ANNALES DE L'I. H. P., SECTION C

STEFAN MÜLLER

On the singular support of the distributional determinant

Annales de l'I. H. P., section C, tome 10, nº 6 (1993), p. 657-696

<http://www.numdam.org/item?id=AIHPC_1993__10_6_657_0>

Ann. Inst. Henri Poincaré,

Vol. 10, n° 6, 1993, p. 657-696. *Analyse non linéaire*

On the singular support of the distributional determinant

by

Stefan MÜLLER

Institut für Angewandte Mathematik,
Universität Bonn, Beringstr. 4, D-5300 Bonn 1, Germany

ABSTRACT. — Let $\Omega\subset\mathbb{R}^n$ be bounded and open, let $p\geqq n^2/(n+1)$ and let $u:\Omega\to\mathbb{R}^n$ be in the Sobolevspace $W^{1,n}(\Omega;\mathbb{R}^n)$. This paper discusses the singular part of the distributional determinant $\mathrm{Det}\,Du$ and shows the existence of functions u for which that singular part is supported in a set of prescribed Hausdorff-dimension $\alpha\in(0,n)$. For $n=2$ and simply connected Ω the problem is equivalent to analyzing $\mathrm{div}(bv)-b.Dv$ where $v\in W^{1,p}(\Omega;\mathbb{R}^2)$ with $\mathrm{div}\,b=0$.

Key words : Compensated compactness, determinant.

RÉSUMÉ. — Soit $\Omega\subset\mathbb{R}^n$ un ouvert borné, soit $p\geqq n^2/(n+1)$ et soit $u:\Omega\to\mathbb{R}^n$ dans l'espace de Sobolev $W^{1,p}(\Omega;\mathbb{R}^n)$. On construit des applications u dont le déterminant au sens de distributions $\mathrm{Det}\,Du$ est une mesure de Radon positive, portée par un ensemble singulier dont la dimension de Hausdorff est arbitraire, strictement entre 0 et n.

A.M.S. Classification: 46 F 10 (26 B 10).

1. INTRODUCTION

In this paper we continue the study, begun in [Mu 90*a*], [Mu 90*b*], [MTY 92] of the validity of certain formal identities for distributions. For illustration consider an open, bounded set $\Omega \subset \mathbb{R}^2$ and a map $u : \Omega \to \mathbb{R}^2$ which is in the Sobolev space $W^{1,p}(\Omega; \mathbb{R}^2)$, $p \geqq 4/3$. Denoting components by upper indices and partial derivatives by lower indices with comma, one defines (a. e.) the *pointwise determinant* of the (distributional) gradient Du by

$$\det Du(x) = u^1_{,1}(x)\, u^2_{,2}(x) - u^1_{,2}(x)\, u^2_{,1}(x),$$

and its *distributional determinant by*

$$\operatorname{Det} Du = (u^1 u^2_{,2})_{,1} - (u^1 u^2_{,1})_{,2},$$

(where the derivatives outside the parentheses are to be understood in the sense of distributions). The distributional determinant is of crucial importance in nonlinear elasticity because it enjoys continuity properties with respect to weak convergence in $W^{1,p}$ (*see* Ball [Ba 77]). For smooth u one has

$$\operatorname{Det} Du = \det Du, \tag{1.1}$$

and by approximation the identity holds in the sense of distributions, if $u \in W^{1,2}(\Omega; \mathbb{R}^2)$. The example $\Omega = B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$, $u(x) = x/|x|$ shows that (1.1) is in general false if $p < 2$. Indeed, one has $u \in W^{1,p}(\Omega; \mathbb{R}^2)$ for all $p < 2$, $\det Du = 0$ a. e. but

$$\operatorname{Det} Du = \pi \delta_0.$$

So far in all examples where (1.1) was known to fail, $\operatorname{Det} Du - \det Du$ was a linear combination of point masses (*see* [Ba 77]). Here examples are constructed for which $\operatorname{Det} Du - \det Du$ is a singular measure whose support is a set of prescribed Hausdorff-dimension $\alpha \in (0, 2)$. Moreover in these examples u is continuous (in fact Hölder-continuous with exponent $\alpha/2$).

Before discussing the generalization to higher dimensions we note that for $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ the problem of analyzing

$$\operatorname{Det} Du - \det Du, \tag{1.2}$$

is equivalent (for simply connected Ω) to studying

$$\operatorname{div}(bv) - b \,.\, Dv, \tag{1.3}$$

for a scalar function $v : \Omega \to \mathbb{R}$ and a divergence free vectorfield $b : \Omega \to \mathbb{R}^2$.

Now consider maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ and recall that the adjungate $\operatorname{adj} F$ of an n by n matrix F is defined as the transpose of the matrix of cofactors of F so that $(\operatorname{adj} F)^i{}_j F^j{}_k = \delta^i{}_k \det F$. Here and throughout this paper the summation convention is used. For $u \in W^{1,p}(\Omega; \mathbb{R}^n)$, $p \geqq n^2/(n+1)$ one defines the distributional determinant by

$$\operatorname{Det} Du = (u^i (\operatorname{adj} Du)^j{}_i)_{,j}.$$

Using the identity $(\operatorname{adj} Du)^j_{i,j}=0$ ([Mo 66], Lemma 4.6.4) one finds by approximation (*see* [Ba 77], [Da 89]) that for $p \geqq n$

$$\operatorname{Det} Du = \det Du, \tag{1.4}$$

in the sense of distributions. As before the map $u(x)=x/|x|$ shows that the identity fails for $p<n$. Our goal is to construct "many" maps for which (1.4) fails.

THEOREM 1.1. – *Let $n \geqq 2$, let $\Omega=(0,1)^n$, and let $\alpha \in (0, n)$. Then there exists a closed subset S of Hausdorff dimension α and a map $\Omega \to \mathbb{R}^n$ such that:*

(i) $u \in W^{1,p}(\Omega; \mathbb{R}^n) \cap C^0(\bar{\Omega})$ *for all* $p \in [0, n)$.

(ii) $\operatorname{Det} Du = \det Du$ *a. e.*

(iii) $\operatorname{Det} Du$ *is a nonnegative measure with support* S.

The functions u in Theorem 1.1 are constructed explicitly, and more precise information on u and Du is available (*see* Theorems 4.1 and 5.1 below). In particular one has not only (ii) but

$$\prod_{i=1}^{n} |Du^i| = 0 \quad \text{a. e.}$$

The ultimate goal would be to understand the range of the map $u \mapsto \operatorname{Det} Du$ (*cf.* the work by Dacorogna and Moser [DM 90] on the solvability of $\operatorname{Det} Du = f$). If $p > n^2/(n+1)$ Coifman, Lions, Meyer and Semmes [CLMS 89] showed that $\operatorname{Det} Du$ lies in the Hardy space $\mathcal{H}^{p/n}$. Even under the additional hypothesis that $\operatorname{Det} Du$ be a Radon measure (e.g. $\operatorname{Det} Du \geqq 0$) this seems to imply no restrictions on the support S of the singular part of $\operatorname{Det} Du$ (by [Mu 90*a*] in this case $\det Du$ is the regular part of $\operatorname{Det} Du$). On the other hand in all examples that I am aware of where $\operatorname{Det} Du$ is a measure one has for any $(n-1)$-dimensional manifold N

$$H^{n-1}(N \cap S) = 0, \tag{1.5}$$

where H^{n-1} denotes the $(n-1)$ dimensional Hausdorff measure. I conjecture that (1.5) holds in general if $\operatorname{Det} Du$ is a measure but I am not aware of a proof even for $n=2$. Below we show that (1.5) may fail if $\operatorname{Det} Du$ is not a (Radon) measure.

THEOREM 1.2. – *Let* $\Omega = (0, 1) \times (-1, 1)$.

There exists a map $u : \Omega \to \mathbb{R}^n$ *with the following properties:*

(i) *For all* $p \in [1, 2)$: $u \in W^{1,p}(\Omega; \mathbb{R}^n) \cap C^{0,1/2}(\bar{\Omega}; \mathbb{R}^2)$.

(ii) *For all* $\varphi \in C_0^\infty(\Omega)$ *one has;*

$$\langle \operatorname{Det} Du, \varphi \rangle = 2 A_0 \int_0^1 \varphi(x, 0)\, dx + \lim_{k \to \infty} \int_{|y| \geqq 2^{-k}} \det Du\, \varphi\, dx\, dy,$$

where $A_0 \neq 0$.

More details are given in Theorem 6.3.

With the analogy of (1.2) and (1.3) in mind we also produce an anisotropic example for which $\mathrm{Det} \neq \det$.

THEOREM 1.3. – *Let* $\Omega = (0, 1) \times (-1, 1)$, $\beta > 0$.
Then there exists a map $u: \Omega \to \mathbb{R}^2$ *such that:*

(i) $\left.\begin{cases} u^1 \in W^{1,q}(\Omega) \cap C^{0,\beta}(\bar{\Omega}) \\ u^2 \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \end{cases}\right\}$ *for all* $q < \dfrac{2-\beta}{1-\beta}$, *and all* $p < (2-\beta)$;

(ii) $\mathrm{Det}\, Du \neq \det Du$ *in the sense of distributions.*

Note that $\dfrac{2-\beta}{1-\beta} > 2$ and that $\left(\dfrac{2-\beta}{1-\beta}\right) + (2-\beta)^{-1} = 1$.

More details can be found in theorem 6.1 below.

The idea of the construction underlying Theorem 1.1 is to use a self-similar set S. Consider first a vaguely analogous situation for $n=1$: Find a function $h:(0, 1) \to \mathbb{R}$ such that h is differentiable a.e. with derivative 0 and such that the distributional derivative h' is a measure supported on a set M of Hausdorff dimension $\beta \in (0, 1)$. It is well known that such functions exist, indeed M may be obtained as a Cantor set. Letting $n=2$ for simplicity, a first guess might then be to choose $\tilde{u}(x, y) = (h(x), h(y))$. Such a choise satisfies (ii) and (iii) of the Theorem 1.1 but neither $\tilde{u}$ nor h is in $W^{1,1}$. The idea is then to choose u such that $u^1(x, y) = h(x)$ if $y \in M$ but such that $u^1(., y)$ is "smoother" if $y \notin M$. Similarily $u^2(x, y) = h(y)$ if $x \in M$, but "smooth" if $x \notin M$.

A construction related to the one given here appeares in [Po 87], where Ponomarev constructs homeomorphisms in $W^{1,p}$ $(p<n)$ which map a set of measure zero to a set of positive measure. Recently Maly and [MM 92] solved a longstanding conjecture by exhibiting a map $u \in W^{1,n}$, satisfying $\det Du = 0$ a.e. (and hence $\mathrm{Det}\, Du = 0$) which also maps a null set to a set of positive measure. De Arcangelis [DA 89] has studied lower semicontinuity properties of integral functionals and used examples for which $\det Du \neq \mathrm{Det}\, Du$ to show that certain of his hypotheses cannot be relaxed. I believe that the examples given here allow one to extend that line of reasoning.

Outline. The properties of the Cantor sets M and the functions h discussed above are reviewed in Section 2, while in Section 3 a sufficiently smooth map $\rho: [0, 1]^2 \to \mathbb{R}$ is constructed which interpolates between h and the identity. Sections 4 and 5 contain the proof of Theorem 1.1 for $n=2$ and $n \geqq 2$, respectively. Although the constructions are very similar in both cases, the case $n=2$ was given a separate treatment because certain technical and notational difficulties are not present. In Section 6 Theorems 1.2 and 1.3 are proved.

Notation. For an open set $\Omega\subset\mathbb{R}^n$, $W^{1,p}(\Omega)$, $C^{0,\beta}(\Omega)$ denote the usual spaces of Sobolev and Hölder continuous functions, respectively, the spaces of corresponding vector-valued functions are denoted by $W^{1,p}(\Omega;\mathbb{R}^n)$ etc. By H^α we denote the α-dimensional Hausdorff measure and by $\mathscr{L}^k$ the k-dimensional Lebesgue measure. The square $(0, 1)^2$ is denoted by Q, while $Q^n=(0, 1)^n$. The letters C and c denote generic constants whose value may change from line to line. For sets A, $B\subset\mathbb{R}^n$ we let

$$A+\lambda B=\{a+\lambda b: a\in A,\ b\in B\}.$$

The euclidian diameter of a set is

$$\operatorname{diam}_2 A=\sup\{|x-y|: x, y\in A\},$$

and the cubic distance of y from A is

$$\operatorname{dist}(y, A)=\inf\{|x-y|_\infty: x\in A\},$$

where $|x-y|_\infty=\sup\limits_{1\leq i\leq n}|x_i-y_i|$.

Additional notation related to Cantor sets is introduced in Section 2.

2. CANTOR SETS

For a set $A\subset\mathbb{R}^n$ the α-dimensional Hausdorff premeasure is defined by

$$H_\delta^\alpha(A)=\inf\left\{\sum_{j=1}^\infty \omega(\alpha)\left(\frac{\operatorname{diam}_2 C_j}{2}\right)^\alpha : A\subset\bigcup_{j=1}^\infty C_j,\ \operatorname{diam}_{2\,j}\leqq\delta\right\},$$

where $\omega(\alpha)=\pi^{\alpha/2}/\Gamma\left(\frac{\alpha}{2}+1\right)$ and where diam_2 is the diameter with respect to the euclidean norm. As H_δ^α is decreasing in δ the α-dimensional Hausdorff measure is defined by

$$H^\alpha(A)=\lim_{\delta\to 0} H_\delta^\alpha(A)=\sup_{\delta>0} H_\delta^\alpha(A).$$

The Hausdorff dimension of a set is given by

$$\dim_H A=\inf\{\alpha>0: H^\alpha(A)=0\}.$$

A famous set of fractional Hausdorff dimension is obtained by the following construction. Let $\gamma\in(0, 1)$. To construct the Cantor set M_γ begin with the closed interval [0, 1] and, in the first step, remove an open set of length γ in its middle. In the second step remove an interval of lenght $\left(\frac{1-\gamma}{2}\right)\gamma$ in the middle of the two remaining intervals. Continue the process, removing 2^{k-1} open intervals of lenght $\left(\frac{1-\gamma}{2}\right)^{k-1}\gamma$ in the k-th step. This eventually leaves a closed (Lebesgue) nullset M_γ.

