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Looking for the Bernoulli shift

by

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ABSTRACT. — We prove a result on the topological entropy of a large class of Hamiltonian systems. This result is obtained variationally by constructing “multibump” homoclinic solutions.

Key words : Hamiltonian systems, convexity, dual variational methods, concentration-compactness, homoclinic orbits, Bernoulli shift, topological entropy, chaos.

RÉSUMÉ. — On démontre un résultat sur l'entropie topologique d'une grande classe de systèmes hamiltoniens. Ce résultat est obtenu par une méthode variationnelle qui permet de construire des solutions homoclines « multi-bosses ».

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I. INTRODUCTION

1. Some history

Homoclinic orbits were first introduced by H. Poincaré (see [M] for a modern exposition). Considering a hyperbolic fixed point p of a diffeomorphism φ in \mathbb{R}^{2N} , we say that a point $r \neq p$ is homoclinic if it belongs to the intersection of the unstable and stable manifolds W^u, W^s associated to (p, φ) ; the orbit of r is called a homoclinic orbit. Assuming that W^u, W^s intersect transversally at r , and that φ is symplectic, Poincaré proved that there are infinitely many homoclinic orbits, geometrically distinct in the following sense:

(the orbits of r, r' are geometrically distinct) $\Leftrightarrow (\forall n \in \mathbb{Z} : \varphi^n(r) \neq r')$.

Birkhoff, Smale and other authors also studied homoclinic orbits, and their relation with Bernoulli shifts. We state here a result of Smale on homoclinics (see [M]): if $r \neq p$ is a point of transverse intersection of W^u, W^s , then there are $l \in \mathbb{N}^*$ and a homeomorphism $\tau : \{0, 1\}^{\mathbb{Z}} \rightarrow I$, where I is an invariant set for φ^l , such that $\varphi^l \circ \tau = \tau \circ \sigma$. Here, $\sigma((a_n)) = (b_n)$ with $b_n = a_{n+1}$ and $\{0, 1\}^{\mathbb{Z}}$ is endowed with the standard metric

$$d(a, b) = \frac{1}{3} \sum_{n \in \mathbb{Z}} \frac{|b_n - a_n|}{2^{|n|}}.$$

This structure is called a Bernoulli shift.

Bernoulli shifts are an important tool in the study of chaotic behavior. For instance, Smale's result given above implies that the topological entropy of $\varphi, h_{\text{top}}(\varphi)$, is greater than $\frac{\text{Ln } 2}{l}$. This is a direct consequence of the following definition (see [O], p. 182-183):

$$h_{\text{top}}(\varphi) = \sup_{R > 0} \lim_{e \rightarrow 0} \left(\limsup_{n \rightarrow \infty} \frac{\text{Log } s(n, e, R)}{n} \right),$$

where

$$s(n, e, R) = \max \{ \text{Card}(E) : E \subset B(0, R), \\ (\forall x \neq y \in E) (\exists k \in \llbracket 0, n \rrbracket) : |\varphi^k(x) - \varphi^k(y)| \geq e \}.$$

2. Variational approach

The results described in the preceding section were proved by dynamical systems methods, with a transversality assumption on W^u, W^s . The question examined in this paper is the following one:

We assume that φ is the time-one map of a Hamiltonian system $x' = J \nabla_x H(t, x)$, H being one-periodic in time. Is it possible to say some-

thing about Bernoulli shifts and topological entropy, using a variational method? We will see that this approach has several advantages:

- The existence of a homoclinic point r is not an assumption any more, but follows from global hypotheses on H that we call (hA), (hR).
- The classical transversality hypothesis can be replaced by a weaker condition, denoted (\mathcal{H}).

3. Main results

We work with the same Hamiltonian system as in the paper [CZ-E-S]:

$$x' = JA x + J \nabla_x R(t, x), \quad x \in \mathbb{R}^{2N}, \quad t \in \mathbb{R}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (1)$$

We are looking for non-zero solutions satisfying $x(\pm \infty) = 0$, *i.e.* solutions homoclinic to 0.

We make the following assumptions on A, R :

$$\left. \begin{array}{l} A^* = A, \text{ and } JA = E \text{ is a constant matrix,} \\ \text{all eigenvalues of which have a non-zero real part.} \end{array} \right\} \quad (\text{hA})$$

- $R(\cdot + 1, \cdot) = R(\cdot, \cdot)$, and R is C^2 .
- $(\forall t \in \mathbb{R}), R(t, \cdot)$ is strictly convex.
- for some $\alpha > 2, 0 < k_1 < k_2 < +\infty$, we have

$$\begin{aligned} \forall (t, x) \in \mathbb{R} \times \mathbb{R}^{2N}, \quad R(t, x) &\leq \frac{1}{\alpha} (\nabla_x R, x), \\ k_1 |x|^\alpha &\leq R(t, x) \leq k_2 |x|^\alpha. \end{aligned} \quad (\text{hR})$$

In [CZ-E-S], it was proved under these assumptions that there are at least two homoclinic orbits x, y , geometrically distinct, *i.e.* such that $\forall n \in \mathbb{Z} : n * x \neq y$, where $n * x(t) = x(t - n)$. One of them was obtained by a mountain-pass argument on a dual action functional. This paper has motivated some related work.

Concerning the existence of at least one homoclinic solution, the convexity assumption was relaxed in [H-W] and [T], by two different methods.

Concerning multiplicity, a novel variational argument was introduced in [S], and the following result was proved:

THEOREM I. — *Assume (hA), (hR) are true. Then there are infinitely many orbits homoclinic to 0, geometrically distinct in the sense*

$$x_1 \neq x_2 \Leftrightarrow (\forall n : n * x_1 \neq x_2).$$

The idea in [S] was to look for solutions near $(-n) * x + n * x$, where x is the homoclinic orbit found in [CZ-E-S] by mountain-pass, and n is large enough. We call them “solutions with two bumps distant of $2n$ ”.

The existence of such solutions is a well-known fact of classical dynamical systems theory, in many particular situations. Let describe briefly one of them (*see* [W]):

Consider the autonomous system associated to the Hamiltonian

$$H(p, q) = p^2 - q^2 + p^4 + q^4, \quad (p, q) \in \mathbb{R}^2.$$

It is integrable, and does not have any solution with two (or more) bumps. But in the autonomous case, we have a continuum of solutions which are the translates of one of them in time, and Theorem I is not contradicted.

By Melnikov's theory, it is possible to find small non-autonomous perturbations $H(p, q) + \varepsilon K(t, p, q)$ of the Hamiltonian such that W^u, W^s intersect transversally. Then, using the implicit function theorem, multi-bump homoclinic solutions can be constructed.

To give more detailed comments on Theorem I, we need some notations:

f is the dual action functional introduced in [CZ-E-S]. It is defined on the space $L^\beta(\mathbb{R}, \mathbb{R}^{2N})$, with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ (the exact form of f will be given in section II). $f^a = \{x/f(x) \leq a\}$, \mathcal{C} is the set of non-zero critical points, and \mathbb{Z} acts by integer translations in time.

$L: L^\beta \rightarrow W^{1, \beta}$ is an isomorphism such that, if $u \in \mathcal{C}$, then Lu is a homoclinic orbit (*see* §II).

c is the mountain-pass level, let us define it precisely:

0 is a strict local minimum for f , and $f(0) = 0$. Moreover, f is not bounded from below (*see* [CZ-E-S]). So we consider

$$\Gamma = \{\gamma \in C^0([0, 1], L^\beta) / \gamma(0) = 0, f \circ \gamma(1) < 0\}.$$

Γ is non-empty, and we choose $c = \inf_{\gamma \in \Gamma} (\max f \circ \gamma) > 0$ as mountain-pass level.

In [S], the variational gluing of two bumps was possible under the following assumption:

(*): There is some $c' > c$ such that $(\mathcal{C} \cap f^{c'})/\mathbb{Z}$ is finite.

The following result, which is a more precise version of Theorem I, is an immediate consequence of the arguments given in [S]:

THEOREM I'. — *Assume that (hA), (hR) and (*) are true. Then there are two critical points u, v such that for any $r, h > 0$ and $n \geq N(r, h)$, exists a critical point u_n , with*

$$\|u_n - [(-n) * u + n * v]\|_{L^\beta} < r \quad \text{and} \quad f(u_n) \in [2c - h, 2c + h].$$

u, v , possibly equal, satisfy $f(u) = f(v) = c$. The homoclinic orbit $y_n = Lu_n$ is called a solution with two bumps distant of $2n$. It satisfies

$$\|y_n - [(-n) * Lu + n * Lv]\|_{W^{1, \beta}} < \|L\| \cdot r.$$

Theorem I is trivial when (*) is not satisfied (“degenerate” situation), and Theorem I’ implies Theorem I when (*) is satisfied (“non-degenerate” situation).

In the later work [CZ-R]¹, Coti Zelati and Rabinowitz apply the ideas of [S] to the case of second order systems, and construct, under assumption (*), solutions with m bumps, *i.e.* located in a ball of center $p^1 * x_1 + \dots + p^m * x_m$ and radius ε , for the norm of the functional space $E = W^{1,2}(\mathbb{R}, \mathbb{R}^N)$. The x_i are in a fixed finite set of critical points of the action functional $\int \frac{x^2}{2} - V$ defined on E . They are found thanks to a mountain-pass. Moreover, for any i , $(p^{i+1} - p^i) \geq K(\varepsilon, m)$. In the construction of [CZ-R]¹, the minimal distance K between bumps goes to infinity as m goes to infinity, for ε fixed.

Other applications, in the domain of partial differential equations, are given in [CZ-R]², [LI]¹, [LI]².

In the paper [C-L] of Chang and Liu, the assumption (*) is replaced by (**): $\mathcal{C} \cap f^{c'}$ contains only isolated points.

In the present work, (**) is replaced by the weaker assumption (\mathcal{H}): $\mathcal{C} \cap f^{c'}$ is at most countable.

Moreover, multibump solutions are constructed for a minimal distance K between bumps independent of m . This last point, whose proof requires many modifications in the arguments of [S], [CZ-R]¹, allows to study the topological entropy of the Hamiltonian system. The main theorem that we will prove can be stated as follows:

THEOREM II. — Assume (hA), (hR) and (\mathcal{H}) are true. Then there exists a homoclinic orbit x such that, for any $\varepsilon > 0$, and any finite sequence of integers $\bar{p} = (p^1, \dots, p^m)$, satisfying

$$(\forall i): (p^{i+1} - p^i) \geq K(\varepsilon),$$

there is a homoclinic orbit $y_{\bar{p}}$, with

$$(\forall t \in \mathbb{R}): \left| y_{\bar{p}}(t) - \sum_{i=1}^m x(t - p^i) \right| \leq \varepsilon.$$

Here, K is a constant independent of m .

Remark 1. — The assumption (\mathcal{H}) cannot be satisfied in the autonomous situation, where the translates of x in time form a continuum. Now, if W^u, W^s intersect transversally, then their intersection is at most countable, and so is the set of homoclinic solutions; but the converse is false.

