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Nonlinear symmetric positive systems

by

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ABSTRACT. — We study the existence and uniqueness of solutions as well as their continuous dependence on given data for the boundary value problem for a general nonlinear symmetric positive system by a Nash-Hörmander iteration scheme. Results on quasilinear systems and the existence of smooth solutions which improve known results on this subject are also presented.

Key words : Nonlinear symmetric positive systems, Nash-Hörmander iteration, singular perturbation.

RÉSUMÉ. — A l'aide de la méthode itérative de Nash-Hörmander, nous étudions un système nonlinéaire positif symétrique et un problème de perturbation singulière. Nous obtenons des résultats d'existence, d'unicité et de dépendance continue par rapport aux données.

In this paper we shall study the nonlinear symmetric positive system

$$F^i(x, u, \nabla u) = f^i, \qquad i = 1, \ldots, N \tag{0.1}$$

where $u(x) = (u^1(x), \ldots, u^N(x))$ is defined in a domain Ω in $\mathbb{R}^n$, $\nabla u = \left(\frac{\partial u^1}{\partial x_1}, \ldots, \frac{\partial u^1}{\partial x_n}, \ldots, \frac{\partial u^N}{\partial x_1}, \ldots, \frac{\partial u^N}{\partial x_n}\right)$, and $F^i(x, z, p)$ is a function

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defined in $\bar{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{Nn}$. For simplicity we shall assume the boundary of Ω is smooth. By a nonlinear symmetric positive system we mean the linearised system of (0.1) at a fixed u

$$A^j(x;\,u)\frac{\partial v}{\partial x_j}+B(x;\,u)\,v=g, \tag{0.2}$$

where A^j, $j=1,\ldots,n$ and B are respectively the $N\times N$ matrices $\frac{\partial F^i}{\partial p_{kj}}(x,\,u,\,\nabla u)$ *and* $\frac{\partial F^i}{\partial z_k}(x,\,u,\,\nabla u)$ is a linear symmetric positive system. Suppose that $F^i(x,\,0,\,0)=0$. One is asked to solve (0.1) for small solutions satisfying certain homogeneous boundary conditions when f is small. We shall study the existence, uniqueness, and continuous dependence on f of the solutions in a rather detailed way. Besides, it is intended to obtain results under the weakest differentiability condition on f. (0.1) was studied by Moser [11] in the periodic case. He proved that when f is continuously differentiable up to l-th order where $l>\max\left(\frac{3n}{2}+6,\,15\right)$ and is uniformly small, then there exists a C^2-solution of (0.1). Although we are concerned with boundary conditions, our results clearly apply to periodic case. We'll show that for $l>\frac{n}{2}+2$ and f is small in the Sobolev space H^1, there exists a solution of (0.1) which is small in $C^2(\bar{\Omega})$.

Before going further, let's review the linear theory of symmetric positive systems which was introduced by K. O. Friedrich in 1958 [1] as a unified treatment for equations of different types and was studied by many authors. See, for instance, [1], [2], [7], [8], [14], and [15]. A first order system on a domain Ω

$$Lu=A^j\frac{\partial u}{\partial x_j}+Bu=f \tag{0.3}$$

is called a *symmetric positive system* (SPS) if the $N\times N$ matrices $A^j(x)$, $j=1,\ldots,n$ are symmetric and

$$\left(\left(B(x)-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial A^j}{\partial x_j}(x)\right)\xi,\,\xi\right)\geqq b\,|\,\xi\,|^2,\qquad \xi\in R^n \tag{0.4}$$

for some $b>0$. If we denote the characteristic matrix $\sum_{j=1}^{n}A^j(x)\,\nu_j(x)$ ($\nu(x)$ is the unit outer normal) by $\beta(x)$, a subspace $N(x)$ defined on $\partial\Omega$, the boundary of Ω, of R^N is called *admissible* to (0.3) if $(\xi,\,\beta(x)\,\xi)\geqq 0$ for all $\xi\in N(x)$ and it is a maximal subspace w.r.t. this property. The boundary value problem of SPS is: To find a solution u of (0.3) such that $u(x)$

belongs to a given admissible $N(x)$. When $\partial\Omega$ is smooth and β is nonsingular, the basic results on the well-posedness of the boundary value problem can be summarised as

(*a*) Given $f \in L^2(\Omega; R^N)$, there exists a unique strong solution u of the boundary value problem. By a strong solution we mean (u, f) lies in the L^2-closure of the graph of (v, Lv) where $v \in C^1(\Omega; R^N)$ and $v(x) \in N(x)$.

(*b*) If $f \in H^l(\Omega; R^N)$, then $u \in H^l(\Omega; R^N)$ provided b in (0.4) is sufficiently large (depending on l and the derivatives of the coefficients). One has

$$\|u\|_l \leqq C_l \|f\|_l. \tag{0.5}$$

Return to the nonlinear problem. Let $\Phi(u) = F(x, u, \nabla u)$. Since $\Phi(0) = 0$ and we are looking for solutions for small f, a first attempt would be try to use the classical implicit function theorem. Let's denote $H^l_N(\Omega; R^N)$ the subspace of $H^l(\Omega; R^N)$ consisting of those satisfy $u(x) \in N(x)$. In the following we'll drop R^N in $H^l(\Omega; R^N)$ when the context is clear. By Moser's inequality, Φ maps $H^l_N(\Omega)$ to $H^{l-1}(\Omega)$, $l > \frac{n}{2} + 1$. Consider the linearised system (0.2) at $u = 0$. If $\beta(x; 0)$ is nonsingular, and $N(x)$ is admissible, from (*a*) and (*b*) one knows that (0.2) is uniquely solvable. However, the inverse map $g \mapsto v$ is in general not bounded in view of (0.5). If we set up the Picard iteration as we did in the proof of implicit function theorem, in each step we lose one derivative and the iteration would terminate after finitely many steps. Thus, a direct application of implicit function theorem doesn't work. A second attempt would be try to reduce the system to a quasi-linear SPS for u and its derivatives by differentiating (0.1). However, because of the presence of the boundary, we don't know how to carry this out. In this paper we shall use a Nash-Hörmander iteration scheme to solve (0.1). This far-reaching generalization of the classical implicit function theorem was developed by Nash [12], Moser [10], [11], Hörmander [5], and others. We refer to the survey article by Hamilton [4] for its other applications. For solving (0.1), a simple scheme due to Moser [10] works as well. However, it doesn't give the optimal result and doesn't yield smooth solutions (*see* Section 5).

We shall use the Nash-Hörmander scheme to construct a sequence of approximate solutions begining with $u_0 = 0$. In doing so it involves solving (0.2) for u near to 0. Since from (0.4) we see that the positivity of (0.2) involves the second derivatives of u and we don't have any relevant *a priori* bounds, in view of (0.5) and Sobolev's inequality we shall require f at least belongs to $H^l(\Omega)$, $l > \frac{n}{2} + 2$.

We denote $\bar{m}$ the smallest integer so that $H^{\bar{m}}(\Omega)$ can be continuously embedded in $C(\bar{\Omega})$. We shall impose the following assmptions:

(A1) $F(x, z, p)$ is continuously differentiable in

$$D=\{(x, z, p)\in\bar{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}: |z|, |p|\leqq 1\}$$

up to $(\bar{m}+5)$-th order.

(A2) $A^j(x; u)=\dfrac{\partial F^i}{\partial p_{kj}}(x, u, \nabla u)$

are symmetric $(j=1, \ldots, n)$ for $|u|_{C^1(\bar{\Omega})}\leqq 1$.

(A3) $\beta(x; u)=A^j(x; u)\nu_j(x)$

is nonsingular for all $u\in C^1(\bar{\Omega})$, $|u|_{C^1(\bar{\Omega})}\leqq 1$.

(A4) $N(x)$ is a smoothly varying subspace of R^N for $x\in\partial\Omega$. It is admissible to $\beta(x; u)$, $|u|_{C^1(\bar{\Omega})}\leqq 1$.

(A3) and (A4) together imply that the dimension of $N(x)$ is equal to the number of positive eigenvalues of $\beta(x; 0)$. Consequently it is constant on each component of the boundary.

(A5) The b in (0.4) is large depending on $|F|_{C^{\bar{m}+5}(D)}$.

MAIN THEOREM. — *(a) Existence. Suppose* $F(x, 0, 0)=0$ *in* Ω *and (A1)-(A5) hold. There exists* $\rho>0$ *such that for any* f *with* $\|f\|_{\bar{m}+2}\leqq\rho$ (0.1) *has a solution u which belongs to* $H_N^{\bar{m}+2-\varepsilon}(\Omega)$, *for all small* $\varepsilon>0$.

(b) Uniqueness. For any given μ *in* $\left(\dfrac{n}{2}+2, \bar{m}+2\right)$, *there corresponds* $r>0$ *such that the solution is unique in* $\{u\in H_N^{\mu}(\Omega): \|u\|_{\mu}\leqq r\}$. *Denote this solution by* $u=u(f)$.

(c) Regularity. Suppose for $l\geqq 0$,

(A5)' F *is in* $C^{\bar{m}+5+2l}(D)$ *and b is sufficiently large depending on* l *and* $|F|_{C^{\bar{m}+5+2l}(D)}$,

holds. Then $u(f)$ *belongs to* $H_N^{\bar{m}+2+l-\varepsilon}(\Omega)$ *for any* $\varepsilon>0$ *when* f *is in* $H^{\bar{m}+2+l}(\Omega)$. *In fact, we have* $\|u(f)\|_{\bar{m}+2+l-\varepsilon}\leqq C_\varepsilon\|f\|_{\bar{m}+2+l}$ *for some constant* C_ε.

(d) Continuous dependence on f. *Moreover, suppose that*

(A5)'' F *is in* $C^{\bar{m}+6+2l}(D)$ *and b is sufficiently large depending on* l *and* $|F|_{C^{\bar{m}+6+2l}(D)}$,

hold. Then for any γ *in* $\left(\dfrac{n}{2}+1, \bar{m}+1\right)$ *there exists* $\rho_1>0$ *such that for* $\|f_1-f_2\|_\gamma\leqq\rho_1$,

$$\|u(f_1)-u(f_2)\|_{\gamma+1-\varepsilon}\leqq C_{\varepsilon,1}\|f_1-f_2\|_{\gamma+1}$$

for all small $\varepsilon>0$.

This paper is arranged as follows. In Section 1 we formulate the Nash-Hörmander iteration scheme [5] where certain changes are necessary for our application. Besides, we prove a general uniqueness result and an estimate on the dependence of the given data. The latter was not treated in [5]. Section 2 consists of a brief review of the linear theory of SPS's. Various points are clarified in order to give a better result in nonlinear theory. The *a priori* estimate derived in Section 2 will then be applied in Section 3 to give an existence theorem for quasilinear systems which improves a previous result of Gu [3]. We shall finish the proof of the main theorem in Section 4 where some related results are presented. Section 5 is devoted to a proof of the existence of smooth solutions for a special class of SPS's. Finally, in Section 6 we give some further comments. In particular, a recent result of Rabinowitz [13] on a singular perturbation problem is discussed. In Appendix A we shall describe a very simple method of constructing smooth operators which preserve homogeneous boundary conditions.

