is also a solution of

$$u_{tt} + Au + \partial\psi\,(u_t) \ni 0, \tag{41}$$

so $(u\,(t),\,u_t\,(t)) = S^B\,(t)\left(-\dfrac{1}{\omega_i}\,v,\,0\right)$. But $(u\,(t),\,u_t\,(t)) \not\to 0$ as $T \nearrow \infty$, whence the contradiction.

In applications, the following corollary is often useful.

COROLLARY 5. – *Assume* $\psi\,(\nu) = \varphi\,(C\,\nu)$, *where* $C \in \mathcal{L}(V,\,U)$, *with* U *a Hilbert space and* $\varphi : U \to [0,\,\infty]$ *convex, l.s.c., satisfying* $\varphi\,(0) = \min\,\varphi = 0$, *and*

$$K_\varphi \cap (-K_\varphi) = \{0\}, \quad \textit{where} \;\; K_\varphi = \{u \in U;\; \varphi\,(u) = \varphi\,(0) = 0\}. \tag{42}$$

Assume moreover that A and B have compact resolvent and (23). *Then strong asymptotic stability holds for $S^B(t)$ if and only if*

$$\forall\, i \in \mathbb{N}, \qquad \operatorname{Ker} C \cap F_i = \{0\}. \tag{43}$$

Proof. – We apply Theorem 4, and thus, have to compute K_ψ.

$$\begin{aligned} K_\psi \cap (-K_\psi) &= \{v \in V;\ C\,v \in K_\varphi\} \cap \{v \in V;\ C(-v) \in K_\varphi\} \\ &= \{v \in V;\ C\,v \in K_\varphi \cap (-K_\varphi)\} = \operatorname{Ker} C \quad \text{by (42)}. \end{aligned}$$

Therefore (43) is equivalent to (27). Clearly (19) is satisfied, and (23) is assumed. Thus Theorem 4 applies.

Remark 1. – Now, we can discuss the meaning of Corollary 5 in a framework familiar in control theory, namely when the observation operator is the dual of the control operator. Besides (42) assume moreover that

$$\partial\psi(v) = C^*\,\partial\varphi(C\,v) \tag{44}$$

which is certainly true is φ is regular or if C is surjective ([LA], [CO-PI$_1$], Appendix 1). Assume also

$$\partial\varphi(0) \subset \operatorname{Ker} C^* \quad (\text{for instance } \partial\varphi(0) = \{0\}!) \tag{45}$$

Then (23) is satisfied and Corollary 5 applies.

Remark 2. – Let us note that (43) is equivalent to a rank condition. Indeed (43) means that C restricted to F_i is injective, or else rank $C_{|F_i} = \dim F_i$. This type of condition appears naturally when one wants to characterize weak observability for the uncontrolled system

$$\begin{cases} w_{tt} + A\,w = 0 \\ z = C\,w_t \end{cases} \tag{46}$$

that is,

$$C\,w_t = 0, \qquad \forall\, t \geqq 0 \quad \Rightarrow \quad w \equiv 0. \tag{47}$$

We refer to [EL JAI-PRI] for a discussion, or [TRI, Theorem 5.5] for a theory in the case where $C \in \mathcal{L}(H, U)$. Here C may be unbounded on H.

In our framework, where A is coercive with compact resolvent, we prove equivalence between weak observability for (46) and the rank condition (43) as follows: we apply C to (31) to obtain

$$C\, w_t\,(t) = \sum_{i=1}^{\infty} [\tilde{b}_i \cos \omega_i\, t - \tilde{a}_i \sin \omega_i\, t]$$

with

$$\tilde{b}_i = \sum_{j=1}^{p_i} b_i^j\, C\, \varphi_i^j, \qquad \tilde{a}_i = \sum_{i=1}^{p_i} a_i^j\, C\, \varphi_i^j.$$

Assume first $C\, w\,(t) = 0$ a.e. $t \geqq 0$. As in the proof of Theorem 4 (but more easily since here K_ψ is replaced by $\{0\}$), we deduce that $\tilde{a}_i = \tilde{b}_i = 0$, $i = 1, 2, \cdots$ By the rank condition (43), this implies $a_i^j = b_i^j = 0$ and therefore $w_t \equiv w \equiv 0$.

Assume conversely that rank $C_{|F_i} < \dim F_i = p_i$; then there exist $(\alpha_i^j)_{j=1, \cdots, p_i} \neq 0$ such that

$$\sum_{j=1}^{p_i} \alpha_i^j\, C\, \varphi_i^j = 0, \qquad \text{but} \sum_{j=1}^{p_i} \alpha_i^j\, \varphi_i^j \neq 0.$$

Set $w_0 = \sum_{j=1}^{p_i} \alpha_i^j\ \varphi_i^j$ and $w_1 = 0$. Then $(w\,(t),\ w_t\,(t)) = S^{B_0}\,(t)\,(w_0,\ w_1)$ satisfies

$$w\,(t) = \sum_{j=1}^{p_i} \alpha_i^j \cos \omega_i\, t \frac{\varphi_i^j}{\omega_i} \not\equiv 0$$

but $C\, w_t\,(t) = -\sum_{j=1}^{p_i} \alpha_i^j \sin \omega_i\, t\, C\, \varphi_i^j \equiv 0.$

Remark 3. – Corollary 5 is an extension of former results found in the literature, in the following sense.

Consider an abstract evolution equation of the form

$$\frac{dy}{dt} + \mathcal{A}\, y + \mathcal{B}^*\, u = 0 \tag{48}$$

where $\mathcal{A}$ generates a strongly continuous semi-group of contractions on a Hilbert space $\mathcal{H}$ and $\mathcal{B}^* \in \mathcal{L}\,(\mathcal{U},\ \mathcal{H})$, with $\mathcal{U}$ another Hilbert space. Assume $\mathcal{A}$ has compact resolvent.

Then following [BE], it is known that system (48) is strongly stabilizable iff the weakly (or strongly, by compactness of the resolvent) unstable states are approximatively controllable (here "unstable" means that it belongs to the orthogonal of the asymptotically stable states for $\mathcal{A}$). In that case, $u = -\mathcal{B}\, y$ (or more generally, $u = -K\, \mathcal{B}\, y$, where $K \in \mathcal{L}(\mathcal{U})$ is coercive), where $\mathcal{B} \in \mathcal{L}(\mathcal{H},\, \mathcal{U})$ is a stabilizing feedback.

