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## **$L^p$ regularity of velocity averages**

by

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**ABSTRACT.** — In this article, we present some general regularity results for “velocity averages” *i. e.* averages in  $v$  of functions  $f(x, v)$  for which  $(v \cdot \nabla_x f)$  has some given regularity. We are able to cover general regularity classes for both  $f$  and  $(v \cdot \nabla_x f)$  and we thus extend various known results. Our methods of proof rely on Littlewood-Paley type decompositions, interpolation arguments and a spectral decomposition adapted to the “velocity direction”.

*Key words :* Velocity averages, Sobolev and Besov spaces, transport equations,  $L^p$  multipliers.

**RÉSUMÉ.** — Nous présentons dans cet article des résultats généraux de régularité de moyennes en vitesse (c'est-à-dire de moyennes en  $v$ ) de fonctions  $f(x, v)$  pour lesquelles  $(v \cdot \nabla_x f)$  a une certaine régularité. Nos résultats permettent des classes de régularité générales pour  $f$  et  $(v \cdot \nabla_x f)$ , généralisant ainsi plusieurs résultats antérieurs. Notre méthode de preuve repose sur les décompositions de type Littlewood-Paley, des arguments d'interpolation et une décomposition spectrale adaptée à la « direction des vitesses ».

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SUMMARY

- I. Introduction.
- II. The time-independent case.
- III. The time-dependent case and applications.

I. INTRODUCTION

In this paper, we discuss the regularity of velocity averages for solutions of transport equations. More precisely, let  $f(x, v)$  [resp.  $f(x, v, t)$ ] satisfy

$$v \cdot \nabla_x f = g \quad \text{for } x \in \mathbb{R}^N, v \in \mathbb{R}^N \tag{1}$$

$$\left( \text{resp. } \frac{\partial f}{\partial t} + v \cdot \nabla_x f = g \text{ for } x \in \mathbb{R}^N, v \in \mathbb{R}^N, t \in \mathbb{R} \right). \tag{2}$$

The general question we address here is: given some “regularity” on  $f$  and  $g$  like, for instance,  $f, g \in L^p$ , what is the regularity of “velocity averages” of  $f$  that is of quantities of the following form

$$\bar{f} = \int_{\mathbb{R}^N} f(x, v) \psi(v) dv, \quad \text{for all } \psi \in \mathcal{D}(\mathbb{R}^N). \tag{3}$$

It was first observed by Agoshkov ([1], [2]), F. Golse, B. Perthame and R. Sentis [16] that, if  $p=2$ , some regularity is gained – more precisely,  $\bar{f}$  was proved to belong to  $H^{1/2}$  for some  $\psi$  in [1], [2] while, in [16],  $H^{1/3}$  regularity was obtained for all  $\psi$ . Later on, it was shown in F. Golse, P. L. Lions, B. Perthame and R. Sentis [15] that, still for  $p=2$ ,  $\bar{f} \in H^{1/2}$  for all  $\psi$  while, if  $1 < p < 2$ ,  $\bar{f} \in W^{s,p}$  for all  $s \in \left(0, \frac{1}{p'}\right)$  with  $\frac{1}{p'} = 1 - \frac{1}{p}$  and for all  $\psi$ . Here and below,  $H^s$  and  $W^{s,p}$  denote the usual Sobolev spaces – for instance

$$W^{s,p} = \left\{ f \in L^p \left/ \iint_{\mathbb{R}^N \times \mathbb{R}^N} |f(x) - f(y)|^p |x - y|^{-(N-sp)} dx dy < \infty \right. \right\}$$

if  $0 < s < 1$ . Several extensions to more general operators than (1) or (2) were then obtained by P. Gérard ([12], [13]), F. Golse and P. Gérard [14]. Another extension was given in R. J. DiPerna and P. L. Lions [9] where  $g \in L^2_x(H_x^{-m})$ ,  $f \in L^2_{x,v}$  (for some  $m \geq 0$ ) and it was proved that  $\bar{f} \in H^{(2(1+m))^{-1}}$ . In all these references, the strategy of proof was always basically the same: Fourier transform  $f, g, \bar{f}$  and the equation (1) [or (2)] in  $x$  [or  $(x, t)$ ] and decompose the partial Fourier transform of  $f$  in order to recover by a “ $L^2$  analysis” the desired regularity.