1

Let $E_a (a \geqq 0)$ be an ascending chain of Banach spaces satisfying $\| u \|_a \leqq \| u \|_b$ if $a \leqq b$. It is said to *admit a smoothing operator* if there exists a family of linear operators $S_\theta : E_0 \to E_\infty = \bigcap E_a$, $a \geqq 0$ for $\theta \geqq \theta_0$ such that

$$\| u - S_\theta u \|_r \leqq C \theta^{-s+r} \| u \|_s, \qquad s \geqq r \tag{1.1}$$

and

$$\| S_\theta u \|_s \leqq C \theta^{(s-r)^+} \| u \|_r \tag{1.2}$$

where the constants C are independent of θ and u. From (1.1) and (1.2) one can deduce that the norms of E_a satisfy for $b \geqq a$, $0 \leqq \lambda \leqq 1$,

$$\| u \|_c \leqq C \| u \|_a^\lambda \| u \|_b^{1-\lambda}, \qquad c = \lambda a + (1-\lambda) b. \tag{1.3}$$

Consider two chains of Banach spaces E_a and F_a which admit smoothing operators S_θ and T_θ respectively. Let $u_0 \in E_\infty$ and N be an $E_\alpha - n'd$ of u_0 for some α. We consider a map $\Phi : N \cap E_\infty \to F_0$. Assume that there is associated with Φ another map $\Phi' : N \cap E_\infty \times E_\infty \to F_0$ such that

$$\begin{aligned} \| \Phi'(u) w - \Phi'(v) w \|_s \leqq C_s [& \| u - v \|_{a_1} \| w \|_{s+b_1} \\ & + \| u - v \|_{s+b_1} \| w \|_{a_1} + \| u - v \|_{a_1} \| w \|_{a_1} (1 + \| u \|_{s+b_2} + \| v \|_{s+b_2})] \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} & \| \Phi(u) - \Phi(v) - \Phi'(v)(u-v) \|_s \\ & \leqq C_s [\| u - v \|_{a_1} \| u - v \|_{s+b_1} + \| u - v \|_{a_1}^2 (1 + \| u \|_{s+b_2} + \| v \|_{s+b_2})] \end{aligned} \tag{1.5}$$

hold for $u, v \in N \cap E_\infty$ and $w \in E_\infty$. Here a_1, b_1, and b_2 are nonnegative numbers. In the following theorems we shall require (1.4) and (1.5) hold for s in a certain range. In practise Φ' is actually the (Fréchet) derivative of Φ. In case Φ is (Fréchet) twice differentiable and its second derivative Φ'' satisfies

$$\|\Phi''(u)(v, w)\|_s \leqq C_s[\|v\|_{a_1}\|w\|_{s+b_1} + \|v\|_{s+b_1}\|w\|_{a_1} + \|v\|_{a_1}\|w\|_{a_1}(1+\|u\|_{s+b_2})]$$

(1.4) and (1.5) can be easily deduced from this estimate by Taylor's formula.

We also assume that $\Phi'(u)$ has a right inverse, that is, a map $\psi: N \cap E_\infty \times F_\infty \to E_\infty$ such that

$$\Phi'(u)\Psi(u)g = g. \tag{1.6}$$

We require that it further satisfies

$$\|\Psi(u)g\|_s \leqq C_s(\|g\|_{s+b_3} + \|g\|_{a_2}\|u\|_{s+b_4}) \tag{1.7}$$

for some non-negative a_2, b_3, and b_4. The range of validity of (1.7) will be specified below.

Under this formulation, given $\beta > \alpha$ and f in F_∞ which is small in $F_{\beta+b_3}$, we shall construct a sequence of approximate solutions $\{u_k\}$ to the equation $\Phi(u) = \Phi(u_0) + f$ in the sense that $u_k \in E_\infty$, u_k tends to u in E_β, for any $\beta' < \beta$ and $\Phi(u_k)$ tends to $\Phi(u_0) + f$ in $F_{\beta+b_3}$ as k tends to ∞. This will be accomplished by a Nash-Hömander scheme.

Setting $\theta_k = 2^{k+K}$, $k \geqq 0$, where K is a large number to be specified and letting

$$\Delta_k = \theta_{k+1} - \theta_k = \theta_k, \tag{1.8}$$

we define $\{u_k\}$ as

$$\left.\begin{aligned}
u_{k+1} &= u_k + \Delta_k w_k, \qquad k \geqq 0 \\
w_k &= \psi(v_k) g_k \qquad \text{where} \quad v_k = S_{\theta_k} u_k \qquad \text{and} \\
g_k &= \Delta_k^{-1}[(T_{\theta_k} - T_{\theta_{k-1}})(f - A_{k-1}) - \Delta_{k-1} T_{\theta_k} e_{k-1}], \\
g_0 &= \Delta_0^{-1} T_{\theta_0} f; \\
e_k &= e'_k + e''_k, \\
e'_k &= \Delta_k^{-1}(\Phi'(u_k) - \Phi'(v_k))\Delta_k w_k, \\
e''_k &= \Delta_k^{-1}(\Phi(u_{k+1}) - \Phi(u_k) - \Phi'(u_k)\Delta_k w_k), \\
A_k &= \sum_0^{k-1} e_j \Delta_j, \qquad A_0 = 0.
\end{aligned}\right\} \tag{1.9}$$

THEOREM 1.1. — *Let:*

(*a*) $a_2 \geqq (a_1 - b_1)^+$ *and* $a_1 \geqq a_2 - b_3$;

(*b*) $\beta > \alpha$ *be a positive number satisfying*

$$\beta > a_1 + b_2 - b_1, \tag{1.10}$$

$$\beta > \max\left\{ b_3 + a_1 + b_1,\ \frac{1}{2}(a_1 + a_2 + b_1 + b_4),\ a_2 + b_1,\ a_2 - b_3 + b_4 \right\}; \quad (1.11)$$

(*c*) (1.4), (1.5) *and* (1.7) *hold for* $0 \leqq s \leqq s^*$ *where* $s^* > 2\beta - a_1$.

Then there exists K_0 *such that for each* $K \geqq K_0$, *there is* $\rho > 0$ *so that for* f *in* F_∞ *with* $\| f \|_{\beta + b_3} \leqq \rho$, *the sequence* $\{u_k\}$ *defined in* (1.9) *with* $\theta_0 = 2^K$ *satisfies*

(i) u_k *belongs to* $N \cap E_\infty$;

(ii) *For any* $\beta' < \beta$, u_k *tends to some* u *in* $E_{\beta'}$, *as* k *goes to* ∞,

(iii) $\| u - u_0 \|_{\beta'} \leqq C \| f \|_{\beta + b_3}$, *and*

(iv) $\Phi(u_k)$ *tends to* $\Phi(u_0) + f$ *in* $F_{\beta' + b_3}$ *as* k *goes to* ∞.

In case Φ can be extended to a continuous map from N to F_0, then u is a solution of $\Phi(u) = \Phi(u_0) + f$. Such iteration scheme was used originally by J. Nash in this study of isometric embedding of riemannian manifolds. Its present form is due to Hörmander [5] where the reader is referred to for a detailed discussion. Theorem 1.1 is somehow a simplified version of Theorem 2.2.2 in [5]. We modify it in such a way that it applies to Sobolev spaces. The main change is due to the fact that we don't have a characterisation of Sobolev space as in Theorem A.11 in [5]. The result is an "infinite loss of derivative" for the solution. Compare Lemma 1.2 with Theorem A.11 in [5]. Also we point out that due to our choice of θ_k the condition (iv) in Theorem A.10 in [5] which involves the derivative of $S_\theta u$ in θ is not needed. Such choice was used in [6].

We begin the proof of Theorem 1.1 with

LEMMA 1.2. — *Suppose for some* $\delta > 0$ *and* $0 \leqq j \leqq k$,

$$\| w_j \|_s \leqq \delta \theta_j^{s - \beta - 1}, \qquad a_1 \leqq s \leqq s^*. \quad (1.12)$$

Letting $U_k = \sum_0^k \Delta_j w_j$, *then for* $\beta'' < \beta$ *we have*

$$\| U_k \|_s \leqq C \delta \theta_{k+1}^{(s - \beta'')^+}, \quad (1.13)$$

and

$$\| U_k - S_{\theta_{k+1}} U_k \|_s \leqq C \delta \theta_{k+1}^{s - \beta''} \quad (1.14)$$

for $0 \leqq s \leqq s^*$. *Here constants* C *only depend on* β, β'', *and the smoothing operators*

Proof. — Clearly (1.13) follows from (1.12) for $s \geqq \beta$ or $s \leqq \beta''$. By (1.3) it holds for $\beta'' < s < \beta$. Similarly, when $s \geqq \beta$

$$\| U_k - S_{\theta_{k+1}} U_k \|_s \leqq \| U_k \|_s + C \| U_k \|_s \leqq C \delta \theta_{k+1}^{s - \beta''}$$

and when $s \leqq \beta''$ the same estimate follows from (1.2). Again by (1.3) (1.14) holds for $\beta' < s < \beta$.

Q.E.D.

Since $u_{j+1} = u_0 + U_j$, we immediately deduce from this lemma that

$$\| u_{j+1} \|_s \leqq C \theta_{j+1}^{(s-\beta'')^+}, \tag{1.15}$$
$$\| u_{j+1} - v_{j+1} \|_s \leqq C \theta_{j+1}^{s-\beta''} \tag{1.16}$$

for $0 \leqq s \leqq s^*$. Here constants C also depend on $\| u_0 \|_s^*$.

Fix β'', $\beta < \beta$, such that it is greater than α and (1.10) and (1.11) holds for β'' in place of β. We shall use induction to show that (1.12) holds for all k with δ being a constant multiple of $\| f \|_{\beta + b_3}$. Let's suppose that (1.12) has been established for k and we are going to prove it for $k+1$. First we observe that for sufficiently small $\| f \|_{\beta + b_3}$ u_{k+1} belongs to N in view of $(1.13)_\alpha$. Hence for sufficiently large K v_{k+1} belongs to N and w_{k+1} is well-defined. We estimate the quantities involved in the definition of w_{k+1} as follows:

CLAIM:

$$\| e_j \|_s \leqq C \delta \theta_j^{L(s)-1}, \qquad j \leqq k, \quad (a_1 - b_1)^+ \leqq s \leqq \tilde{s},$$

where $L(s) = s + a_1 - \beta'' + b_1 - \beta$ and $\tilde{s} = s^* - \max\{b_1, b_2\}$.

For, using (c) and then (1.12), (1.15) and (1.16), we have

$$\| e_k' \|_s = \Delta_k^{-1} \| (\Phi'(u_k) - \Phi'(v_k)) \Delta_k w_k \|_s$$
$$\leqq C \delta \theta_k^{L(s)-1}$$

where $L(s) = \max$

$$\{ a_1 - \beta'' + s + b_1 - \beta,\ a_1 - \beta'' + a_1 - \beta,\ a_1 - \beta'' + a_1 - \beta + (s + b_2 - \beta'')^+ \}.$$

By (1.10) $L(s) = a_1 - \beta'' + s + b_1 - \beta$. Similarly, using (1.5) instead of (1.4) we have the same estimate on e_k''. Here our claim is established.

