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On the Davey-Stewartson systems

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ABSTRACT. — We study the initial value problem for the Davey-Stewartson systems. This model arises generically in both physics and mathematics. Using the classification in [15] we consider the elliptic-hyperbolic and hyperbolic-hyperbolic cases. Under smallness assumption on the data it is shown that the IVP is locally wellposed in weighted Sobolev spaces.

Key words : Initial value problem, smoothing effect, local well-posedness.

RÉSUMÉ. — Nous étudions le problème de Cauchy pour les systèmes de Davey-Stewartson. Ce modèle se présente de façon générique en mathématiques et en physique. Nous utilisons la classification de [15] et considérons les cas elliptique-hyperbolique et hyperbolique-hyperbolique. Sous des conditions (de petites tailles) sur les données nous montrons que le problème est bien posé sur les espaces de Sobolev à poids.

1. INTRODUCTION

Consider the initial value problem IVP for the Davey-Stewartson (D-S) system

$$(1.1) \quad \begin{cases} i\,\partial_t u + c_0\,\partial_x^2 u + \partial_y^2 u = c_1\,|u|^2 u + c_2\,u\,\partial_x \varphi, & x, y \in \mathbb{R}, \quad t > 0 \\ \partial_x^2 \varphi + c_3\,\partial_y^2 \varphi = \partial_x |u|^2 \\ u(x, y, 0) = u_0(x, y) \end{cases}$$

where $u = u(x, y, t)$ is a complex-valued function, $\varphi = \varphi(x, y, t)$ is a real-valued function $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$ and $c_0, \ldots, c_3$ are real parameters.

A system of this kind was first derived by Davey and Stewartson [11] in their work on two-dimensional long waves over finite depth liquids (see also [12]). Independently Ablowitz and Haberman [1] obtained a particular form of (1.1) as an example of a completely integrable model which generalizes the two-dimensional nonlinear Schrödinger equation. Since then several works have been devoted to study special forms of the system (1.1) using the inverse scattering approach. In fact when $(c_0, c_1, c_2, c_3) = (-1, 1, -2, 1)$ or $(1, -1, 2, -1)$ the system in (1.1) is known in inverse scattering as the DSI and DSII respectively. In these cases several remarkable results concerning the associated IVP have been established (see [2]-[5], [10] and their bibliography). On the other hand the above system arises in water waves, plasma physics and nonlinear optics. Moreover, it has been shown that under appropriate asymptotic considerations a large class of nonlinear dispersive models in two dimensions can be reduced to the system (1.1) (see [13], [29] and references therein).

In [15] Ghidaglia and Saut studied the existence problem for solutions of the IVP (1.1). They classified the system as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic and hyperbolic-hyperbolic according to the respective sign of (c_0, c_3): $(+, +)$, $(+, -)$, $(-, +)$ and $(-, -)$. For the elliptic-elliptic and hyperbolic-elliptic cases they obtained a quite complete set of results concerning local and global properties of solutions to the IVP (1.1) in L^2, H^1, H^2. Their main tools were the $L^p - L^q$ estimates of Strichartz type [24] (see [6], [16], [19], [26]) and the good continuity properties of the operator $(-\Delta)^{-1}$ (and its derivatives). Also in the elliptic-hyperbolic case they established the global existence of a weak solution of the IVP (1.1) corresponding to "small" data (see also [25]).

In this case (elliptic-hyperbolic) as well as the hyperbolic-hyperbolic case one has to assume that $\varphi(.)$ satisfies the radiation condition i.e.

$$\varphi(x, y, t) \to 0 \quad \text{as} \quad x + y \quad \text{and} \quad x - y \to \infty$$

[without loss of generality we have taken $c_3 = -1$ in (1.1)]. This guarantees that for $F \in L^1(\mathbb{R}^2)$, $\mathcal{K}^{-1} F$ is well defined where

$$\mathcal{K}\,\varphi = (\partial_x^2 - \partial_y^2)\,\varphi = F.$$

Thus the IVP (1.1) is equivalent to

(1.2)
$$\begin{cases} i\partial_t u + c_0 \partial_x^2 u + \partial_y^2 u = c_1 |u|^2 u + c_2 u \partial_x^2 \mathscr{K}^{-1} |u|^2 \\ u(x, y, 0) = u_0(x, y) \end{cases}$$

with $c_0 > 0$ (resp. $c_0 < 0$) corresponding to the elliptic-hyperbolic case (resp. hyperbolic-hyperbolic case).

As was remarked in [15] and [25] no existence results were known for the hyperbolic-hyperbolic case.

Our main purpose here is to established local well-posedness results for the IVP (1.2) (with $c_0 \neq 0$) (i.e. the elliptic-hyperbolic and hyperbolic-hyperbolic cases) for "small" data. Our notion of well posedness includes existence, uniqueness, persistence [i.e. the solution $u(.)$ describes a continuous curve in the function space X whenever $u_0 \in X$]. The problem (1.2) can be seen as a nonlinear Schrödinger equation involving derivatives and a nonlocal term in the non-linearity. It is interesting to remark that previous approaches used in nonlinear evolution equation ($L^p - L^q$ estimates, energy inequality, L^2-theory, etc.) do not apply in this case.

In [22] Kenig, Ponce, and Vega studied the IVP for nonlinear Schrödinger equation of the form

(1.3)
$$i\partial_t u - \Delta u = Q(u, \nabla_x u, \bar{u}, \nabla_x \bar{u})$$

with $\nabla_x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and $Q: \mathbb{C}^{2n+2} \to \mathbb{C}$ denoting a polynomial having no constant or linear terms. Their arguments rely heavily on sharp versions (see [21], [22]) of the homogeneous and inhomogeneous smoothing effect first established by Kato [18] in solutions of the Korteweg-de Vries equation. This allows them to obtain conditions which guarantee that for "small" data the IVP (1.3) is local wellposed. Here we shall extend this approach to treat the equation in (1.2) which presents a more complicated nonlinear term (i.e. nonlocal term involving an operator with bad continuity properties) than that considered in (1.3).

In the hyperbolic-hyperbolic case (i.e. $c_0 < 0$) after rotation in the xy-plane and rescaling the system (1.2) can be written as

(1.4)
$$\begin{cases} i\partial_t u - \partial_{xy}^2 u = c_1 |u|^2 u + c_2 u \mathscr{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |u|^2 \\ u(x, y, 0) = u_0(x, y) \end{cases}$$

where $\mathscr{K}\varphi = \partial_{xy}^2 \varphi$ (with φ satisfying the appropriate radiation condition) and c_1, c_2, c_3 are arbitrary constants.

To explain our results (in the hyperbolic-hyperbolic case) it is convenient to consider first the associated linear problem to (1.4)

(1.5)
$$\begin{cases} i\partial_t u - \partial_{xy}^2 u = 0 \\ u(x, y, 0) = u_0(x, y). \end{cases}$$

It will be shown (*see* Theorem 2.1) that there exists $c > 0$ such that for any $y \in \mathbb{R}$

$$(1.6) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |D_x^{1/2} e^{it \partial_{xy}^2} u_0(x, y)|^2 \, dx \, dt = c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u_0(x, y)|^2 \, dx \, dy$$

where $\{e^{it \partial_{xy}^2}\}_{-\infty}^{\infty}$ denotes the group associated to the IVP (1.5), $D_x^{1/2} v(x, y, t) = \tilde{c}(|\xi|^{1/2} \hat{v}^{(x)}(\xi, y, t))^{\vee}$ with $\hat{v}^{(x)}$ denoting the Fourier transform in the x-variable. Notice that (1.6) is a global (in space and time) estimate which involves the $L_y^{\infty} L_t^2 L_x^2$-norm. Previous results only provide the gain of half-derivatives in $L_{loc}^2(\mathbb{R}^3)$ (*see* [9], [27], [28]). Roughly speaking (1.6) corresponds to the sharp one dimensional version of the Kato smoothing effect obtained in [21] (Theorem 4.1). Also the estimate (1.6) illustrates one of the key arguments in the proof of the hyperbolic-hyperbolic case (*see* Theorem A below), *i.e.* the use of different L^p-norms for the x and y variables. This kind of estimate also appears in the inhomogeneous version of (1.6) (*see* Theorem 2.3) and when inverting the operator $\mathscr{K}$ [*see* estimate (2.20) in Proposition 2.7].

Our results in the hyperbolic-hyperbolic case are contained in the following theorem.

THEOREM A. — *There exists $\delta > 0$ such that for any*

$$u_0 \in H^s(\mathbb{R}^2) \cap H^3(\mathbb{R}^2 : r^2 \, dx \, dy) \equiv Y_s$$

with $s \geq 6$ and

$$\delta_0 = \|u_0\|_{H_{xy}^6} + \|u_0\|_{H_{xy}^3 (r^2)} < \delta$$

there exist $T = T(\delta_0) > 0$ [with $T(\delta_0) \to \infty$ as $\delta_0 \to 0$] and a unique classical solution $u(.)$ of the IVP (1.4) satisfying

$$(1.7) \qquad\qquad\qquad u \in C([0, T] : Y_s),$$

$$(1.8) \quad \|D_x^{s+1/2} u\|_{L_y^{\infty} L_t^2 L_x^2} \equiv \sup_y \left(\int_0^T \int_{-\infty}^{\infty} |D_x^{s+1/2} u(x, y, t)|^2 \, dx \, dt \right)^{1/2} < \infty$$

and

$$(1.9) \quad \|D_y^{s+1/2} u\|_{L_x^{\infty} L_t^2 L_y^2} \equiv \sup_x \left(\int_0^T \int_{-\infty}^{\infty} |D_y^{s+1/2} u(x, y, t)|^2 \, dy \, dt \right)^{1/2} < \infty.$$

Moreover for any $T' \in (0, T)$ there exists a neighborhood V_{u_0} of u_0 in Y_s such that the map $\tilde{u}_0 \to \tilde{u}(t)$ from V_{u_0} into the class defined by (1.7)-(1.9) with T' instead of T is Lipschitz. —

In Theorem A (and Theorem B below) we shall not optimize the lower bound for the Sobolev exponents given in the hypothesis.