Consider now our case where

$$\mathcal{A} = \begin{pmatrix} 0 & -I \\ A & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{B} = [0,\, C], \quad \mathcal{H} = V \times H, \quad \mathcal{U} = U. \tag{49}$$

Since $\mathcal{A}$ is skew-adjoint, the associated semi-group is conservative and all states are unstable, in the sense made precised above.

We recall that, roughly speaking, approximate controllability of a pair $(\mathcal{A},\, \mathcal{B}^*)$ is equivalent to weak observability of the dual pair $(\mathcal{A}^*,\, \mathcal{B})$, and, as remarked previously, characterized by rank conditions.

Hence we have proved a *nonlinear unbounded* extension of the result of [BE], namely, in the framework of second order systems: the system

$$y_{tt} + A\, y + C^*\, u = 0 \qquad \text{with} \quad C \in \mathcal{L}(V,\, U), \tag{50}$$

is strongly stabilizable iff the pair (A, C) is weakly observable. In that case $u \in \partial\varphi\,(C\, y_t)$ is a stabilizing feedback, provided $\varphi : U \to [0, \infty]$ is convex, l.s.c., proper, and satisfies (42) and (44)-(45) (with compactness of B).

We observe that Theorem 4 goes beyond this formulation, since it does not need the introduction of any observation operator C.

Remark 4. – Another interesting feature of our formulation is that it can handle "unilateral" feedback conditions, since conditions (27) or (42) concern $K\,\psi \cap (-K\,\psi)$, or $K\,\varphi \cap (-K\,\varphi)$, and not separately K_ψ or K_φ. This will be used in examples in next section. Such "unilateral" feedbacks were also considered previously ([HA$_1$]).

Remark 5. – Finally, we would like to remark that Theorem 4 is a way of systematically reducing the problem of stabilization to the verification of an adequate uniqueness property for the operator A, in an abstract "unbounded nonlinear" framework (for the damping term). For a similar point of view, in a linear or nonlinear framework, *see* [DA], [Q-R] for an abstract formulation, and [LAG$_1$], [LA$_1$], [Q-R] for applications. In particular, the formalization and results in [DA] are very similar to ours, though developed for bounded feedbacks.

3. APPLICATIONS

Now we show how the strong stabilization for various nonlinear feedback terms can be deduced from Theorem 4 or Corollary 5, in the case of wave, beam or plate-like equations. Of course, with our technique, we do not obtain any estimate of the decay to zero. Other techniques are needed, together generally with geometric assumptions on the domain, *see* for example [CHE], [LAG], [LA], [KO-ZU], [ZU] for the wave equation with boundary damping, and [LA], [LAG$_1$] for plate-like equations.

3.1. Euler-Bernoulli beam equation

We consider a beam which is clamped at the left end, and controlled at the right end by a force and a moment which are nonlinear functions of the transversal and angular velocities. We assume variable mass density b and flexural rigidity a such that

$$a,\ b \in L^{\infty}(0,\ 1),\ a,\ b \geqq m > 0 \tag{51}$$

$$\begin{cases} b u_{tt} + (a u_{xx})_{xx} = 0, & 0 < x < 1, \quad t > 0 \\ u(0,\ t) = u_x(0,\ t) = 0, & t > 0 \\ -a u_{xx}(1,\ t) \in \beta(u_{xt}(1,\ t)), & t > 0 \\ (a u_{xx})_x(1,\ t) \in \alpha(u_t(1,\ t)), & t > 0, \end{cases} \tag{52}$$

where α and β are maximal monotone graphs in $\mathbb{R}^2$ with $0 \in \alpha(0)$, $0 \in \beta(0)$. This problem has been specifically studied in [CO-PI]. Here we only show that it fits in our general abstract framework.

Let $H = L^2(0,\ 1)$ equipped with the scalar product $(u,\ v)_H = \int_0^1 b(x)\, u(x)\, v(x)\, dx$,

$$V = \{v \in H^2(0,\ 1);\ v(0) = v_x(0) = 0\},$$

$$\langle \tilde{A}\, u,\ v\rangle_{V' \times V} = \int_0^1 a u_{xx}\, v_{xx}\, dx,$$

$$D(A) = \{u \in V;\ (a u_{xx})_{xx} \in H;\ a u_{xx}(1) = (a u_{xx})_x(1) = 0\}$$

$$\forall\, u \in D(A), \qquad A\, u = \frac{1}{b}\,(a u_{xx})_{xx}.$$

Let $j_1, j_2 : \mathbb{R} \to [0, \infty]$ be convex, l.s.c. and proper such that $\partial j_1 = \alpha$, $\partial j_2 = \beta$, $j_1(0) = j_2(0) = 0$. We set $\psi(v) = j_1(v(1)) + j_2(v_x(1)) = \varphi(Cv)$, $\forall v \in V$, where

$$\varphi(\xi_1, \xi_2) = j_1(\xi_1) + j_2(\xi_2), \qquad \forall (\xi_1, \xi_2) \in U = \mathbb{R}^2,$$

and $C \in \mathcal{L}(V, U)$ is defined by $Cv = (v(1), v_x(1))$. Then

$$\partial\psi(v) = \alpha(v(1))\,\delta_1 - \beta(v_x(1))\,\delta_1'$$

[here, C is surjective, so that $\partial\psi(v) = C^*\,\partial\varphi(Cv)$]. Noticing that, for regular v, one has

$$\begin{aligned}\langle \tilde{A}\,v + \partial\psi(h), w\rangle_{V' \times V} =& a v_{xx}(1)\,w_x(1) - (a v_{xx})_x(1)\,w(1)\\ &+ \alpha(h(1))\,w(1) + \beta(h_x(1))\,w_x(1)\\ &+ \int_0^1 (a v_{xx})_{xx}\,w\,dx,\end{aligned}$$

we get immediately from (5)

$$\begin{aligned}D(B) = \{(v, h) \in V \times H;\ h \in V;\ h(1) \in D(\alpha);\ h_x(1) \in D(\beta);\\ a v_{xx}(1) \in \beta(h_x(1));\ (a v_{xx})_x(1) \in \alpha(h(1));\ (a v_{xx})_{xx} \in H\}\end{aligned}$$

and if $(v, h) \in D(B)$

$$B(v, h) = \left(-h, \frac{1}{b}(a v_{xx})_{xx}\right).$$

One can prove that $D(B)$ is dense in $V \times H$, thus (52) is well-posed on $V \times H$, and that A and B have compact resolvent [CO-PI]. The eigenvalues and eigenvectors of A are given by

$$\begin{cases} (a\,\varphi_{xx})_{xx} = \omega^2\,b\,\varphi \\ \varphi(0) = \varphi_x(0) = 0 \\ a\,\varphi_{xx}(1) = (a\,\varphi_{xx})_x(1) = 0. \end{cases}$$