As a consequence we have

$$\| A_k \|_{\tilde{s}} \leqq C \delta \sum_0^{k-1} \theta_j^{L(\tilde{s})}$$
$$\leqq c \delta \theta_k^{L(\tilde{s})}$$

since by the choice of s^* $L(\tilde{s}) > 0$. We claim:

$$\| g_{k+1} \|_s \leqq C (\delta \theta_{k+1}^{L(s)-1} + \theta_{k+1}^{s-\beta-b_3-1} \| f \|_{\beta + b_3}), \qquad s \geqq (a_1 - b_1)^+. \tag{1.17}$$

For, we have

$$\| \Delta_{k+1}^{-1} (T_{\theta_{k+1}} - T_{\theta_k}) A_k \|_s \leqq \Delta_{k+1}^{-1} \| (T_{\theta_{k+1}} - I) A_k \|_s + \Delta_{k+1}^{-1} \| (T_{\theta_k} - I) A_k \|_s$$
$$\leqq C \delta \theta_{k+1}^{L(s)-1}, \qquad 0 \leqq s \leqq \tilde{s},$$

$$\| \Delta_{k+1}^{-1} (T_{\theta_{k+1}} - T_{\theta_k}) A_k \|_s \leqq \Delta_{k+1}^{-1} (\| T_{\theta_{k+1}} A_k \|_s + \| T_{\theta_k} A_k \|_s$$
$$\leqq C \delta \theta_{k+1}^{L(s)-1}, \qquad s \geqq \tilde{s};$$

$$\| \Delta_{k+1}^{-1} \Delta_k T_{\theta_{k+1}} e_k \|_s \leqq C \delta \theta_{k+1}^{L(s)-1}, \qquad s \geqq (a_1 - b_1)^+,$$

and

$$\| \Delta_{k+1}^{-1} (T_{\theta_{k+1}} - T_{\theta_k}) f \|_s \leqq C \theta_{k+1}^{s-\beta-b_3-1} \| f \|_{\beta+b_3}, \qquad s \geqq 0.$$

Combining these estimates (1.17) follows. By (1.7), (1.17), (1.15), (a) and (1.11) we have

$$\| w_{k+1} \|_s \leqq C_1 (\delta \theta_{k+1}^{-\varepsilon+s-\beta-1} + \theta_{k+1}^{s-\beta-1} \| f \|_{\beta+b_3}), \qquad a_1 \leqq s \leqq s^*, \tag{1.18}$$

for some $\varepsilon > 0$. On the other hand, by (1.7) we have

$$\| w_0 \|_s \leqq C_2 \theta_0^{s-\beta-1} \| f \|_{\beta+b_3}, \qquad a_1 \leqq s \leqq s^*$$

where C_2 depends on $\theta_0 = 2^K$. Therefore for those K satisfying $C_1 2^{-\varepsilon K} \leqq 2^{-1}$, $(1.12)_0$ and $(1.12)_{k+1}$ follow after we set $\delta = \max\{2C_1, C_2\} \| f \|_{\beta+b_3}$. By induction we conclude that it holds for all $k \geqq 0$.

Now Theorem 1.1 follows easily. (i) has already been proved above. To prove (ii) notice

$$\| u_k - u_l \|_{a_1} = \left\| \sum_{l+1}^{k} \Delta_j w_j \right\|_{a_1} \leqq \delta \sum_{l+1}^{k} \theta_j^{a_1-\beta}.$$

As $a_1 < \beta$, $\{u_k\}$ is a Cauchy sequence in E_{a_1}. For any $\beta' < \beta$ which satisfies (1.11) when β is replaced by β', take $\beta'' = \frac{1}{2}(\beta + \beta')$ in Lemma 1.2. (1.15) shows that $\{u_k\}$ is bounded in $E_{\beta''}$. Using (1.3) we conclude that it is a Cauchy sequence in $E_{\beta'}$. Letting k go to ∞ in $(1.15)_\beta$, we obtain (iii). Finally to prove (iv) we write

$$\Phi(u_{k+1}) - \Phi(u_0) = T_{\theta_k} f - (T_{\theta_k} - I) A_k + \Delta_k e_k.$$

Hence

$$\begin{aligned} \| \Phi(u_{k+1}) - \Phi(u_0) - f \|_{\beta'+b_3} & \\ \leqq \| (T_{\theta_k} - I) f \|_{\beta'+b_3} &+ \| (T_{\theta_k} - I) A_k \|_{\beta'+b_3} + \| \Delta_k e_k \|_{\beta'+b_3} \\ \leqq C (\theta_k^{\beta'-\beta} \| f \|_{\beta+b_3} &+ \delta \theta_k^{\beta'-\beta''} \sum^{k} \theta_j^{L(\beta''+b_3)} + \delta \theta_k^{L(\beta'+b_3)})). \end{aligned}$$

Since $L(\beta'' + b_3) < 0$, $\| \Phi(u_{k+1}) - \Phi(u_0) - f \|_{\beta'+b_3}$ tends to zero as k goes to ∞. The proof of the theorem is completed.

Remark 1.3. — An examination of the above proof shows that (i)-(iv) still hold if the assumptions are relaxed to

(*a*) The smoothing operators S_θ and T_θ map E_0 and F_0 to E_{m_1} and F_{m_2}, $m_1 \geqq s^* + b_4$, $m_2 \geqq s^* + b_3$ respectively. Note that $N \cap E_\infty$ appearing in (i) should be replaced by $N \cap E_{m_1}$.

(*b*) For u_0 in E_s^* and f in $F_{\beta+b_3}$, $\Phi: N \cap E_s^* \to F_0$ and $\Phi': N \cap E_s^* \times E_s^* \to F_0$. (1.4) and (1.5) hold for $(a_1 - b_1)^+ \leqq s \leqq \tilde{s}$. Moreover, $\Psi: N \cap E_s^* \times F_{m_2} \to E_s^*$ satisfying (1.6) and (1.7) in $a_1 \leqq s \leqq s^*$.

Remark 1.4. – The solution constructed in the above proof actually enjoys a regularity property, namely, $u \in E_{\beta'+\beta_1}$ if $f \in F_{\beta+\beta_1}$ for $\beta_1 \geqq 0$ on the condition that (1.4), (1.5) hold in $a_1 \leqq s \leqq \tilde{s}+2\beta_1$ and (1.7) hold in $a_1 \leqq s \leqq s^*+2\beta_1$. To see this observe that if (1.7) holds in $a_1 \leqq s \leqq s^*+\varepsilon$, instead of obtaining (1.18) we have

$$\| w_{k+1} \|_s \leqq C_1 (\delta + \| f \|_{\beta+\varepsilon+b_3}) \theta_{k+1}^{s-(\beta+\varepsilon)-1} \tag{1.19}$$

for s in $[a_1, s^*+\varepsilon]$. Since we have shown that (1.12) holds for all $k \geqq 0$, (1.19) holds for all $k \geqq 0$. By a further restriction on ε we may assume that $L\varepsilon$ is equal to β_1 for some natural number L. Since we also have

$$\| w_0 \|_s \leqq C_2' \| f \|_{\beta+\varepsilon+b_3} \theta_0^{s-(\beta+\varepsilon)-1}$$

in the same interval, we conclude that

$$\| w_j \|_s \leqq \delta_1 \theta_j^{s-(\beta+\varepsilon)-1} \tag{1.12)'$$

holds for all $j \geqq 0$ in $[a_1, s^*+\varepsilon]$ where

$$\delta_1 = \max \{ C_1 (\delta + \|f\|_{\beta+\varepsilon+b_3}), C_2' \|f\|_{\beta+\varepsilon+b_3} \}.$$

Using (1.12)′ instead of (1.12), we can follow the same line (replacing β, β'' and s^* by $\beta+\varepsilon$, $\beta''+\varepsilon$, and $s^*+2\varepsilon$) leading to (1.19) to obtain

$$\| w_{k+1} \|_s \leqq C_1' (\delta_1 + \|f\|_{\beta+2\varepsilon+b_3}) \theta_{k+1}^{s-(\beta+2\varepsilon)-1} \tag{1.19)'$$

in $[a_1, s^*+2\varepsilon]$. Again we can choose δ_2 such that $\| w_j \|_s \leqq \delta_2 \theta_j^{s-(\beta+2\varepsilon)-1}$ in $[a_1, s^*+2\varepsilon]$. Repeating this argument finitely many times we conclude $\| w_j \|_s \leqq \delta_L \theta_j^{s-(\beta+\beta_1)-1}$ where δ_L is a constant multiple of $\| f \|_{\beta+\beta_1+b_3}$ in $[a_1, s^*+2\beta_1]$ for all $j \geqq 0$. In view of Lemma 1.2 $\{u_k\}$ and u belong to $E_{\beta'+\beta_1}$. Notice that we also have $\| u \|_{\beta'+\beta_1} \leqq C \| f \|_{\beta+\beta_1+b_3}$ for some constant C depending on β' and β_1.

Next, we consider an operator Φ depending on a parameter. We suppose that for small ε, $\Phi(u, \varepsilon)$ satisfies all assumptions in the formulation of Theorem 1.1. In particular, (1.4), (1.5) and (1.7) should hold uniformly in ε, $|\varepsilon| \leqq \varepsilon_0$ for some ε_0. Then by Theorem 1.1, for $\beta' < \beta$, and f_i, $i=1, 2$, which are small in $F_{\beta+b_3}$, the Nash-Hörmander scheme $\{u_k^i\}$, $i=1, 2$, beginning with the same u_0, converges to u^i in E_β, provided K is sufficiently large. In the following we want to estimate u^1-u^2 in terms of $\varepsilon_1-\varepsilon_2$ and f_1-f_2. We need some further assumptions, namely, the inverse $\Psi(u, \varepsilon)$ in (1.6) is also a left inverse, *i.e.*, $\Psi(u, \varepsilon)\Phi'(u, \varepsilon)g=g$, $g \in F_\infty$ and it satisfies

$$\| (\Phi'(u, \varepsilon_1) - \Phi'(u, \varepsilon_2)) w \|_s \leqq C |\varepsilon_1 - \varepsilon_2| [\| w \|_{s+b_1} + \| w \|_{a_1} (1 + \| u \|_{s+b_2})] \tag{1.20}$$

and

$$\begin{aligned} &\| (\Phi(v^1, \varepsilon_1) - \Phi(u^1, \varepsilon_1) - \Phi'(u^1, \varepsilon_1) w^1) \\ &\qquad - (\Phi(v^2, \varepsilon_2) - \Phi(u^2, \varepsilon_2) - (\Phi'(u^2, \varepsilon_2) w^2) \|_s \\ &\qquad \leqq C \{ \| w^1 - w^2 \|_{s+b_1} (\| w^1 \|_{a_1} + \| w^2 \|_{a_1}) \end{aligned}$$

$$+ \|w^1 - w^2\|_{a_1} (\|w^1\|_{s+b_1} + \|w^2\|_{s+b_1}) + \|w^1 - w^2\|_{a_1}$$
$$(\|w^1\|_{a_1} + \|w^2\|_{a_1})(1 + \|u^1\|_{s+b_2} + \|w^1\|_{s+b_2} + \|u^2\|_{s+b_2} + \|w^2\|_{s+b_2})$$
$$+ |\varepsilon_1 - \varepsilon_2| [\|w^1\|_{a_1} \|w^2\|_{s+b_1} + \|w^1\|_{s+b_1} \|w^2\|_{a_1}$$
$$+ \|w^1\|_{a_1} \|w^2\|_{a_1} (1 + \|u^1\|_{s+b_2} + \|w^1\|_{s+b_2})]$$
$$+ (\|u^1 - u^2\|_{a_1} + \|w^1 - w^2\|_{a_1})(\|w^1\|_{a_1} \|w^2\|_{s+b_1} + \|w^1\|_{s+b_1} \|w^2\|_{a_1})$$
$$+ (\|u^1 - u^2\|_{s+b_1} + \|w^1 - w^2\|_{s+b_1}) \|w^1\|_{a_1} \|w^2\|_{a_1}$$
$$+ \|u^1 - u^2\|_{a_1} \|w^1\|_{a_1} \|w^2\|_{a_1} (1 + \|u^1\|_{s+b_2}$$
$$+ \|w^1\|_{s+b_2} + \|u^2\|_{s+b_2} + \|w^2\|_{s+b_2})\} \quad (1.20)'$$

where $w^i = v^i - u^i$ $(i=1, 2)$ in a certain range of s. For the applications in this paper Φ is thrice continuously differentiable and satisfies estimates of the following form

$$\left\| \frac{\partial \Phi'}{\partial \varepsilon}(u, \varepsilon) w \right\|_s \leqq C_s [\|w\|_{s+b_1} + \|w\|_{a_1} (1 + \|u\|_{s+b_2})],$$

$$\|\Phi''(u, \varepsilon)(v, w)\|_s, \left\| \frac{\partial \Phi''}{\partial \varepsilon}(u, \varepsilon)(v, w) \right\|_s$$
$$\leqq C_s [\|v\|_{a_1} \|w\|_{s+b_1}$$
$$+ \|v\|_{s+b_1} \|w\|_{a_1} + \|v\|_{a_1} \|w\|_{a_1} (1 + \|u\|_{s+b_2})],$$

as well as

$$\|\Phi^{(3)}(u, \varepsilon)(v, w, z)\|_s$$
$$\leqq C_s [\|v\|_{s+b_1} \|u\|_{a_1} \|z\|_{a_1} + \|u\|_{a_1} \|v\|_{s+b_1} \|z\|_{a_1}$$
$$+ \|u\|_{a_1} \|v\|_{a_1} \|z\|_{s+b_1} + \|u\|_{a_1} \|v\|_{a_1} \|z\|_{a_1} (1 + \|u\|_{s+b_2})].$$

It is not hard to see that (1.4), (1.5), (1.20) and (1.20)′ are consequences of these estimates and Taylor's formula.