Related to the above procedure is the construction of a nondecreasing function h_γ whose derivative vanishes on $[0, 1]\setminus M_\gamma$ (see *Fig.* 1). Set $h_\gamma = 1/2$

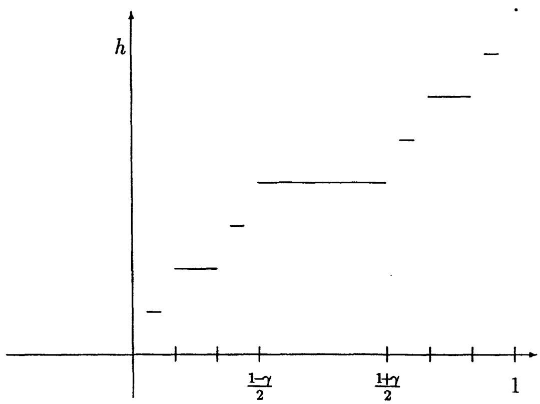

FIG. 1. – The Cantor function h_γ.

on the interval removed on the first step, $h_\gamma = 1/4$ and $3/4$, respectively on the intervals removed on the second step, $h_\gamma = (2j-1)\,2^{-k}$, $j = 1, \ldots, 2^{k-1}$ on the 2^{k-1} intervals removed in the k-th step. This defines h_γ on $[0, 1]\setminus M_\gamma$. One easily checks that there is a unique monotone and continuous extension.

For future reference we give a more formal description of M_γ. Define numbers $b_{k,m}$ recursived by

$$b_{0,1} = 0 \tag{2.1}$$

$$b_{k+1,m} = \begin{cases} \dfrac{1-\gamma}{2} b_{k,m} & \text{if } m \in \{1, \ldots, 2^k\} \\ \dfrac{1+\gamma}{2} + \dfrac{1-\gamma}{2} b_{k,m-2^k} & \text{if } m \in \{2^k+1, \ldots, 2^{k+1}\} \end{cases} \tag{2.2}$$

The sets $I_{k-1,m}$ removed in the k-th step of the construction described above are given by

$$I_{k-1,m} = b_{k-1,m} + \left(\frac{1-\gamma}{2}\right)^{k-1}\left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right), \tag{2.3}$$

And the sets remaining after the k-th step are

$$J_{k,m} = b_{k,m} + \left(\frac{1-\gamma}{2}\right)^{k}[0, 1]. \tag{2.4}$$

One has the disjoint union

$$[0, 1]=\left(\bigcup_{m=1}^{2^k} J_{k, m}\right)\cup\bigcup_{l=0}^{k-1}\bigcup_{m=0}^{2^l} I_{l, m}. \tag{2.5}$$

The Cantor set M_γ is defined by

$$M_\gamma=\bigcap_{k=1}^{\infty}\left(\bigcup_{m=1}^{2^k} J_{k, m}\right). \tag{2.6}$$

To obtain h, consider the sequence of functions $h_\gamma^{(k)}: [0, 1]\to\mathbb{R}$ defined by

$$h_\gamma^{(0)}(x)=x, \tag{2.7}$$

$$h_\gamma^{(k+1)}(x)=\begin{cases}\frac{1}{2}h^{(k)}\left(\left(\frac{1-\gamma}{2}\right)^{-1}x\right), & x\in\left[0, \frac{1-\gamma}{2}\right]\\ \frac{1}{2}, & x\in\left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)\\ \frac{1}{2}+\frac{1}{2}h^{(k)}\left(1-\left(\frac{1-\gamma}{2}\right)^{-1}(1-x)\right), & x\in\left[\frac{1+\gamma}{2}, 1\right].\end{cases} \tag{2.8}$$

For $l>k\geqq 0$ one easily verifies by induction that

$$h_\gamma^{(l)}(x)=\frac{2m-1}{2^{k+1}}, \qquad \text{for} \quad x\in I_{k, m}, \quad m=1, \ldots, 2^k.$$

Thus the sequence $\{h_\gamma^{(l)}\}$ converges pointwise on the dense set $[0, 1]\setminus M_\gamma$.

Again by induction,

$$|h_\gamma^{(k)}(x)-h_\gamma^{(k)}(y)|\leqq|x-y|^\beta \quad \text{for all } x, y\in[0, 1], \tag{2.9}$$

where

$$\beta=\frac{\ln 2}{\ln(2/(1-\gamma))}. \tag{2.10}$$

Thus $h_\gamma^{(k)}\to h_\gamma$ uniformly and

$$h_\gamma(x)=\frac{2m-1}{2^{k+1}}, \qquad \text{for} \quad x\in I_{k, m}, \quad m=1, \ldots, 2^k. \tag{2.11}$$

The following proposition summarizes some well-known properties of h_γ and M_γ (*see* also Falconer [Fa 85], Theorem 1.14; [Ro 70]).

Proposition 2.1. – *Let* β *be given by* (2.10). *The function* h_γ *is non-decreasing continuous and satisfies*

$$|h_\gamma^{(k)}(x)-h_\gamma^{(k)}(y)|\leqq|x-y|^\beta \quad \textit{for all } x, y\in[0, 1], \tag{2.12}$$

Moreover

$$h_\gamma(b_{k, m})=\frac{m-1}{2^k} \qquad \textit{for} \quad m\in\{1, \ldots, 2^k\}, \tag{2.13}$$

and one has the self-similar scaling law

$$\left\{\begin{aligned} & h_\gamma(x)=h_\gamma(b_{k,m})+2^{-k}h_\gamma\left(\left(\frac{1-\gamma}{2}\right)^{-k}(x-b_{k,m})\right) \\ & \text{for all} \\ & \qquad x\in \mathrm{J}_{k,m}=\left[b_{k,m},\, b_{k,m}+\left(\frac{1-\gamma}{2}\right)^{k}\right]. \end{aligned}\right. \tag{2.14}$$

The set M_γ *is a closed set of Hausdorff dimension* β, *the distributional derivative* h'_γ *is a measure supported on* M_γ *and for any Borel set* $\mathrm{U}\subset[0, 1]$ *one has*

$$h'_\gamma(\mathrm{U})=\mathrm{H}^\beta(\mathrm{U}\cap \mathrm{M}_\gamma). \tag{2.15}$$

Finally the set M_γ *is self-similar*, i. e.

$$x\in \mathrm{M}_\gamma\cap \mathrm{J}_{k,m} \quad\Leftrightarrow\quad \left(\frac{1-\gamma}{2}\right)^{-k}(x-b_{k,m})\in \mathrm{M}_\gamma. \tag{2.16}$$

For future reference we also note

PROPOSITION 2.2. – *Let* $k\geqq 0$, *then the open sets*

$$\tilde{\mathrm{J}}_{k,m}=b_{k,m}+\left(\frac{1-\gamma}{2}\right)^{k}\left(-\frac{\gamma}{1-\gamma},\, 1+\frac{\gamma}{1-\gamma}\right), \qquad m=1,\ldots,2^k,$$

form a disjoint cover of M_γ *and*

$$\operatorname{dist}(x, \mathrm{M}_\gamma)\geqq\left(\frac{1-\gamma}{2}\right)^{k}\frac{\gamma}{1-\gamma}, \qquad \textit{for}\quad x\in[0, 1]\setminus\bigcup_{m=1}^{2^k}\tilde{\mathrm{J}}_{k,m}. \tag{2.17}$$

Here we set $\operatorname{dist}(x, \mathrm{A})=\inf\{|x-y|:y\in \mathrm{A}\}$.

Proof. – The $\tilde{\mathrm{J}}_{k,m}$ obviously cover M_γ as $\tilde{\mathrm{J}}_{k,m}\supset \mathrm{J}_{k,m}$. To show that the cover is disjoint let

$$c_{k,m}=b_{k,m}-\left(\frac{1-\gamma}{2}\right)^{k}\frac{\gamma}{1-\gamma}; \qquad d_{k,m}=b_{k,m}+\left(\frac{1-\gamma}{2}\right)\left(1+\frac{\gamma}{1-\gamma}\right).$$

We first verify by induction that

$$d_{k,m}\leqq c_{k,m+1}; \qquad \text{for}\quad m=1,\ldots,2^k-1. \tag{2.18}$$

For $k=0$ there is nothing to show, assume that (2.18) holds for some $k\geqq 0$. Then by (2.2), for $m=1,\ldots,2^k$,

$$c_{k+1,m}=\frac{1-\gamma}{2}c_{k,m}; \qquad d_{k+1,m}=\frac{1-\gamma}{2}d_{k,m}$$

$$c_{k+1,m+2^k}=\frac{1+\gamma}{2}+\frac{1-\gamma}{2}c_{k,m}; \qquad d_{k+1,m+2^k}=\frac{1+\gamma}{2}+\frac{1-\gamma}{2}d_{k,m}.$$

Hence, by (2.18) $d_{k+1,m} \leqq c_{k,m+1}$, for $m \leqq 2^{k+1}-1$, $m \neq 2^k$. By induction one deduces easily from (2.2) that

$$b_{k,2^k} \leqq 1-\left(\frac{1-\gamma}{2}\right)^k, \qquad d_{k,2^k} \leqq 1+\left(\frac{1-\gamma}{2}\right)^k \frac{\gamma}{1-\gamma},$$

and hence

$$d_{k+1,2^k} \leqq \frac{1-\gamma}{2}+\frac{\gamma}{2} \leqq \frac{1}{2}$$

$$c_{k+1,2^k+1} \geqq \frac{1+\gamma}{2}+\frac{1-\gamma}{2} c_{k,1} \geqq \frac{1+\gamma}{2}-\frac{\gamma}{2} \geqq d_{k+1,2^k}.$$

We turn to the proof of (2.17). As the $\tilde{J}_{k,m}$ are disjoint one has for $x \in [0,1] \setminus \bigcup_{m=1}^{2^k} \tilde{J}_{k,m}$,

$$\operatorname{dist}(x, J_{k,m}) \geqq \left(\frac{1-\gamma}{2}\right)^k \frac{\gamma}{1-\gamma},$$

for $m=1, \ldots, 2^k$, and hence (2.17). □

3. INTERPOLATION BETWEEN h AND THE IDENTITY

In this and the following sections we fix $\gamma \in (0, 1)$, we set

$$\beta=\frac{\ln 2}{\ln (2/(1-\gamma))}, \tag{3.1}$$

and we drop the subscript γ from all the quantities defined in Section 2. Similarly all constants appearing in estimates may depend on γ. We let

$$Q=(0,1)^2,$$

and we construct a function $f \in W^{1,2}(Q) \cap C^{0,\beta}(\bar{Q})$ satisfying (see *Fig.* 2)

$$f(x, 0)=h(x), \qquad f(x, 1)=x, \tag{3.2}$$

$$f(0, y)=0, \qquad f(1, y)=1, \tag{3.3}$$

where $h=h_\gamma$ is the Cantor function defined in the previous section.

Define $f_0 : [0, 1] \times \left[\dfrac{1-\gamma}{2}, 1\right] \to \mathbb{R}$ by (see *Fig.* 2)

$$f_0(x, y) = \begin{cases} \left(1 - \dfrac{2\gamma}{1+\gamma}(1-y)\right)^{-1} x, & x \in \left[0, \dfrac{1}{2} - \dfrac{\gamma}{1+\gamma}(1-y)\right], \\ \dfrac{1}{2}, & x \in \left(\dfrac{1}{2} - \dfrac{\gamma}{1+\gamma}(1-y), \dfrac{1}{2} + \dfrac{\gamma}{1+\gamma}(1-y)\right), \\ 1 - \left(1 - \dfrac{2\gamma}{1+\gamma}(1-y)\right)^{-1}(1-x), & \\ & x \in \left[\dfrac{1}{2} + \dfrac{\gamma}{1+\gamma}(1-y), 1\right]. \end{cases} \tag{3.4}$$

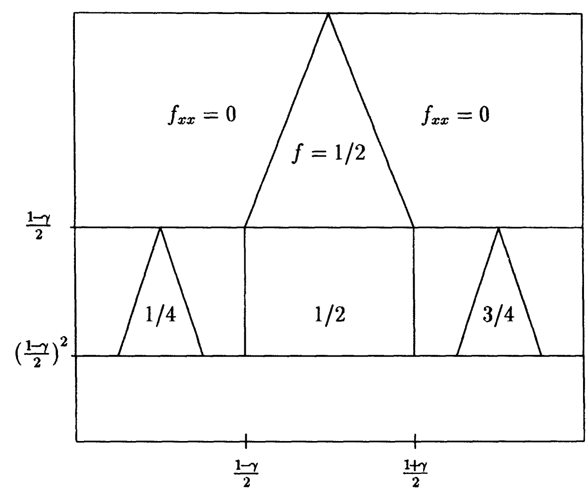

FIG. 2. — Interpolation between h and the identity.