All the eigenvalues ω_k^2, $k = 1, 2, \cdots$, are simple, and $\varphi_k(1)$ and $\varphi_{kx}(1)$ are nonzero for any k [CO-PI].

Now we apply Theorem 4. Here

$$K_\psi = \{v \in V;\ v(1) \in j_1^{-1}(0);\ v_x(1) \in j_2^{-1}(0)\}.$$

Condition (23) is equivalent to $\alpha(0) = \{0\}$ and $\beta(0) = \{0\}$, *i.e.* the two graphs are not "vertical" at the origin.

Let us next study condition (27) of Theorem 4. If both $j_1^{-1}(0)$ and $j_2^{-1}(0)$ contain a neighborhood of the origin, then, for any $v \in V$, there exists $\lambda \in \mathbb{R}$ such that $\pm\lambda v \in K\psi$, thus stabilization is not possible. On the other hand, if $j_1(\xi) > 0$ for $\xi > 0$ or $\xi < 0$, or j_2 has this same property, then

$$K_\psi \cap (-K_\psi) = \{v \in V;\ v(1) = 0\} \quad \text{or} \quad \{v \in V;\ v_x(1) = 0\}.$$

Thus condition (27) amounts to proving the following uniqueness result

$$\text{(53)} \qquad \begin{cases} (au_{xx})_{xx} = \lambda\, bu \\ u(0) = 0, \qquad u_x(0) = 0 \\ au_{xx}(1) = 0, \qquad (au_{xx})_x(1) = 0, \end{cases}$$

and $u(1) = 0$ *or* $u_x(1) = 0$, imply $u \equiv 0$. But this is always true, as a consequence of the simplicity of the eigenvalues (*see* [CO-PI] for a proof).

Thus strong asymptotic stability holds if the two graphs α and β are not vertical at 0, one at least being not "flat" at 0. Note that the result is true for intance with the following unilateral feedback

$$au_{xx}(1, t) = 0, \qquad (au_{xx})_x(1, t) = [u_t(1, t)]^+.$$

It is also possible to generalize equation (52) by taking coupled boundary conditions, that is (with $a = b \equiv 1$ for simplicity) $\psi(v) = \varphi(Cv)$ where

$$\begin{aligned} \varphi :\quad & \mathbb{R}^2 \to [0, \infty] \text{ is convex, l.s.c. and proper} \\ & \varphi(0) = \min \varphi \text{ and } Cv = (v(1), v_x(1)). \end{aligned}$$

Consider for instance

$$\varphi(\xi, \eta) = \frac{1}{2}(\tau\xi - \eta)^2, \qquad \tau > 0.$$

Here Corollary 5 is not applicable since $K_\varphi \cap (-K_\varphi) \neq \{0\}$. However Theorem 4 applies. Indeed,

$$K_\psi = \{v \in V;\ \tau v(1) - v_x(1) = 0\},$$

and

$$\langle \partial\psi(v), h\rangle_{V' \times V} = [\tau v(1) - v_x(1)]\,[\tau h(1) - h_x(1)],$$

thus (23) is obviously true.

For condition (27), one has to prove the following uniqueness result: if u is a solution of (53) in the specific case $a = b \equiv 1$, and if moreover $\tau u(1) - u_x(1) = 0$, then $u \equiv 0$.

Let $u \not\equiv 0$ be a solution of (53). Then $u = \varphi_k$, $\lambda_k = \omega_k^2$, and one has to prove that

$$\tau \varphi_k(1) - \varphi_{kx}(1) \neq 0, \qquad \forall k,$$

$$\text{or else } \varphi_k(1) \left[\tau - \frac{\varphi_{kx}(1)}{\varphi_k(1)}\right] \neq 0, \qquad \forall k = 1, 2, \cdots$$

But for the normalized eigenfunctions, it is known that

$$\varphi_k(1) = 2(-1)^{k+1}, \qquad \text{and} \qquad \lim_{k \nearrow \infty} \left| \frac{\varphi_{kx}(1)}{\varphi_k(1)} \right| = +\infty \quad \text{[CO]}.$$

So strong stabilization occurs for the problem

$$(54) \qquad \begin{cases} u_{tt} + u_{xxxx} = 0 \\ u(0, t) = u_x(0, t) = 0 \\ u_{xx}(1, t) = \tau u_t(1, t) - u_{xt}(1, t) \\ u_{xxx}(1, t) = \tau [\tau u_t(1, t) - u_{xt}(1, t)] \end{cases}$$

for any $\tau > 0$, except for a sequence going to infinity. In particular, (54) is strongly stable for any small $\tau > 0$.