In the elliptic-hyperbolic case (*i.e.* $c_0 = 1$) after a rotation in the *xy*-plane and rescaling, (1.2) becomes

(1.10)
$$\begin{cases} i\,\partial_t u - \Delta u = c_1 \,|u|^2\, u + c_2\, u\, \mathscr{K}^{-1}\,(\partial_x^2 + c_3\,\partial_y^2)\,|u|^2 \\ \qquad\qquad u(x,\,y,\,0) = u_0\,(x,\,y) \end{cases}$$

where $\mathscr{K}\,\varphi = \partial_{xy}^2\,\varphi$ and $c_1,\,c_2,\,c_3$ arbitrary constants.

As was remarked above in this case Ghidaglia and Saut [15] established the global existence of a weak solution corresponding to "small" data. Also in [25] M. Tsutsumi studied the asymptotic behavior of this weak solution. Our results show that the IVP (1.10) is local wellposed for small data u_0.

THEOREM B. — *There exists $\delta > 0$ such that for any*

$$u_0 \in H^s\,(\mathbb{R}^2) \cap H^6\,(\mathbb{R}^2 : r^6\,dx\,dy) \equiv W_s$$

with $s \geq 12$ and

$$\delta_0 = \|\,u_0\,\|_{H_{xy}^{12}} + \|\,u_0\,\|_{H_{xy}^6\,(r^6)} < \delta$$

there exist $T = T(\delta_0) > 0$ [with $T(\delta_0) \to \infty$ as $\delta_0 \to 0$] and a unique classical solution $u(.)$ of the IVP (1.10) satisfying

(1.11)
$$u \in \mathbb{C}\,([0,\,T] : W_s),$$

and

(1.12)
$$\sup_{\alpha,\,\beta}\left(\int_0^T \int_\beta^{\beta+1} \int_\alpha^{\alpha+1} |\,D_{x,\,y}^{s+1}\, u(x,\,y,\,t)\,|^2\, dx\,dy\,dt \right)^{1/2} < \infty.$$

Moreover for any $T' \in (0,\,T)$ there exists a neighborhood $\tilde{V}_{u_0}$ of u_0 in Y_s such that the map $\tilde{u}_0 \to \tilde{u}(t)$ from $\tilde{V}_{u_0}$ into the class defined by (1.11)-(1.12) with T' instead of T is Lipschitz. —

This paper is organized as follows: in section 2 we shall deduce all linear estimates needed in the proof of Theorems A, B. Section 3 contains the essential arguments in the proof of our nonlinear results. Here all the nonlinear estimates to be used in sections 4, 5 are carried out in details. Finally in sections 4 and 5 we prove Theorems A and B respectively.

2. LINEAR ESTIMATES

In this section we shall deduce several estimates concerning the linear IVP

(2.1)
$$\begin{cases} i\,\partial_t u + \varepsilon\,\partial_x^2\, u + \partial_y^2\, u = 0, \qquad \varepsilon = \pm 1 \\ \qquad u(x,\,y,\,0) = u_0\,(x,\,y). \end{cases}$$

First we consider the hyperbolic-hyperbolic case, *i.e.* $\varepsilon = -1$. In this case after changing variable and rescaling (2.1) can be written as

$$(2.2) \qquad \begin{cases} i\,\partial_t u - \partial_{xy}^2\, u = 0 \\ u(x,\, y,\, 0) = u_0\,(x,\, y). \end{cases}$$

We begin by establishing the following sharp versions of the Kato smoothing effect in the group $\{ e^{it\,\partial_{xy}^2} \}_{-\infty}^{\infty}$ commented in the introduction.

THEOREM 2.1. $-$ *There exists $c > 0$ such that if $u_0 \in L^2\,(\mathbb{R}^2)$ then for any $y \in \mathbb{R}$ the solution $u(.,\,.,\,.)$ of the IVP (2.2) satisfies that*

$$(2.3) \quad \| D_x^{1/2}\, u(.,\, y,\, .) \|_{L_t^2\, L_x^2}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | D_x^{1/2}\, e^{it\,\partial_{xy}^2}\, u_0\,(x,\, y) |^2\, dx\, dt \equiv c \, \| u_0 \|_2. \quad -$$

It is clear that the same estimate holds with the roles of x and y interchanged.

Proof. $-$ By Fourier transform it follows that

$$e^{it\,\partial_{xy}^2}\, u_0\,(x,\, y) = c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\,(x\,\xi + y\,\eta)}\, e^{it\,\xi\eta}\, \hat{u}_0\,(\xi,\, \eta)\, d\xi\, d\eta.$$

Hence performing the change of variables $a = \xi\eta$ and $b = \xi$, using Plancherel's theorem in the $(x,\, t)$-variables, returning to the original variables and using again Plancherel's theorem one obtains that

$$\| D_x^{1/2}\, e^{it\,\partial_{xy}^2}\, u_0\,(x,\, y) \|_{L_t^2\, L_x^2}$$

$$= c \left\| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\,(x\,\xi + y\,\eta)}\, e^{it\,\xi\eta}\, | \xi |^{1/2}\, \hat{u}_0\,(\xi,\, \eta)\, d\xi\, d\eta \right\|_{L_t^2\, L_x^2}$$

$$= c \left\| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ixb}\, e^{ita}\, e^{iya/b}\, | b |^{1/2}\, \hat{u}_0\,(.,\,.)\, \frac{1}{|b|}\, da\, db \right\|_{L_t^2\, L_x^2}$$

$$= c \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{1}{|b|^{1/2}}\, \hat{u}_0\,(.,\,.) \right|^2\, da\, db \right)^{1/2}$$

$$= c \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | \hat{u}_0\,(\xi,\, \eta) |^2\, d\xi\, d\eta \right)^{1/2}$$

$$= c \, \| u_0 \|_2.$$

COROLLARY 2.2. $-$ *Let $F \in L_y^1\,(\mathbb{R} : L_{(x,\, t)}^2\,(\mathbb{R}^2))$. Then*

$$(2.4) \qquad \left\| D_x^{1/2} \int_{-\infty}^{\infty} e^{it\,\partial_{xy}^2}\, F\,(.,\,.,\, t)\, dt \right\|_{L_x^2\, L_y^2} \leqq c \, \| F \|_{L_y^1\, L_t^2\, L_x^2}$$

where $D_x^{1/2}\, G\,(x,\, y,\, t) = c\,(| \xi |^{1/2}\, \hat{G}^{(x)}\,(\xi,\, y,\, t))^{\vee}$. $-$

Proof. — It follows from (2.3) by duality.

Next we deduce the inhomogeneous version of the estimate (2.3). Thus we consider the inhomogeneous IVP

$$(2.5) \qquad \begin{cases} i\,\partial_t u - \partial_{xy}^2 u = F(x,\,y,\,t) \\ \qquad u(x,\,y,\,0) = 0, \end{cases}$$

which solution $u(.)$ is given by the formula

$$(2.6) \qquad u(t) = \int_0^t e^{i\,(t-t')\,\partial_{xy}^2}\,F(.,\,.,\,t')\,dt'.$$

THEOREM 2.3. — *If $u(.)$ is the solution of the IVP (2.5) then*

$$(2.7) \qquad \|\partial_x u\|_{L_y^\infty\,L_t^2\,L_x^2} \leq c\,\|F\|_{L_y^1\,L_t^2\,L_x^2}. \qquad -$$

Proof. — We shall follow the argument in [22].

Using Fourier Transform in the time and space variables one formally has that

$$\tilde{u}(x,\,y,\,t) = c\int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i\,(x\,\xi + y\,\eta + t\tau)}\,\frac{1}{\tau - \xi\eta}\,\hat{F}(\xi,\,\eta,\,\tau)\,d\xi\,d\eta\,d\tau.$$

Hence applying Plancherel's theorem it follows that

$$(2.8) \quad \|\partial_x \tilde{u}(.,\,y,\,.)\|_{L_t^2\,L_x^2}$$

$$= c\left\|\int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i\,(x\,\xi + y\,\eta + t\tau)}\,\frac{1}{\tau/\xi - \eta}\,\hat{F}(\xi,\,\eta,\,\tau)\,d\xi\,d\eta\,d\tau\right\|_{L_t^2\,L_x^2}$$

$$= c\left\|\int_{-\infty}^\infty e^{iy\,\eta}\,\frac{1}{\tau/\xi - \eta}\,\hat{F}(\xi,\,\eta,\,\tau)\,d\eta\right\|_{L_t^2\,L_\xi^2}$$

$$= c\left\|\int_{-\infty}^\infty K(y - y',\,\xi,\,\tau)\,\hat{F}^{(x,\,t)}(\xi,\,y',\,\tau)\,dy'\right\|_{L_t^2\,L_\xi^2}$$

where

$$K(y - y',\,\xi,\,\tau) = c\int_{-\infty}^\infty e^{i\,(y - y')\,\eta}\,\frac{1}{\tau/\xi - \eta}\,d\eta$$

and $\hat{F}^{(x,\,t)}$ denotes the Fourier transform of F in the x, t variables. By comparison with the kernel of the Hilbert transform (or its translated) it is easy to see that $K \in L^\infty(\mathbb{R}^3)$. Thus combining (2.8), Minkowski's integral inequality and Plancherel's theorem we find that for any $y \in \mathbb{R}$

$$(2.9) \quad \|\partial_x \tilde{u}(.,\,y,\,.)\|_{L_t^2\,L_x^2} \leq c\int_{-\infty}^\infty \|\hat{F}^{(x,\,t)}(.,\,y',\,.)\|_{L_t^2\,L_x^2}\,dy' = c\,\|F\|_{L_y^1\,L_t^2\,L_x^2}.$$

Using Parseval's identity we find (formally) that the solution $\tilde{u}(x, y, t)$ satisfies the following data

$$\tilde{u}(x, y, 0) = c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{\tau - \xi\eta} \hat{F}(\xi, \eta, \tau) \, d\tau \right) e^{i(x\xi + y\eta)} \, d\xi \, d\eta$$

$$= c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{it'\xi\eta} \operatorname{sgn}(t') \hat{F}^{(x, y)}(\xi, \eta, t') \, dt' \right) e^{i(x\xi + y\eta)} \, d\xi \, d\eta$$

$$= c \int_{-\infty}^{\infty} e^{it' \partial_{xy}^2} \operatorname{sgn}(t') F(., ., t') \, dt'.$$

By (2.4) we can infer that $D_x^{1/2} \tilde{u}(x, y, 0) \in L^2(\mathbb{R}^2)$. Finally since

$$u(x, y, t) = \tilde{u}(x, y, t) - e^{it \partial_{xy}^2} \tilde{u}(x, y, 0)$$

combining (2.9), (2.3) and the above remark we obtain (2.7). The above formal computation can be justified (and the proof stays essentially the same) by using the argument given in [23] (section 3).