In the following we recall that $\beta'' < \beta$ has been fixed in the beginning of the proof of Theorem 1.1.

THEOREM 1.5. – *Let γ be a positive number less than β''. In addition to the hypotheses in Theorem* 1.1 *we assume:*

(*a*) $\beta'' > a_1 + b_4$, $a_2 + b_2$, *and* $\gamma > a_1$.

(*b*) *The function* W *defined in* (1.24) (*replacing γ' by γ*) *satisfies*

$$W(s + b_3) < s - \gamma,$$

and

$$W(a_2) + (s + b_4 - \beta'')^+ < s - \gamma$$

in $a \leqq s \leqq s^*$;

(*c*) (1.26)-(1.29) *hold* (*in* (1.26) *and* (1.27) γ' *appearing on the left hand side of the inequalities should be replaced by* γ) *in* $a_1 \leqq s \leqq s^*$;

(*d*) *Validity of* (1.4) *in* $a_1 \leqq s \leqq s^* + b_3$, (1.7) *in* $a_1 \leqq s \leqq s^* + b_1 + b_3$, (1.20) *in* $(a_1 - b_1)^+ \leqq s \leqq s^* + b_3$, *and* (1.21) *in* $(a_1 - b_1)^+ \leqq s \leqq \tilde{s}$ *for* ε, $|\varepsilon| \leqq \varepsilon_0$;

(*e*) $W(\tilde{s}) > 0$. (*Recall that* $\tilde{s} = s^* - \max\{b_1, b_2\}$.)

Then there exists $\rho_1>0$ *such that for* f_i, $i=1, 2$, *in* F_∞ *with* $\|f_i\|_{\beta+b_3}\leqq\rho$ *and* $|\varepsilon_1-\varepsilon_2|+\|f_1-f_2\|_{\gamma+b_3}\leqq\rho_1$, *we have, for any* $\gamma'<\gamma$,

$$\|u'-u^2\|_{\gamma'}\leqq C(|\varepsilon_1-\varepsilon_2|+\|f_1-f_2\|_{\gamma+b_3}). \tag{1.21}$$

Proof: The proof of this theorem is similar to that of Theorem 1.1. For a given $\gamma'<\gamma$ which satisfies $\gamma'>a_1$, $W(s+b_3)<s-\gamma$, $W(a_2)+(s+b_4-\beta'')^+<s-\gamma$, (1.26) and (1.27), we shall establish the estimate

$$\|w_j^1-w_j^2\|_s\leqq\delta_1\,\theta_j^{s-\gamma-1},\qquad a_1\leqq s\leqq s^*, \tag{1.22}$$

for all $j\geqq 0$ where δ_1 will be chosen as a constant multiple of $|\varepsilon_1-\varepsilon_2|+\|f_1-f_2\|_{\gamma+b_3}$. In case (1.22) has been proved up to k, as before we deduce

$$\|u_{j+1}^1-u_{j+1}^2\|_s\leqq C\,\delta_1\,\theta_j^{(s-\gamma')^+},\qquad 0\leqq s\leqq s^*, \tag{1.23}$$

for $0\leqq j\leqq k$. Taking $s=\gamma'$ and then letting k go to infinity, as $\{u_{k+1}^1\}$ and $\{u_{k+2}^2\}$ tend to u^1 and u^2 respectively in E_γ, we see that the theorem follows. As before (1.21) will be established by induction. Hence assuming $(1.22)_j$, $j\leqq k$, are valid we are going to establish it for $k+1$.

We estimate the difference in errors as follows: Write

$$\begin{aligned} e_k^{1'}-e_k^{2'}=&(\Phi'(u_k^1,\varepsilon_1)-\Phi'(u_k^2,\varepsilon_2))\,w^1\\ &+(\Phi'(v_k^2,\varepsilon_1)-\Phi'(v_k^1,\varepsilon_1))\,w^1\\ &+(\Phi'(u_k^2,\varepsilon_1)-\Phi'(v_k^2,\varepsilon_1))\,w^1\\ &+(\Phi'(v_k^2,\varepsilon_1)-\Phi'(u_k^2,\varepsilon_1))\,w^2\\ &+(\Phi'(u_k^2,\varepsilon_1)-\Phi'(u_k^2,\varepsilon_2))\,w^2\\ &+(\Phi'(v_k^2,\varepsilon_2)-\Phi'(v_k^2,\varepsilon_1))\,w^2. \end{aligned}$$

Using (1.4), (1.12) and (1.23) the F_s-norm of the first four terms in $\|e_k^{1'}-e_k^{2'}\|_s$ are bounded by $C\delta\delta_1\,\theta_k^{M(s)-1}$ in $(a_1-b_1)^+\leqq s\leqq\tilde{s}$ where

$$M(s)=\max\{s+b_1-\beta,\ (s+b_1-\gamma')^++a_1-\beta,\ a_1-\beta+(s+b_2-\beta'')^+\}.$$

Similarly, using (1.20) instead of (1.4), the last two terms are estimated by $C\delta\,|\varepsilon_1-\varepsilon_2|\,\theta_k^{N(s)-1}$ in $(a_1-b_1)^+\leqq s\leqq\tilde{s}$ where

$$N(s)=\max\{s+b_1-\beta,\ a_1-\beta+(s+b_2-\beta'')^+\}.$$

Next, using (1.21) we have

$$\|e_k^{1''}-e_k^{2''}\|_s\leqq C(\delta\delta_1\,\theta_k^{P(s)-1}+\delta^2\,\delta_1\,\theta_k^{Q(s)-1}+|\varepsilon_1-\varepsilon_2|\,\delta^2\,\theta_k^{R(s)-1})$$

where

$$\begin{aligned} P(s)&=\max\{a_1-\beta+s+b_1-\gamma,\ a_1-\gamma+a_1-\beta+(s+b_2-\beta'')^+\},\\ Q(s)&=\max\{a_1-\beta+s+b_1-\beta,\ (s+b_1-\gamma')^++2(a_1-\beta),\ (s+b_2-\beta'')^+\} \end{aligned}$$

and

$$R(s)=\max\{a_1-\beta+s+b_1-\beta,\ 2(a_1-\beta)+(s+b_2-\beta'')^+\}.$$

Letting

$$\begin{aligned}W(s) &= \max\{M, N, P, Q, R\}(s)\\ &= \max\{s+b_1-\beta, (s+b_1-\gamma')^+ + a_1-\beta, (s+b_2-\beta'')^+\}\end{aligned} \quad (1.24)$$

and restricting δ and δ_1 to be less than 1 we conclude

$$\| e_k^1 - e_k^2 \|_s \leqq C(\delta_1 + |\varepsilon_1 - \varepsilon_2|)\,\theta_{k+1}^{W(s)-1}$$

for $(a_1 - \min\{b_1, b_2\})^+ \leqq s \leqq \tilde{s}$. Under (*e*) it follows as before

$$\begin{aligned}&\| g_{k+1}^1 - g_{k+1}^2 \|_s\\ &\quad \leqq C[(\delta_1 + |\varepsilon_1 - \varepsilon_2|)\,\theta_{k+1}^{W(s)-1} + \|f_1 - f_2\|_{\gamma+b_3}\,\theta_{k+1}^{s-\gamma-b_3-1}]\end{aligned} \quad (1.25)$$

for all $s \geqq (a_1 - \min\{b_1, b_2\})^+$. Write

$$\begin{aligned}w_{k+1}^1 - w_{k+1}^2 &= \Psi(v_{k+1}^1, \varepsilon_1)(g_{k+1}^1 - g_{k+1}^2)\\ &\quad + (\Psi(v_{k+1}^1, \varepsilon_1) - \Psi(v_{k+1}^2, \varepsilon_1))\, g_{k+1}^2\\ &\quad + (\Psi(v_{k+1}^2, \varepsilon_1) - \Psi(v_{k+1}^2, \varepsilon_2))\, g_{k+1}^2 = A + B + C.\end{aligned}$$

Using (1.7), (1.25) and then (*b*) we have

$$\begin{aligned}\|A\|_s &\leqq C\{(\delta + |\varepsilon_1 - \varepsilon_2|)\,\theta_{k+1}^{W(s+b_3)-1} + \theta_{k+1}^{s-\gamma-1}\|f_1 - f_2\|_{\gamma+b_3}\\ &\quad + [(\delta + |\varepsilon_1 - \varepsilon_2|)\,\theta_{k+1}^{W(a_2)-1} + \theta_{k+1}^{a_2-b_3-\gamma-1}\|f_1 - f_2|_{\gamma+b_3}]\,\theta_{k+1}^{(s+b_4-\beta'')^+}\}\\ &\leqq C[(\delta + |\varepsilon_1 - \varepsilon_2|)\,\theta_{k+1}^{-\varepsilon} + \|f_1 - f_2\|_{\gamma+b_3}]\,\theta_{k+1}^{s-\gamma-1}\end{aligned}$$

for some $\varepsilon > 0$ in $a_1 \leqq s \leqq s^*$. Also by (1.7), (1.4) and then (1.7) again we have

$$\begin{aligned}\|B\|_s &= \|\Psi(v_{k+1}^1, \varepsilon_1)(\Phi'(v_{k+1}^1, \varepsilon_1) - \Phi'(v_{k+1}^2, \varepsilon_1))\,\Psi(v_{k+1}^2, \varepsilon_1)\, g_{k+1}^2\|_s\\ &\leqq C\,\delta_1(\delta + \|f\|_{\beta+b_3})\,\theta_{k+1}^{s-\gamma-1-\varepsilon}\end{aligned}$$

provided

$$\begin{aligned}\max\{&L(s+b_1+2b_3), L(a_2) + (s+b_1+b_3+b_2-\beta'')^+,\\ &(s+b_1+b_3-\gamma')^+ + L(a_1+b_3), (s+b_1+b_3-\gamma')^+ + L(a_2),\\ &L(a_1+b_3) + (s+b_2+b_3-\beta'')^+, L(a_2) + (s+b_2+b_3-\beta'')^+,\\ &L(a_2+b_1+b_3) + (s+b_4-\beta'')^+,\\ &L(a_2) + (a_2+b_1+b_4-\beta'')^+ + (s+b_4-\beta'')^+, (a_2+b_1-\gamma')^+ + L(a_1+b_3)\\ &+ (s+b_4-\beta'')^+, (a_2+b_1-\gamma')^+ + L(a_2) + (s+b_4-\beta'')^+\}\\ &< s-\gamma\end{aligned} \quad (1.26)$$

and

$$\begin{aligned}\max\{&s+b_1+b_3-\beta, a_2-\beta-b_3 + (s+b_1+b_3+b_4-\beta'')^+,\\ &(s+b_1+b_3-\gamma')^+ + a_1-\beta, (s+b_1+b_3-\gamma')^+ + a_2-\beta-b_3,\\ &a_1-\beta + (s+b_3+b_2-\beta'')^+, a_2-\beta-b_3 + (s+b_3+b_2-\beta'')^+,\\ &a_2+b_1-\beta + (s+b_4-\beta'')^+,\\ &a_2-\beta-b_3 + (a_2+b_1+b_4-\beta'')^+ + (s+b_4-\beta'')^+,\\ &(a_2+b_1-\gamma')^+ + a_1-\beta + (s+b_4-\beta'')^+\}\\ &< s-\gamma.\end{aligned} \quad (1.27)$$