Note that $x \mapsto f(x, y)$ is piecewise linear, that f is Lipschitz and

$$f_0(x, 1) = x,$$

$$f_0\left(x, \frac{1-\gamma}{2}\right) = \begin{cases} (1-\gamma)^{-1} x & \text{if } 0 \leqq x \leqq (1-\gamma)/2, \\ 1/2 & \text{if } (1-\gamma)/2 \leqq x \leqq (1+\gamma)/2, \\ 1 - (1-\gamma)^{-1}(1-x) & \text{if } (1+\gamma)/2 \leqq x \leqq 1. \end{cases}$$

For $k \geqq 0$ define $f_k : [0, 1] \times \left[\left(\frac{1-\gamma}{2}\right)^{k+1}, \left(\frac{1-\gamma}{2}\right)^{k}\right] \to \mathbb{R}$ recursively by

$$f_{k+1}(x, y) = \begin{cases} \frac{1}{2} f_k\left(\left(\frac{1-\gamma}{2}\right)^{-1} x, \left(\frac{1-\gamma}{2}\right)^{-1} y\right), \\ \qquad \text{if } x \in \left[0, \frac{1-\gamma}{2}\right], \\ \frac{1}{2} \quad \text{if } x \in \left[\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right], \\ \frac{1}{2} + \frac{1}{2} f_k\left(1 - \left(\frac{1-\gamma}{2}\right)^{-1}(1-x), \left(\frac{1-\gamma}{2}\right)^{-1} y\right), \\ \qquad \text{if } x \in \left[\frac{1+\gamma}{2}, 1\right]. \end{cases} \tag{3.5}$$

One easily verifies that f_k is Lipschitz and that

$$f_1\left(x, \frac{1-\gamma}{2}\right) = f_0\left(x, \frac{1-\gamma}{2}\right),$$

and hence

$$f_{k+1}\left(x, \left(\frac{1-\gamma}{2}\right)^k\right) = f_k\left(x, \left(\frac{1-\gamma}{2}\right)^k\right).$$

Thus the function f given by

$$f \,|_{[0,\,1] \times [((1-\gamma)/2)^{k+1},\, ((1-\gamma/2)^k]} = f_k, \tag{3.6}$$

is thus well-defined and continuous. We extend f to a continuous function on $[0, 1]^2$ by setting

$$f(x, 0) = h(x). \tag{3.7}$$

LEMMA 3.1. — *Let $b_{k,m}$ and $J_{k,m}$ given by* (2.1), (2.2) *and* (2.4). *Then the function f defined by* (3.6), (3.7) *satisfies:*

(i)

$$f \in W^{1,q}(Q) \cap C^{0,\beta}(\bar{Q}) \quad \textit{for all } q \in \left[1, \frac{2-\beta}{1-\beta}\right), \tag{3.8}$$

(ii)

$$\left.\begin{array}{ll} f(x, 0) = h(x), & f(x, 1) = x, \\ f(0, y) = 0, & f(1, y) = 1, \end{array}\right\} \tag{3.9}$$

(iii)

$$\left.\begin{aligned} & f(x, y)=h(b_{k,m})+2^{-k} f\left(\left(\frac{1-\gamma}{2}\right)^{-k}(x-b_{k,m}), \left(\frac{1-\gamma}{2}\right)^{-k} y\right) \\ & \text{for} \\ & \qquad (x, y)\in J_{k,m}\times\left[0, \left(\frac{1-\gamma}{2}\right)^{k}\right], \end{aligned}\right\} \tag{3.10}$$

(iv)

$$\left.\begin{aligned} & f(x, y)=h(x) \\ & \text{if} \\ & (x, y)\in\left([0, 1]\setminus\bigcup_{j=1}^{2^k} J_{k,m}\right)\times\left[0, \left(\frac{1-\gamma}{2}\right)^{k}\right], \end{aligned}\right\} \tag{3.11}$$

$$\left.\begin{aligned} & Df(x, y)=0 \\ & \text{if} \\ & (x, y)\in\left([0, 1]\setminus\bigcup_{j=1}^{2^k} J_{k,m}\right)\times\left(0, \left(\frac{1-\gamma}{2}\right)^{k}\right), \end{aligned}\right\} \tag{3.12}$$

(v)

$$f(x, y)=h(x) \qquad \text{if } y\leqq\frac{2}{\gamma}\operatorname{dist}(x, M). \tag{3.13}$$

Remark. — The regularity result in (i) is optimal, *i.e.*

$$f\notin W^{1,(2-\beta)/(1-\beta)}(Q),$$

and for all $\lambda>\beta$, $f\notin C^{0,\lambda}(\bar{Q})$.

Proof. — Assertion (ii) follows directly from the definition of f.

Proof of (iii). — In view of (ii) and Proposition 2.1 we may assume $y>0$. It thus suffices to verify the following assertion (P_l) for all $l\in\mathbb{N}_0$.

$$\left.\begin{aligned} & \text{For all} \\ & y\in\left[\left(\frac{1-\gamma}{2}\right)^{l+1}, \left(\frac{1-\gamma}{2}\right)^{l}\right], \quad \text{all } k\leqq l, \quad \text{all } m\in\{1, \ldots, 2^l\} \\ & \text{and all} \\ & \qquad x\in\left[b_{k,m}, b_{k,m}+\left(\frac{1-\gamma}{2}\right)^{k}\right] \\ & \text{one has,} \\ & f(x, y)=h(b_{k,m})+2^{-k} f\left(\left(\frac{1-\gamma}{2}\right)^{-k}(x-b_{k,m}), \left(\frac{1-\gamma}{2}\right)^{-k} y\right). \end{aligned}\right\} \tag{P_l}$$

The case $l=0$ is trivial. Assume (P_{l-1}) was true. We will show that (P_l) is true. Let $0<k\leqq l$, let $x=b_{k,m}+\left(\frac{1-\gamma}{2}\right)^{k}\tilde{x}$, $\tilde{x}\in[0, 1]$. Consider first the

case $m \leqq 2^{k-1}$, then by (2.2) and (2.13)

$$b_{k,m} = \left(\frac{1-\gamma}{2}\right)^{-1} b_{k-1,m}; \qquad h(b_{k,m}) = \frac{1}{2} h(b_{k-1,m}).$$

Moreover $x \leqq (1-\gamma)/2$, so that with the abbreviations

$$\hat{x} = \left(\frac{1-\gamma}{2}\right)^{-1} x, \qquad \hat{y} = \left(\frac{1-\gamma}{2}\right)^{-1} y, \qquad \tilde{y} = \left(\frac{1-\gamma}{2}\right)^{-k} y,$$

one has by definition (3.5) of f_l and (P_{l-1})

$$\begin{aligned} f(x, y) = f_l(x, y) &= \frac{1}{2} f_{l-1}(\hat{x}, \hat{y}) \\ &= \frac{1}{2} f\left(b_{k-1,m} + \left(\frac{1-\gamma}{2}\right)^{k-1} \tilde{x}, \hat{y}\right) \\ &= \frac{1}{2} h(b_{k-1,m}) + \frac{1}{2} 2^{-(k-1)} f(\hat{x}, \tilde{y}) \\ &= h(b_{k,m}) + 2^{-k} f(\tilde{x}, \tilde{y}), \end{aligned}$$

and (P_l) is proved, provided that $m \leqq 2^{k-1}$. The case $m > 2^{k-1}$ is analogous.

Proof of (iv). – Consider $y \in \left[0, \left(\frac{1-\gamma}{2}\right)^k\right]$, $x \in [0, 1] \setminus \bigcup_{m=1}^{2^k} J_{k,m}$.

Thus by (2.5), (2.3), $x \in I_{l,m'}$ for some $l \leqq k-1$ so that

$$x = b_{l,m'} + \left(\frac{1-\gamma}{2}\right)^l \tilde{x}, \quad l \leqq k-1, \qquad \text{with} \quad \tilde{x} \in \left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right).$$

If $l=0$, then $x = \tilde{x}$ and $k \geqq 1$. Thus by definition of f_k,

$$f(x, y) = f(\tilde{x}, y) = \frac{1}{2} = h(x). \tag{3.14}$$

If $l > 0$, by (iii), (3.14) and (2.14)

$$\begin{aligned} f(x, y) &= h(b_{l,m'}) + 2^{-l} f\left(\tilde{x}, \left(\frac{1-\gamma}{2}\right)^{-l} y\right) \\ &= h(b_{l,m'}) + 2^{-l} h(\tilde{x}) \\ &= h(x). \end{aligned}$$

Moreover for x as above $h'(x) = 0$ by (2.11).

Thus $Df = 0$ in the open set $\left([0, 1] \setminus \bigcup_{m=1}^{2^k} J_{k,m}\right) \times \left(0, \left(\frac{1-\gamma}{2}\right)^k\right)$.

Proof of (v). – Assume $y \leqq \frac{2}{\gamma}\operatorname{dist}(x, M)$. If $x \in M$ the assertion follows from (ii). If $x \notin M$ there exist k, m such that

$$x \in I_{k,m} = \left(b_{k,m} + \frac{1-\gamma}{2}\left(\frac{1-\gamma}{2}\right)^k, b_{k,m} + \frac{1+\gamma}{2}\left(\frac{1-\gamma}{2}\right)^k\right).$$

Consider first $k=0$.

Then $x \in \left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)$, and hence

$$\operatorname{dist}(x, M) = \min\left\{\left|x - \frac{1-\gamma}{2}\right|, \left|x - \frac{1+\gamma}{2}\right|\right\} = \frac{\gamma}{2} - \left|x - \frac{1}{2}\right|.$$

If $y \leqq \frac{1-\gamma}{2}$ then $f(x, y) = \frac{1}{2} = h(x)$ by the definition of f_k. If

$$\frac{1-\gamma}{2} < y \leqq \frac{2}{\gamma}\operatorname{dist}(x, M) = 1 - \frac{2}{\gamma}\left|x - \frac{1}{2}\right|,$$

one deduces that

$$\left|x - \frac{1}{2}\right| \leqq \frac{\gamma}{2}(1-y) \leqq \frac{\gamma}{1+\gamma}(1-y),$$

and hence $f(x, y) = \frac{1}{2} = h(x)$ by (3.4).

If $k>0$ one has similarily

$$y \geqq \frac{2}{\gamma}\operatorname{dist}(x, M) \leqq \left(\frac{1-\gamma}{2}\right)^k.$$

Letting

$$\tilde{y} = \left(\frac{1-\gamma}{2}\right)^{-k} y; \qquad \tilde{x} = \left(\frac{1-\gamma}{2}\right)^{-k}(x - b_{k,m}),$$

one has in view of (2.6) and (2.16)

$$\tilde{y} \leqq \frac{2}{\gamma}\operatorname{dist}(\tilde{x}, M); \qquad \tilde{x} \in \left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right),$$

and so by (iii), the result for $k=0$ and (2.14)

$$\begin{aligned} f(x, y) &= h(b_{l,m}) + 2^{-k} f(\tilde{x}, \tilde{y}) \\ &= h(b_{l,m}) + 2^{-k} h(\tilde{x}) = h(x). \end{aligned}$$

Proof of (i). – We show first that $f \in C^{0,\beta}(\bar{Q})$. To this end it suffices to show that there exists a constant C, such that for all pairs (x, y), (ξ, η)

in $[0, 1]^2$ one has

$$|f(\xi, \eta)-f(\xi, y)| \leqq C|y-\eta|^{\beta}, \tag{3.15}$$

$$|f(\xi, y)-f(x, y)| \leqq C|x-\xi|^{\beta}. \tag{3.16}$$

We may assume $\eta \leqq y$. We may also assume $y>0$ since for $y=\eta=0$ the estimates follow from (3.7) and Proposition 2.1. To establish (3.15), choose k such that $y \in \left(\left(\frac{1-\gamma}{2}\right)^{k+1}, \left(\frac{1-\gamma}{2}\right)^{k}\right]$.

If $k=0$ then (3.15) is obvious. Indeed either $\eta>\left(\frac{1-\gamma}{2}\right)^2$ so that (3.15) follows from the fact that $f|_{[0,1]\times[((1-\gamma)/2)^2, 1]}$ is Lipschitz or $\eta \leqq \left(\frac{1-\gamma}{2}\right)$ in which case $|y-\eta| \geqq \frac{1-\gamma}{2}-\left(\frac{1-\gamma}{2}\right)^2$ and (3.15) follows since $0 \leqq f \leqq 1$.

If $k>0$ and $\xi \notin \bigcup_{l=1}^{2^k} J_{k,m}$ then (3.15) follows from (iv). Finally if $\xi \in J_{k,m}$, let $\tilde{\xi}=\left(\frac{1-\gamma}{2}\right)^{-k}(\xi-b_{k,m})$. By (iii) and the result for $k=0$ one has

$$\begin{aligned} |f(\xi, \eta)-f(\xi, y)| &\leqq 2^{-k}\left|f\left(\tilde{\xi}, \left(\frac{1-\gamma}{2}\right)^{-k}\eta\right)-f\left(\tilde{\xi}, \left(\frac{1-\gamma}{2}\right)^{-k}y\right)\right| \\ &\leqq 2^{-k}C\left|\left(\frac{1-\gamma}{2}\right)^{-k}(y-\eta)\right|^{\beta} \leqq C|y-\eta|^{\beta}, \end{aligned}$$

since

$$\left(\frac{1-\gamma}{2}\right)^{\beta}=\frac{1}{2}.$$

This proves (3.15).

We next show by induction that (3.16) holds for all

$$y \in \left(\left(\frac{1-\gamma}{2}\right)^{k+1}, \left(\frac{1-\gamma}{2}\right)^{k}\right].$$

For $k=0$, there is nothing to show since $f|_{[0,1]\times[(1-\gamma/2), 1]}$ is Lipschitz.

To carry out the induction step assume that $y \in \left(\left(\frac{1-\gamma}{2}\right)^{k+2}, \left(\frac{1-\gamma}{2}\right)^{k+1}\right]$.

If $x, \xi \in \left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)$, the result follows from the definition (3.5) of f_{k+1}.

We may thus assume $x \notin \left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)$ and, by symmetry under $x \mapsto 1-x$, $x \in \left[0, \frac{1-\gamma}{2}\right]$.

If $\xi \in \left[0, \frac{1-\gamma}{2}\right]$ then, by the definition of f_{k+1}, and the result for $y \in \left(\left(\frac{1-\gamma}{2}\right)^{k+1}, \left(\frac{1-\gamma}{2}\right)^{k}\right]$

$$\begin{aligned} &|f(\xi, y)-f(x, y)| \\ &\quad = \frac{1}{2}\left|f\left(\left(\frac{1-\gamma}{2}\right)^{-1}\xi, \left(\frac{1-\gamma}{2}\right)^{-1}y\right)-f\left(\left(\frac{1-\gamma}{2}\right)^{-1}x, \left(\frac{1-\gamma}{2}\right)^{-1}y\right)\right| \\ &\quad \leqq \frac{1}{2}C\left|\left(\frac{1-\gamma}{2}\right)^{-1}(x-\xi)\right|^{\beta} = C|x-\xi|^{\beta}. \end{aligned} \tag{3.17}$$

If $\xi \in \left[\frac{1+\gamma}{2}, 1\right]$ then (3.16) holds since $|x-\xi| \geqq \frac{1+\gamma}{2} - \frac{1-\gamma}{2} = \gamma$ and $0 \leqq f \leqq 1$. Finally if $\xi \in \left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)$ then $f(\xi, y) = \frac{1}{2} = f\left(\frac{1-\gamma}{2}, y\right)$ and as in (3.17)

$$|f(\xi, y)-f(x, y)| \leqq C\left|x-\frac{1-\gamma}{2}\right|^{\beta} \leqq C|x-\xi|^{\beta}.$$

Thus the assertion $f \in C^{0, \beta}(\bar{Q})$ is established.