Next we recall some estimates concerning the Kato smoothing effect in the group $\{ e^{it\Delta} \}_{-\infty}^{\infty}$.

It is convenient to introduce the notation:

$$Q_{\alpha, \beta} = I_\alpha \times I_\beta = [\alpha, \alpha+1] \times [\beta, \beta+1],$$

thus $\{ Q_{\alpha, \beta} \}_{\alpha, \beta \in \mathbb{Z}}$ forms a family of cubes of side one with nonoverlapping interiors such that $\mathbb{R}^2 = \bigcup_{\alpha, \beta = -\infty}^{\infty} Q_{\alpha, \beta}$.

THEOREM 2.4. — *Let* $u_0 \in L^2(\mathbb{R}^2)$. *Then*

$$(2.10) \quad \sup_{\alpha, \beta \in \mathbb{Z}} \left(\int_{Q_{\alpha, \beta}} \int_{-\infty}^{\infty} |D_{x, y}^{1/2} e^{it\Delta} u_0(x, y)|^2 \, dt \, dx \, dy \right)^{1/2} \leqq c \| u_0 \|_2$$

where $D_{x, y}^{1/2} G(x, y, t) = (|(\xi, \eta)|^{1/2} \hat{G}^{(x, y)}(\xi, \eta, t))^{\vee}$.

Let $F \in l_{\alpha, \beta}^1(L^2(Q_{\alpha, \beta} \times \mathbb{R}))$. *Then*

$$(2.11) \quad \left\| D_{x, y}^{1/2} \int_{-\infty}^{\infty} e^{it\Delta} F(., ., t) \, dt \right\|_{L_x^2 L_y^2}$$

$$\leqq c \sum_{\alpha, \beta = -\infty}^{\infty} \left(\int_{Q_{\alpha, \beta}} \int_{-\infty}^{\infty} |F(x, y, t)|^2 \, dt \, dx \, dy \right)^{1/2},$$

and

$$(2.12) \quad \sup_{\alpha, \beta \in \mathbb{Z}} \left(\int_{Q_{\alpha, \beta}} \int_{-\infty}^{\infty} \left| \nabla_{x, y} \int_0^t e^{i(t-t')\Delta} F(., ., t') \, dt' \right|^2 \, dt \, dx \, dy \right)^{1/2}$$

$$\leqq c \sum_{\alpha, \beta = -\infty}^{\infty} \left(\int_{Q_{\alpha, \beta}} \int_{-\infty}^{\infty} |F(x, y, t)|^2 \, dt \, dx \, dy \right)^{1/2}$$

where $\nabla_{x, y} = (\partial_x, \partial_y)$. —

Proof. − The estimate (2.10) was basically proven in [9], [27] and [28]. (2.11) is the dual version of (2.10). Finally (2.12) was established in [22].

To complement the previous estimates in the proof of our nonlinear results in section 3 we shall use the following theorems.

LEMMA 2.5. − *Let* $u_0 \in H^4(\mathbb{R}^2) \cap H^3(\mathbb{R}^2 : r^2 \, dx \, dy)$. *Then*

$$(2.13) \quad \int_{-\infty}^{\infty} \sup_{[-T, T]} \sup_x \left| e^{it \partial_{xy}^2} u_0(x, y) \right| dy$$

$$\leq c(1+T^2) \| u_0 \|_{H_{xy}^4} + c(1+T) \| u_0 \|_{H_{xy}^3 (r^2)}$$

where

$$(2.14) \quad \| u_0 \|_{H_{xy}^k (r^l)} \equiv \sum_{|\alpha| \leq k} \left(\int_{\mathbb{R}^2} | \partial_x^{\alpha_1} \partial_y^{\alpha_2} u_0(x, y) |^2 r^l \, dx \, dy \right)^{1/2}$$

with $r = (x^2 + y^2)^{1/2}$. −

Proof. − For simplicity in the exposition, it will be carried out only the details for $T > 0$.

For (y, t) fixed Sobolev's theorem tells us that

$$\sup_x \left| e^{it \partial_{xy}^2} u_0(x, y) \right| \leq x \| e^{it \partial_{xy}^2} u_0 \|_{L_x^2} + c \| e^{it \partial_{xy}^2} \partial_x u_0 \|_{L_x^2}.$$

Similarly for y fixed

$$(2.15) \quad \sup_{[0, T]} \sup_x \left| e^{it \partial_{xy}^2} u_0(x, y) \right|$$

$$\leq \frac{c}{T} \int_0^T \| e^{it \partial_{xy}^2} u_0 \|_{L_x^2} \, dt + c \int_0^T \left| \frac{\partial}{\partial t} \| e^{it \partial_{xy}^2} u_0 \|_{L_x^2} \right| dt$$

$$+ \frac{c}{T} \int_0^T \| e^{it \partial_{xy}^2} \partial_x u_0 \|_{L_x^2} \, dt + c \int_0^T \left| \frac{\partial}{\partial t} \| e^{it \partial_{xy}^2} \partial_x u_0 \|_{L_x^2} \right| dt.$$

now using the inequality

$$\| g \|_{L_y^1} \leq c \left\{ \| g \|_{L_y^2} + \| yg \|_{L_y^2} \right\}$$

together with Minkowski's integral inequality and the identity (*see* [17])

$$y e^{it \partial_{xy}^2} u_0 = e^{it \partial_{xy}^2} y u_0 - it e^{it \partial_{xy}^2} \partial_x u_0$$

we find that

$$\int_{-\infty}^{\infty} \frac{c}{T} \int_0^T \| e^{it \partial_{xy}^2} u_0 \|_{L_x^2} \, dt \, dy$$

$$\leq \frac{c}{T} \int_0^T \| e^{it \partial_{xy}^2} u_0 \|_{L_x^2 L_y^2} \, dt + \frac{c}{T} \int_0^T \| y e^{it \partial_{xy}^2} u_0 \|_{L_x^2 L_y^2} \, dt$$

$$\leq c \left\{ \| u_0 \|_{L_x^2 L_y^2} + \| y u_0 \|_{L_x^2 L_y^2} + T \| \partial_x u_0 \|_{L_x^2 L_y^2} \right\}$$

and

$$\int_{-\infty}^{\infty} \int_{0}^{T} \left| \frac{\partial}{\partial t} \| e^{it\, \partial_{xy}^2} u_0 \|_{L_x^2} \right| dt\, dy$$

$$= \int_{0}^{T} \int_{-\infty}^{\infty} \| e^{it\, \partial_{xy}^2} \partial_{xy}^2 u_0 \|_{L_x^2}\, dy\, dt$$

$$\leq c \int_{0}^{T} \| e^{it\, \partial_{xy}^2} \partial_{xy}^2 u_0 \|_{L_x^2 L_y^2}\, dt + c \int_{0}^{T} \| y e^{it\, \partial_{xy}^2} \partial_{xy}^2 u_0 \|_{L_x^2 L_y^2}\, dt$$

$$\leq c\, T \| \partial_{xy}^2 u_0 \|_{L_x^2 L_y^2} + c\, T \| y\, \partial_{xy}^2 u_0 \|_{L_x^2 L_y^2} + c\, T^2 \| \partial_{xxy}^3 u_0 \|_{L_x^2 L_y^2}.$$

The same argument applied to the last two terms on the right hand side of (2.15) yields (2.13).

Using the notation introduced before the statement of Theorem 2.4 and in (2.14) one has the corresponding result for the group $\{ e^{it\,\Delta} \}_{-\infty}^{\infty}$.

LEMMA 2.6. — *Let* $u_0 \in H^6(\mathbb{R}^2) \cap H^6(\mathbb{R}^2 : r^2\, dx\, dy)$. *Then*

$$(2.16) \qquad \sum_{\alpha,\, \beta = -\infty}^{\infty} \sup_{[0,\, T]} \sup_{Q_{\alpha,\, \beta}} | e^{it\,\Delta} u_0 | \leq c\,(1 + T^5) \{ \| u_0 \|_{H_{xy}^6} + \| u_0 \|_{H_{xy}^6\, (r^6)} \}.$$

Proof (see [22], Proposition 3.7).

Next we deduce some estimates concerning the second equation in (1.1). Thus after a change of variable we need to consider the problem

$$(2.17) \qquad \qquad \partial_{xy}^2 w(x, y) = \mathcal{K}\, w = F(x, y)$$

with $F \in L^1(\mathbb{R}^2)$ and $w(.,.)$ satisfying the radiation condition

$$\lim_{x \to +\infty} w(x, y) = \lim_{y \to +\infty} w(x, y) = 0.$$

Under the above hypotheses the equation (2.17) has a unique solution given by the formulae

$$(2.18) \qquad w(x, y) = \mathcal{K}^{-1} F = \int_{x}^{\infty} \int_{y}^{\infty} F(x', y')\, dy'\, dx'.$$

PROPOSITION 2.7. — *If* $F \in L^1(\mathbb{R}^2)$ *then* $\mathcal{K}^{-1} F \in C(\mathbb{R}^2)$ *and*

$$(2.19) \qquad \qquad \| \mathcal{K}^{-1} F \|_{L_x^\infty L_y^\infty} \leq c \| F \|_{L_x^1 L_y^1}.$$

In addition, if $F \in L_y^1(\mathbb{R} : L_x^2(\mathbb{R}))$ *then*

$$(2.20) \qquad \qquad \| \partial_x \mathcal{K}^{-1} F \|_{L_y^\infty L_x^2} \leq c \| F \|_{L_y^1 L_x^2}. \qquad —$$

We recall the notation

$$\| F \|_{L_y^p L_x^q} = \left(\int_{-\infty}^{\infty} \| F(., y) \|_q^p\, dy \right)^{1/p}.$$

As was commented in the introduction we observe that the estimates (2.3)-(2.7) and (2.20) use different L^p-norms for the x and y variables.

Proof. — (2.19) follows directly from (2.18). To obtain (2.20) from (2.18) one sees that

$$\partial_x \mathcal{K}^{-1} F(x, y) = - \int_y^\infty F(x, y')\, dy'.$$

Thus computing the L^2-norm in x, using Minkowski's integral inequality and taking supremum in the y-variable we obtain (2.20).