Remark 2. — The estimate on $y_{\bar{p}} - \sum_{i=1}^m x(t - p^i)$ is given in L^∞ norm. In [S] and [CZ-R]¹, it was given in global $W^{1,q}(\mathbb{R})$ norm. Without this change,

it seems impossible, or at least very difficult, to choose K independently of m .

Since K does not depend on m , we can study the limit $m \rightarrow \infty$, and get solutions with infinitely many bumps (those are not homoclinic orbits any more). We have

COROLLARY II.1. — *With the hypotheses and notations of Theorem II, for any interval $I \subset \mathbb{Z}$, finite or infinite, and any sequence of integers $\bar{p} = (p^i)_{i \in I}$ such that $(\forall i) : (p^{i+1} - p^i) \geq K(\varepsilon)$, there is a solution $y_{\bar{p}}$ of (1) satisfying*

$$(\forall t \in \mathbb{R}) : \left| y_{\bar{p}}(t) - \sum_{i \in I} x(t - p^i) \right| \leq \varepsilon.$$

If I is infinite, we say that y has infinitely many bumps.

As a consequence, we have an “approximate” Bernoulli shift structure:

COROLLARY II.2. — *Under the hypotheses of Theorem II, there is $x_0 \in \mathbb{R}^{2N} \setminus \{0\}$ such that, for any $\varepsilon > 0$, exist $K = K(\varepsilon) > 0$ and*

$$\tilde{\tau} = \tilde{\tau}(\varepsilon) : (\{0, 1\}^{\mathbb{Z}}, d) \rightarrow (\mathbb{R}^{2N}, |\cdot|),$$

with:

- $\tilde{\tau}$ is injective, and $\tilde{\tau}^{-1}$ is uniformly continuous.
- $(\forall n \in \mathbb{Z}) \|\tilde{\tau} \circ \sigma^n - \Phi^{Kn} \circ \tilde{\tau}\|_{\infty} < 2\varepsilon.$
- $\begin{cases} s_0 = 1 \Rightarrow |\tilde{\tau}(s) - x_0| < \varepsilon \\ s_0 = 0 \Rightarrow |\tilde{\tau}(s)| < \varepsilon. \end{cases}$

Here, φ is the time-one flow of (1), and $\sigma(s)_n = s_{n+1}$. Note that we cannot say that $\tilde{\tau}$ is continuous. We call $(\tilde{\tau}(\{0, 1\}^{\mathbb{Z}}), \Phi^K)$ an approximate Bernoulli shift structure.

Corollary II.2 will be proved in section VI.

Now, we are in a position to state the result on topological entropy.

Choose $\varepsilon \leq \frac{|x_0|}{3}$. If two sequences s, s' are such that $s_k \neq s'_k$ for some k , then

$$|\Phi^{K(\varepsilon)k} \circ \tau(s) - \Phi^{K(\varepsilon)k} \circ \tau(s')| \geq \frac{|x_0|}{3}.$$

So, for $e < \frac{|x_0|}{3}$ and $R > |x_0| + \varepsilon$, we get $s(Kn, e, R) \geq 2^n$, and

$h_{\text{top}}(\varphi) \geq \frac{\text{Ln } 2}{K(\varepsilon)}$. So Corollary II.2 implies

COROLLARY II.3. — *With the hypotheses of Theorem I, the flow of (1) has a positive topological entropy.*

Note: Independently of the present paper, Bessi in [B] constructs variationally an approximate Bernoulli shift for the one-dimensional pendulum,

by a method inspired of [S]. He replaces assumption (*) by a weakening of the classical Melnikov condition, and his result is given for small perturbations of an autonomous system.

**II. VARIATIONAL FRAMEWORK
AND SKETCH OF PROOF OF THEOREM II**

We use a variational formulation based on Clarke's dual action principle (see [CZ-E-S], [E]). Define $G(t, y) = \max \{ (z \cdot y) - R(t, z) / z \in \mathbb{R}^{2N} \}$. G is 1-periodic in time, strictly convex in y , and satisfies, for $\frac{1}{\alpha} + \frac{1}{\beta} = 1$:

$$0 \leq \frac{1}{\beta} (\nabla_y G, y) \leq G(t, y) \leq (\nabla_y G, y),$$

$$(\exists c_1, c_2 > 0) (\forall (y, t)) \quad c_1 |y|^\beta \leq G(t, y) \leq c_2 |y|^\beta,$$

$$|\nabla_y G(t, y)| \leq c_2 |y|^{\beta-1}.$$

We define

$$D: W^{1, \beta}(\mathbb{R}, \mathbb{R}^{2N}) \rightarrow L^\beta(\mathbb{R}, \mathbb{R}^{2N})$$

$$z \mapsto \left(-J \frac{d}{dt} - A \right) z,$$

$$L = D^{-1}.$$

We call \mathcal{C} the set of non-zero critical points of the following functional f :

$$f(u) = \int G(t, u) dt - \frac{1}{2} \int (u, Lu) dt, \quad u \in L^\beta(\mathbb{R}, \mathbb{R}^{2N}).$$

We have (see [CZ-E-S])

LEMMA 1. — *If $u \in \mathcal{C}$, then $x = Lu$ is a non-zero solution of (1) such that $x(\pm \infty) = 0$, i. e. an orbit homoclinic to 0.*

Our task will be to find a large class of elements of \mathcal{C} .

For this purpose, we need some compactness properties of f . Unfortunately, f does not satisfy the Palais-Smale (PS) condition, because it is invariant for the action of the non-compact group $\mathbb{Z} : n * u = u(\cdot - n)$. To deal with this problem, we use the concentration-compactness theory of P. L. Lions (see [LS]).

We have (see [CZ-E-S])

LEMMA 2. — *Suppose (hA), (hR) are true. Then f satisfies the following compactness property:*

Let $(u_n)_{n \geq 0}$ be a sequence such that

$$f(u_n) \rightarrow a > 0, \quad f'(u_n) \rightarrow 0.$$

Then there exist $m > 0$, a subsequence $(n_p)_{p \geq 0}$, and u^1, \dots, u^m in \mathcal{C} , not necessarily distinct, such that

$$\left\| u_{n_p} - \sum_{i=1}^m k_p^i \star u^i \right\|_{p \rightarrow \infty} \rightarrow 0,$$

where $k_p^i \in \mathbb{Z}$, $(k_p^j - k_p^i) \rightarrow +\infty$ as $p \rightarrow +\infty$ if $i < j$.

To simplify notations, we will write

$$\begin{aligned} \bar{k}_p &= (k_p^1 \dots k_p^m) \in \mathbb{Z}^m, & \bar{u} &= (u^1 \dots u^m) \in \mathcal{C}^m, \\ \bar{k}_p \star \bar{u} &= \sum_{i=1}^m k_p^i \star u^i. & \text{Moreover,} & \quad \left(\lim_{k \rightarrow \infty} (k_p^j - k_p^i) = +\infty \text{ if } i < j \right) \end{aligned}$$

will be summarized by

$$(\bar{k}_p \rightarrow \Omega \text{ as } p \rightarrow +\infty).$$

Now, what is special here is that the splittings $\bar{k} \star \bar{u}$ do not vary continuously when \bar{k} varies. This leads to introduce a new compactness condition (see [CZ-E-S], [S]).

CONDITION $\overline{\text{PS}}$ (a). — Let (u_n) be a sequence such that $f(u_n) \leq a \in \mathbb{R}$, $f'(u_n) \rightarrow 0$, $(u_{n+1} - u_n) \rightarrow 0$. Then (u_n) is convergent.

We have:

LEMMA 3. — Assume (hA), (hR) and (\mathcal{H}) are true. Then $\overline{\text{PS}}(c')$ holds.

Lemma 3 will be proved in section III, and will be used in the proof of Lemma 7, section IV.

The interest of $\overline{\text{PS}}$ is that, if f is bounded on a pseudo-gradient line, then one can find a $\overline{\text{PS}}$ sequence on this line. So $\overline{\text{PS}}$ can give the same kind of deformation lemmas as the Palais-Smale condition. If $\overline{\text{PS}}$ is satisfied under level c' , by deforming a particular curve in Γ , one finds at least one critical point u between levels c and c' . When $(*)$ holds, one can impose $f(u) = c$. When only (\mathcal{H}) holds, the best that can be done is to take u with $(f(u) - c)$ arbitrarily small.

In [S], under assumption $(*)$, a “product min-max” is constructed at level $2c$, for the “split” functional $\tilde{f}(x) = f(x \chi_{\mathbb{R}^-}) + f(x \chi_{\mathbb{R}^+})$, where χ_I is the characteristic function of I . Theorems I and I' are then proved by contradiction, thanks to a deformation argument. This argument works because the differentials f' and \tilde{f}' “look the same” near $(-n) \star u + n \star v$, where u, v are critical points associated to the mountain-pass, possibly equal.

The proof of Theorem II is based on the same ideas, but contains several technical improvements.

We first construct, for any $r, h > 0$, a non-trivial homology class in $H_1(f^{\bar{c}+h}, f^{\bar{c}})$, containing a chain included in $B(u, r)$, thanks to assumption

(\mathcal{H}). Here, $\bar{c} = f(u) \in [c, c']$, and $u \in \mathcal{C}$, found thanks to the mountain-pass, is independent of r, h (see § IV).

Then, roughly speaking, we consider a product of m “copies” of this homology class, and find a “product min-max” in a neighborhood of $\sum_{i=1}^m p^i \star u$. This is done in section IV thanks to Künneth’s formula,

$$H_*(X \times Y, (Z \times Y) \cup (X \times T)) = H_*(X, Z) \otimes H_*(Y, T).$$

Note that in [S], [CZ-R]¹, a more elementary procedure (without homology) is used to construct the product min-max. It would be possible to use this procedure in the proof of Theorem II. But the method involving homology seems easier to generalize to situations where the min-max is not of mountain-pass type.

Finally, we find a critical point $u_{\bar{p}}$ in a neighborhood of $\sum_{i=1}^m p^i \star u$, provided $(p^{i+1} - p^i) \geq K$, K depending only on r , not on m . To do this, we assume that $u_{\bar{p}}$ does not exist, construct a more precise version of the deformation used in [S], and apply it to the “product min-max” to obtain a contradiction (see § V).

In the proof of Theorem II, a crucial point is to make a suitable choice of the neighborhood of $\sum_{i=1}^m p^i \star u$ in which we want to find $u_{\bar{p}}$: this choice allows to control K as m increases. The correct neighborhood will be defined in the statement of Theorem III (see the end of section V), after the introduction of some technical notations. Theorem II will be a direct consequence of Theorem III.

III. COMPACTNESS PROPERTIES OF f

We first prove the following result:

LEMMA 4. — *Suppose (hA), (hR) and (\mathcal{H}) are true. Then there is an at most countable compact set D such that:*

If $(u_n)_{n \geq 0}$ satisfies $f(u_n) \leq c', f'(u_n) \rightarrow 0$, then

$$(\forall r > 0) \quad (\exists N > 0), \quad [p > q > N \Rightarrow \|u_p - u_q\| \in B(D, r)].$$

Here, $B(D, r) = \{x \in [0, +\infty) / d(x, D) < r\}$.