Finally, using (1.7), (1.20) and then (1.7) again we have

$$\| \mathrm{C} \|_s = \| \Psi(v_{k+1}^2, \varepsilon_2)(\Phi'(v_{k+1}^2, \varepsilon_2) - \Phi'(v_{k+1}^2, \varepsilon_1)\Psi(v_{k+1}^2, \varepsilon_1)) g_{k+1}^2 \|_s \leqq \mathrm{C} | \varepsilon_1 - \varepsilon_2 | (\delta + \|f\|_{\beta + b_3}) \theta_{k+1}^{s-\gamma-1-\varepsilon}$$

provided

$$\max \{ \mathrm{L}(s+2b_3+b_1), \mathrm{L}(a_2)+(s+b_3+b_2+b_1-\beta'')^+, \mathrm{L}(a_1+b_3)+(s+b_3+b_2-\beta'')^+, \mathrm{L}(a_2+b_1+b_3)+(s+b_4-\beta'')^+, \mathrm{L}(a_2)+(a_2+b_1+b_4-\beta'')^+ + (s+b_4-\beta'')^+, \mathrm{L}(a_1+b_3)+(s+b_4-\beta'')^+ \} < s-\gamma. \quad (1.28)$$

and

$$\max \{ s+b_3+b_1-\beta, a_2-\beta-b_3+(s+b_3+b_1+b_4-\beta'')^+, a_1-\beta+(s+b_3+b_2-\beta'')^+, a_2-\beta-b_3+(s+b_3+b_2-\beta'')^+, a_2+b_1-\beta+(s+b_4-\beta'')^+, a_2-\beta-b_3+(a_2+b_4+b_1-\beta'')^+ + (s+b_4-\beta'')^+, a_1-\beta+(s+b_4-\beta'')^+ \} < s-\gamma. \quad (1.29)$$

Combining these estimates we arrive that

$$\| w_{k+1}^1 - w_{k+1}^2 \|_s \leqq \mathrm{C}_3 [(\delta_1 + | \varepsilon_1 - \varepsilon_2 |) \theta_{k+1}^{-\varepsilon} + \| f_1 - f_2 \|_{\gamma + b_3}] \theta_{k+1}^{s-\gamma-1} \quad (1.30)$$

in $a_1 \leqq s \leqq s^*$. On the other hand, we also have

$$\| w_0^1 - w_0^2 \|_s \leqq \mathrm{C}_4 (| \varepsilon_1 - \varepsilon_2 | \theta_0^{-\varepsilon} + \| f_1 - f_2 \|_{\gamma + b_3}) \theta_0^{s-\gamma-1}$$

in $a_1 \leqq s \leqq s^*$. Therefore, letting

$$\delta = \max \{ 2\mathrm{C}_3, \mathrm{C}_4 \} (| \varepsilon_1 - \varepsilon_2 | + \| f_1 - f_2 \|_{\gamma + b_3})$$

we conclude (1.21) for $j=0$ and $j=k+1$ simultaneously. The proof of Theorem 1.5 is completed. ■

Remark 1.6. – For the applications in Section 4 and Section 6 we shall take $a_1 = a_2 (=a)$, $b_1 = b_2 (=b)$, $b_3 = 0$, and $\gamma' < \beta'' - \max \{b, b_4\}$. Under these conditions (1.26)-(1.29) can be combined into a single inequality:

$$\max \{ s+b, a+(s+b-\gamma')^+, a+b+(s+b_4-\beta'')^+, a+(a+b-\gamma')^+ + (s+b_4-\beta'')^+ \} < s-\gamma+\beta$$

Remark 1.7. – Theorem 1.5 remains valid if some conditions are relaxed. Namely, it is sufficient to assume u_0 belongs to E_s^* and the smoothing operators S_θ and T_θ map E_0 and F_0 to E_{m_3} and F_{m_4} where $m_3 \geqq s^* + b_3 + \max \{b_1 + b_4, b_2\}$ and $m_4 \geqq s^* + 2b_3 + b_1$ respectively.

Remark 1.8. – In case f_i belongs to $\mathrm{F}_{\gamma+\varepsilon+b_3}$, in the above proof instead of (1.30) we have

$$\| w_{k+1}^1 - w_{k+1}^2 \|_s \leqq \mathrm{C}_3 (\delta_1 + | \varepsilon_1 - \varepsilon_2 | + \| f_1 - f_2 \|_{\gamma+\varepsilon+b_3}) \theta_{k+1}^{s-(\gamma+\varepsilon)-1} \quad (1.31)$$

in $a_1 \leqq s \leqq s^* + \varepsilon$. Therefore, assuming (1.4), (1.5), (1.7), (1.20) and (1.20)' hold in suitably larger range (more precisely, replace s^* therein by

$s^*+2\beta_1$), we can argue as in Remark 1.4 that

$$\| u^1-u^2 \|_{\gamma'+\beta_1} \leqq C(|\varepsilon_1-\varepsilon_2| + \|f_1-f_2\|_{\gamma+\beta_1+b_3}).$$

In the following we formulate a general uniqueness theorem based on Moser [10]. Let M an open set in E_μ for some μ. Suppose that $\Phi: M \to F_{\mu-m}$ and $\Phi': M \times E_\mu \to F_{\mu-m'}$, $\mu \geqq m$, satisfies $(1.5)_s$ for

$$0 \leqq s \leqq \tilde{\mu} = \mu - \max\{b_1, b_2\}$$

whenever u and v in M. Furthermore, we suppose that $\Phi'(u)$ has a left inverse $\Psi_1: M \times F_{\mu-m} \to F_0$ such that

$$\Psi_1(u)\Phi'(u)v = v, \quad u \in M, \ v \in E_\mu. \tag{1.32}$$

We have

THEOREM 1.8. — *Let μ, μ' and λ be three positive numbers such that*

(*a*) $\mu \geqq a_2 + \max\{b_1, b_2\}$, $\mu' < \mu$, *and* $\lambda < \mu'$ *satisfy* (1.33), (1.36), *and* (1.38).

(*b*) (1.5) *holds for* $0 \leqq s \leqq \tilde{\mu}$ *and* (1.7) *holds for* $s = \lambda$.

Then there exists $r > 0$ *such that if* $\Phi(u^1) = \Phi(u^2)$ *for* u^1 *and* u^2 *in* $\{u : \|u\|_\mu \leqq r\}$, *then* $u^1 = u^2$.

Proof. — We shall show that if $\|u^1-u^2\|_\mu$ is sufficiently small, then $u^1=u^2$. Applying (1.5) to u^1, u^2, and $w=u^1-u^2$, we have

$$\begin{aligned}\|\Phi'(u^2)w\|_s &= \|\Phi(u^1)-\Phi(u^2)-\Phi'(u^2)w\|_s \\ &\leqq C(\|w\|_{a_1}\|w\|_{s+b_1} + \|w\|_{a_1}^2)\end{aligned}$$

for $0 \leqq s \leqq \tilde{\mu}$ where C depends on R. In case

$$\lambda + b_3 + \max\{b_1, b_2\}, \qquad \lambda + b_4 \leqq \mu, \tag{1.33}$$

(1.7) gives

$$\begin{aligned}\|w\|_\lambda &= \|\Psi_1(u^2)\Phi'(u^2)w\|_\lambda \\ &\leqq C(\|w\|_{a_1}\|w\|_{\lambda+b_3+b_1} + \|w\|_{a_1}^2 + \|w\|_{a_1}\|w\|_{a_2+b_1}).\end{aligned} \tag{1.34}$$

Let $w_j = S_{\theta_j} w$ where $\theta_j = 2^{j+K}$ for some large K to be specified later. We shall prove by induction that for some $\delta > 0$,

$$\|w_j\|_s \leqq \delta\theta_j^{s-\mu'}, \qquad \lambda \leqq s \leqq \mu', \quad j \geqq 0. \tag{1.35}$$

By induction hypothesis, if $(1.35)_k$ holds,

$$\|w\|_s \leqq \|w_k\|_s + \|(I-S_{\theta_k})w\|_s \leqq (\delta + C_5\,\theta_k^{\mu'-\mu}\|w\|_\mu)\,\theta_k^{s-\mu'}$$

for $\lambda \leqq s \leqq \mu'$. Consequently, if

$$\lambda \leqq a_1, \qquad \lambda + b_3 + b_1, \qquad a_2 + b_1 \leqq \mu', \tag{1.36}$$

from (1.34) it follows that

$$\begin{aligned}\|w_{k+1}\|_s &\leqq C_6\,\theta_{k+1}^{s-\lambda}\|w\|_\lambda \\ &\leqq C_6(\delta + C_5\,\theta_k^{\mu'-\mu}\|w\|_\mu)^2\,\theta_{k+1}^{s+a}\end{aligned} \tag{1.37}$$

where $a = \max\{a_1 + b_3 + b_1 - 2\mu', a_1 + a_2 + b_1 - 2\mu' - \lambda, 2(a_1 - \mu') - \lambda)\}$. Thus, if

$$\mu' \geqq \max\left\{a_1 + b_3 + b_1,\ a_1 + a_2 + b_1 - \lambda,\ a_2 - \frac{\lambda}{2}\right\}, \tag{1.38}$$

for $\delta \leqq (4C_6)^{-1}$ and K so large that $2C_5 \theta_0^{\mu'-\mu} \leqq \delta$, (1.35) holds for $k+1$. On the other hand, (1.35) holds for $j=0$ if $\|w\|_\mu$ is sufficiently small. Hence by induction (1.35) is valid for all j. In particular, taking $s=\lambda$ in (1.35) and then let j tend to ∞ we conclude $\|w\|_\lambda = 0$.

Q.E.D.

2. LINEAR SYSTEMS

In this section we collect basic results on linear SPS's for noncharacteristic boundary.

Consider a SPS in a domain Ω:

$$L u = A^j u_j + B u = f \tag{2.1}$$

where

$$\left(\left(B(x) - \frac{1}{2} A_j^j(x)\right)\xi, \xi\right) \geqq b^2 |\xi|^2, \qquad x \in \Omega \tag{2.2}$$

for some $b>0$. Here $u(x)$ is an N-vector, $A^j(x)$ are $N \times N$ symmetric matrices ($j=1, \ldots, n$), $B(x)$ is a $N \times N$-matrix and $f(x)$ an N-vector. Recall that a smoothly varying subspace $N(x)$ defined on $\partial\Omega$ is called *semi-admissible* to (2.1) if

$$(\xi, \beta(x)\xi) \geqq 0, \qquad \xi \in N(x) \tag{2.3}$$

and is *admissible* if it is further a maximal subspace with respect to (2.3).

Throughout this section we shall assume β is nonsingular and A^j, $j=1, \ldots, n$ and B are at least in $C^1(\bar{\Omega})$.

LEMMA 2.1. — *Suppose* $N(x)$ *is semi-admissible. Then for any* $u \in H_N^1(\Omega)$,

$$b\|u\|_0 \leqq \|L u\|_0 \tag{2.4}$$

Proof: Apply Green's theorem and then use (2.3).