We are going to unify and extend here all these results. In fact, we cover much more general situations like (for instance)  $g(I - \Delta_x)^{\tau/2} (I - \Delta_v)^{m/2} \tilde{g}$  and  $\tilde{g} \in L^p(p \geq 1, m \geq 0, \tau \in [0, 1))$  and  $f \in L^q(q > 1)$ . For exemple, if  $q = p \leq 2$ ,

we will prove that  $\bar{f} \in B_2^{s,p}$  with  $s = (1 - \tau)(1 + m)^{-1}(p')^{-1}$  where  $B_q^{s,p}$  denote the usual Besov spaces. Notice that if  $\tau = m = 0$ , we recover  $s = \frac{1}{p'}$  (sharpening thus a bit the result in [15]) while if  $q = p = 2$ ,  $\tau = 0$ ,  $m \geq 0$ , we recover  $s = (2(1 + m))^{-1}$  as in [9]. Let us also emphasize that the methods used in [15], [9] do not allow for the general case mentioned above and that we can also consider the cases when  $p \neq q$  (or even when  $f$  or  $g$  have different integrabilities in  $x$  and  $v \dots$ ).

The method of proof relies upon a Littlewood-Paley decomposition of  $\bar{f}$  where each block is then split into two pieces that correspond to the action of certain convolution operators upon the corresponding dyadic blocks of  $f$  and  $g$ . Careful estimates on each piece yield the desired regularity. In the case when  $p \neq q$  (see above), the argument is a bit more complex and uses appropriately real interpolation theory on each block. The estimates on the convolution operators we introduce in the analysis of the blocks follow from some elementary considerations on  $L^p$  multipliers.

As we will see below, the scalings involved in the decomposition of each dyadic block are far from being obvious and we will also present an argument which gives some clues on these scalings. This argument was originally our first method of proof which is rather different (at least at a technical level), less elementary and leads to slightly worse results ( $B_\infty^{s,p}$  instead of  $B_2^{s,p}$ ). Its only advantage is that it gives a rather clear indication on what should be the right scales for the velocity – spectral decompositions performed in our proofs. This method in order to be fully implemented requires the use of Hardy spaces on product domains [namely,  $\mathcal{H}^1(\mathbb{R}_+^n \times \mathbb{R}_+^2)$ ] and their properties regarding multipliers and interpolation results (see, for instance, the survey paper by S. A. Chang and R. Fefferman [4] and S. A. Chang and R. Fefferman [5], K. C. Lin [18] for interpolation results with Hardy spaces on product domains).

At this stage, we would like to explain the origin and motivations for these results. The linear transport operators in (1) and (2) are the basic operators that are used in all kinetic models like, for instance, Boltzmann equations, Vlasov systems, radiative transfer or neutrons transport... In this context, averages in  $v$  of  $f$  represent the so-called macroscopic quantities (and  $v$  stands for velocities) which are expected to be smoother than  $f$  itself, the precise meaning of this very vague credo being of course the mathematical results we described above. Furthermore, not only these results provide a good mathematical basis for an intuitive statement but also they have turned out to be fundamental for global existence and stability results (in particular) of solutions to these kinetic models. Indeed, the results of [15] play an important role in the global existence proof for Boltzmann equations due to the first two authors ([7], [8]). Similarly, the averaging results of [9] lead to the studies of Vlasov-Maxwell [9] and

Vlasov-Poisson [10] systems. There is no doubt that the general results we present here will have similar applications and probably even more so in view of the great flexibility offered by them. In fact, we already know several such applications: the first one concerns the Landau's model and related Fokker-Planck models (see R. J. DiPerna and P. L. Lions [11]). Another is the convergence of splitting methods (free transport and the spatially homogeneous equation) to solutions of Boltzmann equations by L. Desvillettes [6]. Finally, we also present in section III below an application to convergence properties of sequence of solutions of "generalized kinetic models".

## II. THE TIME-INDEPENDENT CASE

Our first result is the

**THEOREM 1.** — *Let  $g \in L^p(\mathbb{R}^N \times \mathbb{R}^N)$  and let  $f \in L^p(\mathbb{R}^N \times \mathbb{R}^N)$  satisfy (1), where  $1 < p \leq 2$ . Let  $\psi \in \mathcal{D}(\mathbb{R}^N)$ , we denote by  $\bar{f}(x) = \int f(x, v) \psi(v) dv$ . Then,  $\bar{f} \in B_2^{s, p}$  where  $s = \frac{1}{p'}$ .*

*Remarks.* — 1. If  $2 \leq p < \infty$ , the same result (with the same proof) holds provided we replace  $B_2^{1/p'}$  by  $B_p^{1/p, p}$ : it suffices to observe in the proof below that since  $f, g \in L^p$  then  $f, g \in B_p^{0, p}$  (recall that  $2 \leq p < \infty$ ).