3. NONLINEAR ESTIMATES

In this section we shall obtain all the nonlinear estimates needed in the proofs of Theorems A, B. First we have the following inequalities concerning fractional derivatives.

THEOREM 3.1. — *Let* $s > 0$, $0 < \alpha < 1$, α_1, $\alpha_2 \in [0, \alpha]$ *with* $\alpha = \alpha_1 + \alpha_2$. *Let* $p, p' \in (1, \infty)$ *with* $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ *and* $q \in [1, \infty)$. *Then for any* $f, g \in \mathscr{S}(\mathbb{R}^n)$ *(Schwartz class)*

(3.1) $\| (I - \Delta)^{s/2} (fg) \|_{L^p (\mathbb{R}^n)}$

$\leq c \{ \| f \|_{L^\infty (\mathbb{R}^n)} \| (I - \Delta)^{s/2} g \|_{L^p (\mathbb{R}^n)} + \| g \|_{L^\infty (\mathbb{R}^n)} \| (I - \Delta)^{s/2} f \|_{L^p (\mathbb{R}^n)} \}$,

(3.2) $\displaystyle \sum_{|\beta| = k} \| \partial^\beta (fg) \|_{L^q (\mathbb{R}^n)}$

$\leq c \{ \| f \|_{L^\infty (\mathbb{R}^n)} \displaystyle\sum_{|\beta| = k} \| \partial^\beta g \|_{L^q (\mathbb{R}^n)} + \| g \|_{L^\infty (\mathbb{R}^n)} \displaystyle\sum_{|\beta| = k} \| \partial^\beta f \|_{L^q (\mathbb{R}^n)} \}$

and when $n = 1$ *(not essential)*

(3.3) $\| D^\alpha (fg) \|_{L^p} \leq c \{ \| g \|_{L^{p_1}} \| D^\alpha f \|_{L^{p_2}} + \| f \|_{L^\infty} \| D^\alpha g \|_{L^p} \}$

with $\dfrac{1}{p_1} + \dfrac{1}{p_2} = \dfrac{1}{p}$, $p_1, p_2 \in [p, \infty]$, *and*

(3.4) $\| D^\alpha (fg) - f D^\alpha g - g D^\alpha f \|_{L^1} \leq c \| D^{\alpha_1} f \|_{L^p} \| D^{\alpha_2} g \|_{L^{p'}}$.

Proof. — The estimate (3.1) was proven in [20] (Appendix). (3.2) follows by combining Galiardo-Nirenberg, Hölder and Young inequalities. Finally, for (3.3) and (3.4) we refer to Theorems A.12 and A.13 in [23] respectively. As was remarked in [20] and [23] the proof of (3.1), (3.3)-(3.4) relies on ideas of Coifman and Meyer ([7]-[8]).

The following estimates form the essential steps in the proof of Theorems A, B. They combine the linear results obtained in section 2 with the inequalities in Theorem 3.1. Propositions 3.2-3.5 are concerned with Theorem A, and Propositions 3.6-3.9 with Theorem B.

PROPOSITION 3.2. — *If $k = s - 1/2 \in \mathbb{Z}$ with $k \geq 3$ then*

$$(3.5) \quad \left\| \partial_x^{k+1} \int_0^t e^{i(t-t')\partial_{xy}^2} (c_1 |v|^2 v + c_2 v \mathscr{K}^{-1} \right.$$

$$\left. \times (\partial_x^2 + c_3 \partial_y^2) |v|^2)(t') \, dt' \right\|_{L_y^\infty L_T^2 L_x^2}$$

$$\leq c \, T \sup_{[0,T]} \| v(t) \|_{H_{xy}^3}^2 \sup_{[0,T]} \| v(t) \|_{H_{xy}^s}$$

$$+ c \| v \|_{L_y^1 L_T^\infty L_x^\infty}^2 \| \partial_x^{k+1} v \|_{L_y^\infty L_T^2 L_x^2} + \| \partial_x v \|_{L_y^1 L_T^\infty L_x^\infty}^2 \| \partial_x^k v \|_{L_y^\infty L_T^2 L_x^2}.$$

Proof. — To simplify the notation we assume (without loss of generality) that $c_1 = c_2 = c_3 = 1$. Thus

$$(3.6) \quad \left\| \partial_x^{k+1} \int_0^t e^{i(t-t')\partial_{xy}^2} (|v|^2 v + v \mathscr{K}^{-1} \right.$$

$$\left. \times (\partial_x^2 + \partial_y^2) |v|^2)(t') \, dt' \right\|_{L_y^\infty L_T^2 L_x^2}$$

$$\leq \left\| \partial_x^{k+1} \int_0^t e^{i(t-t')\partial_{xy}^2} (|v|^2 v)(t') \, dt' \right\|_{L_y^\infty L_T^2 L_x^2}$$

$$+ \left\| \partial_x^{k+1} \int_0^t e^{i(t-t')\partial_{xy}^2} (v \mathscr{K}^{-1} \partial_y^2 |v|^2)(t') \, dt' \right\|_{L_y^\infty L_T^2 L_x^2}$$

$$+ \left\| \partial_x^{k+1} \int_0^t e^{i(t-t')\partial_{xy}^2} (v \mathscr{K}^{-1} \partial_x^2 |v|^2)(t') \, dt' \right\|_{L_y^\infty L_T^2 L_x^2}$$

$$\equiv A_1 + A_2 + A_3.$$

Using the homogeneous version of the Kato smoothing effect in the group $\{ e^{it\partial_{xy}^2} \}_{-\infty}^\infty$ described in (2.3) together with Minkowski's integral inequality and the estimate in (3.1) it follows that

$$(3.7) \quad A_1 \leq c \int_0^T \| D_x^{k+1/2} (|v|^2 v) \|_{L_x^2 L_y^2} (t) \, dt$$

$$\leq c \int_0^T (\| v \|_{L_x^\infty L_y^\infty}^2 \| v \|_{H_{xy}^2})(t) \, dt$$

$$\leq c \, T \sup_{[0,T]} \| v(t) \|_{L_x^\infty L_y^\infty}^2 \sup_{[0,T]} \| v(t) \|_{H_{xy}^s}.$$

To bound A_2 we first observe that

$$\partial_x^k (v \mathscr{K}^{-1} \partial_y^2 |v|^2) = \partial_x^k v \, \mathscr{K}^{-1} \partial_y^2 |v|^2 + \sum_{j=0}^{k-1} m_j \partial_x^j v \, \partial_x^{k-j-1} \partial_y |v|^2$$

$$= \partial_x^k v \, \mathscr{K}^{-1} \partial_y^2 |v|^2 + B_1,$$

since $\partial_{xy}^2 \mathscr{K}^{-1} \equiv$ identity.

Hence (2.3) and Minkowski's integral inequality lead to

$$(3.8) \quad A_2 \leqq c \int_0^T \| D_x^{1/2} (\partial_x^k v \mathcal{K}^{-1} \partial_y^2 |v|^2) \|_{L_x^2 L_y^2} (t) \, dt$$

$$+ c \int_0^T \| D_x^{1/2} B_1 \|_{L_x^2 L_y^2} (t) \, dt.$$

The estimate for the second term on the right hand side of (3.8) is the same as that in (3.7). To handle the first term on the right hand side of (3.8) we use (3.3), (3.4), (2.19) and (3.2) to obtain

$$(3.9) \quad \int_0^T \| D_x^{1/2} (\partial_x^k v \mathcal{K}^{-1} \partial_y^2 |v|^2) \|_{L_x^2 L_y^2} (t) \, dt$$

$$\leqq c \int_0^T (\| \mathcal{K}^{-1} \partial_y^2 |v|^2 \|_{L_x^\infty L_y^\infty}$$
$$+ \| \mathcal{K}^{-1} D_x^{1/2} \partial_y^2 |v|^2 \|_{L_x^\infty L_y^\infty}) \| v \|_{H_{xy}^s} (t) \, dt$$

$$\leqq c \int_0^T (\| \partial_y^2 |v|^2 \|_{L_x^1 L_y^1} + \| D_x^{1/2} \partial_y^2 |v|^2 \|_{L_x^1 L_y^1}) \| v \|_{H_{xy}^s} (t) \, dt$$

$$\leqq c \int_0^T (\| v \|_{H_{xy}^3}^2 \| v \|_{H_{xy}^s}) (t) \, dt$$

$$\leqq c \, T \sup_{[0, T]} \| v(t) \|_{H_{xy}^3}^2 \sup_{[0, T]} \| v(t) \|_{H_{xy}^s}.$$

Finally we consider the term A_3 in (3.6). Since

$$(3.10) \quad \partial_x^k (v \mathcal{K}^{-1} \partial_x^2 |v|^2) = v \partial_x^k \mathcal{K}^{-1} \partial_x^2 |v|^2 + k \, \partial_x v \, \partial_x^{k-1} \mathcal{K}^{-1} \partial_x^2 |v|^2 + B_2,$$

where $B_2 = \sum_{j=2}^k \tilde{m}_j \, \partial_x^j v \, \partial_x^{k-j} \mathcal{K}^{-1} \partial_x^2 |v|^2.$

From (2.3) and (2.7) we have that

$$(3.11) \quad A_3 \leqq \left\| \partial_x \int_0^t e^{i(t-t') \partial_{xy}^2} (v \partial_x^k \mathcal{K}^{-1} \partial_x^2 |v|^2 \right.$$

$$+ k \, \partial_x v \, \partial_x^{k-1} \mathcal{K}^{-1} \partial_x^2 |v|^2) (t') \, dt' \bigg\|_{L_y^\infty L_T^2 L_x^2}$$

$$+ \left\| D_x^{1/2} \int_0^t e^{i(t-t') \partial_{xy}^2} D_x^{1/2} B_2 (t') \, dt' \right\|_{L_y^\infty L_T^2 L_x^2}$$

$$\leqq c \left\{ \| v \partial_x^k \mathcal{K}^{-1} \partial_x^2 |v|^2 \|_{L_y^1 L_T^2 L_x^2} \right.$$

$$+ \| \partial_x v \, \partial_x^{k-1} \mathcal{K}^{-1} \partial_x^2 |v|^2 \|_{L_y^1 L_T^2 L_x^2} + \int_0^T \| D_x^{1/2} B_2 (t) \|_{L_x^2 L_y^2} dt \bigg\}.$$