Proof. – Consider the set

$$D = \left\{ x \in [0, +\infty) / x = \sum_{i=1}^m \|u_i - v_i\|, m \geq 1, u_i, v_i \in \mathcal{C} \cup \{0\}, \right. \\ \left. \sum_{i=1}^m f(u_i) \leq c', \sum_{i=1}^m f(v_i) \leq c' \right\}.$$

From (\mathcal{H}) , D is at most countable.

Let us prove that D is compact. We know (see [CZ-E-S]) that there is $\Lambda > 0$ such that

$$(\forall u \in \mathcal{C}) \quad f(u) \geq \Lambda.$$

Consider a sequence (d^n) in D , with

$$d^n = \sum_{i=1}^{M_n} \|u_i^n - v_i^n\|, \quad u_i^n, v_i^n \in \mathcal{C} \cup \{0\}, \quad \sum_{i=1}^{M_n} f(u_i^n) \leq c', \\ \sum_{i=1}^{M_n} f(v_i^n) \leq c', \quad (u_i^n = 0 \Rightarrow v_i^n \neq 0).$$

We have $M_n \leq 2c'/\Lambda$.

So, after extraction, we may assume that $M_n = M$ is constant and, by Lemma 2, that, $\forall i \in \llbracket 1, M \rrbracket$:

$$\|u_i^n - \bar{k}_i^n * \bar{U}_i\| \rightarrow 0, \quad \bar{U}_i \in \mathcal{C}^{m(i)}, \quad \bar{k}_i^n \xrightarrow{n \rightarrow \infty} \Omega, \\ \|v_i^n - \bar{l}_i^n * \bar{V}_i\| \rightarrow 0, \quad \bar{V}_i \in \mathcal{C}^{m'(i)}, \quad \bar{l}_i^n \xrightarrow{n \rightarrow \infty} \Omega.$$

One easily sees that

$$d_n \rightarrow \sum_{k=1}^{m'} \|u_k - v_k\| = d_\infty$$

where u_k , resp. v_k , if non-zero, are of the form $n * \bar{U}_i^j$, resp. $n * \bar{V}_i^j$, and $d_\infty \in D$.

We have thus proved that D is compact. The last step is to study (u_n) such that

$$f(u_n) \leq c', \quad f'(u_n) \rightarrow 0.$$

Assume there are two subsequences $(u_{p_m})_{m \geq 0}$, $(u_{q_m})_{m \geq 0}$ satisfying $\|u_{p_m} - u_{q_m}\| \notin B(D, \rho)$ for some $\rho > 0$. After extraction, we may impose

$$\|u_{p_m} - \bar{\kappa}_m * \bar{\mu}\| \rightarrow 0, \quad \bar{\mu} = (\mu^1, \dots, \mu^r) \in \mathcal{C}^r, \\ \bar{\kappa}_m \rightarrow \Omega, \quad \sum f(\mu^i) \leq c' \\ \|u_{q_m} - \bar{\lambda}_m * \bar{v}\| \rightarrow 0, \quad \bar{v} = (v^1, \dots, v^s) \in \mathcal{C}^s, \\ \bar{\lambda}_m \rightarrow \Omega, \quad \sum f(v^i) \leq c'.$$

After a new extraction, each sequence $(\kappa_m^i - \lambda_m^j)$ has a limit $l_{i,j}$ in $\mathbb{Z} \cup \{-\infty, +\infty\}$. Moreover, for each i , $\text{Card}(\{j/|l_{i,j}| < +\infty\}) \leq 1$.

Hence

$$\|u_{p_m} - u_{q_m}\| \rightarrow \sum_{k=1}^t \|l_k * w_k - w'_k\|,$$

where $(w_k)_{1 \leq k \leq t}$ is a reindexing of

$$(\mu^1, \dots, \mu^r, \underbrace{0, \dots, 0}_{(t-r) \text{ terms}}),$$

$(w'_k)_{1 \leq k \leq t}$ is a reindexing of

$$(\nu^1, \dots, \nu^s, \underbrace{0, \dots, 0}_{(t-s) \text{ terms}}),$$

and $l_k \in \mathbb{Z}$.

Clearly, $\sum f(w_k) = \sum f(\mu^i) \leq c'$, $\sum f(w'_k) = \sum f(\nu^j) \leq c'$. So $\sum_{k=1}^t \|w_k - w'_k\| \in D$,

which contradicts the assumption $\|u_{p_m} - u_{q_m}\| \notin B(D, \rho)$. The last assertion of Lemma 4 is thus proved by contradiction. \square

We now give another lemma, that will be used in section V.

LEMMA 5. — Suppose that f satisfies (hA), (hR) and (\mathcal{H}). Then the set

$$F = \left\{ x = \sum_{k=1}^m f(u_k)/m \geq 1, (u_1, \dots, u_m) \in \mathcal{C}^m, (\forall k), f(u_k) \leq c' \right\}$$

is closed and a most countable.

The proof of Lemma 5 is analogous to that of Lemma 4, so we won't give it. Now, we prove Lemma 3 as a consequence of Lemma 4.

Proof. — Consider a sequence (u_n) such that

$$f(u_n) \leq c', \quad f'(u_n) \rightarrow 0, \quad (u_{n+1} - u_n) \rightarrow 0.$$

we want to prove by contradiction that (u_n) is a Cauchy sequence.

Assume the contrary, i. e. $\|u_{q_n} - u_{p_n}\| \rightarrow \delta > 0, p_n < q_n < p_{n+1}$.

The open set $]0, \delta[\setminus D$ contains an interval $[d_1 - d_2, d_1 + d_2]$. And there is P such that

$$\left(p > P \Rightarrow \|u_{p+1} - u_p\| \leq \frac{d_2}{2} \right).$$

So, if $p_n > P$,

$$\|u_{r_n} - u_{p_n}\| \in \left[d_1 - \frac{d_2}{2}, d_1 + \frac{d_2}{2} \right] \text{ for some } r_n \in [p_n, q_n].$$

But this implies $\|u_{r_n} - u_{p_n}\| \notin B(D, d_2/2)$, which is impossible by Lemma 4.

So (u_n) is Cauchy, hence convergent. Lemma 3 is thus proved. \square

We now study the local compactness of \mathcal{C} . We prove

LEMMA 6. — Assume (hA) and (hR) are true. There is $r_0 > 0$ such that, if a sequence (u_n) satisfies

$$\left\{ \begin{array}{l} f'(u_n) \rightarrow 0 \\ (\exists R > 0), (\forall p, q), \quad \|(u_p - u_q) \chi_{\mathbb{R} \setminus [-R, R]}\| \leq 2r_0 \end{array} \right.$$

then (u_n) is precompact.

Proof. — We remark (see [CZ-E-S]) that there is $r_0 > 0$ such that

$$\frac{3r_0}{2} < \|u\| \quad (\forall u \in \mathcal{C})$$

We now apply Lemma 2 to the sequence (u_n) . If $m \geq 2$ or if $(m=1)$ and $\lim_{p \rightarrow \infty} (|k_p^1| = +\infty)$, then for any $P > 0$, there are $p > q > P$ such that

$$\|(\bar{k}_p * \bar{u} - \bar{k}_q * \bar{u}) \chi_{\mathbb{R} \setminus [-R, R]}\| \geq 3r_0.$$

This contradicts $\|(u_p - u_q) \chi_{\mathbb{R} \setminus [-R, R]}\| \leq 2r_0$, for P large enough.

So $m=1$, and we may extract a subsequence $u_{n_{\phi(p)}}$ such that $k_{\phi(p)}^1 = k$ is constant, and $u_{n_{\phi(p)}} \rightarrow k * u^1 \in \mathcal{C}$. Lemma 6 is thus proved. \square

Lemma 6 will be used in the proof of Lemma 12, section V.

IV. THE PRODUCT MIN-MAX

We want to find a min-max at each level kc , $k \geq 2$. This will be done thanks to singular homology over \mathbb{Z} . We first need to “localize” the min-max

$$\inf_{\gamma \in \Gamma} (\max f \circ \gamma) = c.$$

This will be done thanks to (\mathcal{H}) .

We recall some notations:

$$\begin{aligned} f^l &= \{x/f(x) \leq l\}, & f^{<l} &= \{x/f(x) < l\}, \\ f_l &= (-f)^{-l}, & f_a^b &= f_a \cap f^b, \\ \mathbf{B}(x, \rho) &= \{y/\|y-x\| < \rho\}, & \mathbf{S}(x, \rho) &= \{y/\|y-x\| = \rho\}. \end{aligned}$$

We have

LEMMA 7. — Assume (hA), (hR) and (\mathcal{H}) are true. Choose $r \in \mathbb{R}_+^* \setminus \mathbf{D}$, with the notation of Lemma 4.

Then for any $h > 0$, exist $p = p(h, r) \in \mathbb{N}^*$, $(u^1, \dots, u^p) \in (\mathcal{C} \cap f_c^{c+h})^p$, and $\gamma \in \Gamma$, with:

- (i)
$$\text{Im}(\gamma) \cap f_c \subset \bigcup_{i=1}^p B(u^i, r)$$
- (ii)
$$\text{Im}(\gamma) \cap f_{c+h} = \emptyset$$
- (iii)
$$\text{Im}(\gamma) \cap f_c \cap \left[\bigcup_{i=1}^p S(u^i, r) \right] = \emptyset$$

Proof. – Given $r > 0$, we just have to prove the result for h small enough. We take $\gamma^h \in \Gamma$ such that $f \circ \gamma^h < c + h$.

We are going to take γ as a deformation of γ^h . We choose $e > 0$ such that $[r - 2e, r + 2e] \cap D = \emptyset$. For $d \geq 0$, we define

$$\begin{aligned} U^d &= \{x \in f_c^{c+h} / (\forall y \in \mathcal{C} \cap f_c^{c+h}) \|x - y\| > r + d\} \\ V^d &= \{x \in f_c^{c+h} / (\exists y \in \mathcal{C} \cap f_c^{c+h}) \|x - y\| \in [r - d, r + d]\} \\ K^d &= (\{x \in f_c^{c+h} / (\exists y \in \mathcal{C} \cap f_c^{c+h}) \|x - y\| < r - d\} \\ &\quad \cup \{x \in f^{c'} / (\exists y \in \mathcal{C} \cap f^c) \|x - y\| < r - d\}) \setminus V^d \end{aligned}$$

We assume $c + h < c'$. From Lemma 4, there is $\mu > 0$, independent of h , and such that $\inf\{\|f'(x)\| / x \in V^{2e}\} \geq \mu$. We assume, moreover, that $h < \mu e / 2$. We build a locally Lipschitz vector field V on f^{c+h} , such that:

- (j)
$$x \in K^{2e} \cup f^{c-h} \Rightarrow V(x) = 0$$
- (jj)
$$(\forall x) f'(x) \cdot V(x) \leq 0, \quad |V(x)| \leq 2|f'(x)|^{-1}$$
- (jjj)
$$x \in U^e \cup V^e \Rightarrow f'(x) \cdot V(x) \leq -1$$

Consider the flow φ_t defined by

$$(\forall (t, x) \in \mathbb{R}_+ \times f^{c+h}) \begin{cases} \varphi_0(x) = x \\ \frac{\partial}{\partial t} \varphi_t(x) = V \circ \varphi_t(x). \end{cases}$$

Assume that for some $x \in f^{c+h}$, the maximal interval of definition of $t \mapsto \varphi_t(x)$ is $[0, L[$, $L < +\infty$. Then $\int_0^L \|V \circ \varphi_t(x)\| dt = +\infty$. So we can define a sequence (t_n) by

$$\begin{aligned} t_0 &= 0 \\ \int_{t_n}^{t_{n+1}} \|V \circ \varphi_t(x)\| dt &= \sqrt{L - t_n} \end{aligned}$$

So we get

- (α) $\forall (u, v) \in [t_n, t_{n+1}]^2: \|\varphi_u(x) - \varphi_v(x)\| \leq \sqrt{L - t_n}$
- (β) $\exists s_n \in [t_n, t_{n+1}]: \begin{cases} \|f' \circ \varphi_{s_n}(x)\| \leq 2 \|V \circ \varphi_{s_n}(x)\|^{-1} \leq 2 \sqrt{L - t_n} \\ \varphi_{s_n}(x) \in f^{c+h} \setminus K^{2e} \end{cases}$
- (γ) $\int_0^l \|V \circ \varphi_t(x)\| dt = \sum_{n=0}^{+\infty} \sqrt{L - t_n}$, where $l = \lim_{n \rightarrow \infty} t_n$.