2. In fact, the proof below shows that if  $f, g \in B_p^{0, p}$  ( $1 \leq p \leq \infty$ ) then  $\bar{f} \in B_p^{s, p}$  where  $s = \min\left(\frac{1}{p'}, \frac{1}{p}\right)$ .

3. If  $1 < p < 2$ , we do not know if  $s = \frac{1}{p'}$  is "optimal". In the case  $p = 2$ , it is easy to check that  $s = \frac{1}{2}$  is indeed optimal (see for more general results of that sort P. Gérard [13]).

4. The proof below only requires that  $\psi \in L^\infty(\mathbb{R}^N)$ ,  $\text{Supp } \psi$  is compact. It is in fact even possible to replace  $\psi \in L^\infty$  by  $\psi \in L^r$  for  $p' < r \leq \infty$ , we then have to replace  $s = \frac{1}{p'}$  by  $s = \frac{1}{p'} - \frac{1}{r}$ .

5. Of course, we only need to assume that  $f, g \in L^p(\mathbb{R}^N \times \text{Supp } \psi)$ .

6. If  $N = 1$ ,  $\bar{\psi}$  does not need to have compact support as it can be verified from the proof below [therefore,  $\bar{\psi} \in L^\infty(\mathbb{R})$  is enough in view of Remark 4 above].

7. In the situation described in Theorem 1, it is in fact possible to show that  $\bar{f} \in H^{s, p}$  (where we denote by  $H^{s, p}$  the generalized Sobolev spaces

defined through the Bessel potentials). Indeed, one observes first that the informations on  $f, g$  namely that  $f, g \in L^p$  can be summarized by  $f + v \cdot \nabla_x f \in L^p$ . For  $p=2$ , we already know that  $\bar{f} \in H^{1/2}$ . Therefore, if we prove that  $\bar{f} \in \mathcal{H}^1(\mathbb{R}^N)$  if  $f + v \cdot \nabla_x f \in \mathcal{H}^1(\mathbb{R}^N \times \mathbb{R}^N)$  where  $\mathcal{H}^1(\mathbb{R}^N)$  denotes the Hardy space, then we obtain the regularity claimed above by complex interpolation arguments (recall that  $[\mathcal{H}^1, L^2]_{[\theta]} = L^p$  with  $p = \frac{2}{1+\theta}$ ,  $[\mathcal{H}^1, H^{1/2}]_{[\theta]} = H_p^{1/r, p}$ ). Now, if  $f + v \cdot \nabla_x f \in \mathcal{H}^1$ , standard results on multipliers in  $\mathcal{H}^1$  (see for instance E. Stein [22]) imply that  $f \in \mathcal{H}^1(\mathbb{R}^N \times \mathbb{R}^N)$  and we conclude easily. This argument, even if it yields a slightly more precise result than Theorem 1 since  $H^{s,p} \subset B_2^{s,p}$  if  $p < 2$ , does not carry over the various situations considered below for which we do not know if it is possible to replace  $B_2^{s,p}$  by  $H^{s,p}$  (or  $B_p^{s,p}$ ).  $\square$

*Proof of Theorem 1.* — The proof will rely on Littlewood-Paley decompositions into dyadic blocks. After introducing some notations, we will, in a first step, obtain a decomposition of  $f$  involving a splitting into two terms corresponding to decompositions of  $f$  and  $g$  respectively. In a second step, we will estimate carefully each of those terms.

We need to introduce first a few notations: let  $\psi \in C^\infty(\mathbb{R}^N)$  be spherically symmetric, supported in  $\{1 \leq |\xi| \leq 3\}$  and such that

$$\psi_0(\xi)^2 + \sum_{j=1}^{+\infty} (\psi(2^{-j}\xi))^2 = 1 \quad \text{for all } \xi.$$