Remark. – Assume a and b are piecewise regular, and for simplicity, constant on (x_{i-1}, x_i) with $x_0 = 0$, $x_N = 1$. With the previous notations, consider

$$\psi(v) = \frac{1}{2} \sum_{i=1}^{N} [\alpha_i v^2(x_i) + \beta_i v_x^2(x_i)], \qquad \alpha_i \geqq 0, \quad \beta_i \geqq 0.$$

Then the abstract formulation covers the following problem (A is the same as previously)

$$b_i u_{tt} + a_i u_{xxxx} = 0 \quad \text{on } (x_{i-1}, x_i), \qquad i = 1, \cdots, N$$

$$u(0,t) = 0, \qquad u_x(0, t) = 0$$

$$u(x_{i-}, t) = u(x_{i+}, t), \qquad u_x(x_{i-}, t) = u_x(x_{i+}, t)$$

$$\begin{pmatrix} a_i u_{xxx}(x_{i-}, t) - a_{i+1} u_{xxx}(x_{i+}, t) \\ -a_i u_{xx}(x_{i-}, t) + a_{i+1} u_{xx}(x_{i+}, t) \end{pmatrix} = \begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix} \begin{pmatrix} u_t(x_i, t) \\ u_{tx}(x_i, t) \end{pmatrix}, \qquad i = 1, \cdots, N-1$$

$$\begin{pmatrix} a_N u_{xxx}(1, t) \\ -a_N u_{xx}(1, t) \end{pmatrix} = \begin{pmatrix} \alpha_N & 0 \\ 0 & \beta_N \end{pmatrix} \begin{pmatrix} u_t(1, t) \\ u_{tx}(1, t) \end{pmatrix}.$$

This problem has been considered in [CHE-DE], where uniform exponential decay is proved under the assumption $\alpha_N > 0$, $b_i \leqq b_{i+1}$, $a_i \geqq a_{i+1}$, $\forall\, i$.

Applying our Theorem 4, we obtain the strong stability under the assumptions α_i, $\beta_i \geqq 0$, $i = 1, \cdots, N$, and α_N *or* $\beta_N > 0$, as in the previous case. If $\alpha_N = \beta_N = 0$, but α_{i_0} or $\beta_{i_0} > 0$ for some i_0, then, due to the simplicity of all eigenvalues, x_{i_0} has to be "strategic" in the sense that it has to be different from all the zeros of the eigenfunctions or of the derivatives of the eigenfunctions.

With our method, it is not hard to consider nonlinear feedback laws as previously, and even to combine this with coupled interaction between the transversal and angular velocities at the nodes x_i, $i = 1, \cdots, N$.

Finally the same analysis for coupled vibrating strings instead of beams would lead to the result in Theorem 3.4 of the recent paper [HO].

3.2. Hybrid system (for Euler-Bernouilli beams)

We consider an homogeneous Euler-Bernouilli beam clamped at the left end, and controlled at the right end by a moment, but now, there is a mass and inertia at this end. Normalizing the constants, the model is the following

$$\begin{cases} w_{tt} + w_{xxxx} = 0, & 0 < x < 1, \quad t > 0, \\ w(0, t) = w_x(0, t) = 0, & t > 0, \\ -w_{xx}(1, t) = w_{xtt}(1, t) + f(t), & t > 0, \\ w_{xxx}(1, t) = w_{tt}(1, t), & t > 0. \end{cases}$$

With $f(t) = w_{xt}(1, t)$ if $|w_{xt}(1, t)| \leqq r$, $f(t) = r \operatorname{sgn}(w_{xt}(1, t))$ otherwise, we get the problem studied in [SLE], where strong asymptotic stability has been proved.

Let us show that we can also handle this problem with our technique. We set

$$H = L^2(0, 1) \times \mathbb{R} \times \mathbb{R},$$

$$\begin{aligned} V = \{(w, a, b) \in H;\ & w \in H^2(0, 1); \\ & w(0) = w_x(0) = 0;\ a = w(1);\ b = w_x(1)\}, \end{aligned}$$

$$\left(\tilde{A}\begin{pmatrix} w \\ a \\ b \end{pmatrix}, \begin{pmatrix} \varphi \\ \alpha \\ \beta \end{pmatrix}\right)_{V' \times V} = \int_0^1 w_{xx}\, \varphi_{xx},$$

$$D(A) = \{(w, a, b) \in V;\ w \in H^4(0, 1)\},$$

and, if $(w, a, b) \in D(A)$,

$$A \begin{pmatrix} w \\ a \\ b \end{pmatrix} = \begin{pmatrix} w_{xxxx} \\ -w_{xxx}(1) \\ w_{xx}(1) \end{pmatrix}.$$

Let $\varphi : \mathbb{R} \times \mathbb{R}$ be the function

$$\begin{cases} \varphi(\xi) = \xi & \text{if} \quad |\xi| \leqq r \\ \varphi(\xi) = r \operatorname{sgn}(\xi) & \text{if} \quad |\xi| \geqq r \end{cases}$$

and j the primitive of φ such that $j(0) = 0$. For $(w, a, b) \in V$, we set $\psi(w, a, b) = j(b)$. Then it is not difficult to see that the abstract formulation

$$u_{tt} + \tilde{A}\, u + \partial\psi(u_t) = 0, \qquad u = (w, a, b),$$

recovers the initial problem of [SLE]. So this problem is well-posed on $V \times H$, as a first order equation. Omitting the details concerning the precise definition of B, the density of $D(B)$ in $V \times H$, and the compactness of the resolvents (obtained using [CO-PI$_1$], Appendix 2), the asymptotic stability amounts, through Theorem 4, to proving the following "uniqueness" result.

Let φ be an "eigenvalue" of A, $\varphi = (w, a, b)$, such that $\psi(\varphi) = 0$, then $\varphi \equiv 0$. Or else, if w satisfies

$$\begin{cases} w_{xxxx} = \lambda\, w \\ w_{xxx}(1) = \lambda\, a = \lambda\, w(1) \\ w_{xxx}(1) = \lambda\, b = \lambda\, w_x(1) \\ w(0) = w_x(0) = 0 \end{cases}$$

and $w_x(1) = 0$

then $w \equiv 0$ (thus also $a = b = 0$). But, in [SLE] it is proved that the eigenvalues of this equation are simple and that, moreover $w_x(1) \not\equiv 0$ (*see* [SLE, (8.29)]. Whence the strong asymptotic stabilization.

Remark 1. – Instead of controlling by a moment, one can control by a force, that is, boundary conditions are now

$$\begin{cases} -w_{xx}(1, t) = w_{xtt}(1, t) \\ w_{xxx}(1, t) = w_{tt}(1, t) + g(t), \end{cases}$$

where $g(t) = w_t(1, t)$, possibly also truncated as before.