It remains to show $f \in W^{1, q}(Q)$, for all $q < \frac{2-\beta}{1-\beta}$. Note that f is a absolutely continuous along the lines $y = y_0$ (if $y_0 \neq 0$) and, by (iv), along the lines $x = x_0$ for $x \notin M$. By [Mo 66], Theorem 3.1.2, it thus suffices to show $|Df| \in L^q(Q)$.

Let $Q_{k, m} = J_{k, m} \times \left[\left(\frac{1-\gamma}{2}\right)^{k+1}, \left(\frac{1-\gamma}{2}\right)^{k}\right]$. By (iv) $Df = 0$ a.e. outside $\bigcup_{k=0}^{\infty} \bigcup_{m=1}^{2^k} Q_{k, m}$. By (iii)

$$\begin{aligned} \int_{Q_{k, m}} |Df|^q \, dx\, dy &= \int_{Q_{0, 0}} 2^{kq}\left(\frac{1-\gamma}{2}\right)^{-kq} |Df_0|^q \left(\frac{1-\gamma}{2}\right)^{2k} dx\, dy \\ &= 2^{-qk}\left(\frac{1-\gamma}{2}\right)^{(2-q)k} C, \end{aligned}$$

and hence

$$\int_Q |Df|^q \, dx\, dy = C \sum_{k=0}^{\infty} 2^k 2^{-qk} \left(\frac{1-\gamma}{2}\right)^{(2-q)k}$$
$$= C \sum_{k=0}^{\infty} \left(\frac{1-\gamma}{2}\right)^{[(2-q)-\beta(1-q)]k}.$$

The sum converges if and only if $(2-q)-\beta(1-q)>0$ or, equivalently, $q<\frac{2-\beta}{1-\beta}$.

The proof of the Lemma 3.1 is finished. □

4. CONSTRUCTION FOR $n=2$

As before we fix $\gamma\in(0, 1)$, we let

$$\beta=\frac{\ln 2}{\ln(2/(1-\gamma))}$$

and consider $M=M_\gamma$, $h=h_\gamma$ as defined in Section 2. In the following we do not indicate the dependence of the various functions, sets, etc. on γ. Throughout this section f denotes the function defined by (3.6), (3.7). We let $Q=(0, 1)^2$ and

$$\operatorname{dist}(x, M)=\inf\{|x-y|_\infty : y\in M\}, \qquad |x-y|_\infty=\sup\{|x_i-y_i| : i=1, 2\}$$

We define a map $u:\bar{Q}\to\mathbb{R}^2$ by (see *Fig.* 3)

$$u^1(x, y)=f\left(x, \frac{2}{\gamma}\operatorname{dist}(y, M)\right), \tag{4.1}$$

$$u^2(x, y)=f\left(y, \frac{2}{\gamma}\operatorname{dist}(x, M)\right)=u^1(y, x). \tag{4.2}$$

Observe that for $y\in M$, $u^1(x, y)=h(x)$, while $u^1\left(x, \frac{1}{2}\right)=x$. Note that for $y\in[0, 1]$ [*cf.* the proof of Lemma 3.1 (v)]

$$\frac{2}{\gamma}\operatorname{dist}(y, M)\leqq 1.$$

THEOREM 4.1. – *Let u be given by* (4.1), (4.2). *Then:*

(i) *for all* $p\in[1, 2)$, $u\in W^{1,p}(Q; \mathbb{R}^2)\cap C^{0,\beta}(\bar{Q}; \mathbb{R}^2)$*;*

(ii) $|Du^1|\cdot|Du^2|=0$ *a.e. and in particular* $\det Du=0$ *a.e.;*

(iii) $\operatorname{Det} Du$ *is a nonnegative measure given by*

$$\operatorname{Det} Du=h'\otimes h'.$$

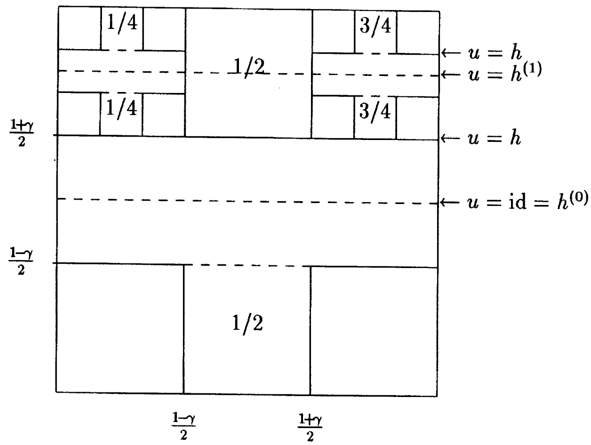

FIG. 3. – Values of u^1, *cf.* (2.7), (2.8).

In particular Det Du *is supported on the set* M × M *of Hausdorff dimension* 2β *and there are constants* c, C *such that for any Borel set* U

$$c\,\mathrm{H}^{2\beta}(\mathrm{U}\cap(\mathrm{M}\times\mathrm{M}))\leqq \mathrm{Det}\,\mathrm{D}u(\mathrm{U})\leqq \mathrm{C}\mathrm{H}^{2\beta}(\mathrm{U}\cap(\mathrm{M}\times\mathrm{M})). \tag{4.3}$$

One may choose $c=2^{-\beta}/\omega(2\beta)$, $\mathrm{C}=1/\omega(2\beta)$.

Remarks:

1. For the analogous result in higher dimensions, *see* Section 5.
2. The assertion $\det \mathrm{D}u=0$ could be deduced from (iii) by appealing to [Mu 90*a*], Theorem 1 and Remark 2.

Proof of Theorem 4.1. – To prove (i) it suffices to consider u^1. The assertion $u^1\in \mathrm{C}^{0,\beta}(\bar{\mathrm{Q}})$ is obvious since $f\in \mathrm{C}^{0,\beta}(\bar{\mathrm{Q}})$ [*see* (3.8)] and $\mathrm{dist}(\cdot,\mathrm{M})$ is Lipschitz. One easily verifies that for $x\notin \mathrm{M}$, $y\notin \mathrm{M}$, u is absolutely continuous along the lines $t\mapsto(t,y)$, $t\mapsto(x,t)$ with derivatives

$$\frac{\partial u^1}{\partial x}=f_{,1}\left(x,\frac{2}{\gamma}\mathrm{dist}(y,\mathrm{M})\right)$$

$$\frac{\partial u^1}{\partial y}=f_{,2}\left(x,\frac{2}{\gamma}\mathrm{dist}(y,\mathrm{M})\right)\frac{\partial}{\partial y}\frac{2}{\gamma}\mathrm{dist}(y,\mathrm{M})),$$

where $f_{,i}$ denotes the partial derivative of f with respect to the i-th argument. Note that $\left|\frac{\partial}{\partial y}\mathrm{dist}(y,\mathrm{M})\right|\leqq 1$ a.e. To prove $u^1\in \mathrm{W}^{1,p}(\mathrm{Q})$ it thus suffices to show $g\in \mathrm{L}^p(\mathrm{Q})$ where

$$g(x,y)=\left|\mathrm{D}f\left(x,\frac{2}{\gamma}\mathrm{dist}(y,\mathrm{M})\right)\right|. \tag{4.4}$$

For $\eta \in \left[\left(\frac{1-\gamma}{2}\right)^{k+1}, \left(\frac{1-\gamma}{2}\right)^{k}\right)$ one has by Lemma 3.1 (iii) and (iv)

$$|\mathrm{D}f(\xi, \eta)| = 0, \quad \text{if} \quad \xi \notin \bigcup_{m=1}^{2^k} \left[b_{k,m}, b_{k,m} + \left(\frac{1-\gamma}{2}\right)^k\right],$$

whereas for $\xi \in \bigcup_{m=1}^{2^k} \left(b_{k,m}, b_{k,m} + \left(\frac{1-\gamma}{2}\right)^k\right)$ one deduces

$$|\mathrm{D}f(\xi, \eta)| \leqq \mathrm{C}\, 2^k \left(\frac{1-\gamma}{2}\right)^{-k} = \mathrm{C} \left(\frac{1-\gamma}{2}\right)^{-k(1-\beta)} \leqq \mathrm{C}\, \eta^{-(1-\beta)}.$$

Thus for all y satisfying $\frac{2}{\gamma} \operatorname{dist}(y, \mathrm{M}) \in \left[\left(\frac{1-\gamma}{2}\right)^{k+1}, \left(\frac{1-\gamma}{2}\right)^{k}\right)$ one obtains

$$\begin{aligned} \int_0^1 |g|^p(x, y)\, dx &\leqq \mathrm{C}\, 2^k \left(\frac{1-\gamma}{2}\right)^k (\operatorname{dist}(y, \mathrm{M}))^{-(1-\beta)p} \\ &\leqq \mathrm{C}\, (\operatorname{dist}(y, \mathrm{M}))^{-(1-\beta)(p-1)}. \end{aligned} \tag{4.5}$$

Let $\mathrm{Q}' = \{(x, y) : \operatorname{dist}(y, \mathrm{M}) \in (0, \gamma/2)$. Then

$$\mathrm{Q} \setminus \mathrm{Q}' = \{(x, y) : \operatorname{dist}(y, \mathrm{M}) \in \{0, \gamma/2\}\} = ([0, 1] \times \mathrm{M}) \cup ([0, 1] \times \{1/2\})$$

has measure zero, and by (4.5)

$$\int_{\mathrm{Q}} |g|^p = \int_{\mathrm{Q}'} |g|^p\, dx\, dy \leqq \mathrm{C} \int_0^1 (\operatorname{dist}(y, \mathrm{M}))^{-(1-\beta)(p-1)}\, dy.$$

It remains to show that the last integral is finite for all $p<2$. To this end consider the distribution function

$$\varphi(\lambda) = \mathscr{L}^1 \{ y \in (0, 1) : (\operatorname{dist}(y, \mathrm{M}))^{-(1-\beta)(p-1)} \geqq \lambda \}.$$

To estimate φ from above we cover M by intervalls as follows. Let

$$\tilde{\mathrm{J}}_{k,m} = b_{k,m} + \left(\frac{1-\gamma}{2}\right)^k \left(-\frac{\gamma}{1-\gamma}, 1 + \frac{\gamma}{1-\gamma}\right),$$

$$\mathrm{U}_k = \bigcup_{m=1}^{2^k} \tilde{\mathrm{J}}_{k,m}.$$

By Proposition 2.2

$$\operatorname{dist}((0, 1) \setminus \mathrm{U}_k, \mathrm{M}) \geqq c \left(\frac{1-\gamma}{2}\right)^k.$$

Moreover

$$\mathscr{L}^1(\mathrm{U}_k) \leqq \mathrm{C}\, 2^k \left(\frac{1-\gamma}{2}\right)^{(1-\beta)k}.$$

Thus

$$\varphi\left(c \left(\frac{1-\gamma}{2}\right)^{-k(1-\beta)(p-1)}\right) \leqq \mathrm{C} \left(\frac{1-\gamma}{2}\right)^{(1-\beta)k},$$

or

$$\varphi(\lambda) \leqq C\lambda^{-1/(p-1)}.$$

Now $\varphi(\lambda) \leqq 1$ for all λ, and $\dfrac{1}{p-1} > 1$ for $p \in (1, 2)$ so that

$$\int_0^1 (\operatorname{dist}(y, M))^{-(1-\beta)(p-1)}(y)\, dy = \int_0^\infty \varphi(\lambda)\, d\lambda < \infty.$$

To prove (ii) and (iii) of the theorem we will use the following

LEMMA 4.2. – *Let* $A = \{(x, y) : \operatorname{dist}(x, M) = \operatorname{dist}(y, M)\}$ *and let u given by* (4.1) *and* (4.2). *Then:*

(i) A *is closed and* $\mathscr{L}^2(A) = 0$;

(ii) *for a.e.* $(x, y) \in [0, 1]^2$ *one has*

$$u^1(x, y) = h(x) \quad \textit{or} \quad Du^2(x, y) = 0; \tag{4.6}$$

(iii) *for a.e.* (x, y)

$$Du^1(x, y) = 0 \quad \textit{or} \quad Du^2(x, y) = 0. \tag{4.7}$$

Proof of the Lemma. – Let $h(x, y) = \operatorname{dist}(x, M) - \operatorname{dist}(y, M)$.

The set A is closed since h is continuous. Moreover h is Lipschitz and $\left|\dfrac{\partial h}{\partial x}\right| = 1$, $\left|\dfrac{\partial h}{\partial y}\right| = 1$ a.e. since M is closed and $\mathscr{L}^1(M) = 0$. Let

$$A_k = \{(x, y) \in (0, 1)^2;\ |h(x, y)| < k^{-1}\}.$$

By the coarea formula (*see* e.g. Giusti [Gi 84]) the function $t \mapsto H^1(h^{-1}(t))$ is an L^1 function and

$$\sqrt{2}\,\mathscr{L}^2(A_k) = \int_{A_k} |\nabla f|\, dH^2 = \int_{-1/k}^{1/k} H^1(h^{-1}(t))\, dt.$$

Since the primitive of an L^1 function is (absolutely) continuous one sees that

$$\mathscr{L}^2(A_k) \to 0 \quad \text{as } k \to \infty,$$

and assertion (i) follows.

To establish assertions (ii) and (iii) consider $(x_0, y_0) \in (0, 1)^2 \setminus A$. Thus either $\operatorname{dist}(x_0, M) > \operatorname{dist}(y_0, M)$ or $\operatorname{dist}(y_0, M) > \operatorname{dist}(x_0, M)$. If the former holds Lemma 3.1 (v) implies that

$$u^1(x, y) = f\left(x, \frac{2}{\gamma}\operatorname{dist}(y, M)\right) = h(x),$$

for (x, y) in a neighbourhood of (x_0, y_0) in particular $Du^1=0$ a.e. in a neighbourhood of (x_0, y_0). If $\operatorname{dist}(y_0, M)>\operatorname{dist}(x_0, M)$ then

$$u^2(x, y)=h(y) \qquad \text{and} \qquad Du^2=0$$

a.e. in a neighbourhood of (x_0, y_0).