If $l < L$, the left term of (γ) is finite, and the right one infinite. So we have $l = L$, and

$$(\varphi_{s_{n+1}}(x) - \varphi_{s_n}(x)) \rightarrow 0, \quad f' \circ \varphi_{s_n}(x) \rightarrow 0.$$

Since f satisfies property $\overline{PS}(c')$, we get

$$u_\infty = \lim_{n \rightarrow \infty} \varphi_{s_n}(x) \in (f^{c+h} \setminus K^{2e}) \cap \mathcal{C}.$$

But this intersection is empty. So we have proved that φ_t is defined on $\mathbb{R}_+ \times f^{c+h}$.

Now, suppose that $f(x) < c + h$, and that $\varphi_h(x) \in U^0 \cup V^0$. Then three situations may occur:

- $(\forall t \in [0, h]), \varphi_t \in U^e \cup V^e$

apply (ijj), and conclude $f \circ \varphi_h(x) < c$: contradiction.

- $(\exists y \in \mathcal{C} \cap f_c^{c+h}) (\exists [\alpha, \beta] \subset [0, h]),$
 $\|\varphi_\alpha(x) - y\| = r - e, \quad \|\varphi_\beta(x) - y\| = r,$
 $(\forall t \in [\alpha, \beta]), \|\varphi_t(x) - y\| \in [r - e, r].$
- $(\exists y \in \mathcal{C} \cap f_c^{c+h}) (\exists [\alpha, \beta] \subset [0, h]),$
 $\|\varphi_\alpha(x) - y\| = r + e, \quad \|\varphi_\beta(x) - y\| = r,$
 $(\forall t \in [\alpha, \beta]), \|\varphi_t(x) - y\| \in [r, r + e].$

In the second and third situations, we have $\|\varphi_\beta(x) - \varphi_\alpha(x)\| \geq e$, and from (ij), (ijj), $f'_y \cdot V_y \leq -\frac{1}{2} \|f'_y\| \cdot \|V_y\| \leq -\frac{\mu}{2} \|V_y\|$ if $y \in \varphi_{[\alpha, \beta]}(x) \cap f_{c-h}$.

Since $h < \mu e/2$, we also conclude $f \circ \varphi_h(x) < c$: contradiction.

So we have proved that if $f(x) < c + h$, then either $f \circ \varphi_h(x) < c$, or $\varphi_h(x) \in K^0$.

Finally, $\gamma = \varphi_h \circ \gamma^h$ is such that

$$\begin{aligned} \text{Im } \gamma \cap \left[\bigcup_{y \in \mathcal{C} \cap f_c^{c+h}} S(y, r) \right] \cap f_c &= \emptyset, \\ (\text{Im } \gamma \cap f_c) &\subset \bigcup_{y \in \mathcal{C} \cap f_c^{c+h}} B(y, r). \end{aligned}$$

Since $\text{Im } \gamma \cap f_c$ is compact, we can extract a finite subcovering:

$$(\text{Im } \gamma \cap f_c) \subset \bigcup_{i=1}^p \mathbf{B}(u^i, r). \quad u^i \in \mathcal{C} \cap f_c^{c+h}.$$

Lemma 7 is thus proved. \square

Lemma 7 has a direct consequence:

COROLLARY 7.1. — Assume (\mathcal{H}) is true. Choose $r > 0, h > 0$. Then there is $u = u(r, h) \in \mathcal{C} \cap f_c^{c+h}$ such that $i_* \neq 0$, where

$$i_*: H_1(f^{<(c+h)} \cap \mathbf{B}(u, r), f^{<c} \cap \mathbf{B}(u, r)) \rightarrow H_1(f^{<(c+h)}, f^{<c})$$

is the morphism induced by the canonical injection

$$i: \mathbf{B}(u, r) \rightarrow L^\beta.$$

Proof. — We juste have to prove the result when $r \in \mathbb{R}_+^* \setminus \mathbf{D}$: it will then be true for any $r' \geq r$.

Let p_0 be the minimal value of p such that there are $(u^1, \dots, u^p) \in \mathcal{C} \cap (f_c^{c+h})^p$ and $\gamma \in \Gamma$ satisfying the conclusion of Lemma 7. $\text{Im } \gamma \cap \mathbf{B}(u^{p_0}, r)$ is the image of a 1-dimensional complex $\omega \in C_1(f^{<(c+h)})$, with $\omega \in \bar{\omega}$, for some $\bar{\omega} \in H_1(f^{<(c+h)} \cap \mathbf{B}(u^{p_0}, r), f^{<c} \cap \mathbf{B}(u^{p_0}, r))$.

If $i_* \bar{\omega} = 0$, then there is a singular 2-dimensional complex $\Omega \in C_2(f^{<(c+h)})$ such that $\partial\Omega = \omega - \alpha$, with $\alpha \in C_1(f^{<c})$. So, replacing the curves of ω by curves of α in γ , we get $\bar{\gamma}$ satisfying the conclusion of Lemma 7 with u^1, \dots, u^{p_0-1} . This contradicts the minimality of p_0 . So $i_* \bar{\omega} \neq 0$. Corollary 7.1 is thus proved, with $u = u^{p_0}$. \square

Corollary 7.1 gives the existence of at least one critical point $u \neq 0$. The hypothesis (\mathcal{H}) seems too weak to get u independent of r, h , and we cannot say that $f(u) = c$. The fundamental reason for this is that the Palais-Smale condition is not satisfied. To overcome this difficulty, we shall make use of Lemma 6 which gives a local Palais-Smale condition.

We first choose $\rho^0 \in]0, r_0[$, $d^0 > 0$, such that $[\rho^0 - d^0, \rho^0 + d^0] \cap \mathbf{D} = \emptyset$, r_0 being defined in Lemma 6.

We define

$$\mu^0 = \frac{1}{2} \inf \{ \|f'(x)\| \mid x \in f^{c'}, (\exists y \in \mathcal{C} \cap f^{c'}) : \|x - y\| \in [\rho^0, \rho^0 + d^0] \}.$$

We take $0 < h < \min(\mu^0 d^0, c' - c)$. By Corollary 7.1, there are

$$u^0 \in \mathcal{C} \cap f_c^{c'}, \quad \bar{\omega} \in H_1(\mathbf{B}(u^0, \rho^0) \cap f^{<c+h}, \mathbf{B}(u^0, \rho^0) \cap f^{<c}),$$

such that $i_* \bar{\omega} \neq 0$, where

$$i_*: H_1(f^{<c+h} \cap \mathbf{B}(u^0, \rho^0), f^{<c} \cap \mathbf{B}(u^0, \rho^0)) \rightarrow H_1(f^{<c+h}, f^{<c})$$

is the morphism induced by the canonical injection

$$i: \mathbf{B}(u^0, \rho^0) \rightarrow L^\beta.$$

We define

$$\begin{aligned}
 X &= (f^{c+h} \cap B(u^0, \rho^0)) \\
 &\cup \left\{ x \in L^B / \|x - u^0\| \in [\rho^0, \rho^0 + d^0], f(x) < c + h \left(1 - \frac{\|x - u^0\| - \rho^0}{d^0} \right) \right\}, \\
 Y &= f^c \cap B(u^0, \rho^0 + d^0).
 \end{aligned}$$

We call

$$j_* : H_1(f^{<c+h} \cap B(u^0, \rho^0), f^{<c} \cap B(u^0, \rho^0)) \rightarrow H_1(X, Y)$$

the morphism induced by the canonical injections

$$\begin{aligned}
 j_+ : f^{<c+h} \cap B(u^0, \rho^0) &\rightarrow X, \\
 j_- : f^{<c} \cap B(u^0, \rho^0) &\rightarrow Y.
 \end{aligned}$$

Clearly, we have $j_* \bar{\omega} \neq 0$.

$$\text{We define } \bar{c} = \inf_{z \in j_* \bar{\omega}} (\max f(z)) \in [c, c + h].$$

By arguments similar to those proving Lemma 7 and Corollary 7.1, we find, for any $n \in \mathbb{N}^*$, a critical point $u^n \in \mathcal{C} \cap f_{\bar{c}}^{\bar{c}+(1/n)} \cap B(u^0, \rho^0 - d^0)$, such that $i_*^n \neq 0$, where

$$\begin{aligned}
 i_*^n : H_1 \left(f^{<\bar{c}+(1/n)} \cap B \left(u^n, \frac{d^0}{n} \right), f^{<\bar{c}} \cap B \left(u^n, \frac{d^0}{n} \right) \right) \\
 \rightarrow H_1 (f^{<(\bar{c}+(1/n))} \cap B(u^n, d^0), f^{<\bar{c}} \cap B(u^n, d^0))
 \end{aligned}$$

is the morphism induced by the canonical injection

$$i_*^n : B \left(u^n, \frac{d^0}{n} \right) \rightarrow B(u^n, d^0).$$

By Lemma 6, the sequence (u^n) is precompact (recall that $\rho^0 < r_0$). Considering one of its limit points, and taking $r_1 = d^0/2$, we get

LEMMA 8. — Assume that (hA), (hR) and (\mathcal{H}) are true.

Then there are $u \in \mathcal{C}$ with $f(u) = \bar{c} \in [c, c']$ and $r_1 > 0$, such that, for any $r \in]0, r_1]$ and $h > 0$, we have $i_* \neq 0$ where

$$\begin{aligned}
 i_* : H_1 (f^{<(\bar{c}+h)} \cap B(u, r), f^{<\bar{c}} \cap B(u, r)) \\
 \rightarrow H_1 (f^{<(\bar{c}+h)} \cap B(u, r_1), f^{<\bar{c}} \cap B(u, r_1))
 \end{aligned}$$

is the morphism induced by the canonical injection

$$i : B(u, r) \rightarrow B(u, r_1).$$

The great difference with Corollary 7.1 is that u does not depend on r , h any more.