Combining Minkowski's integral inequality and the estimates (2.20), (3.2), it is not hard to obtain the following string of inequalities

$$(3.12) \quad \| v \, \partial_x^k \, \mathscr{K}^{-1} \, \partial_x^2 \, |v|^2 \|_{L_y^1 \, L_T^2 \, L_x^2}$$

$$\leqq c \, \| v \|_{L_y^1 \, L_T^\infty \, L_x^\infty} \| \partial_x^k \, \mathscr{K}^{-1} \, \partial_x^2 \, |v|^2 \|_{L_y^\infty \, L_T^2 \, L_x^2}$$

$$\leqq c \, \| v \|_{L_y^1 \, L_T^\infty \, L_x^\infty} \| \partial_x^{k+1} \, \partial_x \, \mathscr{K}^{-1} \, |v|^2 \|_{L_T^2 \, L_y^\infty \, L_x^2}$$

$$\leqq c \, \| v \|_{L_y^1 \, L_T^\infty \, L_x^\infty} \| \partial_x^{k+1} \, |v|^2 \|_{L_T^2 \, L_y^1 \, L_x^2}$$

$$\leqq c \, \| v \|_{L_y^1 \, L_T^\infty \, L_x^\infty} \| \partial_x^{k+1} \, |v|^2 \|_{L_y^1 \, L_T^2 \, L_x^2}$$

$$\leqq c \, \| v \|_{L_y^1 \, L_T^\infty \, L_x^\infty} \| \cdot \| v \|_{L_x^\infty} \| \partial_x^{k+1} \, v \|_{L_x^2} \|_{L_y^1 \, L_T^2}$$

$$\leqq c \, \| v \|_{L_y^1 \, L_T^\infty \, L_x^\infty}^2 \| \partial_x^{k+1} \, v \|_{L_y^\infty \, L_T^2 \, L_x^2}.$$

A similar argument shows that

$$(3.13) \quad \| \partial_x v \, \partial_x^{k-1} \, \mathscr{K}^{-1} \, \partial_x^2 \, |v|^2 \|_{L_y^1 \, L_T^2 \, L_x^2} \leqq c \, \| \partial_x v \|_{L_y^1 \, L_T^\infty \, L_x^\infty}^2 \| \partial_x^k v \|_{L_y^\infty \, L_T^2 \, L_x^2}.$$

On the other hand, using (3.3), (2.19), (3.4) and (3.2) we obtain for j fixed in B_2 that

$$\int_0^T \| D_x^{1/2} (\partial_x^j v \, \partial_x^{k-j} \, \mathscr{K}^{-1} \, \partial_x^2 \, |v|^2)(t) \|_{L_x^2 \, L_y^2} \, dt$$

$$\leqq c \int_0^T (\| D_x^{1/2} \, \partial_x^j v \|_{L_x^2 \, L_y^2} \| \partial_x^{k-j} \, \mathscr{K}^{-1} \, \partial_x^2 \, |v|^2 \|_{L_x^\infty \, L_y^\infty}$$

$$+ \| \partial_x^j v \|_{L_x^2 \, L_y^2} \| D_x^{1/2} \, \partial_x^{k-j} \, \mathscr{K}^{-1} \, \partial_x^2 \, |v|^2 \|_{L_x^\infty \, L_y^\infty}) \, dt$$

$$\leqq c \int_0^T (\| D_x^{1/2} \, \partial_x^j v \|_{L_x^2 \, L_y^2} \| \partial_x^{k-j+2} \, |v|^2 \|_{L_x^1 \, L_y^1}$$

$$+ \| \partial_x^j v \|_{L_x^2 \, L_y^2} \| D_x^{1/2} \, \partial_x^{k-j+2} \, |v|^2 \|_{L_x^1 \, L_y^1}) \, dt$$

$$\leqq c \int_0^T (\| v \|_{H_{xy}^3}^2 \| v \|_{H_{xy}^s})(t) \, dt$$

$$\leqq c \, T \sup_{[0, \, T]} \| v(t) \|_{H_{xy}^3}^2 \sup_{[0, \, T]} \| v(t) \|_{H_{xy}^s}.$$

Therefore

$$(3.14) \quad \int_0^T \| D_x^{1/2} B_2 \|_{L_x^2 \, L_y^2}^2 \, dt \leqq c \, T \sup_{[0, \, T]} \| v(t) \|_{H_{xy}^3}^2 \sup_{[0, \, T]} \| v(t) \|_{H_{xy}^s}.$$

Combining (3.10)-(3.14) one has a bound for A_3. By inserting this bound and those in (3.7)-(3.9) for A_1, A_2 respectively in (3.6) we obtain (3.5).

PROPOSITION 3.3. — *If* $k = s - 1/2 \in \mathbb{Z}$ *with* $k \geq 3$ *then*

$$(3.15) \quad \sup_{[0,T]} \left\| \int_0^t e^{i(t-t')\partial_{xy}^2} \left(c_1 |v|^2 v + c_2 v \mathcal{H}^{-1}(\partial_x^2 + c_3 \partial_y^2)|v|^2\right)(t')\,dt' \right\|_{H^s_{xy}}$$

$$\leq c\, T \sup_{[0,T]} \| v(t) \|_{H^3_{xy}}^2 \sup_{[0,T]} \| v(t) \|_{H^s_{xy}}$$

$$+ c \| v \|_{L^1_y\, L^p_T\, L^\infty_x}^2 \| \partial_x^{k+1} v \|_{L^\infty_y\, L^4_T\, L^2_x}$$

$$+ c \| \partial_x v \|_{L^1_y\, L^p_T\, L^\infty_x}^2 \| \partial_x^k v \|_{L^\infty_y\, L^4_T\, L^2_x}. \quad —$$

Proof. — The argument for highest derivatives is the same as that given in the previous proof where instead of (2.3) and (2.7) one uses the group properties and (2.4). The proof for the lowest derivatives is simpler and similar to that to be used in coming propositions, hence it will be omitted.

PROPOSITION 3.4:

$$(3.16) \quad \sup_{[0,T]} \left\| \int_0^t e^{i(t-t')\partial_{xy}^2} \left(c_1 |v|^2 v + c_2 v \mathcal{H}^{-1}\right.\right.$$

$$\left.\left. \times (\partial_x^2 + c_3 \partial_y^2)|v|^2\right)(t')\,dt' \right\|_{H^3_{xy}(r^2)}$$

$$\leq c\, T \sup_{[0,T]} \| v(t) \|_{H^3_{xy}(r^2)} \sup_{[0,T]} \| v(t) \|_{H^5_{xy}}^2$$

$$+ c\, T^2 \sup_{[0,T]} \| v(t) \|_{H^3_{xy}}^2 \sup_{[0,T]} \| v(t) \|_{H^6_{xy}}. \quad —$$

We recall the notation for the $\| \cdot \|_{H^k_{xy}(r^l)}$-norm:

$$\| f \|_{H^k_{xy}(r^l)} \equiv \sum_{|\alpha| \leq k} \left(\int_{\mathbb{R}^2} |\partial_x^{\alpha_1} \partial_y^{\alpha_2} f(x,y)|^2 r^l \, dx\, dy \right)^{1/2}$$

with $r = (x^2 + y^2)^{1/2}$.

Proof. — Combining Minkowski's integral inequality, the identity (*see* [17])

$$y e^{it\,\partial_{xy}^2} f = e^{it\,\partial_{xy}^2} yf - it e^{it\,\partial_{xy}^2} \partial_x f$$

and the one obtained by reversing the roles of x and y together with the group properties and the estimates (2.19)-(2.20) and (3.2) it is not hard to see that for $|\alpha| \leq 3$

$$\sup_{[0,T]} \left\| r \int_0^t e^{i(t-t')\partial_{xy}^2} \partial_x^{\alpha_1} \partial_y^{\alpha_2} \left(c_1 |v|^2 v + c_2 v \mathcal{H}^{-1}(\partial_x^2 + c_3 \partial_y^2)|v|^2\right)(t')\,dt' \right\|_{L^2_x\, L^2_y}$$

$$\leq \sup_{[0,T]} \int_0^T \left\| x e^{i(t-t')\partial_{xy}^2} \partial_x^{\alpha_1} \partial_y^{\alpha_2} (.)(t') \right\|_{L^2_x\, L^2_y} dt'$$

$$+ \sup_{[0,T]} \int_0^T \left\| y e^{i(t-t')\partial_{xy}^2} \partial_x^{\alpha_1} \partial_y^{\alpha_2} (.)(t') \right\|_{L^2_x\, L^2_y} dt'$$

$$\leq c \int_0^T \left\| x \, \partial_x^{\alpha_1} \partial_y^{\alpha_2} (\,.\,)(t') \right\|_{L_x^2 L_y^2} dt' + c \, T \int_0^T \left\| \partial^{\alpha_\pm} \partial_y^{\alpha_2 + 1} (\,.\,)(t') \right\|_{L_x^2 L_y^2} dt'$$

$$+ c \int_0^T \left\| y \, \partial_x^{\alpha_1} \partial_y^{\alpha_2} (\,.\,)(t') \right\|_{L_x^2 L_y^2} dt' + c \, T \int_0^T \left\| \partial_x^{\alpha_1 + 1} \partial_y^{\alpha_2} (\,.\,)(t') \right\|_{L_x^2 L_y^2} dt'$$

$$\leq c \, T \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^3 (r^2)} \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^5}^2$$

$$+ c \, T^2 \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^3}^2 \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^6}$$

which yields (3.16).