Thus the set of point where (4.6) or (4.7) fail has density zero in $(0, 1)^2 \setminus A$ and hence in $[0, 1]^2$ as $\mathscr{L}^2(A)=0$. Assertions (ii) and (iii) follow. □

Proof of Theorem 4.1 (*continued*). – Assertion (ii) is an immediate consequence of Lemma 4.2 (iii). To prove (iii) recall that for $\varphi \in C_0^\infty(Q)$

$$\langle \operatorname{Det} Du, \varphi \rangle := \int_Q (-u^1 u^2_{,2} \varphi_{,1} + u^1 u^2_{,1} \varphi_{,2})\, dx.$$

With $b_{k,m}$ given by (2.1) and (2.2) and

$$I_{k,m}=b_{k,m}+\left(\frac{1-\gamma}{2}\right)^k \left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right),$$

let

$$S_{k,m}=I_{k,m}\times[0, 1]. \tag{4.8}$$

The sets $S_{k,m}$ are disjoint and $Q \setminus \bigcup_{k=0}^{\infty} \bigcup_{m=1}^{2^k} S_{k,m}$ is a nullset. Consider first $S=S_{0,1}=\left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)\times[0, 1]$. Note that

$$h_{|((1-\gamma)/2,\, (1+\gamma)/2)} \equiv \frac{1}{2},$$

and that by Lemma 4.2 (ii) for $j=1, 2$,

$$(u^1 u^2_{,j})(x, y)=h(x)\, u^2_{,j}(x, y) \quad \text{a.e.}$$

Thus

$$\begin{aligned}\int_S (-u^1 u^2_{,2} \varphi_{,1} + u^1 u^2_{,1} \varphi_{,2})\, dx\, dy &= \int_S \left(-\frac{1}{2} u^2_{,2} \varphi_{,1} + \frac{1}{2} u^2_{,1} \varphi_{,2}\right) dx\, dy \\ &= \int_S \left(\frac{1}{2} u^2 \varphi_{,12} - \frac{1}{2} u^2 \varphi_{,21}\right) dx\, dy \\ &\quad + \int_0^1 \frac{1}{2} u^2(x, y)\, \varphi_{,2}(x, y) \Big|_{x=(1-\gamma)/2}^{x=(1+\gamma)/2} dy.\end{aligned}$$

Now $\frac{1\pm\gamma}{2}\in M$ so that $u^2\left(\frac{1\pm\gamma}{2}, y\right)=h(y)$ by (4.2) and (3.9). Thus the last integral equals

$$\int_0^1 \frac{1}{2} h(y)\,\varphi_{,2}(x, y)\Big|_{x=(1-\gamma)/2}^{x=(1+\gamma)/2} dy = \int_S \frac{1}{2} h(y)\,\varphi_{,12}(x, y)\,dx\,dy$$
$$= \int_S h(x)\,h(y)\,\varphi_{,12}(x, y)\,dx\,dy.$$

Using the fact that h is constant on $I_{k,m}$ one obtains similarily

$$\int_{S_{k,m}} (-u^1 u^2_{,1}\,\varphi_{,1} + u^1 u^2_{,1}\,\varphi_{,2})\,dx\,dy = \int_{S_{k,m}} h(x)\,h(y)\,\varphi_{,12}(x, y)\,dx\,dy.$$

Letting $\varphi(x, y)=\psi(x)\,\eta(y)$ one finally has

$$\begin{aligned}\langle \operatorname{Det} Du, \varphi\rangle &= \int_Q h(x)\,h(y)\,\psi'(x)\,\eta'(x)\,dx\,dy\\ &= \langle h, \psi'\rangle\langle h, \eta'\rangle = \langle h', \psi\rangle\langle h', \eta\rangle\\ &= \langle h'\otimes h', \varphi\rangle.\end{aligned}$$

Thus $\operatorname{Det} Du = h'\otimes h'$ as claimed.

It remains to show that $M\times M$ has Hausdorff dimension 2β and that (4.3) holds. The former follows from the latter. To show (4.3) let $\mu=h'\otimes h'$, $\nu(A)=H^{2\beta}(A\cap(M\times M))$. Extend the Radon measure μ to an outer measure on all subsets of $[0, 1]^2$. We first show that (4.3) holds for all rectangles

$$R=[a_1, b_1]\times[a_2, b_2].$$

To show the upper bound let $\mathscr{F}=\{F_m\}_{m\in\mathbb{N}}$ be a covering of $R\cap(M\times M)$. Choose a cube $\tilde{F}_m$ parallel to the co-ordinate axes with side $l_m=2\operatorname{diam}_2 F_m$ such that $\tilde{F}_m\supset F_m$. By (2.12)

$$\mu(F_m)\leqq\mu(\tilde{F}_m)\leqq l_m^{2\beta}\leqq 2^{2\beta}(\operatorname{diam}_2 F_m)^{2\beta}.$$

Since $\mu|_{Q\setminus(M\times M)}=0$ one has

$$\begin{aligned}\mu(R)=\mu(R\cap(M\times M)) &\leqq \sum_m \mu(F_m)\\ &\leqq 2^{2\beta}\sum_m (\operatorname{diam}_2 F_m)^{2\beta}.\end{aligned}$$

Thus

$$\mu(R)\leqq\frac{2^{2\beta}}{\omega(2\beta)}H^{2\beta}(R\cap(M\times M)). \tag{4.9}$$

To show the converse estimate let $k \in \mathbb{N}$ and choose the integers $\underline{j}_1, \underline{j}_2, \bar{j}_1, \bar{j}_2$ such that (*cf.* Proposition 2.2)

$$b_{k,\underline{j}_i} \leqq a_i < b_{k,\underline{j}_i+1}; \qquad b_{k,\bar{j}_i-1} + \left(\frac{1-\gamma}{2}\right)^k < b_i \leqq b_{k,\bar{j}_i}.$$

One has $h(b_{k,j}) = \dfrac{j-1}{2^k}$ [*see* (2.13)] and thus

$$\bar{j}_i - \underline{j}_i \leqq 2^k (h(b_i) - h(a_i)) + 2.$$

The collection

$$\mathcal{F} = \{ J_{k,j_1} \times J_{k,j_2} : \underline{j}_1 \leqq j_1 \leqq \bar{j}_1, \underline{j}_2 \leqq j_2 \leqq \bar{j}_2 \},$$

of cubes forms a cover of $R \cap (M \times M)$ and each cube has diameter $\sqrt{2}\left(\dfrac{1-\gamma}{2}\right)^k$. Thus

$$\begin{aligned}\sum_{F \in \mathcal{F}} (\operatorname{diam} F)^{2\beta} &\leqq (2^k (h(b_1) - h(a_1)) + 2) \\ &\qquad \times (2^k (h(b_2) - h(a_2)) + 2) \left[\sqrt{2}\left(\frac{1-\gamma}{2}\right)^k\right]^{2\beta} \\ &= \sqrt{2}^{2\beta} (h(b_1) - h(a_1))(h(b_2) - h(a_2)) + \mathcal{O}(2^{-k}) \\ &\leqq \sqrt{2}^{2\beta} \mu(R) + \mathcal{O}(2^{-k}).\end{aligned}$$

Letting $k \to \infty$ it follows that

$$\frac{1}{\omega(2\beta)} H^{2\beta}(R \cap (M \times M)) \leqq \sqrt{2}^{2\beta} \mu(R).$$

Thus (4.3) is proved if U is a rectangle parallel to the co-ordinate axes. A particular consequence is that for any hyperplane H parallel to the co-ordinate axes ones has $\nu(H) = \mu(H) = 0$, as h is continuous. Now let U be an arbitrary open set. Then U can be written as a countable disjoint union of open rectangles and (portions of) hyperplanes, both parallel to the co-ordinate axes. Hence (4.3) follows for open sets and thus for Borel sets.

The proof of Theorem 4.1 is finished. □

5. EXTENSIONS TO $n > 2$

Let $n \geqq 2$ and let $\Omega \subset \mathbb{R}^n$ be open and bounded. In this section we generalize Theorem 4.1 to maps $u : \Omega \to \mathbb{R}^n$. Recall that for any n by n matrix F, adj F denotes the transpose of the matrix of cofactors F so that $F^i{}_j (\operatorname{adj} F)^j{}_k = \delta^i{}_k \det F$. For a smooth map u one has

$$(\operatorname{adj} Du)^j{}_{i,j} = 0 \qquad \text{for} \quad i = 1, \ldots, n, \tag{5.1}$$

(*see* Morrey [Mo 66], Lemma 4.6.4) and by density (5.1) holds in the sense of distributions if $u \in W^{1,n-1}(\Omega; \mathbb{R}^n)$. If the products $u^1 (\operatorname{adj} Du)^j{}_1$ are in $L^1_{loc}(\Omega)$ $\left[\text{it suffices, e.g., that } u \in W^{1,p}(\Omega; \mathbb{R}^n),\ p \geqq \frac{n^2}{n+1}\right]$ one defines the distributional determinant by

$$\operatorname{Det} Du = (u^1 (\operatorname{adj} Du)^j{}_1)_{,j}. \tag{5.2}$$

Let $\det Du(x)$ denote the (pointwise) determinant of $Du(x)$.

Using (6.1) one easily verifies that for $u \in C^2$

$$\operatorname{Det} Du = \det Du. \tag{5.3}$$

By approximation one shows that (5.3) holds in the sense of distributions if $u \in W^{1,n}(\Omega; \mathbb{R}^n)$ (*see* Ball [Ba 77], Dacorogna [Da 89]). We construct examples where this identity fails and where the singular set of $\operatorname{Det} Du$ has prescribed Hausdorf dimension.

As above we fix $\gamma \in (0, 1)$ and we let

$$\beta = \frac{\ln 2}{\ln (2/(1-\gamma))}.$$

We suppress all dependencies on γ in the following. The Cantor set $M = M_\gamma$ and the functions $h = h_\gamma$ are as in Section 2 and f denotes the function given by (3.6) and (3.7). We let furthermor

$$Q = (0, 1)^n,$$

and for $x = (x_1, \ldots, x_n)$ we use the notation

$$\hat{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

For $y \in \mathbb{R}^N$ define

$$\begin{aligned} \operatorname{dist}(y, M^N) &:= \inf \{ |y - z|_\infty : z \in M^N \} \\ &= \inf \{ \max_{i=1,\ldots,N} |y_i - z_i|_\infty : z_i \in M \}. \end{aligned} \tag{5.4}$$

Generalizing (4.1), (4.2) we define $u : [0, 1]^n \to \mathbb{R}^n$ by

$$u^i(x) = f\left(x_i, \frac{2}{\gamma} \operatorname{dist}(\hat{x}_i, M^{n-1})\right). \tag{5.5}$$

THEOREM 5.1. – *Let $n \geqq 2$ and let u given by* (5.5):

(i) *for all $p < n$, $u \in W^{1,p}(Q, \mathbb{R}^n) \cup C^{0,\beta}(\bar{Q}; \mathbb{R}^n)$;*

(ii) $|Du^1| \ldots |Du^n| = 0$ *a.e., in particular* $\operatorname{Det} Du = 0$ *a.e.;*

(iii) $\operatorname{Det} Du$ *is a nonnegative measure given by*

$$\operatorname{Det} Du = h' \otimes \ldots \otimes h'. \tag{5.6}$$

and for any Borel set U *one has*

$$c\, H^{n\beta}(U \cap M^n) \leqq \operatorname{Det} Du(U) \leqq C H^{n\beta}(U \cap M^n). \tag{5.7}$$

where c, C depend only on n and β.

The proof of Theorem 5.1 is analogous to the one for the special case $n=2$ discussed in Section 4. Some technical difficulties, however, do appear and we first state some preliminary results.

LEMMA 5.2. — *For all* $p<n$, $u \in W^{1,p}(Q, \mathbb{R}^n)$.

LEMMA 5.3. — *Consider the hyperplane* $H=\{x \in Q : x_j = a\}$.

Then for all $p \in [1, n-1)$, $u \in W^{1,p}(H; \mathbb{R}^n)$ *and there exists a constant* $C_{p,n}$ *depending on p and n (and on* γ*) but not on a or j such that*

$$\|u\|_{W^{1,p}(H;\mathbb{R}^n)} \leqq C_{p,n}. \tag{5.8}$$

COROLLARY 5.4. — *There exists a sequence* $u^{(\nu)} \in C^\infty(Q; \mathbb{R}^n)$ *such that for all* $p<n$ *and for all hyperplanes* $H=\{x \in Q : x_j = a\}$

$$u^{(\nu)} \to u, \qquad W^{1,p}(Q; \mathbb{R}^n),$$
$$u^{(\nu)} \to u, \qquad W^{1,p-1}(H; \mathbb{R}^n).$$

Proof of the Corollary. — Extend u to $\hat{Q}=(-1, 2)^n$ by successive reflection at the planes $x_i=0$ and $x_i=1$, $i=1, \ldots, n$. This yields $\hat{u}$ such that for all $p<n$ and all hyperplanes $\hat{H}=\{x \in \hat{Q} : x_j = a\}$ one has

$$\hat{u} \in W^{1,p}(\hat{Q}; \mathbb{R}^n); \qquad \hat{u} \in W^{1,p-1}(\hat{H}; \mathbb{R}^n),$$

with a bound analogous to (5.8). The corollary follows by applying mollifying kernels of the form $\varphi(x)=\varphi(x_1)\ldots\varphi(x_n)$.

To prove Lemma 5.2 we will make use of the following

PROPOSITION 5.5. — *Let* $g(x)=\operatorname{dist}(x, M^N)$. *Then, for all* $p \in [1, N+1)$

$$\int_{(0,1)^N} g^{-(p-1)(1-\beta)}\,dx < \infty.$$

Remark. — The case $N=1$ was already used in the previous section.