We denote by  $\Delta_j$  the convolution operator defined by

$$\Delta_j \varphi = 2^{Nj} \Psi(2^j \cdot) * \varphi \quad (\forall j \geq 1), \quad \Delta_0 \varphi = \Psi_0 * \varphi \tag{4}$$

where  $\psi$  and  $\psi_0$  are respectively the Fourier transforms of  $\Psi$  and  $\Psi_0$ . Then, we have for all  $f \in L^p (1 < p < \infty)$

$$f = \sum_0^{+\infty} f_j \quad \text{where } f_j = \Delta_j^2 f. \tag{5}$$

We will denote by  $\hat{\varphi}$  the Fourier transform of a function  $\varphi(x)$  or indifferently the partial Fourier transform in  $x$  of a function  $\varphi(x, v)$ : with this convention we have

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} \hat{f}(\xi, v) \bar{\Psi}(v) \cdot dv$$

Next, we introduce a cut-off function  $\varphi_0 \in \mathcal{D}(\mathbb{R})$ , even, such that  $\varphi_0 = 1$  near 0 and we set  $\varphi_1 = 1 - \varphi_0$ .

With these notations, we are going to obtain the afore mentioned decomposition of  $\bar{f}$ . Indeed, we first write

$$\hat{f}(\xi, v) = \sum_1^{+\infty} \hat{f}_j(\xi, v) + \hat{f}_0(\xi, v).$$

or

$$\hat{f}(\xi, v) = \sum_1^{+\infty} \varphi_0\left(2^j \frac{\xi \cdot v}{|\xi|}\right) \hat{f}_j(\xi, v) + \sum_1^{+\infty} \varphi_1\left(2^j \frac{\xi \cdot v}{|\xi|}\right) \hat{f}_j(\xi, v) + \hat{f}_0(\xi, v). \tag{6}$$

Since  $\varphi_1$  vanishes if  $(\xi \cdot v)$  is small, we are going to use the differential information on  $f$  i. e.  $g = v \cdot \nabla_x f$  in the second term of the right-hand side. Indeed, since  $\Delta_j$  and  $v \cdot \nabla_x$  commute, we have obviously

$$\hat{f}_j(\xi, v) = (i(\xi \cdot v))^{-1} \hat{g}_j(\xi, v). \tag{7}$$

Therefore, combining (6) and (7) we obtain

$$\hat{f}(\xi, v) = \sum_1^{+\infty} \varphi_0\left(2^j \frac{\xi \cdot v}{|\xi|}\right) \hat{f}_j(\xi, v) - i \sum_1^{+\infty} \frac{2^j}{|\xi|} \varphi_2\left(2^j \frac{\xi \cdot v}{|\xi|}\right) \hat{g}_j(\xi, v) + \hat{f}_0(\xi, v). \tag{8}$$

where  $\varphi_2(s) = \frac{1}{s} \varphi_1(s)$ . Observe that  $\varphi_2$  is odd,  $\varphi_2 \in C^\infty(\mathbb{R})$ ,  $\varphi_2(s) = \frac{1}{s}$  for  $|s|$  large and  $\varphi_2$  vanishes near 0.

In particular, if we denote by  $A_j$  and  $B_j$  the operators [clearly defined on  $L^p(\mathbb{R}^N \times \mathbb{R}^N)$  with values in  $L^p(\mathbb{R}^N)$  for all  $p$ ] defined by

$$A_j f = \mathcal{F}_x^{-1} \left\{ \int_{\mathbb{R}^N} \varphi_0\left(2^j \frac{\xi \cdot v}{|\xi|}\right) \psi\left(\frac{\xi}{2^j}\right) \hat{f}(\xi, v) \bar{\Psi}(v) dv \right\} \tag{9}$$

$$B_j f = \mathcal{F}_x^{-1} \left\{ \int_{\mathbb{R}^N} \varphi_2\left(2^j \frac{\xi \cdot v}{|\xi|}\right) \frac{2^j}{|\xi|} \psi\left(\frac{\xi}{2^j}\right) \hat{g}(\xi, v) \bar{\Psi}(v) dv \right\} \tag{10}$$

then the identity (8) yields

$$\bar{f}(x) = \sum_1^{+\infty} A_j \Delta_j f - i \sum_1^{+\infty} B_j \Delta_j g + \int_{\mathbb{R}^N} \hat{f}_0(\xi, v) \bar{\Psi}(v) dv. \tag{11}$$

It is important to notice that the Fourier transforms of  $A_j \Delta_j f$  and  $B_j \Delta_j g$  are supported in  $\{2^j \leq |\xi| \leq 3 \cdot 2^j\}$ . Let us also point out that the operator  $A_j$  and  $B_j$  are basically of the same type, the main difference being in the “replacement of  $\varphi_0$  by  $\varphi_2$ ” which is not compactly supported but has a decay at infinity which will be sufficient for our purpose as we shall see below.