Remark 2. – One can combine the model studied in the Remark of Section 3.1 with the previous one. Then one can study the case of serially connected Euler-Bernouilli beams with, at each node (of possibly nonzero mass and nonzero inertia) control by force and/or moment, that is, at each node

$$\begin{cases} a_i\, u_{xxx}(x_{i-}, t) - a_{i+1}\, u_{xxx}(x_{i+}, t) = m_i\, u_{tt}(x_i, t) + g_i(t) \\ -a_i\, u_{xx}(x_{i-}, t) + a_{i+1}\, u_{xx}(x_{i+}, t) = J_i\, u_{xtt}(x_i, t) + f_i(t), \end{cases}$$

where f_i and g_i are of the form just presented (with, or without, truncation). But of course the reduced "uniqueness" result becomes rather hard to handle...

3.3. Wave equation

We consider the wave equation on a smooth bounded domain Ω of $\mathbb{R}^N$, $N \geqq 1$. Control is exerted by means of a force which is a nonlinear function of the observed velocity, on a part Γ_0 of the boundary Γ, assumed to be regular. In the sequel (Γ_0, Γ_*) is a partition of Γ, and we assume meas $(\Gamma_0) > 0$, int $\Gamma_* \neq \varnothing$.

The system is the following

$$(55) \qquad \begin{cases} u_{tt} - \Delta u = 0, & x \in \Omega, \quad t > 0 \\ u = 0, & x \in \Gamma_*, \quad t > 0 \\ \dfrac{\partial u}{\partial \nu} = -a(x)\, g(u_t), & x \in \Gamma_0, \quad t > 0, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is monotone, continuous (just for simplification, one could actually take a maximal monotone graph instead), such that $g(0) = 0$, $a > 0$ is continuous, and ν is the normal unit vector on Γ pointing outwards Ω.

When $\Gamma_0 = \{x \in \Gamma;\ (x - x_0) \cdot \nu > 0\}$, $\Gamma_* = \Gamma \setminus \Gamma_0$, where $x_0 \in \mathbb{R}^N$, int $\Gamma_* \neq 0$, and, if $N > 3$, $\overline{\Gamma}_0 \cap \Gamma_* = \varnothing$, strong stability holds for problem (55), with estimates for the decay depending on the behaviour of g (*see* for instance [ZU]). Strong stability has also been proved in [LA$_1$] in this framework, for more general g.

Here we obtain strong stabilization for very general partitions (Γ_0, Γ_*) of the boundary. First, we put (55) in our abstract framework. We set

$$g = \partial j, \qquad H = L^2(\Omega), \qquad V = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_*\},$$
$$\langle \tilde{A}\, u,\ v\rangle_{V' \times V} = \int_\Omega \nabla u \cdot \nabla v \, dx,$$
$$D(A) = \left\{ u \in V;\ \Delta u \in L^2(\Omega);\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\},$$
$$\forall u \in D(A), \qquad A\, u = -\Delta u.$$

Let $\psi(v) = \displaystyle\int_{\Gamma_0} a(x)\, j(v(x))\, d\sigma(x), \forall v \in V$. Assume g satisfies a suitable growth condition. Then $\forall v \in V$, $\partial \psi(v) = \{f\}$, where (*see* [CO-PI$_1$],

Appendix 4)

$$\langle f,\, h\rangle_{V'\times V} = \int_{\Gamma_0} a(x)\, g(v(x))\, h(x)\, d\sigma(x),$$

so that, from (5), one deduces

$$D(B) = \left\{(v,\, h) \in V \times V;\ \Delta v \in L^2(\Omega);\ \frac{\partial v}{\partial \nu} = -ag(h) \text{ on } \Gamma_0\right\}$$

and, if $(v,\, h) \in D(B)$, $B(v,\, h) = (-h,\, -\Delta v)$. Since $D(B) \supset D(A) \times \{h \in V;\ h = 0 \text{ on } \Gamma_0\}$, it follows that $D(B)$ is dense in $V \times H$, so that (55) is well-posed on $V \times H$. Compactness of A is obvious and compactness of B follows from the adequate growth condition on g (*see* [CO-PI$_1$], Appendix 4).

Thus we can apply Theorem 4. We first observe that (23) is obviously satisfied.

Let us now show that (27) is satisfied. Assume $K_j \cap (-K_j) = \{0\}$ that is, g is not "flat" at 0, as usual. Then

$$\begin{aligned} K_\psi \cap -K_\psi &= \{v \in V;\ j(v(\sigma)) = 0,\ j(-v(\sigma)) = 0, \text{ a.e. on } \Gamma_0\} \\ &= \{v \in V;\ v(\sigma) \in K_j \cap (-K_j) \text{ a.e. on } \Gamma_0\} \\ &= \{v \in V;\ v = 0 \text{ a.e. on } \Gamma_0\}. \end{aligned}$$

So condition (27) amounts to proving the following "uniqueness" result:

$$\text{(56)} \qquad \begin{cases} -\Delta\varphi = \omega^2\varphi \\ \varphi = 0 \quad \text{on } \Gamma_* \\ \dfrac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \Gamma_0 \end{cases}$$

and

$$\text{(57)} \qquad \varphi = 0 \quad \text{on } \Gamma_0$$

imply $\varphi \equiv 0$ in Ω.

This uniqueness result holds for very general situations where Γ_0 is not too "thin". For instance, if Γ_0 contains $B(x_0, \varepsilon) \cap \Gamma$, where $x_0 \in \Gamma$, and Γ is regular. The proof is elementary and proceeds by extending φ by 0 outside Γ (and near Γ_0), using analyticity properties. This can also be viewed as a very particular case of Holmgren's unique continuation theorem. So, we have an extension of the results of [LA$_1$].

When $\Gamma_0 = (x \in \Gamma;\ (x - x_0) \cdot \nu > 0\}$, one can prove the uniqueness result by the usual multipliers's technique. However, in that case, we get much more than uniqueness (or equivalently weak observability), namely strong observability.

One can easily be convinced that stabilization holds for more general partitions by taking for instance a rectangular membrane $\Omega = (0, a) \times (0, b)$ with $u = 0$ on the vertical edges Γ_*, and $\dfrac{\partial u}{\partial \nu} \in -\beta(u_t)$ on the horizontal edges Γ_0, where β is a maximal monotone graph satisfying adequate growth conditions, so that the compactness assumptions hold. In that case, the uniqueness result can be proved in an elementary way, using Fourier expansions.