Proof. — Consider the distribution function

$$\varphi(\lambda)=\mathscr{L}^N\{x \in (0, 1)^N : g^{-(p-1)(1-\beta)}(x) \geqq \lambda\}.$$

To estimate $\varphi(\lambda)$ from above we cover M^N by 2^{kN} cubes as follows. Recall that

$$\tilde{J}_{k,m}=b_{k,m}+\left(\frac{1-\gamma}{2}\right)^k\left(-\frac{\gamma}{1-\gamma}, 1+\frac{\gamma}{1-\gamma}\right).$$

Let

$$U_k=\bigcup_{m_i=1}^{2^k} \tilde{J}_{k,m_1}\times\ldots\times\tilde{J}_{k,m_N}.$$

By Proposition 2.2 one has

$$\operatorname{dist}(x, M^N) \geqq c\left(\frac{1-\gamma}{2}\right)^k \qquad \text{if} \quad x \in (0, 1) \setminus U_k.$$

Thus

$$\varphi\left(c\left(\frac{1-\gamma}{2}\right)^{-k(p-1)(1-\beta)}\right) \leqq C\left(\frac{1-\gamma}{2}\right)^{kN(1-\beta)}$$

and hence

$$\varphi(\lambda) \leqq C^{-(N/(p-1))}.$$

Since $\frac{N}{p-1} > 1$ by assumption and $\varphi(\lambda) \leqq 1$ for all λ one has

$$\int_{(0,1)^N} g^{-(p-1)(1-\beta)}\, dx = \int_0^\infty \varphi(\lambda)\, d\lambda < \infty. \quad \square$$

Proof of Lemma 5.2. — By symmetry it suffices to consider u^1. Observe first that u^1 is absolutely continuous along a. e. line parallel to the co-ordinate axis [consider lines $t \mapsto (x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$, $x_j \notin M$ for $j \neq i$] and that a. e. on these lines

$$\frac{\partial u^1}{\partial x_1}(x) = f_{,1}\left(x_1, \frac{2}{\gamma}\operatorname{dist}(\hat{x}_1, M^{n-1})\right)$$

$$\frac{\partial u^1}{\partial x_i}(x) = f_{,2}\left(x_1, \frac{2}{\gamma}\operatorname{dist}(\hat{x}_1, M^{n-1})\right)\frac{2}{\gamma}\frac{\partial}{\partial x_i}\operatorname{dist}(\hat{x}_1, M^{n-1}), \qquad \text{if} \quad i \neq 1.$$

Here $f_{,i}$ denote the partial derivatives of f. Since the distance function is Lipschitz continuous with Lipschitz constant 1 it suffices to show (*cf.* [Mo 66], Theorem 3.1.12) that the function G given by

$$G(x) = \left|Df\left(x_1, \frac{2}{\gamma}\operatorname{dist}(\hat{x}_1, M^{n-1})\right)\right|, \tag{5.9}$$

satisfies $G \in L^p(Q)$.

Let $\eta \in \left[\left(\frac{1-\gamma}{2}\right)^{k+1}, \left(\frac{1-\gamma}{2}\right)^k\right)$. In view of Lemma 3.1 (iii) and (iv) one has

$$|Df(\xi, \eta)| = 0 \qquad \text{if} \quad \xi \notin \bigcup_{m=1}^{2^k} J_{k,m},$$

and for a. e. $\xi \in J_{k,m}$

$$|Df(\xi, \eta)| \leqq C\, 2^{-k}\left(\frac{1-\gamma}{2}\right)^k \leqq C\left(\frac{1-\gamma}{2}\right)^{-k(1-\beta)}.$$

Moreover

$$\mathscr{L}^1\left(\bigcup_{m=1}^{2^k} J_{k,m}\right)=2^k\left(\frac{1-\gamma}{2}\right)^k=\left(\frac{1-\gamma}{2}\right)^{k(1-\beta)}.$$

Let $\hat{x}_1$ be such that $\frac{2}{\gamma}\operatorname{dist}(\hat{x}_1, M^{n-1})\in\left[\left(\frac{1-\gamma}{2}\right)^{k+1}, \left(\frac{1-\gamma}{2}\right)^k\right)$. Then

$$\begin{aligned}\int_0^1 |G(x_1, \hat{x}_1)|^p\,dx_1 &\leqq C\left(\frac{1-\gamma}{2}\right)^{k(1-\beta)}\left(\frac{1-\gamma}{2}\right)^{-k(1-\beta)p}\\ &\leqq C\left(\frac{1-\gamma}{2}\right)^{-k(p-1)(1-\beta)}\\ &\leqq C(\operatorname{dist}(\hat{x}_1, M^{n-1}))^{-(p-1)(1-\beta)}.\end{aligned}$$

Since the set

$$\left\{z\in(0, 1)^{n-1}:\operatorname{dist}(z, M^{n-1})\notin\left(0, \frac{\gamma}{2}\right)\right\}$$
$$=\left\{z\in(0, 1)^{n-1}:\operatorname{dist}(z, M^{n-1})\in\left\{0, \frac{\gamma}{2}\right\}\right\}$$

is an $\mathscr{L}^{n-1}$ null set we deduce

$$\int_Q |G|^p\,dx\leqq C\int_{(0, 1)^{n-1}}\operatorname{dist}(y, M^{n-1})^{-(p-1)(1-\beta)}\,dy.$$

The assertion follows from Proposition 5.5, applied with $N=n-1$. □

Proof of Lemma 5.3. – Again it suffices to consider u^1.

First case: $j\neq 1$. – By symmetry we may assume $j=n$. Observe that $u^1|_H$ is absolutely continuous on a. e. co-ordinate line

$$t\mapsto(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, a)$$

and as in the proof of Lemma 5.2

$$|D_H u^1(x)|\leqq CG(x)=C\left|Df\left(x_1, \frac{2}{\gamma}\operatorname{dist}(\hat{x}_1, M^{n-1})\right)\right|.$$

Here D_H denotes the tangential gradient in the directions $x_1, \ldots, x_{n-1}$. Arguing as in the proof of Lemma 5.2 one obtains

$$\begin{aligned}&\int_H |G|^p\,dH^{n-1}\\ &\quad\leqq C\int_{(0, 1)^{n-2}}[\operatorname{dist}((y_1, \ldots, y_{n-2}, a), M^{n-1})]^{-(p-1)(1-\beta)}\,dy_1\ldots dy_{n-2}.\end{aligned}$$

Now

$$\operatorname{dist}((y_1, \ldots, y_{n-2}, a), M^{n-1}) \geqq \operatorname{dist}((y_1, \ldots, y_{n-2}), M^{n-2}).$$

Thus

$$\int_H |G|^p dH^{n-1} \leqq C \int_{(0,1)^{n-2}} (\operatorname{dist}((y_1, \ldots, y_{n-2}), M^{n-2}))^{-(p-1)(1-\beta)} dy.$$

The last expression is independent of a and is finite for $p<n-1$ by Proposition 5.5.

Second case: $j=1$. — Again one only has to estimate $\|G\|_{L^p(H)}$. Now from Lemma 3.1 one deduces easily that for fixed a and a.e. $\eta \in \left[\left(\frac{1-\gamma}{2}\right)^{k+1}, \left(\frac{1-\gamma}{2}\right)^k\right)$ one has

$$|Df(a, \eta)| \leqq C \left(\frac{1-\gamma}{2}\right)^{-k(1-\beta)} \leqq C \eta^{-(1-\beta)},$$

with C independent of a. Thus

$$\int_H |G|^p dH^{n-1} \leqq C \int_{(0,1)^{n-1}} (\operatorname{dist}(y, M^{n-1}))^{-(1-\beta)p} dy.$$

The right hand side is independent of a and, in view of Proposition 5.5 (applied with $N=n-1$), remains finite for $p<n-1$. □

Next we aim for a counterpart of Lemma 4.2 which will be used to establish (ii) and (iii) of Theorem 5.1. Define sets

$$A_{ij} = \{x : \operatorname{dist}(x_i, M) = \operatorname{dist}(x_j, M)\},$$
$$A = \bigcup_{i \neq j} A_{ij}.$$

LEMMA 5.6. — *One has:*

(i) A *is closed and* $\mathscr{L}^n(A)=0$;

(ii) *for a.e.* $x \in Q$ *one has*

$$u^1(x) = h(x_1) \quad or \quad (\operatorname{adj} Du)^j{}_1(x) = 0, \qquad \forall j=1, \ldots n; \tag{5.10}$$

(iii) *for a.e.* $x \in Q$

$$Du^j(x) = 0 \quad for\ some\ j \in \{1, \ldots, n\}. \tag{5.11}$$

Proof. — The proof of (i) is completely analogous to the one given for Lemma 4.2 (i). To prove (ii) and (iii) consider $x^0 \in Q \setminus A$. If $\operatorname{dist}(x_1^0, M) > \operatorname{dist}(x_j^0, M)$ for $j=2, \ldots, n$, then $u^1(x) = h(x_1)$ in a neighbourhood of x^0 by (5.5) and (3.13). In particular $Du^1(x)=0$ a.e. in a neighbourhood of x^0. If there is a $k \neq 1$ such that

$$\operatorname{dist}(x_k^0, M) > \operatorname{dist}(x_j^0, M), \quad \text{for all } j \neq k,$$

then $Du^k(x)=0$ a. e. in a neighbourhood of x^0. Hence the set where (5.10) or (5.11) fails has density zero in $Q \setminus A$ and is thus a null set as claimed. □

To compute $\operatorname{Det} Du$ we shall use the following result from calculus.

PROPOSITION 5.7. – *Let $v \in C^2(Q; \mathbb{R}^n)$, let $v_t:(0, 1)^{n-1} \to \mathbb{R}^{n-1}$ be given by*

$$v_t(y_1, \ldots, y_{n-1}) = (v^2, \ldots, v^n)(t, y_1, \ldots, y_{n-1}). \tag{5.12}$$

Let $(a, b) \subset (0, 1)$. Then, for all $\varphi \in C_0^\infty(Q)$

$$\int_{(a, b) \times (0, 1)^{n-1}} (\operatorname{adj} Dv)^j{}_1 \, \varphi_{,j} \, dx = \langle \operatorname{Det} v_t, \varphi_t \rangle \big|_{t=a}^{b},$$

where $\varphi_t(y) = \varphi(t, y)$.

Proof. – Since $\dfrac{\partial}{\partial x^j}(\operatorname{adj} Dv)^j{}_1 = 0$ (*see* [Mo 66], Lemma 4.6.4) and $\varphi_{|\partial Q} = 0$ one has

$$\begin{aligned}
&\int_{(a, b) \times (0, 1)^{n-1}} (\operatorname{adj} Dv)^j{}_1 \, \varphi_{,j} \, dx \\
&\quad = \int_{(0, 1)^{n-1}} ((\operatorname{adj} Dv)^1{}_1 \, \varphi)(t, x_2, \ldots, x_n) \, dx_2 \ldots dx_n \big|_{t=a}^{t=b} \\
&\quad = \int_{(0, 1)^{n-1}} \det Dv_t(y) \, \varphi_t(y) \, dy \big|_{t=a}^{t=b}.
\end{aligned}$$

Since v_t is C^2 and $\varphi_t(.) \in C_0^\infty((0, 1)^{n-1})$ it follows from (5.3) that the last expression equals

$$\langle \operatorname{Det} v_t, \varphi_t \rangle \big|_{t=a}^{b},$$

as claimed. □

Proof of Theorem 5.1. – Assertion (i) follows from Lemma 5.2, Lemma 3.1 (i) and the fact that the distance function is Lipschitz. Assertion (ii) follows from Lemma 5.6 (iii). To prove (iii) we argue by induction over n. For $n=2$ the result has been proved in Section 4. To carry out the induction step from $n-1$ to n we let (*cf.* 4.8)

$$\begin{aligned}
S_{k, m} &= I_{k, m} \times [0, 1]^{n-1} \\
&= \left(b_{k, m} + \frac{1-\gamma}{2}\left(\frac{1-\gamma}{2}\right)^k, \; b_{k, m} + \frac{1+\gamma}{2}\left(\frac{1-\gamma}{2}\right)^k \right) \times [0, 1]^{n-1}.
\end{aligned}$$

We have for $\varphi \in C_0^\infty(Q)$

$$\langle \operatorname{Det} Du, \varphi \rangle = -\int_Q u^1 (\operatorname{adj} Du)^j{}_1 \varphi_{,j}\, dx$$
$$= -\sum_{k=0}^{\infty} \sum_{j=1}^{2^k} \int_{S_{k,j}} u^1 (\operatorname{adj} Du)^j{}_1 \varphi_{,j}\, dx. \qquad (5.13)$$

We first compute the term with $S_{0,1} = S = \left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right) \times (0, 1)^{n-1}$. For $z \in \left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)$ one has $h(z) = \frac{1}{2}$. In view of Lemma 5.6 it follows that

$$-\int_S u^1 (\operatorname{adj} Du)^j{}_1 \varphi_{,j}\, dx = -\frac{1}{2}\int_S (\operatorname{adj} Du)^j{}_1 \varphi_{,j}\, dx.$$

At this point we would like to apply Proposition 5.7. Now u fails to be in C^2 but according to Corollary 5.4 we can find a sequence $u^{(\nu)}$ of smooth functions such that, for all $p<n$

$$u^{(\nu)} \to u \quad \text{in } W^{1,p}(Q; \mathbb{R}^n)$$
$$u_t^{(\nu)} \to u_t \quad \text{in } W^{1,p-1}((0, 1)^{n-1}; \mathbb{R}^{n-1}).$$

Thus

$$\int_S (\operatorname{adj} Du^{(\nu)})^j{}_1 \varphi_{,j}\, dx \to \int_S (\operatorname{adj} Du)^j{}_1 \varphi_{,j}\, dx.$$

By the definition of Det and the Sobolev imbedding theorem one has

$$\langle \operatorname{Det} Du_t^{(\nu)}, \psi \rangle \to \langle \operatorname{Det} Du_t, \psi \rangle \quad \text{for all } \psi \in C_0^\infty((0, 1)^{n-1}).$$

Since Proposition 5.7 applies to $u^{(\nu)}$ we obtain

$$-\int_S u^1 (\operatorname{adj} Du)^j{}_1 \varphi_{,j}\, dx = -\frac{1}{2}\langle \operatorname{Det} Du_t, \varphi_t \rangle \Big|_{t=(1-\gamma)/2}^{t=(1+\gamma)/2}. \qquad (5.14)$$