We now turn to the second step of the proof which consists in estimating  $A_j$  and  $B_j$ . The estimates we need are contained in the following two results.

LEMMA 1. — The operators A<sub>j</sub> and B<sub>j</sub> are bounded from L<sup>1</sup>(ℝ<sup>N</sup> × ℝ<sup>N</sup>) into L<sup>1</sup>(ℝ<sup>N</sup>), uniformly in j.

LEMMA 2. — The operators A<sub>j</sub> and B<sub>j</sub> are bounded from L<sup>2</sup>(ℝ<sup>N</sup> × ℝ<sup>N</sup>) into L<sup>2</sup>(ℝ<sup>N</sup>), with a norm bounded from above by C2<sup>-j/2</sup> for some positive constant C independent of j.

Admitting temporarily those estimates, we may now easily conclude the proof of Theorem 1 as follows. Let us mention that C will denote here after various positive constants independent of j, f, g. By interpolating Lemmas 1 and 2, we obtain for 1 ≤ p ≤ 2

$$\|\bar{f}_j\|_{L^p} \leq C 2^{-j/p'} (\|\Delta_j f\|_{L^p} + \|\Delta_j g\|_{L^p}) \tag{12}$$

where  $\bar{f}_j = A_j \Delta_j f - i B_j \Delta_j g$ .

Next, we recall the famous result (Littlewood-Paley [19])

$$\left( \sum_1^{+\infty} \|\Delta_j f(\cdot, v)\|_{L_x^p}^2 \right)^{1/2} \leq C \|f(\cdot, v)\|_{L_x^p}. \tag{13}$$

Therefore, if we raise this inequality to the power p and we integrate in v we obtain

$$\left\{ \int_{\mathbb{R}^N} dv \left( \sum_1^{+\infty} \|\Delta_j f(\cdot, v)\|_{L_x^p}^2 \right)^{p/2} \right\}^{1/p} \leq C \|f\|_{L_x^p(\mathbb{R}^N \times \mathbb{R}^N)}.$$

At this point, we recall that since p/2 ≤ 1, we may use the reverse Minkowski's inequality and deduce

$$\left( \sum_1^{+\infty} \|\Delta_j f\|_{L_x^p(\mathbb{R}^N \times \mathbb{R}^N)}^2 \right)^{1/2} \leq C \|f\|_{L_x^p(\mathbb{R}^N \times \mathbb{R}^N)}. \tag{14}$$

Of course, the same inequality holds for Δ<sub>j</sub>g and we deduce from (12) and (14)

$$\|\bar{f}_j\|_{L^p} = \varepsilon_j 2^{-j/p'}, \text{ where } \varepsilon_j \in l^2. \tag{15}$$

And this allows to conclude that  $\bar{f} = \sum_1^{+\infty} \bar{f}_j \in B_2^{s,p}$  where  $s = \frac{1}{p'}$ , since  $\hat{f}$  is supported in  $\{2^j \leq |\xi| \leq 3 \cdot 2^j\}$ .

There only remains to prove Lemmas 1 and 2 and we begin by the simplest one namely Lemma 2.

*Proof of Lemma 2.* — In order to prove Lemma 2, it is enough to show

$$\left\| \int_{\mathbb{R}^N} \varphi_0 \left( 2^j \frac{\xi \cdot v}{|\xi|} \right) \psi \left( \frac{\xi}{2^j} \right) \hat{f}(\xi, v) \bar{\Psi}(v) dv \right\|_{L^2(\mathbb{R}_\xi^N)} \leq C 2^{-j/2} \|\hat{f}\|_{L^2} \tag{16}$$

and

$$\left\| \int_{\mathbb{R}^N} \varphi_2 \left( 2^j \frac{\xi \cdot v}{|\xi|} \right) \frac{2^j}{|\xi|} \psi \left( \frac{\xi}{2^j} \right) \hat{g}(\xi, v) \bar{\Psi}(v) dv \right\|_{L^2(\mathbb{R}_\xi^N)} \leq C 2^{-j/2} \|\hat{g}\|_{L^2}. \tag{17}$$