PROPOSITION 3.5.

$$(3.17) \quad \left\| \int_0^t e^{i(t-t') \partial_{xy}^2} \left(c_1 |v|^2 v + c_2 v \mathcal{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2 \right)(t') dt' \right\|_{L_y^1 L_T^\infty L_x^\infty}$$
$$\leq c \, T(1 + T^2) \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^2}^2 \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^6}$$
$$+ c \, T(1 + T) \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^3 (r^2)} \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^5}^2. \quad -$$

Proof. — Using Minkowski's integral inequality, (2.13), (3.1), (3.2) and Sobolev's theorem it follows that

$$(3.8) \quad \left\| \int_0^t e^{i(t-t') \partial_{xy}^2} \left(c_1 |v|^2 v + c_2 v \mathcal{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2 \right)(t') dt' \right\|_{L_y^1 L_T^\infty L_x^\infty}$$

$$\leq \int_0^T \left\| e^{i(t-t') \partial_{xy}^2} (\,.\,)(t') \right\|_{L_y^1 L_T^\infty L_x^\infty} dt'$$

$$= T \sup_{t' \in [0, T]} \left\| e^{i(t-t') \partial_{xy}^2} (\,.\,) \right\|_{L_y^1 L_T^\infty L_x^\infty}$$

$$\leq T \left\| e^{i(t-t') \partial_{xy}^2} (\,.\,) \right\|_{L_y^1 L_t^\infty L_T^\infty L_x^\infty}$$

$$\leq c \, T(1 + T^2) \left\| c_1 |v|^2 v + c_2 v \mathcal{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2 \right\|_{H_{xy}^4}$$
$$+ c \, T(1 + T) \left\| c_1 |v|^2 v + c_2 v \mathcal{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2 \right\|_{H_{xy}^3 (r^2)}$$

$$\leq c \, T(1 + T^2) \sup_{[0, T]} \left\| v(t) \right\|_{L_x^\infty L_y^\infty}^2 \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^4}$$
$$+ c \, T(1 + T^2) \left\| v \mathcal{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2 \right\|_{H_{xy}^4}$$
$$+ c \, T(1 + T) \sup_{[0, T]} \left\| rv \right\|_{L_x^2 L_y^2} \sup_{[0, T]} \left\| v(t) \right\|_{L_x^\infty L_y^\infty} \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^5}$$
$$+ c \, T(1 + T) \left\| v \mathcal{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2 \right\|_{H_{xy}^3 (r^2)}.$$

From (3.2) and (2.19) it is not hard to see that

$$(3.19) \quad \left\| v \mathcal{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2 \right\|_{H_{xy}^4} \leq c \sup_{[0, T]} \left\| v(t) \right\|_{L_x^2 L_y^2}^2 \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^6}$$

and

$$(3.20) \quad \left\| v \, \mathscr{K}^{-1} (\partial_x^2 + c_3 \, \partial_y^2) |v|^2 \right\|_{H_{xy}^3 (r^2)} \leq c \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^3 (r^2)} \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^5}^2.$$

Inserting (3.19)-(3.20) in (3.18) and using Sobolev's theorem we obtain the desired inequality (3.17).

PROPOSITION 3.6. − If $k = s - 1/2 \in \mathbb{Z}$ with $k \geq 3$ then

$$(3.21) \quad \sup_{\alpha, \, \beta \in \mathbb{Z}} \left(\int_0^T \int_{I_\beta} \int_{I_\alpha} \left/ \partial_x^{k+1} \int_0^t e^{i(t-t')\Delta} (c_1 |v|^2 v + c_2 \, v \, \mathscr{K}^{-1} \right. \right.$$

$$\left. \left. \times (\partial_x^2 + c_3 \, \partial_y^2) |v|^2) (t') \, dt' \right|^2 dx \, dy \, dt \right)^{1/2}$$

$$\leq c \, T \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^3}^2 \sup_{[0, T]} \left\| v(t) \right\|_{H_{xy}^s}$$

$$+ c \sum_{k \leq |\gamma| \leq k+1} \sup_\alpha \sup_\beta \left(\int_0^T \int_{I_\beta} \int_{I_\alpha} | \partial_{x, \, y}^\gamma v |^2 \, dx \, dy \, dt \right)^{1/2}$$

$$\times \sum_{|l| \leq 1} \left(\sum_{\alpha, \, \beta = -\infty}^{\infty} \sup_{[0, T]} \sup_{Q_{\alpha, \beta}} | \partial_{x, \, y}^l v | \right)^2$$

$$+ c \, T^{1/2} \sum_{|l| \leq 1} \left(\sum_{\alpha, \, \beta = -\infty}^{\infty} \sup_{[0, T]} \sup_{Q_{\alpha, \beta}} | \partial_{x, \, y}^l v | \right)^3 \equiv D. \quad -$$

We recall the notation:

$$Q_{\alpha, \, \beta} = I_\alpha \times I_\beta = [\alpha, \, \alpha + 1] \times [\beta, \, \beta + 1]$$

with $\alpha, \, \beta \in \mathbb{Z}$.

Proof. − Without loss of generality we can assume $c_1 = c_2 = 1$. Thus

$$(3.22) \quad \sup_{\alpha, \, \beta \in \mathbb{Z}} \left(\int_0^T \int_{I_\beta} \int_{I_\alpha} \left/ \partial_x^{k+1} \int_0^t e^{i(t-t')\Delta} (|v|^2 v + v \, \mathscr{K}^{-1} \right. \right.$$

$$\left. \left. \times (\partial_x^2 + c_3 \, \partial_y^2) |v|^2) (t') \, dt' \right|^2 dx \, dx \, dy \right)^{1/2}$$

$$\leq c \sup_{\alpha, \, \beta \in \mathbb{Z}} \left(\int_0^T \int_{I_\beta} \int_{I_\alpha} \left| \partial_x^{k+1} \right. \right.$$

$$\left. \left. \times \int_0^t e^{i(t-t')\Delta} (|v|^2 v) (t') \, dt' \right|^2 dx \, dy \, dt \right)^{1/2}$$

$$+ c \sup_{\alpha, \, \beta \in \mathbb{Z}} \left(\int_0^T \int_{I_\beta} \int_{I_\alpha} \left| \partial_x^{k+1} \right. \right.$$

$$\left. \left. \times \int_0^t e^{i(t-t')\Delta} (v \, \mathscr{K}^{-1} \, \partial_x^2 |v|^2) (t') \, dt' \right|^2 dx \, dy \, dt \right)^{1/2}$$

$$+ c \sup_{\alpha,\, \beta \in \mathbb{Z}} \left(\int_0^T \int_{I_\beta} \int_{I_\alpha} \left| \partial_x^{k+1} \right. \right.$$

$$\left. \left. \times \int_0^t e^{i(t-t')\Delta} (v \,\mathcal{H}^{-1} \partial_y^2 |v|^2)(t')\, dt' \right|^2 dx\, dy\, dt \right)^{1/2}$$

$$\equiv \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3.$$

From the homogeneous version of the Kato smoothing effect in $\{e^{it\Delta}\}_{-\infty}^{\infty}$ (2.10), the Minkowski's integral inequality and the estimate (3.1) it follows that

$$(3.23) \quad \tilde{A}_1 \leq c \int_0^T \left\| D_x^{k+1/2} (|v|^2 v) \right\|_{L_x^2 L_y^2}(t)\, dt$$

$$\leq c \int_0^T \left(\|v\|_{L_x^\infty L_y^\infty}^2 \|v\|_{H_{xy}^s} \right)(t)\, dt$$

$$\leq c\, T \sup_{[0,\,T]} \|v(t)\|_{L_x^\infty L_y^\infty}^2 \sup_{[0,\,T]} \|v(t)\|_{H_{xy}^s}.$$

To bound $\tilde{A}_2$ we notice that

$$(3.24) \quad \partial_x^k (v \,\mathcal{H}^{-1} \partial_x^2 |v|^2) = v \,\partial_x^k \,\mathcal{H}^{-1} \partial_x^2 |v|^2 + k \,\partial_x v \,\partial_x^{k-1} \,\mathcal{H}^{-1} \partial_x^2 |v|^2 + \tilde{B}_1,$$

Using an argument similar to that given in (3.23) [based in estimate (2.10)] together with the inequality (2.19) one easily sees that

$$(3.25) \quad \sup_{\alpha,\, \beta \in \mathbb{Z}} \left(\int_0^T \int_{I_\alpha} \int_{I_\beta} \left| D_x^{1/2} \int_0^t e^{i(t-t')\Delta} D_x^{1/2} H_x \tilde{B}_1 (t')\, dt' \right|^2 dx\, dy\, dt \right)^{1/2}$$

$$\leq c\, T \sup_{[0,\,T]} \|v(t)\|_{H_{xy}^3}^2 \sup_{[0,\,T]} \|v(t)\|_{H_{xy}^s},$$

where H_x denotes the Hilbert transform in the x-variable.

To handle the first term on the right hand side of (3.24) we use (2.12) to see that

$$(3.26) \quad \sup_{\alpha,\, \beta \in \mathbb{Z}} \left(\int_0^T \int_{I_\alpha} \int_{I_\beta} \left| \partial_x \int_0^t e^{i(t-t')\Delta} \right. \right.$$

$$\left. \left. \times (v \,\partial_x^k \,\mathcal{H}^{-1} \partial_x^2 |v|^2)(t')\, dt' \right|^2 dx\, dy\, dt \right)^{1/2}$$

$$\leq c \sum_{\alpha,\, \beta = -\infty}^{\infty} \left(\int_0^T \int_{I_\alpha} \int_{I_\beta} |(v \,\partial_x^k \,\mathcal{H}^{-1} \partial_x^2 |v|^2)(t)|^2 \, dx\, dy\, dt \right)^{1/2}$$

$$\leq c \left(\sum_{\alpha,\, \beta = -\infty}^{\infty} \sup_{[0,\,T]} \sup_{x \in I_\alpha} \sup_{y \in I_\beta} |v| \right)$$

$$\times \sup_\alpha \sup_\beta \left(\int_0^T \int_{I_\beta} \int_{I_\alpha} |\mathcal{H}^{-1} \partial_x^{k+2} |v|^2|^2 \, dx\, dy\, dt \right)^{1/2}.$$