Q.E.D.

THEOREM 2.2. — *Suppose* A^j *and* B *are in* $H^l(\Omega)$ *where* l *is an integer* $\geqq \bar{m}+1$ *and* $N(x)$ *is semi-admissible. Then for any* H_N^l*-solution* u *of* (2.1) *for* $f \in H^l(\Omega)$, *we have*

$$b\|u\|_m \leqq C_m \left[\|f\|_m + |u|_{C^1}\left(\sum_{j=1}^n \|A^j\|_m + \|B\|_m\right)\right] \tag{2.5}$$

for $0 \leqq m \leqq l$ *provided* b *is sufficiently large (depending on* l, $|A^j|_1$ *and* $|B|_1$). *The constants* C_m *also depend on the same quantities.*

Proof. – (2.4) provides a stronger result for (2.5) in case $m=0$.

For $m>0$ we first localise the problem. Cover $\bar{\Omega}$ by open sets (in $\bar{\Omega}$) V_i ($i \geqq 0$) where $\bar{V}_0 \subseteqq \Omega$ and $V_i \cap \partial\Omega \neq \varnothing$ for $i \neq 0$ in such a way that each V_i ($i \neq 0$) intersects at most M many other V_j's. Suppose further that each V_i has been chosen so thin that a normal coordinate can be introduced. In other words, there is a diffeomorphism φ_i from a rectangle $R=\{(x', y) \in R^n: |x|<1, -\rho<y \leqq 0\}$ to V_i such that $-y$ is the distance of $\varphi_i(x', y)$ to $\partial\Omega$. If we set $\bar{F}=F \circ \varphi_j$, then $\bar{u}$ satisfies

$$A_1^j \bar{u}_j + \bar{B}\bar{u} = \bar{f}$$

in R for some symmetric matrices A_1^j, $j=1, \ldots, n$. Since $\bar{N}(x)=N(\varphi(x))$ is smooth, we can find an orthonormal matrix $O(x', 0)$ such that $\xi \in \bar{N}(x)$ iff $\eta^{l+1}=\ldots\eta^N=0$ where $\xi=O(x', 0)\eta$ and l is the dimension of $\bar{N}(x)$. If we extend $O(x', 0)$ to V_i by setting $O(x', y')=O(x', 0)$ and then change the dependent variable $\bar{u}(x)$ to $\tilde{u}(x)=O(x)u(x)$, $\tilde{u}$ satisfies

$$\tilde{A}^j \tilde{u}_j + \tilde{B}\tilde{u} = \bar{f}$$

in R. For sufficiently large b, this system is still a SPS. Furthermore, the boundary condition which is now simply $\tilde{u}^{l+1}(x)=\ldots\tilde{u}^N(x)=0$ is still semi-admissible. Consequently we may assume (2.1) is defined in R and $N(x)=\{(u^1, \ldots, u^N): u^{l+1}=\ldots u^N=0\}$.

Let $R'=\{(x', y) \in R^n: |x'|<d<1, -\rho'<y \leqq 0, \rho'<\rho\}$. We claim

$$b\|u\|_{m, R'} \leqq C_m\left[\|u\|_{m, R}+\|f\|_{m, R} + |u|_{C^1}\left(\sum_{j=1}^{n}\|A^j\|_{m, R}+\|B\|_{m, R}\right)\right], \qquad 0 \leqq m \leqq 1. \quad (2.6)$$

For, let φ be a nonnegative smooth function compactly supported in R and equal to 1 on R'. Applying

$$D^{\alpha, 0}=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial x_{n-1}}\right)^{\alpha_{n-1}}, \quad |\alpha=(\alpha_1, \ldots, \alpha_{n-1})|=m,$$

to (2.1), we find that $w=\varphi D^{\alpha, 0} u$ satisfies

$$A^j w_j + B w = \eta D^{\alpha, 0} f + h$$

where

$$h=A^j[\partial_j, \varphi] D^{\alpha, 0} u + [B, \varphi D^{\alpha, 0}] u + \varphi[A^j \partial_j, D^{\alpha, 0}] u.$$

Since w remains in $N(x)$, we may apply Lemma 2.1 to obtain

$$b\|D^{\alpha, 0} u\|_{0, R'} \leqq C(\|f\|_{m, R}+\|h\|_{0, R}) \leqq C\|u\|_{m, R}+\text{R.H.S. of } (2.6)_m. \quad (2.7)$$

The last step follows from the following well-known inequality: For any f, g in $H^l(\Omega)$,

$$\|fg\|_1 \leqq C_1(|f|_C \|g\|_1 + |g|_C \|f\|_1).$$

To estimate the normal derivatives we apply $D^{0,m-1}$ to the equation and use the nonsingularity of β to express $D^{0,m}u$ in terms of the other terms which consists of derivatives up to m-th order (but up to $(m-1)$-th order in y) of u, derivatives of the coefficients and f up to m-th-order. By using interpolation inequality [7], p. 465, in a standard way, we have

$$\| D^{0,m} u \|_{0,R'} \leqq C \Sigma \| D^{\alpha,0} u \|_{0,R'} + \text{R.H.S. of } (2.6)_m. \qquad (2.8)$$

Combining (2.7) and (2.8) we obtain (2.6). A corresponding interior estimate for V_0 holds by a similar argument. Going back to Ω, we see that (2.5) holds if b is sufficiently large (so that the $\|u\|_m$ in the right can be absorbed to the left).

Q.E.D.

Remark 2.3. — If l in the above theorem is replaced by a real number $s>1$ then

$$b\|u\|_t \leqq C(t, s_0)\left[\|f\|_t + \|u\|_{s_0+1}\left(\sum_{j=1}^{n}\|A^j\|_t + \|B\|_t\right)\right], \qquad (2.5)'$$

$1 \leqq t \leqq s$ and $s_0 > n/2$. (2.5)′ can be obtained by the above argument except we now use a fractional Leibniz rule [16], Lemma 1.1, to estimate h and then apply (4.3).

THEOREM 2.4. — *Suppose* A^j *and* B *are in* $H^l(\Omega)$, $l \geqq \bar{m}+1$ *and* $N(x)$ *is admissible. Then for* $f \in H^l(\Omega)$, (2.1) *has a unique solution* u *in* $H^l_N(\Omega)$ *provided b is sufficiently large (depending on* l, $|A^j|_{C^1}$ *and* $|B|_{C^1}$).

Proof. — This theorem was proved in [2] under the assumptions that the coefficients are smooth and $N(x)$ is stable. By stable we mean all subspaces close to $N(x)$ is also admissible to (2.1). However, these two additional conditions are not necessary. In view of the *a priori* estimate (2.5) the conditions on coefficients can be removed by an approximation argument. On the other hand, it is easy to construct symmetric matrices X^j, $j=1, \ldots, n$ such that $N(x)$ is admissible and stable with respect to the system

$$(A^j + \varepsilon X^j) u_j + B u = f, \qquad \varepsilon \text{ small}.$$

Apply Gu's result to this system we obtain H_N-solution u_ε. By (2.5) we may pass to weak limit and conclude the solution u of (2.1) is in $H^l_N(\Omega)$.

Q.E.D.

Remark 2.5. — There are other proofs of the differentiability of solutions, e.g. [7], [14], and [15]. However, in all of those arguments the largeness of b involes the bounds of the derivatives of coefficients with order higher than one. Thus, they don't give the best result.

3. QUASILINEAR SYSTEMS

In the following we study quasi-linear systems. We'll follow the formulation of Gu [3]: Consider

$$\left.\begin{aligned} A^j(x, u)\,u_j + b^2\,I\,u = f(x, u), \qquad & x \in \Omega \\ u(x) \in N(x), \qquad & x \in \partial\Omega, \end{aligned}\right\} \tag{3.1}$$

where A^j are symmetric and I is the $N \times N$ identity matrix.

THEOREM 3.1. – *Suppose* $u=0$, $\beta(x; 0)$ *is nonsingular and* $N(x)$ *is admissible to* (3.1), *for all u small in* $C^1(\bar{\Omega})$. *Then there exists a solution u for* (3.1) *in* $H^l_N(\Omega)$, $l \geqq \bar{m}+1$ *provided b is sufficiently large depending on* $|A^j(x, z)|_{C^1}$ *and* $|f(x, z)|_{C^1}$ *in x in* $\bar{\Omega}$ *and* $|z| \leqq 1$.

Remark 3.2. – This theorem is an improvement of Theorem 1 in [3] where the largeness of b is required to depend on $|A^j(x, z)|_{C^{1+\bar{m}}}$ and $|f(x, z)|_{C^{1+\bar{m}}}$.

Proof. – By Theorem 2.4, for every $u \in H^l(\Omega)$, $|u|_1$ small, the system

$$\begin{aligned} A^j(x, u)\,v_j + b^2\,I\,v = f(x, u), \qquad & x \in \Omega \\ v(x) \in N(x), \qquad & x \in \partial\Omega \end{aligned}$$

admits a unique solution $v \in H^l_N(\Omega)$ when b is sufficiently large. Denote the map $u \to v$ by $v = T(u)$. Then using the mean value formula and Moser's inequality it is not hard to verify that (*a*) $T(B) \subseteqq B$ where $B = \{ u \in H^l_N(\Omega) : \| u \|_l \leqq \delta \}$, $\delta = \delta(b^{-1})$ small and (*b*) $\| T(u_1) - T(u_2) \|_{l-1} \leqq \gamma \| u_1 - u_2 \|_{l-1}$ for some $0 < \gamma < 1$ if b is large enough. Hence, by extending T to a continuous map on the H^{l-1}-closure of B and then applying contraction mapping principle, we conclude that there exists a fixed point u of T in $H^{l-1}_N(\Omega)$ which is obviously a stong solution of (3.1). Furthermore from (*a*) we see that u actually belongs to $H^l_N(\Omega)$.

Q.E.D.

4. PROOF OF MAIN THEOREM

LEMMA 4.1. – *Let* $N(x)$ *be a smooth subspace of* R^N *for* $x \in \partial\Omega$. *Suppose that locally it can be represented as* $\{(\xi^1, \ldots, \xi^N) : \xi^{l+1} = \ldots = \xi^N = 0\}$

where $l=\dim N(x)$. *Then for* $s \geqq 1$,

$$\{ H_N^1(\Omega; R^N), H_N^s(\Omega; R^N) \}_\theta = H_N^{\theta s}(\Omega; R^N).$$

Therefore, there exists $S_\theta : H_N^1(\Omega; R^N) \to H_N^s(\Omega; R^N)$ *so that* (1.1) *and* (1.2) *hold.* (*For notation* see *Appendix A.*)

Proof. – It follows from the fact that

$$\{ H^1(\Omega), H^s(\Omega) \} = H^{\theta s}(\Omega)$$

and

$$\{ H_0^1(\Omega), H_0^1(\Omega) \cap H^s(\Omega) \} = H_0^1(\Omega) \cap H^{\theta s}(\Omega)$$

plus a partition of unity argument. (Here $H_0^1(\Omega)$ is the completion of all continuously differentiable functions which vanish on $\partial\Omega$ under the H^1-norm.) The last assertion follows from Lemma A in Appendix A.

Q.E.D.