The proofs of (16) and (17) are similar and we will only give the one of (17). We use Cauchy-Schwarz inequality and we find for all  $\xi$

$$\left| \int_{\mathbb{R}^N} \varphi_2 \left( 2^j \frac{\xi \cdot v}{|\xi|} \right) \psi \left( \frac{\xi}{2^j} \right) \hat{g}(\xi, v) \bar{\Psi}(v) dv \right| \leq C \|g(\xi, \cdot)\|_{L^2(\mathbb{R}_v^N)} \left| \psi \left( \frac{\xi}{2^j} \right) \right| \left( \int_{\mathbb{R}^N} \left| \varphi_2 \left( 2^j \frac{\xi \cdot v}{|\xi|} \right) \bar{\Psi}(v) \right|^2 dv \right)^{1/2}.$$

Since  $\bar{\Psi} \in L^\infty$  and is compactly supported, the last term may be estimated by

$$C \left( \int |\varphi_2(2^j t)|^2 dt \right)^{1/2} \leq C 2^{-j/2}$$

since  $\varphi_2 \in L^2(\mathbb{R})$ . At this point, (17) follows easily.  $\square$

*Proof of Lemma 1.* – Since  $\bar{\Psi} \in L^\infty$ , it is clearly enough to show that for each  $v$  the convolution operators defined by the Fourier multipliers  $\left[ \varphi_0 \left( 2^j \frac{\xi \cdot v}{|\xi|} \right) \psi \left( \frac{\xi}{2^j} \right) \right]$  and  $\left[ \varphi_2 \left( 2^j \frac{\xi \cdot v}{|\xi|} \right) \frac{2^j}{|\xi|} \psi \left( \frac{\xi}{2^j} \right) \right]$  are bounded from  $L^1(\mathbb{R}_x^N)$  into  $L^1(\mathbb{R}_x^N)$ , uniformly in  $v$  and  $j$ . Again, we will only treat the second operator since the first one is analyzed in a similar and even simpler way. Due to the invariance of multiplier norms (in  $L^p$ ) by rotations and (partial) dilations, we only have to consider the multiplier

$$\bar{m}_\lambda(\xi) = \varphi_2 \left( \lambda \frac{\xi_1}{|\xi|} \right) \psi_2(\xi) \tag{18}$$

and to prove that it is bounded in  $L^1$  independently of  $\lambda > 0$ , where  $\psi_2(z) = \frac{1}{|z|} \psi(z)$  – notice that  $\psi_2 \in \mathcal{D}(\mathbb{R}^N)$  is radial and supported in  $\{1 \leq |z| \leq 3\}$ . If  $0 < \lambda \leq 4$ ,  $m_\lambda$  lies in a bounded subset of  $\mathcal{D}(\mathbb{R}^N)$  and the bound is obvious. If  $\lambda \geq 4$ , we set  $\varepsilon = 1/\lambda$  and by a further dilation in  $\xi_1$ , we have to estimate the multiplier

$$m_\varepsilon(\eta) = \varphi_2(\eta_1 [\varepsilon^2 \eta_1^2 + \eta_2^2 + \dots + \eta_N^2]^{-1/2}) \times \psi_2((\varepsilon^2 \eta_1^2 + \eta_2^2 + \dots + \eta_N^2)^{1/2}), \quad \text{for } 0 < \varepsilon < \frac{1}{4}. \tag{19}$$

Clearly, this function is supported in  $(\eta_2^2 + \dots + \eta_N^2)^{1/2} \leq 3$  but its ‘‘support’’ in  $\eta_1$  is unbounded. In addition, without loss of generality, we may assume that  $\varphi_2$  is supported in  $\{|s| \geq 1\}$  and  $\varphi_2(s) = \frac{1}{s}$  for  $|s| \geq 2$ . Therefore, for  $\varepsilon \in \left(0, \frac{1}{4}\right)$ ,  $m_\varepsilon$  is supported in  $|\eta_1| \geq \frac{1}{2}$ ,  $(\eta_2^2 + \dots + \eta_N^2) \leq 3$ .

We then decompose  $m_\varepsilon$  using a partition of unity  $1 = \theta_0(\eta_1) + \theta_2(\eta_1)$ , where  $\theta_0 \in \mathcal{D}(\mathbb{R})$ ,  $\theta_1$  is supported in  $|\eta_1| \geq 10$ .

The multiplier  $\theta_0(\eta_1)m_\varepsilon(\eta_1, \dots, \eta_N)$  is clearly bounded since it has a uniform compact support and it is uniformly bounded together with all its derivatives.