Combining the argument used in the proof of (2.20) with the corresponding version of (3.2) for bounded domains it follows that

$$(3.27) \quad \left(\int_{I_\alpha} \int_{I_\beta} \int_0^T | \mathscr{K}^{-1} \partial_x \partial_x^{k+1} |v|^2 |^2 \, dt \, dx \, dy \right)^{1/2}$$

$$\leq \sup_{y \in I_\beta} \left(\int_{I_\alpha} \int_0^T | \partial_x \mathscr{K}^{-1} \partial_x^{k+1} |v|^2 |^2 \, dt \, dx \right)^{1/2}$$

$$\leq \sup_{y \in \mathbb{R}} \left(\int_{I_\alpha} \int_0^T \left(\int_y^\infty | \partial_x^{k+1} |v|^2 | \, dy' \right)^2 dt \, dx \right)^{1/2}$$

$$\leq \int_{-\infty}^\infty \left(\int_{I_\alpha} \int_0^T | \partial_x^{k+1} |v|^2 |^2 \, dt \, dx \right)^{1/2} dy$$

$$\leq \sum_{\beta' = -\infty}^\infty \left(\int_0^T \int_{I_{\beta'}} \int_{I_\alpha} | \partial_x^{k+1} |v|^2 |^2 \, dx \, dy \, dt \right)^{1/2}$$

$$= c \sum_{\beta' = -\infty}^\infty \left(\int_0^T \int_{I_{\beta'}} \sup_{x \in I_\alpha} |v|^2 \left(\int_{I_\alpha} |v|^2 \, dx + \int_{I_\alpha} | \partial_x^{k+1} v |^2 \, dx \right) dy \, dt \right)^{1/2}$$

$$\leq c \sum_{\alpha, \beta = -\infty}^\infty \sup_{[0, T]} \sup_{y \in I_{\beta'}} \sup_{x \in I_\alpha} |v| \left\{ \int_0^T \int_{I_{\beta'}} \int_{I_\alpha} |v|^2 \, dx \, dy \, dt \right.$$

$$\left. + \int_0^T \int_{I_{\beta'}} \int_{I_\alpha} | \partial_x^{k+1} v |^2 \, dx \, dy \, dt \right\}^{1/2}$$

$$\leq c \, T^{1/2} \left(\sum_{\alpha, \beta = -\infty}^\infty \sup_{[0, T]} \sup_{y \in I_\beta} \sup_{x \in I_\alpha} |v| \right)^2$$

$$+ c \sup_\alpha \sup_\beta \left(\int_0^T \int_{I_\beta} \int_{I_\alpha} | \partial_x^{k+1} v |^2 \, dx \, dy \, dt \right)^{1/2}$$

$$\times \left(\sum_{\alpha, \beta = -\infty}^\infty \sup_{[0, T]} \sup_{y \in I_\beta} \sup_{x \in I_\alpha} |v| \right).$$

Inserting (3.27) in (3.26) we obtain the bound for the first term on the right hand side of (3.24). The proof for the second term follows the same argument.

Finally, to bound $\tilde{A}_3$, we notice that

$$\partial_x^k (v \mathscr{K}^{-1} \partial_y^2 |v|^2) = \partial_x^k v \, \mathscr{K}^{-1} \partial_y^2 |v|^2 + \sum_{j=0}^{k-1} m_j \partial_x^j v \, \partial_x^{k-j-1} \partial_y |v|^2$$

$$= \partial_x^k v \, \mathscr{K}^{-1} \partial_y^2 |v|^2 + \tilde{B}_2$$

since $\partial_{xy}^2 \mathscr{K}^{-1} \equiv$ identity.

Then the smoothing effect (2.10) together with similar arguments as those in (3.23) and (3.25) yield

$$(3.28) \quad \tilde{A}_3 \leqq \int_0^T (\| D_x^{1/2} \partial_x^k (v \mathscr{K}^{-1} \partial_y^2 |v|^2) \|_{L_x^2 L_y^2} + \| D_x^{1/2} \tilde{B}_2 \|_{L_x^2 L_y^2})(t) \, dt$$

$$\leqq c \, T \sup_{[0, T]} \| v(t) \|_{H_{xy}^3}^2 \sup_{[0, T]} \| v(t) \|_{H_{xy}^s}.$$

Collecting all these bounds we get (3.21).

PROPOSITION 3.7. — *If $k = s - 1/2 \in \mathbb{Z}$ with $k \geqq 3$ then*

$$(3.29) \quad \sup_{[0, T]} \left\| \int_0^t e^{i(t-t')\Delta} (c_1 |v|^2 v + c_2 v \mathscr{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2)(t') \, dt' \right\|_{H_{xy}^s} \leqq D,$$

where D was defined in (3.21).

Proof. — The part of the proof of (3.29) involving the highest derivatives is similar to that used to obtain (3.21) where instead of (2.10) and (2.12) one needs the group properties and (2.11). The argument for the lowest derivatives is a straight application of the group properties.

PROPOSITION 3.8:

$$(3.30) \quad \sum_{\alpha, \beta = -\infty}^{\infty} \sup_{[0, T]} \sup_{Q_{\alpha, \beta}}$$

$$\times \left| \int_0^t e^{i(t-t')\Delta} (c_1 |v|^2 v + c_2 v \mathscr{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2))(t') \, dt' \right|$$

$$\leqq c \, T (1 + T^5) \left\{ (1 + T^3) \sup_{[0, T]} \| v(t) \|_{H_{xy}^{11}}^3 \right.$$

$$\left. + \sup_{[0, T]} \| v(t) \|_{H_{xy}^6 (r^6)} \sup_{[0, T]} \| v(t) \|_{H_{xy}^{11}}^2 \right\}$$

where $Q_{\alpha, \beta} = I_\alpha \times I_\beta = [\alpha, \alpha + 1] \times [\beta, \beta + 1]$. —

Proof. — From Minkowski's integral inequality and (2.16) it follows that

$$(3.31) \quad \sum_{\alpha, \beta = -\infty}^{\infty} \sup_{[0, T]} \sup_{Q_{\alpha, \beta}}$$

$$\times \left| \int_0^t e^{i(t-t')\Delta} (c_1 |v|^2 v + c_2 v \mathscr{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2)(t') \, dt' \right|$$

$$\leqq c (1 + T^5) \left\{ \int_0^T \| |v|^2 v \|_{H_{xy}^6} \, dt \right.$$

$$\left. + \int_0^T \| v \mathscr{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2 \|_{H_{xy}^6} \, dt \right.$$

$$+ \int_0^T \left\| e^{-it'\Delta} |v|^2 v \right\|_{H^6_{xy}(r^6)} dt'$$

$$+ \int_0^T \left\| e^{-it'\Delta} v \mathcal{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2 \right\|_{H^6_{xy}(r^6)} dt' \Bigg\}$$

$$\equiv M_1 + M_2 + M_3 + M_4.$$

From (3.2) one has that

$$(3.32) \qquad M_1 \leq c\, T\, (1+T^5) \sup_{[0,T]} \|v\|^2_{L^\infty_x L^\infty_y} \sup_{[0,T]} \|v(t)\|_{H^6_{xy}}$$

and

$$(3.33) \quad M_3 \leq c\, T\, (1+T^5) \Big\{ \sup_{[0,T]} \|v\|_{H^6_{xy}(r^6)} \sup_{[0,T]} \|v(t)\|^2_{H^{11}_{xy}}$$

$$+ (1+T^3) \sup_{[0,T]} \|v(t)\|^2_{H^6_{xy}} \sup_{[0,T]} |v(t)|_{H^{11}_{xy}} \Big\}.$$

Similarly from (2.19) and (3.2) it is easy to find that

$$(3.34) \qquad M_2 \leq c\, T\, (1+T^5) \sup_{[0,T]} \|v\|^4_{H^4_{xy}} \sup_{[0,T]} \|v(t)\|_{H^8_{xy}}$$

and

$$(3.35) \quad M_4 \leq c\, T\, (1+T^5) \Big\{ \sup_{[0,T]} \|v\|_{H^6_{xy}(r^6)} \sup_{[0,T]} \|v(t)\|^2_{H^{11}_{xy}}$$

$$+ (1+T^3) \sup_{[0,T]} \|v(t)\|^2_{H^6_{xy}} \sup_{[0,T]} \|v(t)\|_{H^{11}_{xy}} \Big\}.$$

Inserting (3.32)-(3.35) in (3.31) and then using Sobolev's theorem we obtain (3.30).

PROPOSITION 3.9:

$$(3.36) \quad \sup_{[0,T]} \left\| \int_0^t e^{i(t-t')\Delta} (c_1 |v|^2 v + c_2 v \mathcal{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |v|^2)(t')\, dt' \right\|_{H^6_{xy}(r^6)}$$

$$\leq c\, T\, (1+T^3) \Big\{ \sup_{[0,T]} \|v(t)\|_{H^6_{xy}(r^6)}$$

$$\times \sup_{[0,T]} \|v(t)\|^2_{H^{21}_{xy}} + \sup_{[0,T]} \|v(t)\|^3_{H^{11}_{xy}} \Big\}. \quad -$$

Proof. — The proof is similar to that provided in detail for (3.16) (Proposition 3.4). Hence it will be omitted.

4. PROOF OF THEOREM A

To simplify our exposition we fix s such that $s-1/2=k\in\mathbb{Z}$. By hypothesis $k \geq 6$.

For $v \in L^\infty([0, T] : H^s(\mathbb{R}^2)$ define

(4.1) $\lambda_1^T(v) = \| \partial_x^{k+1} v \|_{L_y^\infty L_T^2 L_x^2} + \| \partial_x^k v \|_{L_y^\infty L_T^2 L_x^2}$
$$+ \| \partial_y^{k+1} v \|_{L_x^\infty L_T^2 L_y^2} + \| \partial_y^k v \|_{L_x^\infty L_T^2 L_y^2},$$

(4.2) $$\lambda_2^T(v) = \sup_{[0, T]} \| v(t) \|_{H_{xy}^s},$$

(4.3) $$\lambda_3^T(v) = \sup_{[0, T]} \| v(t) \|_{H_{xy}^3 (r^2)},$$

(4.4) $\lambda_4^T(v) = \| v \|_{L_y^1 L_T^\infty L_x^\infty} + \| \partial_x v \|_{L_y^1 L_T^\infty L_x^\infty} + \| v \|_{L_x^1 L_T^\infty L_y^\infty} + \| \partial_y v \|_{L_x^1 L_T^\infty L_y^\infty},$

(4.5) $$\Lambda_T(v) = \max_{j=1,\ldots,4} \lambda_j^T(v)$$

and

$$X_T = \{ v \in L^\infty([0, T] : H^s(\mathbb{R}^2) \cap H^3(\mathbb{R}^2 : r^2 \, dx \, dy)) / \Lambda_T(v) < \infty \}.$$

For $u_0 \in H^s(\mathbb{R}^2) \cap H^3(\mathbb{R}^2 : r^2 \, dx \, dy)$ we denote by $\Phi_{u_0}(v) = u$ the solution of the linear IVP

(4.6) $$\begin{cases} i \, \partial_t u - \partial_{xy}^2 u = c_1 |v|^2 v + c_2 v \mathcal{K}^{-1}(\partial_x^2 + c_3 \partial_y^2) |v|^2 \\ u(x, y, 0) = u_0(x, y) \end{cases}$$

where $v \in X_T^a = \{ v \in X_T / \Lambda_T(v) \leqq a \}$.