LEMMA 4.2. – (*a*) *For* $f, g \in H^s(\Omega)$, $(s \geqq 0)$ and $s_0 > \frac{n}{2}$,

$$\| fg \|_s \leqq C(s, s_0)(\| f \|_{s_0} \| g \|_s + \| g \|_{s_0} \| f \|_s). \tag{4.3}$$

(*b*) (*Moser's inequality*). *Suppose* $F(x, z)$ *is defined for* $(x, z) \in \bar{\Omega} \times B_R(R^m)$. *Then for* $u \in H^s(\Omega)$ $(s > 1)$, $\| u \|_{s_0} \leqq R$,

$$\| F(x, u) \|_s \leqq C(s, s_0)(1 + \| u \|_s) \tag{4.4}$$

where $C(s, s_0)$ *also depends on* $\| u \|_{s_0+1}$ *and the derivatives of* F *up to order* $[s]+1$ *over tis domain.* (*Here* $[s]=s-1$ *when s is an integer.*)

See Appendix B for a proof.

Proof of the Main Theorem. – We shall take $E_s = H_N^s(\Omega)$, $F_s = H^s(\Omega)$ and apply the results in Section 1. By Lemma 4.1, E_s and F_s both admit smoothing operators. Let $\Phi(u) = F(x, u, \nabla u)$. We choose α in $\left(\frac{n}{2}+2, \bar{m}+2\right)$. By Sobolev's inequality we may fix a H_N^α-neighborhood of 0, N, such that $|u|_{C^2(\bar{\Omega})} \leqq 1$ for u in N. Hence (A 1) and (A 5) are applicable.

Let $\beta = \bar{m}+2$. We verify (1.4)-(1.7) as follows: By (4.4), we have

$$\| \Phi''(u)(v, w) \|_s \leqq C_s [\| v \|_{a_1} \| w \|_{s+1} + \| v \|_{s+1} \| w \|_{a_1} + \| v \|_{a_1} \| w \|_{a_1} (1 + \| u \|_{s+1})]$$

where $a_1 > n/2+1$. By Taylor's formula (1.4) and (1.5) are valid for $b_1 = b_2 = 1$ and $a_1 > n/2+1$. We pick a_1 in $(\bar{m}, \bar{m}+1)$ and $s^* = \bar{m}+4$. By (A 1) (1.4) and (1.5) hold in $[0, \bar{m}+3]$. On the other hand, since $|u|_{C^2(\bar{\Omega})} \leqq 1$, we infer from Theorem 2.2, Theorem 2.4, and Remark 2.3

that the inverse of $\Phi'(u)$, $u \in N$, exists and satisfies

$$b \| v \|_s \leqq C_s [\| g \|_s + \| v \|_{a_2} (1 + \| u \|_{s+1})],$$

$$0 \leqq s \leqq \beta + 3 \quad \text{and} \quad a_2 > \frac{n}{2} + 1 \tag{4.5}$$

if $\Phi'(u) v = g$. Notice that Moser's inequality has been used in the last step. Taking $s = a_2 \leqq \alpha - 1$, we have

$$\| v \|_{a_2} \leqq C \| g \|_{a_2}, \qquad u \in N. \tag{4.6}$$

Substituting (4.6) into (4.5) we see that $(1.7)_s$ holds for $0 \leqq s \leqq \bar{m} + 4$ where $b_3 = 0$ and $b_4 = 1$. We can fix $a_1 = a_2$ and verify that the hypotheses of Theorem 1.1 and Remark 1.3 are fulfilled, concluding the existence of a solution u in $H_N^{\bar{m}+2-\varepsilon}(\Omega)$, $\varepsilon > 0$, for any sufficiently small $\| f \|_{\bar{m}+2}$. The solution further satisfies $\| u \|_{\bar{m}+2-\varepsilon} \leqq C_\varepsilon \| f \|_{\bar{m}+2}$.

Now, for μ in $(n/2+2, \bar{m}+2)$, μ' close to μ and $\lambda = a_1$ in Theorem 1.9 we can choose $a_1 = a_2$ and $a_1 < \mu - 1$. For a further restriction on ρ, we deduce that there is $r > 0$ such that (0.1) has a unique solution in $\{ u : \| u \|_\mu \leqq r \}$.

To prove the regularity of u we appeal to Remarks 1.3 and 1.4. It suffices to make sure (1.4) and (1.5) hold in $0 \leqq s \leqq \tilde{s} + 21$ and (1.7) holds in $0 \leqq s \leqq s^* + 21$. But this follows from (A 5)'.

Finally, to prove the continuous dependence of f we use Theorem 1.5 and Remarks 1.6-1.8. This is because, due to uniqueness, $u(f)$ is the solution constructed by the Nash-Hörmander iteration scheme. By (A 5)'' $(1=0)$ and Taylor's formula we know that (1.21) holds in $0 \leqq s \leqq \bar{m} + 3$. Notice that (1.20) is not needed. Thus, for any γ in $(a_1, \bar{m}+1)$ we can choose a suitable β'' so that all conditions in Theorem 1.5 are satisfied. Consequently for a further restriction on the smallness of $\| f \|_{\bar{m}+2}$ we have $\| u(f_1) - u(f_2) \|_{\gamma-\varepsilon} \leqq C_\varepsilon \| f_1 - f_2 \|_\gamma$, $\varepsilon > 0$, whenever $\| f_1 - f_2 \|_\gamma$ is small. Finally be Remark 1.8 we further obtain

$$\| u(f_1) - u(f_2) \|_{\gamma+l-\varepsilon} \leqq C_{\varepsilon, l} \| f_1 - f_2 \|_{\gamma+l} \qquad \text{for} \quad l \geqq 0.$$

The proof of the Main Theorem is completed.

Remark 4.3. – Sometimes it is also interesting to look at Problem (0.1) in a different way. One may consider it as a perturbation of a linear SPS: $\varphi(u, \varepsilon) = L u + \varepsilon G(x, u, \nabla u)$ where it is assumed that $\Phi(u, \varepsilon)$ satisfies (A 2)-(A 4) as well as

(A 6) b is large depending on the $C^{\beta+4}$-norm of the coefficients of L; and

(A 7) G belongs to $C^{\beta+4}(D)$.

When $\varepsilon = 0$, we have a trivial solution $u = 0$. We would like to know whether there is a unique solution for $\Phi(u, \varepsilon) = 0$. To this end we choose $\beta > \alpha = \bar{m} + 2$, $\mu = \bar{m} + 2$, and $\gamma = \bar{m} + 1$. By a suitable choice of a_i's and λ

Theorems 1.1, 1.5, and 1.9 apply provided L and G are in $C^{s^*+2}(\bar{\Omega})$, $s^*=\beta+2$. Thus, for sufficiently small $\varepsilon_0>0$, there is a family of solutions $u(\varepsilon)$, $|\varepsilon|\leqq\varepsilon_0$, starting from $u(0)=0$ and is unique in $\{u:\|u\|_{\bar{m}+2}\leqq r\}$ for some small r. In particular, when $G(x, 0, 0)$ is not equal to zero, there is no zero solution in $\{u:\|u\|_{\bar{m}+2}\leqq r\}$ for ε in $[-\varepsilon_0, \varepsilon_0]$, ε not equal to zero. By embedding theorem each $u(\varepsilon)$ belongs to $C^3(\bar{\Omega})$ and the map $\varepsilon\mapsto u(\varepsilon)$ from $[-\varepsilon_0, \varepsilon_0]$ to $C^1(\bar{\Omega})$ is continuous.

5. SMOOTH SOLUTIONS

As it is well-known, even for linear SPS's the smoothness of solutions depends on the positivity of b. In general, when b is larger, the solution is more regular [11], p. 293. One doesn't expect to have smooth solutions. However, in [11] Moser studied a special class of SPS and established analyticity of their solutions. In this section we show a corresponding result for smooth solutions.

Consider

$$F(x, u, \nabla u)=f \tag{5.1}$$

under the assumptions

(H_1) (5.1) is positive symmetric at $u=0$. There exists $b>0$ such that

$$\left(B(x; 0)-\frac{1}{2}A_j^j(x; 0)\,\xi, \xi\right)\geqq b^2|\xi|^2, \qquad \xi\in R^N$$

(H_2) $\beta(x; 0)$ is positive definite.

(H_3) $A_k^j(x; 0)_{ml}\,\xi_j\,\xi_k\,\eta_m\,\eta_l\geqq a|\xi|^2|\eta|^2$

for some $a>0$, $\xi\in R^n$ and $\eta\in R^N$. By Gårding's inequality, it follows from (H_3) that there exist C_1, $C_2>0$ such that

(H_3') $(A^j(x; u)\,w_j, w_k)\geqq C_1\|w\|_1^2-C_2\|w\|_0^2$

for all small C^2-functions u. It is also noted that from (H_2) no boundary condition is needed.

THEOREM 5.1. — *Suppose that* (H_1)-(H_3) *hold for* $u=0$ *and that* $F(x; 0)=0$. *Then for* $f\in C^\infty(\Omega)$ *with small* $\|f\|_s$, $s\geqq\bar{m}+2$, *there exists a* C^∞*-solution* u *for* (5.1).

Proof. — Choose N to be a small H^γ-*n'd* of 0 for some $\gamma\geqq\bar{m}+2$ in which (H_1)-(H_5) hold uniformly. We consider the linearised problem for (5.1):

$$Lv=A^j(x; u)\,v_j+B(x; u)=g, \tag{5.2}$$

$u\in N\cap C^\infty(\bar{\Omega})$ and $g\in C^\infty(\bar{\Omega})$. In view of Remark 1.5 it suffices to show (*a*) (5.2) is uniquely solvable in $C^\infty(\bar{\Omega})$ and (*b*) the solution v satisfies

$$\|v\|_m\leqq C[\|g\|_m+\|g\|_\gamma\|u\|_{m+1}], \qquad m\geqq 0. \tag{5.3}$$

Differentiating (5.2) and then taking L^2-product with derivatives of v, one can derive without difficulty that

$$b\|v\|_m^2 \leqq (g, v)_m + C_m[\|v\|_{m-1} + |v|_C(1+\|u\|_{m+1})]\|v\|_m \tag{5.4}$$

after using (H'_3), of course. Then (5.3) follows from (5.4) in a familiar way. To prove that (5.2) has a C^∞-solution we shall show that for any l, (5.2) has a H_0^l-solution. From the uniqueness of solution [*see* (2.4)] and Sobolev embedding theorem we conclude that this solution must be in $C^\infty(\bar{\Omega})$. In view of Theorem 2.2 we may fix a sufficiently large $b_1 = b_1(l)$ such that $L + b_1 I$ is uniquely solvable in $H^l(\Omega)$. Define v_n by

$$\begin{gathered} L v_n + b_1 v_n = g + b_1 v_{n-1}, \qquad n \geqq 1 \\ v_0 = 0 \end{gathered} . \tag{5.5}$$

We claim that there exist R_m, $0 \leqq m \leqq l$, such that

$$\|v_n\|_m \leqq R_m, \qquad n \geqq 0.$$

For, by (2.4)

$$(b+b_1)\|v_n\|_0 \leqq \|g\|_0 + b_1\|v_{n-1}\|_0.$$

We may take $R_0 = b^{-1}\|g\|$. Suppose now that $R_{m'}$, $m' \leqq m-1$ has been chosen. Applying (5.4) to (5.5) we have

$$(b+b_1)\|v_n\|_m \leqq \|g\|_m + b_1\|v_{n-1}\|_m + K_m R_{m-1}.$$

So we can take $R_m = b^{-1}(\|g\|_m + K_m R_{m-1})$. The claim is proved. Since $\{v_n\}$ is uniformly bounded in $H^l(\Omega)$, by passing to a weak limit we conclude that the solution v of (5.2) belongs to $H^l(\Omega)$.

Q.E.D.