Next, the remaining term namely  $\theta_1(\eta_1)m_\varepsilon(\eta_1, \dots, \eta_N)$  is also given by

$$\frac{\theta_1(\eta_1)}{\eta_1} \sqrt{\varepsilon^2 \eta_1^2 + \eta_2^2 + \dots + \eta_N^2} \psi_2(\sqrt{\varepsilon^2 \eta_1^2 + \eta_2^2 + \dots + \eta_N^2}).$$

Then, again by the partial dilation invariance of multipliers norm, we observe that the term

$$\sqrt{\varepsilon^2 \eta_1^2 + \eta_2^2 + \dots + \eta_N^2} \psi_2(\sqrt{\varepsilon^2 \eta_1^2 + \eta_2^2 + \dots + \eta_N^2})$$

has the same multiplier norm than  $|\eta| \psi_2(|\eta|)$  which is a given function in  $\mathcal{D}(\mathbb{R}^N)$ . In order to conclude, we just need to observe that

$$\frac{\theta_1(\eta_1)}{\eta_1} \in \mathcal{F} L^1(\mathbb{R}) \text{ since } 1 - \theta_1 \in \mathcal{S}(\mathbb{R}) \text{ and } \theta_1(0) = 0. \quad \square$$

*Remark.* – In fact, the above proof shows that  $\bar{f}$  can be written as

$$\bar{f} = \sum_{j=1}^{+\infty} A_j \Delta_j f + \sum_{j=1}^{+\infty} B_j \Delta_j g + \bar{f}_0 \tag{20}$$

where  $\bar{f}_0 = \int_{\mathbb{R}^N} \Delta_0^2 f \bar{\Psi}(v) dv$  and

$$A_j f = \int_{\mathbb{R}^N} a_j^v \star_x f(\cdot, v) \bar{\Psi}(v) dv$$

$$B_j g = \int_{\mathbb{R}^N} b_j^v \star_x g(\cdot, v) \bar{\Psi}(v) dv$$

and  $a_j^v, b_j^v$  are bounded in  $L^1(\mathbb{R}_x^N)$  uniformly in  $j, v$ . In addition, the operators  $A_j, B_j$  satisfy the estimates stated in Lemma 2. Let us also point out that it is possible to incorporate  $\Delta_j$  in  $A_j$  and  $B_j$ , a fact which yields another representation with the same properties.  $\square$

We now want to consider more general situations and in particular the ones mentioned in the Introduction. And we will see that the splitting (6) – which was relatively natural – of each block  $\hat{f}_j$  will be considerably modified. In order to explain how this splitting has to be chosen, we would like to present another method to study the regularity of  $\bar{f}$  which, even if it yields slightly worse results (*i.e.*  $B_\infty^{s,p}$  instead of  $B_2^{s,p}$ ), has the advantage to be simpler at least at a formal level. In fact, in order to be rigorous, the formal argument we present below requires a quite sophisticated machinery. This formal argument will yield the correct “ $s$ ” and will also shed some light on the scaling we will need for the splitting of the dyadic blocks  $\hat{f}_j$ .

The argument goes as follows: we write

$$\bar{f} = \mathcal{F}^{-1} \left\{ \int_{\mathbb{R}^N} \bar{\Psi}(v) dv \left\{ \varphi \left( \frac{1}{\varepsilon} \frac{v \cdot \xi}{|\xi|} \right) \hat{f} - i(v \cdot \xi)^{-1} (1 - \varphi) \left( \frac{1}{\varepsilon} \frac{v \cdot \xi}{|\xi|} \right) \hat{g} \right\} \right\} \quad (21)$$

where  $\varepsilon > 0$  is arbitrary,  $\varphi$  is a given (even) function in  $\mathcal{D}(\mathbb{R})$  [or in  $\mathcal{S}(\mathbb{R}) \dots$ ] such that  $\varphi \equiv 1$  in a neighborhood of 0. Denoting by  $\bar{f}_1^\varepsilon, \bar{f}_2^\varepsilon$  respectively the two terms appearing in the right-hand side, corresponding respectively to  $\hat{f}$  and  $\hat{g}$ , one can prove the following estimates

$$\|\bar{f}_1^\varepsilon\|_{L^p_x} \leq C \varepsilon^{1/\bar{p}} \|f\|_{L^p_{x,v}} \quad (22)$$