Now by definition of u_t [*see* (5.12)]

$$u_t^1(y_1, \ldots, y_{n-1}) = u^2(t, y_1, \ldots, y_{n-1})$$
$$= f(y_1, \operatorname{dist}((t, y_2, \ldots, y_{n-1}), M^{n-1})).$$

For $t=(1\pm\gamma)/2$

$$u_t^1(y) = f(y_1, \operatorname{dist}(\hat{y}_1, M^{n-2})),$$

where $\hat{y}_1 = (y_2, \ldots, y_{n-1})$, and similarily

$$u_t^i(y) = f(y_i, \operatorname{dist}(\hat{y}_i, M^{n-2})).$$

Therefore the induction hypothesis implies that

$$\langle \operatorname{Det} Du_t, \varphi_t \rangle \Big|_{t=(1-\gamma)/2}^{t=(1+\gamma)/2} = \langle \underbrace{h' \otimes \ldots \otimes h'}_{n-1 \text{ factors}}, \varphi_t \rangle \Big|_{t=(1-\gamma)/2}^{t=(1+\gamma)/2}.$$

Now let $\varphi_i \in C_0^\infty(0, 1)$ and let

$$\varphi(x) = \varphi_1(x_1) \ldots \varphi_n(x_n) = \varphi(x_1)\,\psi(\hat{x}_1). \tag{5.15}$$

Then

$$\begin{aligned}\langle \operatorname{Det} Du_t, \varphi_t \rangle \big|_{t=(1-\gamma)/2}^{t=(1+\gamma)/2} &= \langle h' \otimes \ldots \otimes h', \psi \rangle\, \varphi_1(t) \big|_{t=(1-\gamma)/2}^{t=(1+\gamma)/2} \\ &= \langle h' \otimes \ldots \otimes h', \psi \rangle \int_{(1-\gamma)/2}^{(1+\gamma)/2} \varphi_1'(s)\, ds,\end{aligned}$$

and by (5.14)

$$\begin{aligned}-\int_S u^1 (\operatorname{adj} Du)^j{}_1\, \varphi_{,j}\, dx &= -\frac{1}{2} \langle h' \otimes \ldots \otimes h', \psi \rangle \int_{(1-\gamma)/2}^{(1+\gamma)/2} \varphi_1'(s)\, ds \\ &= -\left(\int_{(1-\gamma)/2}^{(1+\gamma)/2} h(s)\, \varphi_1'(s)\, ds \right) \langle h' \otimes \ldots \otimes h', \psi \rangle.\end{aligned}$$

Applying the above arguments to $S_{k,m}$ instead of S and using the fact that $h_{|I_{k,m}}$ is constant (*see* (2.11)) one obtains

$$\begin{aligned}\langle \operatorname{Det} Du, \varphi \rangle &= \left(-\int_0^1 h(s)\, \varphi_1'(s)\, ds \right) \langle h' \otimes \ldots \otimes h', \psi \rangle \\ &= \langle h', \varphi_1 \rangle \langle h' \otimes \ldots \otimes h', \psi \rangle \\ &= \langle \underbrace{h' \otimes \ldots \otimes h'}_{n \text{ factors}}, \varphi \rangle.\end{aligned}$$

This holds for all φ of the form (5.15). Since these functions are dense in $C_0^\infty(Q)$ one has

$$\operatorname{Det} Du = \underbrace{h' \otimes \ldots \otimes h'}_{n \text{ factors}},$$

in the sense of distributions.

The proof of (5.6) is completely analogous to the one for (4.3) and will not be repeated here. □

6. FURTHER EXAMPLES

First we construct a map $u = (u^1, u^2) : (0, 1) \times (-1, 1) \to \mathbb{R}^2$ with $u^1 \in W^{1,2}$ and $\det Du \neq \operatorname{Det} Du$. Such examples are relevant in particular in view of the correspondence between (1.2) and (1.3). Secondly (*see* Theorem 6.3 below) we construct a map $u = (u^1, u^2) : (0, 1) \times (-1, 1) \to \mathbb{R}^2$ where the singular support of $\operatorname{Det} Du$ is a line. In this case, however, $\det Du \notin L^1$ and $\operatorname{Det} Du$ is not a Radon measure.

As before fix $\gamma \in (0, 1)$, let

$$\beta = \frac{\ln 2}{\ln(2/(1-\gamma))},$$

let $h = h_\gamma$, as defined before Proposition 2.1 and let f denote the function defined by (3.6), (3.7). We suppress dependence on γ in the following.

THEOREM 6.1. – *Let* $\Omega = (0, 1) \times (-1, 1)$. *Then there exists a map* $u = (u^1, u^2) : \Omega \to \mathbb{R}^2$ *such that:*

(i) *for all* $p \in \left[1, \dfrac{2-\beta}{1-\beta}\right)$, $q \in [1, 2-\beta)$;

$$u^1 \in W^{1,p}(\Omega) \cap C^{0,\beta}(\bar{\Omega}); \qquad u^2 \in W^{1,q}(\Omega) \cap L^\infty;$$

(ii) $\det Du = 0$ *a.e;*

(iii) $\operatorname{Det} Du = 2h' \otimes \delta_0$.

Remark. – Note that $\dfrac{2-\beta}{1-\beta}$ and $2-\beta$ are dual exponents, *i.e.* $\dfrac{1-\beta}{2-\beta} + \dfrac{1}{2-\beta} = 1$. In particular $u^1 \in W^{(2-\beta)/(1-\beta)}(\Omega)$, $u^2 \in W^{1,(2-\beta)}(\Omega)$ would imply $\operatorname{Det} Du = \det Du$ in $L^1(\Omega)$.

An immediate consequence of the theorem is

COROLLARY 6.2. – *Let* $n \geqq 2$, $\Omega = (0, 1) \times (-1, 1) \times (0, 1)^{n-2}$, $r \in (1, \infty)$, *with dual exponent* $r' = r/(r-1)$. *Then there exists* $v : \Omega \to \mathrm{R}$ *and* $\sigma : \Omega \to \mathrm{R}^n$ *such that*

$$v \in W^{1,p}(\Omega), \qquad \forall p < r$$
$$\sigma \in L^q(\Omega; \mathbb{R}^n), \qquad \forall q < r'$$
$$\operatorname{div} \sigma = 0$$

and

$$|\sigma|\,|Dv| = 0 \quad a.e.$$

but

$$\operatorname{div}(v\sigma) \neq 0.$$

Proof. – If $r > 2$ it suffices to let $u = (u^1, u^2)$ as in Theorem 6.1 and to choose

$$v(x) = u^1(x_1, x_2),$$
$$\sigma^1(x) = \frac{\partial u^2}{\partial x_1}(x_1, x_2),$$
$$\sigma^2(x) = -\frac{\partial u^2}{\partial x_1}(x_1, x_2).$$

One has specifically

$$\operatorname{div}(v\,\sigma)=2\,h'\otimes\delta_0\otimes\underbrace{1\otimes\ldots\otimes 1}_{n-2\text{ times}}.$$

For $r<2$ one can reverse the rôles of u^1 and u^2 upon observing that

$$\begin{aligned}\operatorname{Det} Du &= (u^1, u^2_{,2})_{,1}-(u^1\, u^2_{,1})_{,2}\\ &= -(u^1_{,2}\, u^2)_{,1}+(u^1_{,1}\, u^2)_{,2}.\end{aligned}$$

Note that the last identity holds in the sense of distributions as $u\in L^\infty$, $\nabla u\in L^q$, $q>1$. Finally, for $r=2$ one may choose $u^i(z)=z^i/|z|$, $i=1,2$, $z\in\mathbb{R}^2$ and define v and σ as above. □

To prove Theorem 6.1 we shall choose

$$u^1(x, y)=f(x, |y|), \tag{6.1}$$

where f is given by (3.6), (3.7). The function u^2 will be defined by means of a related self-similar construction.

To this end choose a Lipschitz function

$$g:\ \left[-\frac{\gamma}{1-\gamma},\,1+\frac{\gamma}{1-\gamma}\right]\times\left[\frac{1-\gamma}{2},\,1\right]$$

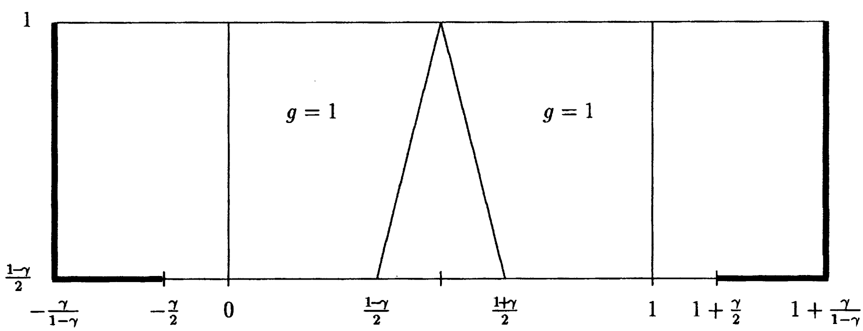

FIG. 4. — On the bold lines $g=0$.

with the following properties (see *Figs.* 4 and 5)

$$g(x, y)=1 \quad \text{if} \quad \frac{\gamma}{1+\gamma}(1-y)<\left|x-\frac{1}{2}\right|<\frac{1}{2}, \tag{6.2}$$

$$g(x, y)=0 \quad \text{if} \quad x=-\frac{\gamma}{1-\gamma}, \quad \text{or} \quad x=1+\frac{\gamma}{1-\gamma}, \tag{6.3}$$

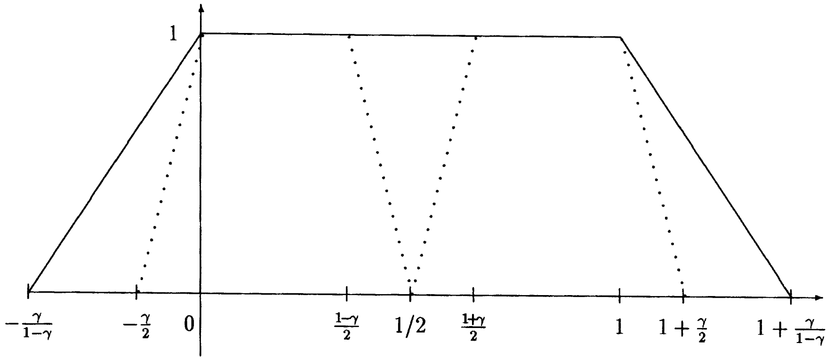

FIG. 5. – Possible choices for $g(\cdot, 1)$ (solid line) and $g\left(\cdot, \frac{1-\gamma}{2}\right)$ (dotted line).

$$g\left(x, \frac{1-\gamma}{2}\right)=\begin{cases} 0 \quad \text{if} \quad x<-\frac{\gamma}{2}, \\ g\left(\left(\frac{1-\gamma}{2}\right)^{-1} x, 1\right) \quad \text{if} \quad -\frac{\gamma}{2}<x<\frac{1}{2}, \\ g\left(1-\left(\frac{1-\gamma}{2}\right)^{-1}(1-x), 1\right) \\ \quad \text{if} \quad \frac{1}{2} \leqq x<1+\frac{\gamma}{2}, \\ 0 \quad \text{if} \quad x \geqq 1+\frac{\gamma}{2}. \end{cases} \tag{6.4}$$

Recall from Section 2 that

$$\tilde{\mathrm{J}}_{k, m}:=b_{k, m}+\left(\frac{1-\gamma}{2}\right)^k\left(-\frac{\gamma}{1-\gamma}, 1+\frac{\gamma}{1-\gamma}\right),$$

and extend g to $[0, 1] \times\left[\left(\frac{1-\gamma}{2}\right)^{k+1},\left(\frac{1-\gamma}{2}\right)^k\right)$ by letting (see *Fig.* 6)

$$g(x, y)=\begin{cases} g\left(\left(\frac{1-\gamma}{2}\right)^{-k}(x-b_{k, m}),\left(\frac{1-\gamma}{2}\right)^{-k} y\right) \\ \quad \text{if} \quad x \in \tilde{\mathrm{J}}_{k, m}; \\ 0 \quad \text{else}. \end{cases} \tag{6.5}$$

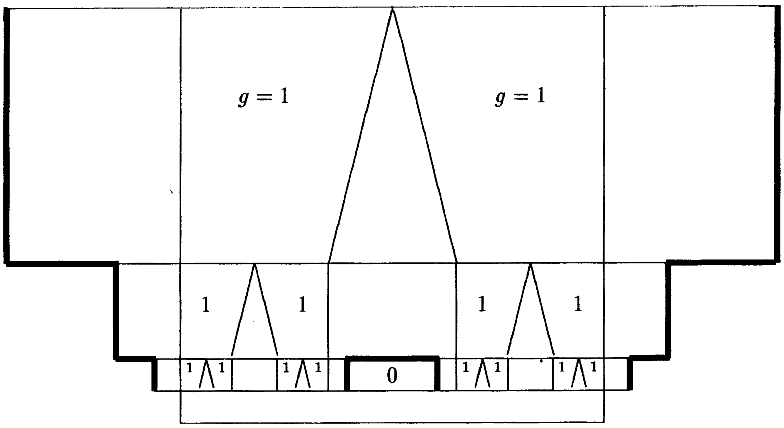

FIG. 6. – Extension of g.

Note that g is well defined as the intervalls $\tilde{J}_{k,m}$ are disjoint (*see* Proposition 2.2) and g vanishes on $\partial\tilde{J}_{k,m}$ by (6.3). Using (6.4) one easily verifies that $g:(0,1)$ is continuous across $y=\left(\frac{1-\gamma}{2}\right)^k$.