Lemma 8 gives a min-max localized around u . To get our multiplicity result, we are going to make products of several “copies” of this min-max. At each product will be associated a new critical point. We first

enounce:

COROLLARY 8.1. — Assume that (hA), (hR) and (\mathcal{H}) are true. Choose $r \in]0, r_1[$, $h > 0$.

Then there is $N = N(r, h)$ such that

$$(\forall (a, b) \in [N, +\infty]^2): I_* \neq 0,$$

where

$$I_*: H_1(f^{<(\bar{c}+h)} \cap B(u, r) \cap L^{\beta}_{(-a, b)}, f^{<\bar{c}} \cap B(u, r) \cap L^{\beta}_{(-a, b)}) \\ \rightarrow H_1(f^{<(\bar{c}+h)} \cap B(u, r_1) \cap L^{\beta}_{(-a, b)}, f^{<\bar{c}} \cap B(u, r_1) \cap L^{\beta}_{(-a, b)})$$

is the morphism induced by

$$I: B(u, r) \cap L^{\beta}_{(-a, b)} \rightarrow B(u, r_1) \cap L^{\beta}_{(-a, b)}$$

and

$$L^{\beta}_{(-a, b)} = \{x \in L^{\beta} / \text{supp}(x) \subset [-a, b]\}.$$

Proof. — We choose $\bar{\omega} \in H_1(f^{<(\bar{c}+h)} \cap B(u, r), f^{<\bar{c}} \cap B(u, r))$ such that

$$i_* \bar{\omega} \neq 0,$$

with the notations of Lemma 8.

The class $\bar{\omega}$ has an element of the form $\sum_{i=1}^r \lambda_i \sigma_i$, satisfying

(P) [$\lambda_i \in \mathbb{R}$, and $\sigma_i: S^1 \rightarrow L^{\beta}$ continuous or $\sigma_i: [0, 1] \rightarrow L^{\beta}$ continuous, with $\sigma_i(0), \sigma_i(1) \in f^{<\bar{c}}$, and $\text{Im}(\sigma_i) \subset f^{<(\bar{c}+h)} \cap B(u, r)$ in both cases].

For $t_1, t_2 \in \mathbb{R}$, we define

$$K_{t_1, t_2}: L^{\beta}(\mathbb{R}, \mathbb{R}^{2N}) \rightarrow L^{\beta}(\mathbb{R}, \mathbb{R}^{2N}) \\ x(t) \mapsto \chi_{[t_1, t_2]}(t) x(t)$$

We note that $\bigcup_{i=1}^r \text{Im} \sigma_i$ is compact, so that

$$\lim_{(t_1, t_2) \rightarrow (-\infty, +\infty)} \left(\sup \left\{ \|x - K_{t_1, t_2}(x)\|; x \in \bigcup_{i=1}^r \text{Im} \sigma_i \right\} \right) = 0.$$

Moreover, $f^{<(\bar{c}+h)} \cap B(u, r)$ and $f^{<\bar{c}} \cap B(u, r)$ are open.

So there is $N = N(r, e, h) \in \mathbb{N}$ such that, if $(a, b) \in [N, +\infty]^2$, then

$$\sum_{i=1}^r \lambda_i (K_{-a, b} \circ \sigma_i) \in \bar{\omega}.$$

As a consequence, there is

$$\tilde{\omega} \in H_1(f^{<(\bar{c}+h)} \cap B(u, r) \cap L^{\beta}_{(-a, b)}, f^{<\bar{c}} \cap B(u, r) \cap L^{\beta}_{(-a, b)})$$

such that $\sum \lambda_i (K_{-a, b} \circ \sigma_i) \in \tilde{\omega}$, and $i_*(\tilde{\omega}) \neq 0$ implies $I_*(\tilde{\omega}) \neq 0$. So I_* cannot be zero.

Corollary 8.1 is thus proved. \square

We now have to introduce some notations.

Take $x \in L^\beta$, $\bar{p} = (p^1, \dots, p^m) \in \mathbb{Z}^m$, $m \geq 1$, $p^i < p^{i+1}$. Denote

$$x_i = x \chi_{[(p^{i-1} + p^i)/2, (p^i + p^{i+1})/2]}, \quad f_i(x) = f(x_i),$$

with χ_I the characteristic function of I , $p^0 = -\infty$, $p^{m+1} = +\infty$.

We have $x = \sum_{i=1}^m x_i$, but $f \neq \sum_{i=1}^m f_i$.

Consider the sets

$$\mathcal{L}_+(h) = \bigcap_{i=1}^m (f_i)^{<(\bar{c}+h)}, \quad \mathcal{L}_-(h) = \bigcup_{i=1}^m (f_i)^{<(\bar{c}-h)},$$

and the “product” ball

$$B_{\bar{p}, \rho}^u = \{x \in L^\beta / (\forall i) \|(x - p^i * u)_i\|_{L^\beta} < \rho\}$$

for $\rho > 0$, $u \in \mathcal{C}$.

From Künneth’s formula,

$$H_*(X \times Y, (Z \times Y) \cup (X \times T)) = H_*(X, Z) \otimes H_*(Y, T),$$

immediately follows

LEMMA 9. — Assume that (hA), (hR) and (\mathcal{H}) are true. u, r_1 are the same as in Lemma 8. Choose $r \in]0, r_1]$, $h > 0$.

Then there is $N = N(r, h)$ such that, if $m \geq 1$ and $\bar{p} = (p^1 \dots p^m)$ satisfy $p^{i+1} - p^i \geq N$ for $1 \leq i \leq m - 1$, then

$$J_* \neq 0,$$

where

$$J_*: H_m(\mathcal{L}_+(h) \cap B_{\bar{p}, r}^u, \mathcal{L}_-(0) \cap \mathcal{L}_+(h) \cap B_{\bar{p}, r}^u) \rightarrow H_m(\mathcal{L}_+(h) \cap B_{\bar{p}, r_1}^u, \mathcal{L}_-(0) \cap \mathcal{L}_+(h) \cap B_{\bar{p}, r_1}^u)$$

is the morphism associated to the canonical injection

$$J: B_{\bar{p}, r} \rightarrow B_{\bar{p}, r_1}.$$

Lemma 9 gives the desired product min-max.

V. A DEFORMATION ARGUMENT

In what follows, we assume once again that (hA), (hR) and (\mathcal{H}) are true. D, F are the same as in Lemmas 4, 5, r_0 is the same as in Lemma 6, u, \bar{c}, r_1 are the same as in Lemmas 8, 9.

5.1. Construction of a vector field

From (hA) (hR), we know that $(\exists \theta, C_1 > 0) (\forall (X, Y) \in (L^\beta)^2)$:

$$\left| \int (X, LY) \right| \leq C_1 \exp(-\theta \delta(X, Y)) \|X\|_\beta \|Y\|_\beta,$$

for $\delta(X, Y) = \text{dist}(\text{supp } X, \text{supp } Y)$.

From (hR), we know that

$$\begin{aligned} (\exists c_1 > 0) \quad (\forall (y, t) \in \mathbb{R}^{2N} \times \mathbb{R}), \quad c_1 |y|^\beta \leq G(y, t) \leq (\nabla G(y, t), y), \\ (\exists c_2 > 0) \quad (\forall (y, t) \in \mathbb{R}^{2N} \times \mathbb{R}), \quad |\nabla G(y, t)| \leq c_2 |y|^{\beta-1}. \end{aligned}$$

We choose $0 < r_2 < \min(1, r_1)$ such that

$$\frac{c_1}{2} (r_2)^\beta > 6 C_1 (r_2)^2, \quad \text{and} \quad B(u, r_2) \subset f^{c'}.$$

We are going to use these technical conditions in the proof of the following Lemma:

LEMMA 10. — Assume that (hA), (hR) and \mathcal{H} are true, and to $0 < r < \frac{r_2}{2}$, associate $e = e(r)$ such that

$$r + 2e \leq \frac{r_2}{2} \quad \text{and} \quad [r - 2e, r + 2e] \cap D = \emptyset.$$

There are $\mu = \mu(r) > 0, A = A(r) > 0$ such that:

If $m \geq 2$, and if $\bar{p} \in \mathbb{Z}^m$ satisfies $(\forall i): p^{i+1} - p^i > A$, then:

$(\forall x \in B_{\bar{p}, r+e}^u \setminus B_{\bar{p}, r-e}^u) (\exists V_x \in B_{\bar{p}, 1}^0)$:

- 1) $f'(x) \cdot V_x > \mu$;
- 2) $(\forall i): (f_i)'(x) \cdot V_x \geq 0$;
- 3) $\|y_i\| \geq r - e \Rightarrow (f_i)'(x) \cdot V_x > \mu$,

with the notation $y_i = (x - p^i * u)_i$. \square

Proof. — Define

$$\bar{\mu} = \frac{1}{2} \inf \{ \|f'(x)\|_u / x \in B(u, r + 2e(r)) \setminus B(u, r - e(r)) \}.$$

$\bar{\mu}$ depends only on r , and $\bar{\mu} > 0$ by Lemma 4. Let $x \in B_{\bar{p}, r+e}^u \setminus B_{\bar{p}, r-e}^u, i \in [1, m]$, and $y_i = (x - p^i * u)_i$. Impose $A > 64$.

We always have $\|x_i\| \leq \|u\| + r_2$. So there is $\tau^i \in [2\sqrt{A}, A/2 - 2\sqrt{A}]$ such that

$$\|x_i \chi_{(\tau^i - \sqrt{A} \leq |t - p^i| \leq \tau^i + \sqrt{A})}\|_\beta \leq \frac{C_2}{A^{1/2\beta}}.$$

Here, C_2 is a constant, but τ^i depends on x, i, A, \bar{p} .

Now, impose $\|u\chi_{\{|t|>\sqrt{A}\}}\| \leq \frac{e}{3}$, and $\frac{C_2}{A^{1/2\beta}} \leq \frac{e}{3}$, which is possible for $A \geq A^0(e)$.