The solutions of the eigenvalue problem (56) are

$$\omega_{mn} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right), \qquad \varphi(x, y) = \sum_{r,\, s} \alpha_{rs} \sin \frac{\pi\, rx}{a} \cos \frac{\pi\, sy}{b}$$

where $r \in \mathbb{N}^*$, $s \in \mathbb{N}$, $\dfrac{r^2}{a^2} + \dfrac{s^2}{b^2} = \dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}$,, with $m \in \mathbb{N}^*$, $n \in \mathbb{N}$.

For $y = 0$, or $y = b$, one gets $\varphi_{|\Gamma_0} = \sum_{r,\, s} \pm\, \alpha_{rs} \sin \dfrac{\pi\, rx}{a} = 0$. This implies $\alpha_{rs} = 0$ for all r, s. Thus (56) and (57) imply $\varphi \equiv 0$.

Clearly, the uniqueness result holds also for any Γ_0 which contains an arbitrarily small horizontal interval.

Instead of a boundary feedback, one can also consider an interior feedback, in a fairly general framework. For simplicity, consider again the rectangular membrane with Dirichlet conditions on Γ_* and Γ_0 (or Neumann conditions on Γ_0).

Let $j : \mathbb{R} \to [0, \infty]$ be a convex, l.s.c. proper function such that min $j = j(0)$, set $\beta = \partial j$, and consider

$$\psi(v) = \int_\Omega j(\tilde{v})\, d\mu, \qquad D(\psi) \subset V = H_0^1(\Omega), \tag{58}$$

where μ is a positive Radon measure on Ω, of finite energy, and $\tilde{v}$ is the quasi-continuous representative of v, with respect to the V-capacity.

In particular, one can consider the case where the support of μ is a "thin" set E of positive capacity (a piece of curve for instance), or a closed set E with non empty interior. With μ the length or area measure in that case, one solves in fact the formal problem

$$u_{tt} - \Delta u + \beta(u_t) \ni 0.$$

Since μ is of finite energy, it follows that $v_n \to v$ in $V \Rightarrow \tilde{v}_n \to \tilde{v}$, μ a.e. up to a subsequence, and hence ψ is l.s.c. Therefore, our theory applies and, provided the compactness assumptions are true, strong stability amounts to verifying (23) and (27).

The usual assumption $\beta(0) = \{0\}$ ensures (23) is true. For (27), we assume $K_j \cap (-K_j) = \{0\}$. Then we have to prove the following uniqueness result:

$$\left\{ \begin{array}{ll} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{array} \right. \tag{59}$$

and

$$\tilde{u} = 0 \ \mu \text{ a.e.} \tag{60}$$

imply $u \equiv 0$.

For instance if μ is the length or area measure on a piece of curve or on a closed set E of non empty interior, stabilization "usually" holds. Assume as above that $\Omega = (0, a) \times (0, b)$; then $\lambda = \omega_{mn} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$ and if $\varphi_{rs}(x, y) = \sin \frac{\pi r x}{a} \sin \frac{\pi s x}{b}$, the solution of (59) can be written

$$u(x, y) = \sum_{r, s} \alpha_{rs} \varphi_{rs}(x, y), \qquad \text{where } \frac{r^2}{a^2} + \frac{s^2}{b^2} = \frac{m^2}{a^2} + \frac{n^2}{b^2}.$$

If $E \supset]\alpha, \beta[\times \{y_0\}$, with $\frac{y_0}{b} \notin \mathbb{Q}$, or if $E \supset \{x_0\} \times]\alpha, \beta[$, with $\frac{x_0}{a} \notin \mathbb{Q}$, then (60) implies $\alpha_{rs} = 0$ for all r, s and $u \equiv 0$. Hence (27) is true.

If the above conditions on E are not true, it may happen that a solution u of (59) is identically zero on E, so stabilization does not hold. We observe that if E contains an open set, strong stabilization holds.

3.4. Rectangular Kirchhoff plates

We consider a simply supported rectangular plate $\Omega = (0, a) \times (0, b)$. We can apply our general formalism to the equation

$$u_{tt} + \Delta^2 u + \partial \psi(u_t) = 0 \quad \text{in } \Omega, \qquad u = \frac{\partial^2 u}{\partial n^2} = 0 \quad \text{on } \partial\Omega$$

to obtain stabilization results. Moreover, one can consider functionals of the form (58) where $D(\psi) \subset H^2(\Omega)$, the capacity being now the one associated with the H^2-norm.

In that way, one can choose Dirac masses for μ, and thus consider inner point control. Let us study one particular case, where the external force acting on the plate is exerted at the points $(p_i,\, q_i) \in \Omega$. This problem has been studied in [Y]. Normalizing the constants, we get the following evolutionary system

$$(61) \quad \begin{cases} u_{tt} + \Delta^2 u = \sum_{i=1}^{1} f_i(t)\, \delta(x - p_i,\, y - q_i), & x \in \Omega, \quad t > 0, \\ u = \dfrac{\partial^2 u}{\partial n^2} = 0 \quad \text{on } \partial\Omega = \Gamma, & t > 0, \\ u(x,\, y,\, 0) = u_0(x,\, y), & \\ u_t(x,\, y,\, 0) = u_1(x,\, y), & \end{cases}$$

where $\Delta^2 = \dfrac{\partial^4}{\partial x^4} + 2\dfrac{\partial^4}{\partial x^2\, \partial y^2} + \dfrac{\partial^4}{\partial y^4}$, $f_i \in L^1_{\text{loc}}(\mathbb{R}_+,\, \mathbb{R})$. We set

$$H = L^2(\Omega), \qquad V = \{v \in H^2(\Omega);\ v = 0 \text{ on } \partial\Omega\}.$$

We consider the control operator $C^* = \{\delta(x - p_i,\, y - q_i)\}_{i=1,\cdots,1} \in \mathcal{L}(R^1,\, V')$.