We next show that

$$g\in W^{1,p}((0,1)^2;\mathbb{R}),\qquad \forall p\in[1,2-\beta). \tag{6.6}$$

Note that

$$\begin{aligned}\int_{((1-\gamma)/2)^{k+1}}^{((1-\gamma)/2)^k}\int_0^1 |Dg|^p\,dx\,dy &\leqq 2^k\int_{(1-\gamma)/2}^{1}\int_{-(\gamma/(1-\gamma))}^{1+(\gamma/(1-\gamma))}\left(\frac{1-\gamma}{2}\right)^{-kp}|Dg|^p\left(\frac{1-\gamma}{2}\right)^{2k}dx\,dy\\ &\leqq C\left(2\left(\frac{1-\gamma}{2}\right)^{2-p}\right)^k = C\left(\frac{1-\gamma}{2}\right)^{(2-\beta-p)k}\end{aligned}$$

Thus

$$\int_{(0,1)^2}|Dg|^p\,dx\,dy<\infty \qquad \text{if}\quad p<2-\beta.$$

We claim that moreover

$$g=0 \quad \text{at } y=0 \quad \text{(in the sense of trace)}. \tag{6.7}$$

Inded, $|g|\leqq 1$ and

$$\mathscr{L}^1\{x:g(x,y)\neq 0\}\leqq 2^k\left(\frac{1-\gamma}{2}\right)^k\left(1+\frac{2\gamma}{1-\gamma}\right),$$

if $y<\left(\frac{1-\gamma}{2}\right)^k$. Thus $g(\cdot, y)\to 0$ in $L^p(0, 1)$ as $y\to 0$ and (6.7) follows. Let u^2 be the antisymmetric extension of g, *i.e.*

$$u^2(x, y)=\begin{cases} g(x, y) & \text{if } y\geqq 0 \\ -g(x, y) & \text{if } y<0. \end{cases} \tag{6.8}$$

Proof of Theorem 6.1. – Let u^1, u^2 be given by (6.1) and (6.8), respectively. Assertion (i) follows from Lemma 3.1 (i), (6.7) and (6.8).

To show (ii) we will prove that

$$\begin{gathered} Du^1(x, y)=0 \quad \text{or} \quad u^2(x, y)=\operatorname{sgn} y \\ \text{for a.e. } (x, y)\in(0, 1)\times(-1, 1). \end{gathered} \tag{6.9}$$

By symmetry we may assume $y>0$. The points with $y=\left(\frac{1-\gamma}{2}\right)^k$ form a nullset. Consider $y\in\left(\left(\frac{1-\gamma}{2}\right)^{k+1}, \left(\frac{1-\gamma}{2}\right)^k\right)$.

If $x\notin\bigcup_{j=1}^{2^k} J_{k,j}$ then $Du^1(x, y)=0$ by (3.12). The points (x, y) with $(x, y)\in\partial J_{k,j}$ form an $\mathscr{L}^2$ nullset. It thus remains to consider

$$x\in \overset{0}{J}_{k,j}=\left(b_{k,j}, b_{k,j}+\left(\frac{1-\gamma}{2}\right)^k\right).$$

Let $\tilde{x}=\left(\frac{1-\gamma}{2}\right)^{-k}(x-b_{k,j})$, $\tilde{y}=\left(\frac{1-\gamma}{2}\right)^{-k} y$. It follows from (3.10), (6.5) and (6.8) that

$$Du^1(x, y)=(1-\gamma)^{-k} Du(\tilde{x}, \tilde{y}), \qquad u^2(x, y)=u^2(\tilde{x}, \tilde{y})$$

It only remains to verify (6.9) for $y\in\left(\frac{1-\gamma}{2}, 1\right)$. For these values of y, (6.9) is an immediate consequence of (3.4) and (6.2).

To prove (iii) let $\varphi(x, y)=\psi(x)\eta(y)$, $\psi\in C_0^\infty((0, 1))$, $\eta\in C_0^\infty((-1, 1))$. Using (6.9) integration by parts and $u^1(x, 0)=h(x)$ one finds that

$$\begin{aligned} \langle \operatorname{Det} Du, \varphi\rangle &= \int_\Omega (-u^1_{,1} u^2 \psi\eta' + u^1_{,2} u^2 \psi'\eta)\, dx\, dy \\ &= \int_{(0, 1)^2} (-u^1_{,1} \psi\eta' + u^1_{,2} \psi'\eta)\, dx\, dy \\ &\quad - \int_{(0, 1)\times(-1, 0)} (-u^1_{,1} \psi\eta' + u^1_{,2} \psi'\eta)\, dx\, dy \end{aligned}$$

$$= -\int_0^1 u^1(x, 0)\,\psi'(x)\,\eta(0)\,dx$$

$$-\int_0^1 u^1(x, 0)\,\psi'(x)\,\eta(0)\,dx$$

$$= 2\langle h', \psi\rangle\langle\delta_0, \eta\rangle.$$

Hence $\operatorname{Det} Du = 2h' \otimes \delta_0$. $\square$

We close this section by an example in two dimensions where the singular support of $\operatorname{Det} Du$ is a line. Consider smooth functions $u^1, u^2 : \mathbb{R} \times \left[\frac{1}{2}, 1\right] \to \mathbb{R}$ satisfying

$$u^i(x+1, y) = u^i(x, y), \qquad i = 1, 2, \tag{6.10}$$

$$u^1(x, 1) = \sin 2\pi x, \qquad u^1\left(x, \frac{1}{2}\right) = \sqrt{\frac{1}{2}} \sin 4\pi x, \tag{6.11}$$

$$u^2(x, 1) = \cos 2\pi x, \qquad u^2\left(x, \frac{1}{2}\right) = \sqrt{\frac{1}{2}} \cos 4\pi x. \tag{6.12}$$

Here the sine and cosine functions are merely chosen for definiteness, suitable other periodic functions would do just as well. Extend u^1, u^2 to $\mathbb{R} \times (-1, 1)$ as follows

$$\left.\begin{aligned} u^i(x, y) = \left(\sqrt{\frac{1}{2}}\right)^k u^i(2^k x, 2^k y) \quad &\text{for} \quad y \in (2^{-(k+1)}, 2^{-k}], \\ i = 1, 2, \quad & \end{aligned}\right\} \tag{6.13}$$

$$u^i(x, 0) = 0 \qquad \text{for} \quad i = 1, 2, \tag{6.14}$$

$$u^1(x, y) = u^1(x, -y) \qquad \text{for} \quad y \in (-1, 0), \tag{6.15}$$

$$u^2(x, y) = -u^2(x, -y) \qquad \text{for} \quad y \in (-1, 0). \tag{6.16}$$

THEOREM 6.3. – *Let* $\Omega = (0, 1) \times (-1, 1)$. *Then the map*

$$u = (u^1, u^2) : \Omega \to \mathbb{R}^2$$

given by (6.10) *to* (6.16) *has the following properties:*

(i) *for all* $p \in [1, 2)$, $u \in W^{1, p}(\Omega; \mathbb{R}^2) \cap C^{0, 1/2}(\bar{\Omega}; \mathbb{R}^2)$;

(ii) *for all* $\varphi \in C_0^\infty(\Omega)$

$$\langle \operatorname{Det} Du, \varphi\rangle = 2A_0 \int_0^1 \varphi(x, 0)\,dx + \lim_{k \to \infty} \int_{|y| \geq 2^{-k}} \varphi \det Du\,dx\,dy, \tag{6.17}$$

where $A_0 = -\int_0^1 (u^1 u^2_{,1})(x, 1)\,dx = \pi$.

Remark. – Examples where u^1, u^2 lie in different Sobolov spaces are easily constructed by replacing (6.13) by

$$u^1(x, y) = \left(\frac{1}{2}\right)^{k\alpha} u^1(2^k x, 2^k y), \qquad u^2(x, y) = \left(\frac{1}{2}\right)^{k(1-\alpha)} u^2(2^k x, 2^k y),$$

$\alpha \in (0, 1)$, and modifying (6.11), (6.12) accordingly.

Proof. – One easily sees that u is absolutely continuous along every co-ordinate line and by (6.13),

$$|Du(x, y)| \leqq C y^{-1/2}.$$

Hence $u \in W^{1,p}(\Omega; \mathbb{R}^2)$ for $p \in [1, 2)$. Hölder continuity also follows easily from (6.13).

To prove (ii) note that it suffices to check (6.17) for test functions φ which are symmetric in y, since both sides vanish for antisymmetric φ. Note furthermore that for $b > a > 0$ one has

$$\int_a^b \int_0^1 (u^1 u^2_{,1} \varphi_{,2} - u^1 u^2_{,2} \varphi_{,1})\, dx\, dy$$
$$= \int_a^b \int_0^1 \varphi \det Du\, dx\, dy - \int_0^1 u^1 u^2_{,1} \varphi\, dx \,\Big|_{y=a}^{y=b}\Big|. \quad (6.18)$$

Indeed (6.18) is obvious for smooth u. Now $u_{|[0,1]\times[a,b]}$ is Lipschitz and (6.19) follows by approximation. Thus, for symmetric φ,

$$\frac{1}{2}\langle \operatorname{Det} Du, \varphi \rangle = \int_{2^{-k}}^1 \int_0^1 (u^1 u^2_{,1} \varphi_{,2} - u^1 u^2_{,2} \varphi_{,1})\, dx\, dy$$
$$+ \int_0^{2^{-k}} \int_0^1 (u^1 u^2_{,1} \varphi_{,2} - u^1 u^2_{,2} \varphi_{,1})\, dx\, dy$$
$$= \int_{2^{-k}}^1 \int_0^1 \varphi \det Du\, dx\, dy - \int_0^1 (u^1 u^2_{,1})(x, 2^{-k})\, \varphi(x, 2^{-k})\, dx$$
$$+ \int_0^{2^{-k}} \int_0^1 (u^1 u^2_{,1} \varphi_{,2} - u^1 u^2_{,2} \varphi_{,1})\, dx\, dy.$$

The last term converges to zero as $k \to \infty$ since $|u^1|\,|Du^2| \in L^1(\Omega)$ [*cf.* (i)]. It only remains to show that

$$- \int_0^1 (u^1 u^2_{,1})(x, 2^{-k})\, \varphi(x, 2^{-k})\, dx$$
$$\to \quad A_0 \int_0^1 \varphi(x, 0)\, dx \quad \text{as } k \to \infty \quad (6.19)$$

It follows from (6.13) that $|(u^1 u^2_{,1})(x, 2^{-k})| \leqq C$. Since

$$|\varphi(x, 2^{-k}) - \varphi(x, 0)| \leqq C\,2^{-k}$$

it suffices to consider

$$\begin{aligned}
&\int_0^1 (u^1 u^2_{,1})(x, 2^{-k})\,\varphi(x, 0)\,dx \\
&= \int_0^1 (u^1 u^2_{,1})(2^k x, 1)\,\varphi(x, 0)\,dx \\
&= 2^{-k} \int_0^{2^k} (u^1 u^2_{,1})(x, 1)\,\varphi(2^{-k} x, 0)\,dx \\
&= 2^{-k} \sum_{j=0}^{2^k-1} \int_0^1 (u^1 u^2_{,1})(x, 1)\,\varphi(2^{-k} j + 2^{-k} x, 0)\,dx,
\end{aligned}$$

by periodicity of $(u^1 u^2_{,1})(., 1)$. The last term is estimated by

$$\begin{aligned}
&2^{-k} \sum_{j=0}^{2^k-1} \left(\int_0^1 (u^1 u^2_{,1})(x, 1)\,dx \right) \varphi(2^{-k} j, 0) + \mathcal{O}(2^{-k}) \\
&= -A_0\, 2^{-k} \sum_{j=0}^{2^k-1} \varphi(2^{-k} j, 0) + \mathcal{O}(2^{-k}) \\
&= -A_0 \int_0^1 \varphi(x, 0)\,dx + \mathcal{O}(2^{-k})
\end{aligned}$$

and (6.19) follows. □

ACKNOWLEDGEMENTS

During the course of this work I profited greatly from the discussions with J. M. Ball, P. L. Lions and L. Tartar. Part of this research was carried out whille I enjoyed the hospitality of the I.M.A. at the University of Minneapolis. Additional support was provided by the N.S.F. under grant DMS-9002679 and by the SFB 256 at the University of Bonn.

REFERENCES

[Ba 77] J. M. BALL, Convexity Conditions and Existence Theorems in Nonlinear Elasticity, *Arch. Rat. Mech. Anal.*, Vol. **63**, 1977, pp. 337-403.

[CLMS 89] R. COIFMAN, P.-L. LIONS, Y. MEYER and S. SEMMES, Compacité par compensation et espaces de Hardy, *C. R. Acad. Sci. Paris*, T. **309**, Series I, 1989, pp. 945-949.

[Da 89] B. DACOROGNA, *Direct Methods in the Calculus of Variations*, Springer, Berlin, New York, 1989.

[DM 90] B. DACOROGNA and J. MOSER, On a Partial Differential Equation Involving the Jacobian Determant, *Ann. I.H.P., Analyse non linéaire*, Vol. **7**, 1990, pp. 1-26.

[DA 89] R. De ARCANGELIS, Some Remarks on the Identity Between a Variational Functional and its Relaxed Functional, *Ann. Univ. Ferrera*, Sez. VII, *Sc. Math.*, Vol. **35**, 1989, pp. 135-145.

[Fa 85] K. J. FALCONER, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1985.

[Gi 84] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Basel, 1984.

[MM 92] J. MALY and O. MARTIO, Lusin's Condition (N) and Mappings of the Class $W^{1,n}$, Preprint.

[Mo 66] C. B. MORREY, *Multiple Integrals in the Calculus of Variations*, Springer, Berlin, New York, 1966.

[Mu 90*a*] S. MÜLLER, Det = det. A Remark on the Distributional Determinant, *C. R. Acad. Sci. Paris*, T. **311**, Series I, 1990, pp. 13-17.

[Mu 90*b*] S. MÜLLER, A Counterexample to Formal Integration by Parts, *C. R. Acad. Sci. Paris*, T. **312**, Series I, 1990, pp. 45-49.

[MTY 92] S. MÜLLER, Q. TANG and B. S. YAN, On a New Class of Elastic Deformations not Allowing for Cavitation, *Ann. I.H.P., Analyse non linéaire*, to appear.

[Po 87] S. P. PONOMAREV, Property N of Homeomorphisms of the Class $W^{1,p}$, *Sib. Math. J.*, Vol. **28**, (1), 1987, pp. 291-298.

[Ro 70] C. A. ROGERS, *Hausdorff Measure*, Cambridge University Press, Cambridge 1970.

(*Manuscript received September* 21, 1992;
accepted November 2, 1992.)