Then, three possibilities may occur:

First case:

$$\|x_i \chi_{\{|t-p^i| \geq \tau^i + \sqrt{A}\}}\| \geq \frac{e}{3}.$$

We take

$$V_{x,i} = x_i (h_- \chi_{]-\infty, p^i - \tau^i - \sqrt{A}[} + h_+ \chi_{]p^i + \tau^i + \sqrt{A}, +\infty[})$$

with

$$\begin{aligned} h_+ &= 1 & \text{if } \|x_i \chi_{]p^i + \tau^i + \sqrt{A}, +\infty[}\| \geq \frac{e}{6}, & \quad h_+ = 0 & \text{ otherwise,} \\ h_- &= 1 & \text{if } \|x_i \chi_{]-\infty, p^i - \tau^i - \sqrt{A}[}\| \geq \frac{e}{6}, & \quad h_- = 0 & \text{ otherwise.} \end{aligned}$$

We have

$$\begin{aligned} (f'_i)'(x) \cdot V_{x,i} &\geq c_1 \|V_{x,i}\|_\beta^\beta - C_1 \|V_{x,i}\|_\beta^2 \\ &\quad - C_1 \|x \chi_{\{\tau^i - \sqrt{A} \leq |t-p^i| \leq \tau^i + \sqrt{A}\}}\|_\beta \cdot \|V_{x,i}\|_\beta \\ &\quad - C_1 \|x \chi_{\{|t-p^i| \leq \tau^i - \sqrt{A}\}}\|_\beta \cdot \|V_{x,i}\|_\beta \exp(-2\theta\sqrt{A}) \\ &\geq \frac{3c_1}{4} \|V_{x,i}\|_\beta^\beta - C_1 \frac{e}{3} \|V_{x,i}\|_\beta \\ &\quad - C_1 (\|u\|_\beta + r_2) \|V_{x,i}\|_\beta \exp(-2\theta\sqrt{A}) \\ &\geq \frac{3c_1}{4} \|V_{x,i}\|_\beta^\beta - C_1 e \|V_{x,i}\|_\beta \quad \text{for } A \geq A^1(e) \\ &\geq \frac{3c_1}{4} \|V_{x,i}\|_\beta^\beta - 6C_1 \|V_{x,i}\|_\beta^2 \\ &\geq \frac{c_1}{4} \|V_{x,i}\|_\beta^\beta \geq \frac{c_1}{4} \left(\frac{e}{6}\right)^\beta. \end{aligned}$$

[We recall that $\frac{e}{6} \leq \|V_{x,i}\|_\beta \leq \|u\chi_{\{|t| \geq \sqrt{A}\}}\| + (r+e) \leq r_2 < 1$, and that $\frac{c_1}{2}(r_2)^\beta > 6C_1(r_2)^2$.]

Second case: $\|x_i \chi_{\{|t-p^i| \geq \tau^i + \sqrt{A}\}}\| < \frac{e}{3}$, and $\|y_i\| < r - e$. Then we take $V_{x,i} = 0$.

Third case: $\|x_i \chi_{\{|t-p^i| \geq t^i + \sqrt{A}\}}\| < \frac{e}{3}$, and $\|y_i\| < r - e$. Then

$$\begin{aligned} \|x \chi_{\{|t-p^i| \leq t^i - \sqrt{A}\}} - p^i * u\| &\geq \|y_i\| - \|x_i \chi_{\{t^i - \sqrt{A} \leq |t-p^i| \leq t^i + \sqrt{A}\}}\| \\ &\quad - \|u \chi_{\{|t| \geq \sqrt{A}\}}\| - \|x_i \chi_{\{|t-p^i| \geq t^i + \sqrt{A}\}}\| \\ &\geq r - e - \frac{e}{3} - \frac{e}{3} - \frac{e}{3} = r - 2e. \end{aligned}$$

Finally,

$$\begin{aligned} r - 2e &\leq \|x \chi_{\{|t-p^i| \leq t^i - \sqrt{A}\}} - p^i * u\| \\ &\leq \|y_i \chi_{\{|t-p^i| \leq t^i - \sqrt{A}\}}\| + \|u \chi_{\{|t| \geq \sqrt{A}\}}\| \\ &\leq r + e + \frac{e}{3} \\ &\leq r + 2e. \end{aligned}$$

So there is $W_{x,i} \in L^\beta$ such that $\|W_{x,i}\| \leq 1$, and

$$f'(x \chi_{\{|t-p^i| \leq t^i - \sqrt{A}\}}) \cdot W_{x,i} > \bar{\mu}.$$

Now,

$$\begin{aligned} f'(x) &= f'(x_i \chi_{\{|t-p^i| \leq t^i - \sqrt{A}\}}) + f'(x_i \chi_{\{t^i - \sqrt{A} \leq |t-p^i| \leq t^i + \sqrt{A}\}}) \\ &\quad + f'(x_i \chi_{\{|t-p^i| \geq t^i + \sqrt{A}\}}) + \sum_{j \neq i} f'(x_j) \\ &= f'(x^a) + f'(x^b) + f'(x^c) + \sum_{j \neq i} f'(x_j). \end{aligned}$$

But $\|x^b\| \leq \frac{C_2}{A^{1/2\beta}}$, and $\max\{\|x^a\|, \|x^c\|, \|x_j\| (j \neq i)\} \leq \|u\| + r_2$.

We choose $V_{x,i} = W_{x,i} \chi_{\{|t-p^i| \leq t^i\}}$. Clearly, $\|V_{x,i}\| \leq 1$. Moreover, we have:

$$\begin{aligned} f'(x) \cdot V_{x,i} &\geq f'(x^a) \cdot W_{x,i} - |f'(x^a) \cdot (V_{x,i} - W_{x,i})| \\ &\quad - |f'(x^b) \cdot V_{x,i}| - |f'(x^c) \cdot V_{x,i}| - \sum_{j \neq i} |f'(x_j) \cdot V_{x,i}| \\ &\geq \bar{\mu} - C_1 (\|u\| + r_2) \exp(-\theta \sqrt{A}) \\ &\quad - c_2 \left(\frac{C_2}{A^{1/2\beta}}\right)^{\beta-1} - C_1 \frac{C_2}{A^{1/2\beta}} - C_1 (\|u\| + r_2) \exp(-\theta \sqrt{A}) \\ &\quad - \sum_{j \neq i} C_1 (\|u\| + r_2) \exp(-\theta \sqrt{A}) \exp[-\theta (|i-j| - 1)A] \\ &\geq \bar{\mu} - c_2 \left(\frac{C_2}{A^{1/2\beta}}\right)^{\beta-1} - C_1 \frac{C_2}{A^{1/2\beta}} \\ &\quad - C_1 (\|u\| + r_2) \cdot \left(2 + \frac{2}{1 - \exp(-\theta A)}\right) \exp(-\theta \sqrt{A}) \\ &\geq \bar{\mu}/2 \quad \text{for } A \geq A^2(r). \end{aligned}$$

Identically,

$$\begin{aligned}
 (f'_i)'(x) \cdot V_{x,i} &= f'(x^a + x^b + x^c) \cdot V_{x,i} \\
 &\geq \bar{\mu} - c_2 \left(\frac{C_2}{A^{1/2\beta}} \right)^{\beta-1} - C_1 \frac{C_2}{A^{1/2\beta}} - 2C_1 (\|u\| + r_2) \exp(-\theta \sqrt{A}) \\
 &\geq \bar{\mu}/2 \quad \text{for } A \geq A^2.
 \end{aligned}$$

Conclusion. — We now take $V_x = \sum_i V_{x,i}$. By construction, $V_x \in B_{\bar{p}, 1}^0$.

Denote by I^1, I^2, I^3 the sets of indices i corresponding to Cases 1, 2, 3 respectively. We write

$$\begin{aligned}
 f'(x) \cdot V_x &= \sum_{i \in I^1} f'(x) \cdot V_{x,i} + \sum_{i \in I^3} f'(x) \cdot V_{x,i} \\
 &\geq \sum_{i \in I^1} f'(x) \cdot V_{x,i} + \frac{\bar{\mu}}{2} \text{card}(I^3).
 \end{aligned}$$

Now, there is a family $J^1 \subset \llbracket 0, m \rrbracket$ such that

$$\sum_{i \in I^1} V_{x,i} = \sum_{j \in J^1} X^j,$$

where

$$\begin{aligned}
 X^j &= (\xi_+^j \chi_{[(p^j + p^{j+1})/2, p^{j+1} - \tau^{j+1} - \sqrt{A}]} + \xi_-^j \chi_{[p^j + \tau^j + \sqrt{A}, (p^j + p^{j+1})/2]}) X \\
 &= \xi_+^j X_+^j + \xi_-^j X_-^j
 \end{aligned}$$

with $\xi_{\pm}^j \in \{0, 1\}$, and

$$\begin{aligned}
 &(\forall s \in \{+, -\}) \quad (\forall j \in \llbracket 0, m \rrbracket) \\
 &\left(\xi_s^j = 1 \Rightarrow \|X_s^j\| \geq \frac{e}{6}, \xi_s^j = 0 \Rightarrow \|X_s^j\| < \frac{e}{3} \right)
 \end{aligned}$$

So there are three possible situations

$$(\xi_-^j = \xi_+^j = 1), \quad (\xi_-^j = 0 \text{ and } \xi_+^j = 1), \quad (\xi_-^j = 1 \text{ and } \xi_+^j = 0).$$

First situation: $\xi_-^j = \xi_+^j = 1$.

Denote

$$\begin{aligned}
 Y^j &= X \chi_{[p^j + \tau^j - \sqrt{A}, p^j + \tau^j + \sqrt{A}]} \cup [p^{j+1} - \tau^{j+1} - \sqrt{A}, p^{j+1} - \tau^{j+1} + \sqrt{A}] \\
 Z^j &= x_j + x_{j+1} - X^j - Y^j.
 \end{aligned}$$

We have

$$\begin{aligned}
 f'(x) \cdot X^j &= f'(X^j) \cdot X^j + f'(Y^j) \cdot X^j + f'(Z^j) \cdot X^j \\
 &\quad + \sum_{k \neq j, j+1} f'(x_k) \cdot X^j \\
 &\geq \frac{3c_1}{4} \|X^j\|^\beta - C_1 \frac{2e}{3} \|X^j\| - 2C_1 \|X^j\| (\|u\| + r_2) \exp(-2\theta \sqrt{A}) \\
 &\quad - 2\|X^j\| (\|u\| + r_2) \sum_{l \geq 0} \exp(-2\theta \sqrt{A}) \exp(-\theta l A)
 \end{aligned}$$

$$\begin{aligned} &\geq \frac{3c_1}{4} \|X^j\|^\beta - C_1 e \|X^j\| \quad \text{for } A \geq A^3(e) \\ &\geq \frac{3c_1}{4} \|X^j\|^\beta - 6C_1 \|X^j\|^2 \\ &\geq \frac{c_1}{4} \|X\|^j \geq \frac{c_1}{4} \frac{e^\beta}{6^\beta} \end{aligned}$$

since $\frac{e}{6} \leq \|X^j\| \leq 2 \|u \chi_{\{|t| \geq \sqrt{A}\}}\| + 2(r+e) \leq r_2$.

Second situation: $\xi_-^j = 0, \xi_+^j = 1$.

We now take

$$\begin{aligned} Y^j &= x(\chi_{[p^j + r^j + \sqrt{A}, ((p_j + p_{j+1})/2)]} + \chi_{[p^{j+1} - r^{j+1} - \sqrt{A}, p^{j+1} - r^{j+1} + \sqrt{A}]}) \\ Z^j &= x_j + x_{j+1} - X^j - Y^j. \end{aligned}$$

We have $\|Y^j\| \leq \frac{e}{3} + \frac{e}{3} = \frac{2e}{3}$, $\text{dist}(\text{supp } Z^j, \text{Supp } X^j) \geq \sqrt{A}$. As in the first situation, we get

$$\begin{aligned} f^r(x) \cdot X^j &\geq \frac{3c_1}{4} \|X^j\|^\beta - C_1 e \|X^j\| \quad \text{for } A \geq A^4(u, e) \\ &\geq \frac{3c_1}{4} \|X^j\|^\beta - 6C_1 \|X^j\|^2 \\ &\geq \frac{c_1}{4} \|X^j\|^\beta \geq \frac{c_1}{4} \frac{e^\beta}{6^\beta}. \end{aligned}$$

The third situation is identical to the second one. Since $I^1 \cup I^3$ is non-empty, we take

$$A(r) = \max(A^0, A^1, A^2, A^3, A^4) \quad \text{and} \quad \mu(r) = \min\left(\bar{\mu}, \frac{c_1}{4} \frac{e^\beta}{6^\beta}\right),$$

and Lemma 10 is proved. \square

LEMMA 11. — Suppose f satisfies (hA), (hR) and (\mathcal{H}) . To $l < c'$, associate $\eta = \eta(l) > 0$ such that $l + 2\eta \leq c'$, and $[l - 2\eta, l + 2\eta] \cap F = \emptyset$.