We shall prove that there exists $\delta > 0$ such that if

$$\delta_0 = \| u_0 \|_{H_{xy}^s} + \| u_0 \|_{H_{xy}^3 (r^2)} < \delta$$

then there exist $T > 0$ and $a > 0$ [with $T = T(\delta_0) \to \infty$ as $\delta_0 \to 0$] such that if $v \in X_T^a$ then $u = \Phi(v) \in X_T^a$ and

$$\Phi: \quad X_T^a \to X_T^a$$

is a contraction. For this purpose we rely on the integral equation version of (4.6) to write

(4.7) $\Phi_{u_0}(v) = e^{it \partial_{xy}^2} u_0$
$$+ \int_0^t e^{i(t-t') \partial_{xy}^2} (c_1 |v|^2 v + c_2 v \mathcal{K}^{-1}(\partial_x^2 + c_3 \partial_y^2) |v|^2)(t') \, dt'$$

Thus combining (2.3), (3.5) and Sobolev's theorem we find that

(4.8) $\lambda_1^T(\Phi_{u_0}(v)) \leqq \delta_0 + c \, T (\lambda_2^T(v))^3 + c \, \lambda_1^T(v) (\lambda_4^T(v))^2.$

Similarly, from the group properties and (3.15) it follows that

(4.9) $\lambda_2^T(\Phi_{u_0}(v)) \leqq \delta_0 + c \, T (\lambda_2^T(v))^3 + c \, \lambda_1^T(v) (\lambda_4^T(v))^2,$

from (3.16)

(4.10) $\lambda_3^T(\Phi_{u_0}(v)) \leqq \delta_0 + c \, T (\lambda_2^T(v))^2 \lambda_3^T(v) + c \, T^2 (\lambda_2^T(v))^3$

and from (2.13), (3.17)

(4.11) $\lambda_4^T(\Phi_{u_0}(v)) \leqq c (1 + T^2) \delta_0 + c \, T (1 + T^2) \{ (\lambda_2^T(v))^3 + (\lambda_2^T(v))^2 \lambda_3^T(v) \}.$

Thus (4.8)-(4.11) yield the inequality

(4.12) $\Lambda_T (\Phi_{u_0} (v)) \leqq c (1+T^2) \delta_0 + c (1+T^2) (\Lambda_T (v))^3.$

Choosing $a = 2 c (1+T^2) \delta_0$ with T satisfying

(4.13) $8 c^3 (1+T^2)^3 \delta_0^2 \leqq 1/2.$

The same argument shows that

(4.14) $\Lambda_T (\Phi_{u_0} (v) - \Phi_{u_0} (\tilde{v})) \leqq c (1+T^2) \{ (\Lambda_T (v))^2 + (\Lambda_T (\tilde{v}))^2 \} \Lambda_T (v - \tilde{v}),$

and that for $T_0 \in (0, T)$

(4.15) $\Lambda_{T_0} (\Phi_{u_0} (v) - \Phi_{\tilde{u}_0} (\tilde{v})) \leqq c (1+T_0^2) ||| u_0 - \tilde{u}_0 |||$
$$+ c (1+T_0^2) \{ (\Lambda_{T_0} (v))^2 + (\Lambda_{T_0} (\tilde{v}))^2 \} \Lambda_{T_0} (v - \tilde{v})$$

when $||| u_0 - \tilde{u}_0 ||| = || u_0 - \tilde{u}_0 ||_{H^s_{xy}} + || u_0 - \tilde{u}_0 ||_{H^3_{xy} (r^2)}$ is sufficiently small.

Therefore $\Phi_{u_0} (X_T^a) \subset X_T^a$ for a and T as above and $\Phi_{u_0} |_{X_T^a}$ is a contraction. Thus there exists a unique $u \in X_T^a$ such that $\Phi_{u_0} (u) = u$, i.e.

(4.16) $u (t) = e^{it \partial_{xy}^2} u_0$
$$+ \int_0^t e^{i (t-t') \partial_{xy}^2} (c_1 |u|^2 u + c_2 u \mathscr{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) |u|^2) (t') dt'.$$

Inserting the argument used for (4.8) in (4.16) we obtain that

$$\lambda_1^T (u) \leqq c \lambda_1^T (e^{it \partial_{xy}^2} u_0) + c T (\lambda_2^T (u))^3 + c \lambda_1^T (u) (\lambda_4^T (u))^2.$$

Since $\lambda_1^T (e^{it \partial_{xy}^2} u_0) = o (1)$ as $T \to 0$ by (4.13) it follows that $\lambda_1^T (u) - o (1)$ as $T \to 0$.

Combining this result with the arguments in (4.9)-(4.10) and the integral equation (4.16) we conclude that

$$u \in C ([0, T] : H^s (\mathbb{R}^2) \cap H^3 (\mathbb{R}^2 : r^2 dx dy)).$$

Now using the continuity properties is not hard to extend the uniqueness result to the class $X_T \cap C ([0, T] : H^s (\mathbb{R}^2) \cap H^3 (\mathbb{R}^2 : r^2 dx dy))$ (see [22]). This observation completes the proof of Theorem A.

5. PROOF OF THEOREM B

As in Theorem A we fix s satisfying $s - 1/2 = k \in \mathbb{Z}$ with $k \geqq 12$. It will be clear from our proof below that this does not represent a loss of generality.

For $v \in L^\infty ([0, T] : H^s (\mathbb{R}^2))$ define

(5.1) $\omega_1^T = \sup\limits_{k \leqq |\gamma| \leqq k+1} \sup\limits_{\alpha, \beta \in \mathbb{Z}} \left(\int_0^T \int_{I_\beta} \int_{I_\alpha} | \partial_{x, y}^\gamma v (x, y, t) |^2 dx dy dt \right)^{1/2},$

$$(5.2) \qquad \omega_2^T (v) = \sup_{[0,\,T]} \| v(t) \|_{H_{xy}^s},$$

$$(5.3) \qquad \omega_3^T (v) = \sup_{[0,\,T]} \| v(t) \|_{H_{xy}^6 (r^6)},$$

$$(5.4) \qquad \omega_4^T (v) = \sum_{|\gamma| \leq 1} \sum_{\alpha,\,\beta = -\infty}^{\infty} \sup_{[0,\,T]} \sup_{\alpha,\,\beta \,\in\, \mathbb{Z}} | \partial_{x,\,y}^\gamma v(x,\,y,\,t) |,$$

$$(5.5) \qquad \Omega_T (v) = \max_{j=1,\ldots,4} \omega_j^T (v)$$

and

$$Z_T = \{ v \in L^\infty ([0,\,T] : H^s (\mathbb{R}^2) \cap H^6 (\mathbb{R}^2 : r^6 \, dx \, dy)) / \Omega_T (v) < \infty \}.$$

For $u_0 \in H^s (\mathbb{R}^2) \cap H^6 (\mathbb{R}^2) : r^6 \, dx \, dy)$ we denote by $\Psi_{u_0} (v) = u$ the solution of the IVP

$$(5.6) \qquad \begin{cases} i \partial_t u - \Delta u = c_1 | v |^2 v + c_2 v \mathcal{K}^{-1} (\partial_x^2 + \partial_y^2) | v |^2 \\ u(x,\,y,\,0) = u_0 (x,\,y) \end{cases}$$

where $v \in Z_T^a = \{ v \in Z_T / \Omega_T (v) \leq a \}$.

It will be established that there exists $\delta > 0$ such that if

$$\delta_0 = \| u_0 \|_{H_{xy}^s} + \| u_0 \|_{H_{xy}^6 (r^6)} < \delta$$

then there exist $t > 0$ and $a > 0$ [with $T (\delta_0) \to \infty$ as $\delta_0 \to 0$] such that if $v \in Z_T^a$ then $u = \Psi_{u_0} (v) \in Z_T^a$ and

$$\Psi : \quad Z_T^a \to Z_T^a$$

is a contraction. As in the proof of Theorem A we rely on the integral form p (5.6)

$$(5.7) \quad \Psi_{u_0} (v) = e^{it\Delta} u_0$$
$$+ \int_0^t e^{i(t-t')\Delta} (c_1 | v |^2 v + c_2 v \mathcal{K}^{-1} (\partial_x^2 + c_3 \partial_y^2) | v |^2) (t') \, dt'$$

from (2.10) and (3.21) it follows that

$$(5.8) \quad \omega_1^T (\Psi_{u_0} (v)) \leq c \, \delta_0 + c \, T (\omega_2^T (v))^3 + c \, \omega_1^T (v) (\omega_4^T (v))^4 + c \, T^{1/2} (\omega_4^T (v))^3.$$

Similarly, from the group properties and (3.29)

$$(5.9) \quad \omega_1^T (\Psi_{u_0} (v)) \leq c \, \delta_0 + c \, T (\omega_2^T (v))^3 + c \, \omega_1^T (v) (\omega_4^T (v))^4 + c \, T^{1/2} (\omega_4^T (v))^3,$$

from (3.6)

$$(5.10) \quad \omega_3^T (\Psi_{u_0} (v)) \leq c \, \delta_0 + c \, T (1 + T^3) \{ (\omega_2^T (v))^2 \, \omega_3^T (v) + (\omega_2^T (v))^3 \}$$

and from (2.16) and (3.30)

$$(5.11) \quad \omega_4^T (\Psi_{u_0} (v)) \leq c (1 + T^5) \delta_0 + c \, T (1 + T^5)$$
$$\times \{ (1 + T^3) (\omega_2^T (v))^3 + (\omega_2^T (v))^2 \, \omega_3^T (v) \}.$$

Collecting the information in (5.8)-(5.11) and using the notation in (5.5) one finds that

$$(5.12) \qquad \Omega_T(\Psi_{u_0}(v)) \leqq c(1+T^5)\delta_0 + c(1+T^8)(\Omega_T(v))^3.$$

Once that the estimate (5.12) has been established the rest of the proof of Theorem B follows an argument similar to that used for Theorem A. Hence it will be omitted.

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