In [Y], problem (61) has been considered on the state space V', with Δ^2 as an operator defined on V', with domain V, so that control becomes "distributed". The main result of [Y] can be summarized as follows:

• if $\dfrac{a^2}{b^2} \in \mathbb{Q}$ then (61) is neither approximately controllable, nor strongly (or weakly) stabilizable, by any bounded linear feedback $\in \mathcal{L}(v,\, \mathbb{R}^1)$, acting on velocity,

• if $\dfrac{a^2}{b^2} \notin \mathbb{Q}$ then (61) is approximately controllable and strongly stabilizable by the feedback $f(t) = -C\, u_t$, if and only if the following (rank) condition is satisfied.

$$\sum_{j=1}^{1} \frac{1}{\lambda_{mn}\, |e_{mn}|_H} |e_{mn}(p_j,\, q_j)| \neq 0, \qquad \forall m,\, n \geqq 1, \ i.e.$$

$$e_{mn}(p_j,\, q_j) \neq 0 \text{ for some } j, \text{ for any } m,\, n \geqq 1.$$

Here λ_{mn} are the eigenvalues of the free vibrating plate, and e_{mn} are the corresponding eigenmodes (the eigenspaces are one-dimensional in the case considered, and for general a and b, approximate controllability is characterized by a rank condition, *see* [Y], Lemma 8). Moreover, in the particular case of one actuator $\left(1 = 1,\ \frac{a^2}{b^2} \notin \mathbb{Q}\right)$, (61) is approximately controllable iff $\frac{p}{a} \notin \mathbb{Q}$ and $\frac{q}{b} \notin \mathbb{Q}$, and $f(t) = -u_t(t, p, q)$ is a stabilizing feedback.

The proofs of the necessary and sufficient conditions given in [Y] are based on theorems in number theory and Diophantine equations.

Here we just show that the feedback $f(t) = -C\,u_t$ is stabilizing, by means of our abstract results, where the state space is $H = L^2(\Omega)$, so that the control operator C^* is unbounded. For simplicity, let us also consider the case of one pointwise actuator. We set

$$\langle \tilde{A}\,u,\ v\rangle_{V'\times V} = \int_\Omega \Delta u\,\Delta v\,dx, \qquad \forall\, u,\ v \in V,$$

$$D(A) = \left\{u \in V;\ \Delta^2 u \in H;\ \frac{\partial^2 u}{\partial n^2} = 0\right\}, \qquad \forall\, u \in D(A),\ \ A\,u = \Delta^2 u.$$

Let $\psi : V \to \mathbb{R}_+$ be defined by $\psi(v) = \frac{1}{2} v^2(p, q)$, so that ψ is convex (regular), defined on the whole space V, and

$$\partial\psi(v) = v(p, q)\,\delta_{pq}.$$

Hence, we study the following equation (61) with $f(t) = -u_t(t, p, q)$

$$u_{tt} + \tilde{A}\,u + \partial\psi(u_t) = 0. \tag{62}$$

For regular v, we get immediately, $\forall\, h \in V$, $\forall\, \varphi \in V$

$$\langle \tilde{A}\,v + \partial\psi(h),\ \varphi\rangle_{V'\times V} = \int_\Gamma \Delta u\,\frac{\partial\varphi}{\partial n} + \int_\Omega \Delta^2 u\,\varphi + h(p, q)\,\varphi(p, q).$$

Choosing first $\varphi \in \mathcal{D}(\Omega)$, then $\varphi \in V$ arbitrary, we get from (5)

$$D(B) = \left\{(v,\ h) \in V \times V;\ \Delta^2 u + h(p, q)\,\delta_{pq} \in H;\ \frac{\partial^2 u}{\partial n^2} = 0 \text{ on } \Gamma\right\}$$

and by (6), if $(v,\ h) \in D(B)$, $B(v,\ h) = \{-h,\ \Delta^2 u + h(p, q)\,\delta_{pq}\}$

Thus, formally, (61) and (62) are equivalent. We observe that $\Delta^2 u \in L^2(\Omega \setminus B(p,\ q;\ \varepsilon))$, so that $u \in H^4(\Omega \setminus B(p,\ q;\ \varepsilon))$, and the trace $\Delta u = \partial^2 u/\partial n^2$ on $\partial\Omega$ makes sense. We also observe that

$$\{(v,\ h);\ v \in H^4(\Omega) \cap V;\ h \in V;\ h(p, q) = 0\} \subset D(B),$$

so that $D(B)$ is dense in $V \times H$.

Also, A has compact resolvent in H by classical regularity (*see* also [Y], Lemma 3). Moreover $\partial\psi : V \to V'$ is obviously compact since Range $(\partial\psi)$ is one dimensional. Using [CO-PI$_1$, Appendix 2], it follows that B has also compact resolvent and so our whole theory on stabilization is valid.

Now we apply Theorem 4. Here $K\psi = \{v \in V;\ v(p,\ q) = 0\}$. Since $\partial\psi(v) = v(p,\ q)\,\delta_{pq} = 0$, condition (23) is satisfied.

For condition (27), one has to prove that for any eigenfunction φ, one has $\varphi(p,\ q) \neq 0$, where $\varphi(x,\ y) = \sum_{r,\,s} \alpha_{rs} \sin \frac{m\pi x}{a} \sin \frac{n\pi x}{b}$, the eigenvalue being

$$\omega_{mn}^4 = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \pi^4, \qquad \text{and} \qquad \frac{r^2}{a^2} + \frac{s^2}{b^2} = \frac{m^2}{a^2} + \frac{n^2}{b^2}.$$

If $\frac{a^2}{b^2} \notin \mathbb{Q}$, the eigenvalues are all simple (by contradiction) and clearly $\sin \frac{m\pi p}{a} \sin \frac{n\pi q}{b} \neq 0$ if and only if $\frac{p}{a}$ and $\frac{q}{b} \notin \mathbb{Q}$.

Hence, in case $\frac{a^2}{b^2} \notin \mathbb{Q}$, (61) is strongly stabilizable with the feedback $f(t) = -C\,u_t$, iff $\frac{p}{a}$ and $\frac{q}{b} \notin \mathbb{Q}$.