Then there are $\mathcal{A} = \mathcal{A}(l)$ and $v = v(l)$ such that for any $m \geq 2, \bar{p} \in \mathbb{Z}^m$, with $(\forall i) p^{i+1} - p^i > \mathcal{A}$, we have:

- $$\left(\forall x \in B_{\bar{p}, (r_2/2)}^u \cap \bigcup_{i=1}^m (f_i)_{i-\eta}^{l+\eta} \right) (\exists \mathcal{V}_x \in B_{\bar{p}, 1}^0):$$
- $f^r(x) \cdot \mathcal{V}_x > v$;
 - $(\forall i \in \llbracket 1, m \rrbracket) : (x \in (f_i)_{i-\eta}^{l+\eta} \Rightarrow (f_i)'(x) \cdot \mathcal{V}_x > v)$;
 - $(\forall i) : (f_i)'(x) \cdot \mathcal{V}_x > 0$.

Proof. — We know that f is uniformly continuous on any bounded part of L^β . So there is $\mathcal{E}(\eta) > 0$ such that, if $X, Y \in B(0, \|u\| + r_2)$, then

$$\|X - Y\| \leq \mathcal{E} \Rightarrow |f(x) - f(y)| \leq \eta.$$

Now, consider $\bar{v} = \frac{1}{2} \inf \{ \|f'(x)\|; x \in f_i^{-1}[\frac{2\eta}{3}] \}$. From Lemma 5, $\bar{v} > 0$. The proof of Lemma 11 is similar to that of Lemma 10, replacing V by \mathcal{V} , $\bar{\mu}$ by \bar{v} , A by \mathcal{A} , e by \mathcal{E} . So we just sketch it. The three possibilities are:

First case: $\|x_i \chi_{\{|t-p^i| \geq \tau^i + \sqrt{\mathcal{A}}\}}\| \geq \frac{\mathcal{E}}{3}$, then

$$\begin{aligned} \mathcal{V}_{x,i} &= x_i (h_- \chi_{[-\infty, p^i - \tau^i - \sqrt{\mathcal{A}}]} + h_+ \chi_{[p^i + \tau^i + \sqrt{\mathcal{A}}, +\infty[}), \\ (f_i)'(x) \cdot \mathcal{V}_{x,i} &\geq \frac{c_1}{2} \frac{\mathcal{E}^\beta}{6^\beta} \quad \text{for } \mathcal{A} \geq \max(\mathcal{A}^0, \mathcal{A}^1). \end{aligned}$$

Second case: $\|x_i \chi_{\{|t-p^i| > \tau^i + \sqrt{\mathcal{A}}\}}\| < \frac{\mathcal{E}}{3}$, and $f_i(x) \notin [l - \eta, l + \eta]$, then $\mathcal{V}_{x,i} = 0$.

Third case: $\|x_i \chi_{\{|t-p^i| > \tau^i + \sqrt{\mathcal{A}}\}}\| < \frac{\mathcal{E}}{3}$, and $f_i(x) \in [l - \eta, l + \eta]$, then

$$f(x \chi_{\{|t-p^i| \leq \tau^i - \sqrt{\mathcal{A}}\}}) \in [l - 2\eta, l + 2\eta] \quad \text{for } \mathcal{A} \geq \mathcal{A}^0,$$

hence $f'(x \chi_{\{|t-p^i| \leq \tau^i - \sqrt{\mathcal{A}}\}}) \cdot \mathcal{W}_{x,i} > \bar{v}$,

$$\begin{aligned} \|\mathcal{W}_{x,i}\| &\leq 1, \quad \mathcal{V}_{x,i} = \mathcal{W}_{x,i} \chi_{\{|t-p^i| \leq \tau^i\}}, \\ f'(x) \cdot \mathcal{V}_{x,i} &\geq \bar{v}/2, \quad (f_i)'(x) \cdot \mathcal{V}_{x,i} \geq \bar{v}/2, \quad \text{for } \mathcal{A} \geq \mathcal{A}^2. \end{aligned}$$

The final study of $f'(x) \cdot \mathcal{V}_x$ is the same as in Lemma 10, and 11 is proved with $\mathcal{A} = \max(\mathcal{A}^0, \dots, \mathcal{A}^4)$, $v = \min\left(\frac{\bar{v}}{2}, \frac{c_1}{2} \frac{\mathcal{E}^\beta}{6^\beta}\right)$. \square

LEMMA 12. — Suppose f satisfies (hA), (hR) and (\mathcal{H}) .

$r, e(r), A(r), \mu(r)$ are the same as in Lemma 10. We impose, moreover, $r < r_0$, with the notation of Lemma 6.

Choose $\lambda > 0$ such that $\bar{c} + \lambda < c'$,

$$\text{and } \begin{cases} \bar{c} + \lambda \notin F \\ \bar{c} - \lambda \notin F. \end{cases}$$

Suppose $m \geq 2, \bar{p} \in \mathbb{Z}^m$,

$$\begin{aligned} (p^{i+1} - p^i) &\geq \max(A(r), \mathcal{A}(\bar{c} - \lambda), \mathcal{A}(\bar{c} + \lambda)) \\ &= \mathcal{B}(r, \lambda) \end{aligned}$$

(\mathcal{A} has been defined in Lemma 11).

If $\mathcal{C} \cap \mathbf{B}_{\bar{p}, r}^u \cap \mathcal{L}_+(\lambda) \setminus \mathcal{L}_-(\lambda) = \emptyset$, then there are $\xi = \xi(\bar{p}, r, \lambda) > 0$ and a locally Lipschitz vector field $V(x)$ such that:

- (i) $(\forall x) : V(x) \in \mathbf{B}_{\bar{p}, 1}^0$, and $(x \notin \mathbf{B}_{\bar{p}, (r_2/2)}^u \Rightarrow V(x) = 0)$;
- (ii) $\forall x \in [\mathbf{B}_{\bar{p}, r}^u \setminus \mathbf{B}_{\bar{p}, (r-e)}^u], \forall i \in \llbracket 1, m \rrbracket,$
 $\left(\|y_i\| \in [r-e, r] \Rightarrow (f_i)'(x) \cdot V(x) > \frac{\mu(r)}{3} \right).$
- (iii) $(\forall x \in \mathbf{B}_{\bar{p}, r}^u \cap (\mathcal{L}_+(\lambda) \setminus \mathcal{L}_-(\lambda))) : f'(x) \cdot V(x) > \xi.$
- (iv) $(\forall x \in \mathbf{B}_{\bar{p}, (r_2/2)}^u) (\forall i \in \llbracket 1, m \rrbracket) :$
 $(f_i(x) \in \{ \bar{c} + \lambda, \bar{c} - \lambda \} \Rightarrow (f_i)'(x) \cdot V(x) > 0).$

Proof. – In Lemma 6, take $R = \max(|p^1|, |p^m|)$. Consider a sequence

$$(u_n) \in \mathbf{B}_{\bar{p}, r}^u \cap \mathcal{L}_+(\lambda - \eta(\bar{c} + \lambda)) \setminus \mathcal{L}_-(\lambda - \eta(\bar{c} - \lambda)).$$

(u_n) satisfies

$$(\forall p, q), \quad \|(u_p - u_q) \chi_{\mathbb{R} \setminus [-R, R]}\| < 2r_2 < 2r_0.$$

So, if $\mathcal{C} \cap \mathbf{B}_{\bar{p}, r}^u \cap \mathcal{L}_+(\lambda) \setminus \mathcal{L}_-(\lambda) = \emptyset$, we cannot have $f'(u_n) \rightarrow 0$, and there is $\alpha(\bar{p}, u, r, \lambda) > 0$ such that

$$\forall x \in \mathbf{B}_{\bar{p}, r}^u \cap \mathcal{L}_+(\lambda - \eta(\bar{c} + \lambda)) \setminus \mathcal{L}_-(\lambda - \eta(\bar{c} - \lambda)) : \|f'(x)\| \geq 2\alpha.$$

Now, if $x \in [\mathbf{B}_{\bar{p}, (r+e)}^u \setminus \mathbf{B}_{\bar{p}, (r-e)}^u]$, we find V_x satisfying the conclusion of Lemma 10, and we choose $V_x = 0$ otherwise.

For $s \in \{-, +\}$. if $x \in \mathbf{B}_{\bar{p}, (r_2/2)}^u \cap \bigcup_i (f_i)_{\bar{c} + s\lambda - \eta}^{\bar{c} + s\lambda + \eta}$, we find \mathcal{V}_x^s satisfying the conclusion of Lemma 11 with $l = c + s\lambda$, and we choose $\mathcal{V}_x^s = 0$ otherwise.

If $x \in \mathbf{B}_{\bar{p}, r}^u \cap \mathcal{L}_+(\lambda) \setminus \mathcal{L}_-(\lambda)$ and if $V_x = \mathcal{V}_x^+ = \mathcal{V}_x^- = 0$, we find $\bar{V}_x \in \mathbf{B}_{\bar{p}, 1}^0$ such that $f'(x) \cdot \bar{V}_x > \alpha$, and we choose $\bar{V}_x = \frac{1}{3}(V_x + \mathcal{V}_x^+ + \mathcal{V}_x^-)$ otherwise.

$$\text{We take } \xi = \min \left\{ \alpha, \frac{1}{3}(\mu(r) + v(\bar{c} + \lambda) + v(\bar{c} - \lambda)) \right\}.$$

\bar{V}_x satisfies:

$$(I) (\forall x) : \bar{V}_x \in \mathbf{B}_{\bar{p}, 1}^0, \text{ and } (x \notin \mathbf{B}_{\bar{p}, (r_2/2)}^u \Rightarrow \bar{V}_x = 0).$$

$$(II) \left\{ \begin{array}{l} \forall x \in [\mathbf{B}_{\bar{p}, r+e}^u \setminus \mathbf{B}_{\bar{p}, (r-e)}^u], \forall i \in \llbracket 1, m \rrbracket, \\ \|\|y_i\| \in [r-e, r+e] \Rightarrow (f_i)'(x) \cdot \bar{V}_x > \frac{\mu(r)}{3}. \end{array} \right.$$

$$(III) \quad \left\{ \begin{array}{l} (\forall x \in \mathbf{B}_{\bar{p}, r+e}^u \cap (\mathcal{L}_+ (\lambda + \eta) (\bar{c} + \lambda)) \setminus \mathcal{L}_- (\lambda + \eta) (\bar{c} - \lambda)) : \\ f'(x) \cdot \bar{\nabla}_x > \xi. \end{array} \right.$$

$$(IV) \quad \left\{ \begin{array}{l} (\forall x \in \mathbf{B}_{\bar{p}, (r_2/2)}^u) (\forall i \in \llbracket 1, m \rrbracket) : \\ (f_i(x) \in \{\bar{c} + \lambda, \bar{c} - \lambda\} \Rightarrow (f_i)'(x) \cdot \bar{\nabla}_x > 0). \end{array} \right.$$

But $\bar{\nabla}_x$ is not continuous. A classical pseudo-gradient construction ends the proof. \square

5.2. The contradiction

We suppose (hA), (hR) and (\mathcal{H}) are true. r , $e(r)$, $\mu(r)$, λ are the same as in Lemma 12. On λ , we impose one more condition:

$$\lambda \leq \frac{\mu(r)e(r)}{6}.$$

As in Lemma 12, we suppose that

$$\mathcal{C} \cap \mathbf{B}_{\bar{p}, r}^u \cap (\mathcal{L}_+ \setminus \mathcal{L}_-) (\lambda) = \emptyset,$$

and we take $m \geq 2$, $\bar{p} \in \mathbb{Z}^m$ with

$$(\forall i) \quad (p^{i+1} - p^i) \geq \mathcal{B}(r, \lambda).$$

We define $\varphi(t, x)$ for $(t, x) \in \mathbb{R} \times \mathbf{L}^\beta$ by

$$\begin{aligned} \varphi(0, x) &= x \\ \frac{\partial \varphi}{\partial t}(t, x) &= -\mathbf{V} \circ \varphi(t, x), \end{aligned}$$

where $\mathbf{V}(x)$ is the vector field of Lemma 12.