6. FURTHER COMMENTS

6.1. In [13] Rabinowitz studied the singular perturbation problem

$$L u + \varepsilon F(x, u, \nabla u, \nabla^2 u, \nabla^3 u) = 0 \tag{4.1}$$

where $L u = -(a^{ij} u_i)_j + cu$ is a uniformly elliptic operator where c is positive. The coefficients F and u are supposed to be periodic in $x = (x_1, \ldots, x_n)$. Adapting a method from [11] he proved that if F and a^{ij}'s are in H^l, $l > 2n + 28$, there is an $\varepsilon_0 > 0$ such that of all $|\varepsilon| < \varepsilon_0$, (4.1) has a solution $u(\varepsilon)$ which is C^3 in x and continuous in ε with $u(0) = 0$. In fact, we may regard (6.1) as a perturbation problem for a third-order symmetric positive equation $L u = 0$. Comparing with (0.1), we see that the positive definiteness of a^{ij} and the positivity of c correspond to the positivity of b in (A 5). The largeness of b now is replaced by the smallness of ε. The third order terms, which correspond to A^j's, vanish identically. This does no harm since no boundary conditions are imposed. Actually, it should be kept in mind that such results hold only when the boundary is empty. Otherwise, we have to restrict the class of perturbation to avoid boundary layer phenomena as we have done before. Along the same line as in the proof of our main result, Rabinowitz's result can be sharpened, namely, it is required that L and F are l many times continuously differentiable in D, the unit ball in $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \mathbb{R}^{n^2}$, for $l > \frac{n}{2} + 16$, and also are of period one, say, in x. Then the solution $u(\varepsilon)$ would be in $C^8(\mathbb{R}^n)$ and the map $\varepsilon \to u(\varepsilon)$ from $[-\varepsilon_0, \varepsilon_0]$ to $C^4(\mathbb{R}^n)$ is continous.

Proof. – Let $E_s = F_s$ be the subspace of $H^s(\mathbb{R}^n)$ consisting all periodic functions of period 1. We take $\Phi(u, \varepsilon) = Lu + \varepsilon F(x; u)$. It is readily seen that for $b_1 = b_2 = 3$, $a_1 > \frac{n}{2} + 3$, (1.4), (1.5), (1.20) and (1.21) hold.

The linearized equation of (4.1) is

$$\Phi'(u) v = g. \tag{6.2}$$

By Proposition 2.36 in [13], for sufficiently small ε_0 which depends on k, the uniform ellipticity of a^{ij}, and the C^{k+1}-norms of L and F in $|u|_{C^3} \leqq 1$, we have

$$\|v\|_k \leqq C_k [\|g\|_{k-2} + |\varepsilon| |v|_{C^3} (1 + \|u\|_{k+1})], \qquad k \geqq 2. \tag{6.3}$$

Let's take $\alpha = \bar{m} + 4$ and let N be a H^α-neighborhood of the origin which contains $\{u : |u|_{C^3} \leqq 1\}$. Taking $k = \bar{m} + 3$ in (6.3) we have

$$\|v\|_k \leqq C_k [\|g\|_{k-2} + \|g\|_{\bar{m}+1} (1 + \|u\|_k)], \qquad k \geqq 2. \tag{6.4}$$

Hence we may take $b_3 = 0$, $b_4 = 1$, and $a_2 \geqq \bar{m} + 1$ in (1.7). To solve (6.2) we use elliptic regularization. Add $\nu \Delta$ where Δ is the Laplacian to (6.2). From elliptic theory the modified equation has a H^{k+2}-solution v_ν which satisfies (6.4) uniformly in ν for small ν. See [13] for the proof of this fact. Letting ν go to 0 we obtain a solution for (6.2). Thus, (1.6) holds. Now for $\beta \geqq \bar{m} + 8$, we choose $\gamma = \beta - 4$, $\mu = \beta - 1$ and a_i's all equal to $\beta - 5$. One can verify that all assumptions in Theorems 1.1, 1.5, and 1.9 are satisfied. Therefore, for some small $\varepsilon_0 > 0$, there exists a family of solutions for (6.1) starting from $u(0) = 0$ which satisfies (1) each $u(\varepsilon)$ belongs to $E_{\beta'}$, (2) $u(\varepsilon)$ is unique in $\{u : \|u\|_{\beta-1} \leqq r\}$ for some small r, and (3) $\varepsilon \to u(\varepsilon)$ is continuous from $[-\varepsilon_0, \varepsilon_0]$ to $E_{\beta'-4}$ for any $\beta' < \beta$. By embedding theorem we deduce the desired result.

6.2. In general, we may use the same method to study the perturbation problem for non-coercive boundary value problems introduced by Kohn and Nirenberg [7] provided the perturbations do not change the "type" of the boundary value problems.

6.3 So far we have only considered the case of non-characteristic boundary. When the boundary is characteristic, the situation would be much more complicated. It is because we do not have much information for the linearized problem, espectially the differentiability of solutions, on which the Nash-Hörmander scheme relies heavily. However, some results are still available for a simple non-coercice boundary value problems, namely, degenerate elliuptic-parabolic equations. *See* [18].

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APPENDIX A
SMOOTHING OPERATORS

We begin with a description of an interpolation method used in [9].

Let X and Y be a pair of Hilbert spaces with X densely embedded in Y. It is known that then there exists a strictly positive self-adjoint operator S such that

$$(Sx, y)_Y = (x, y)_X, \qquad x \in D(S),$$

the domain of S. Let Λ be the square root of S. Define interpolation spaces between X and Y by

$$\{Y, X\}_s = D(\Lambda^s), \qquad 0 \leqq s \leqq 1.$$

Then $\{Y, X\}_0 = Y$ and $\{Y, X\}_1 = X$. Each $\{Y, X\}_s$ is a Hilbert space under the inner product $(x, y)_s = (\Lambda^s x, \Lambda^s y)_Y$. For the proofs of the above facts we refer to [9] and [17].

Let $S = \int_1^\infty \lambda \, dE(\lambda)$ be the spectral resolution of S. We define a map T from Y to X for $\theta \geqq 1$ by $T_\theta x = \int_1^{\theta^2} dE(\lambda)\, x$.

LEMMA A.1. – *For all* $\theta > 1$,

(*a*) $\| T_\theta x \|_s \leqq \theta^{(s-r)^+} \| x \|_r$, $0 \leqq s, r \leqq 1$;

(*b*) $\| (I - T_\theta) x \|_r \leqq \theta^{r-s} \| x \|_s$, $0 \leqq r \leqq s \leqq 1$.

Proof:

$$\| T_\theta x \|_s^2 = \int_1^{\theta^2} \lambda^{2s} \| dE(\lambda) \|_Y^2 \leqq \theta^{2(s-r)^+} \int_1^{\theta^2} \lambda^{2r} \| dE(\lambda)\, x \|_Y^+ \leqq \theta^{2(s-r)^+} \| x \|_r^2.$$

The proof of (*b*) is similar.

Q.E.D.

An Example. — Let Ω be a bounded domain with smooth boundary. Let

$$B_j u(x) = \sum_{|\alpha| \leqq m_j} b_{j,\alpha}(x) D^\alpha f$$

where $b_{j,\alpha}(x) \in C^\infty(\partial\Omega)$, $j=1, \ldots, k$, be differential operators on $\partial\Omega$. $\{B_j\}$ is called a *normal system* if $0 \leqq m_1 < m_2 < \ldots < m_k$ and if for any normal vector $\nu(x)$,

$$\sum_{|\alpha| = m_j} b_{j,\alpha}(x) \nu^\alpha(x) \neq 0, \; j=1, \ldots, k.$$

Denote

$$H^m_{\{B_j\}}(\Omega) = \left\{ u \in H^m(\Omega) : B_j u|_{\partial\Omega} = 0, \; m_j < m - \frac{1}{2} \right\}.$$

Then we have

$$\{ H^l_{\{B_j\}}(\Omega), L^2(\Omega) \}_\theta = H^{l\theta}_{\{B_j\}}(\Omega)$$

if there doesn't exist a number m_j, $j=1, \ldots, k$ such that $l\theta - \frac{1}{2} = m_j$. *See* [17], 1.15 and 4.33.

By applying Lemma A.1 we conclude that there exist smoothing operators which preserves normal boundary conditions.

APPENDIX B

Let $(A_0, \| \;\|_0)$ and $(A_1, \| \;\|_1)$ be two Banach spaces. $A_1 \subseteqq A_0$ continuously. For $t > 0$ and $a \in A_0$, define

$$K(t, a) = \inf \{ \| a_0 \|_0 + t \| a_1 \|_1 : a = a_0 + a_1, \; a_0 \in A_0, \; a_1 \in A_1 \}$$

and the interpolation spaces $(0 < \theta < 1, \; 1 \leqq p < \infty)$

$$[A_0, A_1]_{\theta,p} = \left\{ a \in A_0 : \| a \|_{\theta,p} = \left[\int_0^\infty (t^{-\theta} K(t, a))^p \frac{dt}{t} \right]^{1/p} < \infty \right\}. \tag{B.1}$$

LEMMA B.1 [9]. — *Let* $a \in A_0$, $0 < \theta < 1$ *and* $1 \leqq p < \infty$. *Suppose for each* $t > 0$, *there exists* $a = a_0(t) + a_1(t)$, $a_i(t) \in A_i$, $i = 0, 1$ *with* $\| a_i(t) \|_i \leqq \Phi_i(t)$ *such that*

$$M_i = \left[\int_0^\infty (\Phi_i(t) \, t^{i-\theta})^p \frac{dt}{t} \right]^{1/p} < \infty.$$

Then $a \in [A_0, A_1]_{\theta,p}$ *and*

$$\| a \|_{\theta,p} \leqq M_0^{1-\theta} M_1^\theta. \tag{B.2}$$

LEMMA B.2 [19]. — *Let* $a \in A_0$ *and* $a_t \in A_1$ *satisfy*

$$\| a - a_t \|_0 + t \| a_t \|_1 \leqq 2 K(t, a) \tag{B.3}$$

for some $t > 0$. *If* $a \in [A_0, A_1]_{\theta, p}$ $(0 < \theta < 1,\ 1 \leqq p < \infty)$, *then*

$$\| a_t \|_{\theta, p} \leqq 3 \| a \|_{\theta, p}. \tag{B.4}$$

Proof of (4.3). — When s is integral, (4.3) follows from Gagliardo-Nirenberg inequality. When s is not an integer, let $l = \max([s]+1, \bar{m})$. For each $t > 0$, choose f_t and g_t in $H^l(\Omega)$ according to Lemma B.2. Then

$$\begin{aligned} \| fg - f_t g_t \|_0 &\leqq \| g \|_{s_0} \| f - f_t \|_0 + \| f_t \|_{s_0} \| g - g_t \|_0 \\ &\leqq \Phi_0(t) := C(\| g \|_{s_0} K(t, f) + \| f \|_{s_0} K(t, g)), \\ \| f_t g_t \|_l &\leqq C(\| f_t \|_{s_0} \| g_t \|_l + \| g_t \|_{s_0} \| f_t \|_l) \\ &\leqq \Phi_1(t) := C t^{-1} (\| f \|_{s_0} K(t, g) + \| g \|_{s_0} K(t, f)) \end{aligned}$$

by (B.4) and the interpolative characterisation of Sobolev spaces. Using Lemma B.1 (4.3) follows.

Q.E.D.

Proof of (4.4). — For simplicity we may assume $F(x, z) = F(z)$ and z is a scalar. For $s > 1$,

$$D^s F(u) = D^{s-1} F'(u) . \nabla u.$$

Hence

$$\| F(u) \|_s \leqq C(\| F'(u) \|_{s-1} \| u \|_{s_0+1} + \| u \|_s \| F'(u) \|_{s_0}) \tag{B.5}$$

by (4.3). If $s \leqq \bar{m}$, then (4.4) follows after applying the integral Moser inequality to the right hand side of (B.5). Otherwise keep applying (4.3) to $F^{(i)}$, $i = 1, 2, \ldots$ finitely many times we again obtain (4.4).

Q.E.D.

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