Remark 1. – The same result is true with $\psi(v) = \varphi(v(p,\ q))$, for any strictly convex regular φ. In that case, one controls with $f(t) = \varphi'(u_t(p,\ q,\ t))$, with the assumptions $\frac{a^2}{b^2}, \frac{p}{a}, \frac{q}{b} \notin \mathbb{Q}$.

Remark 2. – One can consider (61) as a problem with boundary feedback at $(p,\ q) \in \partial(\Omega \setminus (p,\ q))$. This shows that one can stabilize a plate with a pointwise feedback (on the boundary, if we want). This result is in contrast with the case of membranes, where points have $V = H^1$-capacity zero (here the $V = H^2$-capacity of a point is $\neq 0$). In the case of membranes, control on a regular set E means that E must contain at least a line, roughly speaking, which is of positive capacity.

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REFERENCES

[BA-PRE] V. BARBU and Th. PRECUPANU, *Convexity and Optimization in Banach Spaces*, D. Reidel Pub. Co., Dordrecht, 1986

[BE] C. D. BENCHIMOL, A Note on Weak Stabilizability of Contraction Semi-groups, *SIAM J. Contr. Opt.*, Vol. **16**, 3, 1978, pp. 373-379.

[BRE] H. BREZIS, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, 1973.

[BRE$_1$] H. BREZIS, *Analyse fonctionnelle, théorie et applications*, Masson, 1983.

[CHE] G. CHEN, Energy Decay Estimates and Exact Boundary Controllability for the Wave Equation in a Bounded Domain, *J. Math. Pures Appl.*, Vol. **58**, 1979, pp. 249-274.

[CHE-DE] G. CHEN, M. DELFOUR, A. M. KRALL and G. PAYRE, Modelling, Stabilization and Control of Serially Connected Beams, *SIAM J. Contr. Opt.*, Vol. **25**, 1987, pp. 526-546.

[CO] F. CONRAD, Stabilization of Beams by Pointwise Feedback Control, *SIAM J. Contr. Opt.*, Vol. **28**, 1990, pp. 423-437.

[CO-PI] F. CONRAD and M. PIERRE, Stabilization of Euler-Bernoulli Beam by Nonlinear Boundary Feedback, *INRIA Report RR # 1235,* 1990.

[CO-PI$_1$] F. CONRAD and M. PIERRE, Stabilization of Second Order Evolution Equations by Unbounded Nonlinear Feedback, *INRIA Report RR #* 1679, 1992.

[DA] C. M. DAFERMOS, Application of the Invariance Principle for Compact Processes, *J. Diff. Eqs.*, Vol. **9**, 1971, pp. 291-299.

[DA$_1$] C. M. DAFERMOS, Asymptotic Behavior of Solutions of Evolution Equations, in *Nonlinear Evolution Equations,* M. G. Crandall, ed., Ac. Press, 1978, pp. 103-123.

[DA-SLE] C. M. DAFERMOS and M. SLEMROD, Asymptotic Behaviour of Nonlinear Contraction Semi-groups, *J. Funct. Anal.*, Vol. **13**, 1973, pp. 97-106.

[EL JAI-PRI] A. EL JAI and T. PRITCHARD, Sensors and Actuators on Distributed Systems, *Int. J. Control,* Vol. **46**, 1987, pp. 1139-1153.

[HA] A. HARAUX, Stabilization of Trajectories for Some Weakly Damped Hyperbolic Equations, *J. Diff. Eqs.*, Vol. **59**, 2, 1985, pp. 145-154.

[HA$_1$] A. HARAUX, Comportement à l'infini pour certains systèmes dissipatifs non linéaires, *Proc. of Royal Soc. Edinburgh,* 84A, 1979, pp. 213-234.

[HO] L. F. HO, Controllability and Stabilizability of Coupled Strings with Control Applied at the Coupled Points, *SIAM J. Control and Optimization,* Vol. **31**, 6, 1993, pp. 1416-1437.

[KO-ZU] V. KOMORNIK and E. ZUAZUA, A Direct Method for the Boundary Stabilization of the Wave Equation *J. Math. Pures et Appl.,* Vol. **69**, 1990, pp. 33-54.

[LA] I. LASIECKA, Stabilization of Wave and Plate-like Equations with Nonlinear Dissipation on the Boundary, *J. Diff. Eqs.*, Vol. **79**, 2, 1989, pp. 340-381.

[LA$_1$] I. LASIECKA, Asymptotic Behaviour of the Solution to Wave Equation with Nonlinear Damping on the Boundary, 8th Int. Conf. Analysis and Optimization of Systems, Antibes, *Lect. Notes in Contr. and Inf. Sci.,* Vol. **111**; A. Bensoussan, J. L. Lions Eds., 1988, pp. 472-483.

[LAG] J. E. LAGNESE, Decay of Solutions of the Wave Equation in a Bounded Region with Boundary Dissipation, *J. Diff. Eqs,* Vol. **50**, 1983, pp. 163-182.

[LAG$_1$] J. E. LAGNESE, *Boundary Stabilization of Thin Plates,* Siam (Studies in Applied Mathematics), 1989.

[Q-R] J. P. QUINN and D. L. RUSSELL, Asymptotic Stability and Energy Decay Rates for Solution of Hyperbolic Equations with Boundary Damping, *Proc. Roy. Soc. Edinburgh,* Sect. A, 77, 1977, pp. 97-127.

[RO] R. T. ROCKAFELLAR, *Convex Analysis,* Princeton Univ. Press, 1972.

[SLE] M. SLEMROD, Feedback Stabilization of a Linear Control System in Hilbert Space with an a priori Bounded Control, *Math. Control Signals Systems,* 1988, pp. 265-285.

[TRI] R. TRIGGIANI, Extension of Rank Conditions for Controllability and Observability to Banach Spaces and Unbounded Operators, *SIAM J. Contr. Opt.,* Vol. **14**, 2, 1976, pp. 313-338.

[Y] Y. YOU, Controllability and Stabilizability of Vibrating Simply Supported Plate With Pointwise Control, *Adv. in Appl. Math.,* Vol. **10**, 1989, pp. 324-343.

[ZU] E. ZUAZUA, Uniform Stabilization of the Wave Equation by Nonlinear Boundary Feedback, *SIAM J. Contr. Opt.,* Vol. **28**, 1990, pp. 466-477.

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