We have

LEMMA 13. — *With the notations and hypotheses above, there is $\mathcal{F} = \mathcal{F}(r, \lambda, \bar{p})$ such that*

$$\varphi(\mathcal{F}, \cdot) [\mathbf{B}_{\bar{p}, r-e}^u \cap \mathcal{L}_+(\lambda)] \subset \mathcal{L}_-(\lambda) \cap \mathcal{L}_+(\lambda).$$

Proof. — Take $x \in \mathbf{B}_{\bar{p}, r-e}^u \cap \mathcal{L}_+(\lambda)$. Then

$$(\forall t \geq 0), \quad \varphi(t, x) \in \mathbf{B}_{\bar{p}, (r_2/2)}^u \cap \mathcal{L}_+(\lambda),$$

by (i) and (iv) of Lemma 12. Moreover, if $\varphi(t, x) \in \mathcal{L}_-(\lambda)$, then for any $t' \geq t$, $\varphi(t', x) \in \mathcal{L}_-(\lambda)$, by (iv). Now, define

$$\mathbf{S} = \mathbf{S}(\bar{p}) = \sup \{ |f(\mathbf{X}) - f(\mathbf{Y})|; (\mathbf{X}, \mathbf{Y}) \in (\mathbf{B}_{\bar{p}, r_2}^u)^2 \}.$$

Define

$$\mathcal{F} = \frac{2\mathbf{S}(\bar{p})}{\xi(\bar{p}, r, \lambda)}.$$

By (iii) of Lemma 12, there is $t_x \in [0, \mathcal{T}]$ such that

$$\varphi(t_x, x) \notin B_{\bar{p}, r}^u \cap (\mathcal{L}_+(\lambda) \setminus \mathcal{L}_-(\lambda)).$$

By (i), (ii) of Lemma 12, this implies $\varphi(\mathcal{T}, x) \in \mathcal{L}_-(\lambda)$ (we recall that $2\lambda \leq \mu(r)e(r)/3$).

Lemma 13 is thus proved. \square

Now, we impose

$$(\forall i) \quad (p^{i+1} - p^i) \geq N(r - e(r), \lambda),$$

with the notations of Lemma 9.

The conclusion of Lemma 13 clearly implies $J_* = 0$, which contradicts the conclusion of Lemma 9.

Now, for any $h > 0$, we may choose $\lambda < h$ satisfying all the conditions above.

So, by contradiction, we have proved the following result:

THEOREM III. — Assume that (hA), (hR) and (\mathcal{H}) are true.

Then there is $u \in \mathcal{C}$, with $f(u) = \bar{c} \in [c, c']$, and such that for any $r, h > 0$, for all $m \geq 1$ and $\bar{p} = (p^1, \dots, p^m) \in \mathbb{Z}^m$:

$$[(\forall i) : (p^{i+1} - p^i) \geq M(r, h)] \Rightarrow [\mathcal{C} \cap U_{\bar{p}, r, h} \neq \emptyset].$$

$M(r, h)$ is a constant independent of m , and $U_{\bar{p}, r, h}$ is a neighborhood of

$\sum_{i=1}^m p^i * u$ defined as follows:

$$U_{\bar{p}, r, h} = B_{\bar{p}, r}^u \cap (\mathcal{L}_+(h) \setminus \mathcal{L}_-(h)), \text{ with the notations of Lemma 9.}$$

We now prove Theorem II:

We take a fixed value of h , and we write $M(r)$ instead of $M(r, h)$. We may choose $K > M(r)$ large enough to get $\|u \chi_{\{|i| \geq K/2\}}\| \leq r$, which implies

$\sum_{i=1}^m p^i * u \in B_{\bar{p}, r}^u$ for any $m \geq 2$, and $\bar{p} \in \mathbb{Z}^m$ such that $(\forall i) (p^{i+1} - p^i) \geq K$. So,

from Theorem III, there is $u_{\bar{p}} \in \mathcal{C}$ such that

$$(\forall i \in \mathbb{Z}) : \left\| \left(u_{\bar{p}} - \sum_{i=1}^m p^i * u \right) \chi_{\{((p^{i-1} + p^i)/2); ((p^i + p^{i+1})/2)\}} \right\|_{\beta} \leq 2r.$$

So, defining $y_{\bar{p}} = L u_{\bar{p}}$:

$$\begin{aligned} \left\| y_{\bar{p}} - \sum_{i=1}^m p^i * x \right\|_{\infty} &\leq 3 C_3 \sum_{n \geq 0} 2r \exp[-2\theta^n M(r)] \\ &= \frac{6 C_3 r}{1 - \exp(-2\theta^K)} \leq \varepsilon, \end{aligned}$$

for $K(\varepsilon)$ large enough. So Theorem II is a direct consequence of Theorem III. \square

We are now going to study the limit ($m \rightarrow +\infty$).

VI. THE APPROXIMATE BERNOULLI SHIFT

Our first task here is to prove Corollary II.1 of Theorem II. We consider a sequence $\bar{p} = (p^i)_{i \in I}$ of integers with $I \subset \mathbb{Z}$ a finite or infinite interval, and $p^{i+1} - p^i \geq K(\varepsilon)$.

The case $0 \leq \text{Card}(I) < \infty$ is clear. So we just consider the case of an infinite I . We may write $I = \bigcup_{k \geq 0} I^k$, each I^k being finite. From Theorem II, we get an orbit y^k such that

$$\|y^k - \sum_{i \in I^k} p^i \star x\|_\infty \leq \varepsilon.$$

The y^k 's being orbits, $\|y^k\|_\infty + \left\| \frac{d}{dt} y^k \right\|_\infty$ is a bounded sequence. So, after extraction, by Ascoli's theorem, y^k converges to some orbit $y_{\bar{p}}$ in the C_{loc}^0 topology, and Corollary II.1 is proved.

Now, we take $s \in \{0, 1\}^{\mathbb{Z}}$ arbitrary (*i.e.* with possibly infinitely many 1's). There are an interval I of integers and a sequence $(q^i)_{i \in I} \subset \mathbb{Z}$, with $(\forall i) q^{i+1} > q^i$, and $s_n = \chi_{\{q^i, i \in \mathbb{Z}\}}(n)$.

We denote $p^i = K(\varepsilon) q^i$, and we define $\mathcal{F}(s) = y_{\bar{p}}$, using Corollary II.1.

We recall that $\{0, 1\}^{\mathbb{Z}}$ may be given the topology associated to the metric $d(s, s') = \frac{1}{3} \sum_{n \in \mathbb{Z}} \frac{|s_n - s'_n|}{2^{|n|}}$.

We define

$$\begin{aligned} \tilde{\tau}: \{0, 1\}^{\mathbb{Z}} &\rightarrow \mathbb{R}^{2\mathbb{N}} \\ s &\mapsto \mathcal{F}(s)(0). \end{aligned}$$

Since

$$\|\mathcal{F}(s) - \sum_n s_n (K n \star x)\|_\infty \leq \varepsilon,$$

we have $\limsup_{d(s, s') \rightarrow 0} |\tilde{\tau}(s') - \tilde{\tau}(s)| \leq 2\varepsilon$.

Now, we take $\delta > 0$. There is $I(\delta) > 0$ such that if $d(s, s') \geq \delta$, then $s^I \neq (s')^I$.

So, taking $K(\varepsilon)$ large enough in Corollary II.1, there is $\rho > 0$ independent of s, s', ε , with

$$\left\| \left(\sum_n s_n (K n \star x) - \sum_n s'_n (K n \star x) \right) \chi_{[-2I, 2I]} \right\|_\infty \geq 2\rho.$$

So

$$\|(\mathcal{F}(s) - \mathcal{F}(s')) \chi_{[-2I, 2I]}\|_\infty \geq \rho$$

for $\varepsilon < \frac{\rho}{2}$.

Now, define

$$\begin{aligned} \mathcal{O}: \mathbb{R}^{2N} &\rightarrow C^0([-2I, 2I], \mathbb{R}^{2N}) \\ x &\mapsto \mathcal{O}(x) \end{aligned}$$

where

$$\begin{aligned} \frac{d}{dt} \mathcal{O} - \mathbf{J} \mathbf{A} \mathcal{O} &= \mathbf{J} \nabla \mathbf{R}(t, \mathcal{O}) \\ \mathcal{O}(x)(0) &= x. \end{aligned}$$

By the classical continuity results on the Cauchy problem, \mathcal{O} is uniformly continuous on any bounded part of \mathbb{R}^{2N} . So there is $\rho'(\delta) > 0$, independent of s, s', r , such that

$$\tilde{d}(s, s') \geq \delta \Rightarrow \|\tilde{\tau}(s) - \tilde{\tau}(s')\| \geq \rho'.$$

So $\tilde{\tau}$ is injective, and $\tilde{\tau}^{-1}$ is uniformly continuous. The other assertions of Corollary II.2 are easy to check, if we choose $x_0 = x(0)$. Corollary II.2 is thus proved. One would like $\tilde{\tau}$ to give a Bernoulli shift structure, *i.e.* $\tilde{\tau}$ homeomorphism, and $\tilde{\tau} \circ \sigma = \varphi^K \circ \tilde{\tau}$ (see [M], [W]). Unfortunately, this is not the case. We only have the estimate $\|\mathcal{T}(s) - \sum_n s_n(n * x)\|_\infty \leq \varepsilon$. The

points s such that $s_n = 0$ except for a finite number of n 's correspond to homoclinic orbits passing through $\tilde{\tau}(s)$ at time 0: there are infinitely many of them.

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