where  $1 < p < \infty, \bar{p} = \max(p, p')$

$$\|\bar{f}_2^\varepsilon\|_{H^{\sigma,p}} \leq C \varepsilon^{-1/\mu} \|G\|_{L^p_{x,v}} \quad (23)$$

where  $g = (I - \Delta_x)^{\tau/2} (I - \Delta_0)^{m/2} G, \tau \in [0, 1), m \geq 0, 1 < p < \infty$  and  $\sigma = 1 - \tau, \mu = \underline{p} (m \underline{p} + 1)^{-1}$  (or  $\frac{1}{\mu} = m + \frac{1}{\underline{p}}$ ) where  $\underline{p} = \min(p, p')$ . Once these estimates are proven, we use real interpolation theory (the K-method for instance - see J. Bergh and J. Löfstrom [3]) observing that we have for all  $t > 0$

$$\bar{f} = \bar{f}_1^\varepsilon + \bar{f}_2^\varepsilon, \quad \|\bar{f}_1^\varepsilon\|_{L^p} + t \|\bar{f}_2^\varepsilon\|_{H^{\sigma,p}} \leq C t^\alpha \|G\|_{L^p}^\alpha \|f\|_{L^p}^{1-\alpha}$$

where

$$\alpha = (m + 1)^{-1} \bar{p}^{-1},$$

choosing  $\varepsilon = (t \|G\|_{L^p} \|f\|_{L^p}^{-1})^{\alpha \bar{p}}$ . Hence,  $\bar{f} \in [L^p, H^{\sigma,p}]_{\alpha, \infty} = B_{\infty}^{s,p}$  with

$$s = (1 - \tau) \bar{p}^{-1} (m + 1)^{-1}. \quad (24)$$

In particular, if  $1 < p \leq 2, s = (1 - \tau) \left( \frac{p-1}{p} \right) (m + 1)^{-1}$  and if  $\tau = m = 0$  we

recover  $s' = \frac{1}{p'}$ . Observe also that if  $G \in L^q_{x,v}$  the above argument still applies and yields  $\bar{f} \in B_{\infty}^{s,r}$  with

$$s = (1 - \tau) \alpha, \quad \left. \frac{1}{r} = \frac{\alpha}{q} + \frac{(1 - \alpha)}{p}, \quad \alpha = \frac{1}{\underline{p}} \left( m + \frac{1}{\underline{p}} + \frac{1}{\underline{q}} \right)^{-1} \right\} \quad (25)$$

As we will see later on, we will keep the idea of using real interpolation theory in order to treat by the method of proof of Theorem 1 (*i.e.* via Littlewood-Paley decompositions) the general cases when  $f \in L^p, G \in L^p$  and  $q$  may be different from  $p$ . In fact, even when  $p = q$ , the above argument gives an indication on the right choice of scales in the decomposition (6) - or in other words, why choosing  $2^j$  in (6) or more complicated expressions for the general situations we will investigate below. Indeed,

one should choose (in the case  $p=2$ )  $\varepsilon$  like  $|\xi|^\beta$  i.e.  $2^{\beta j}$  with  $\beta=2\alpha = \frac{1}{m+1}$  when  $\tau=0$  and  $|\xi|^{\beta(1-\tau)}$  i.e.  $2^{\beta(1-\tau)j}$  when  $\tau \in (0, 1)$ .

Before we present precise results on such general situations, we would like to make a few remarks on the above argument. Clearly, the heart of the matter lies in the inequalities (22), (23) which are straightforward for  $p=2$  by similar arguments to the proof of Lemma 2. It is then enough to treat the case when  $1 < p < 2$ , which is more delicate than the case  $p=2$  if we want a rigorous proof. The first step consists in observing that, in view of classical results on  $L^p$  multipliers, one can easily show (using once more the partial dilations invariance of  $L^p$  multipliers norms)  $\varphi\left(\frac{1}{\varepsilon} \frac{v \cdot \xi}{|\xi|}\right)$  or  $\frac{\xi_\alpha}{(v \cdot \xi)}(1-\varphi)\left(\frac{1}{\varepsilon} \frac{v \cdot \xi}{|\xi|}\right)$  (for  $1 \leq \alpha \leq N$ ) define  $L^p$  multipliers whose norms are bounded independently of  $\varepsilon > 0, v \in \mathbb{R}^N$ . This argument breaks down for  $p=1$  where we have, for each  $v \neq 0$ , to replace  $L^1$  by a space  $\mathcal{H}_v^1$  which depends in fact only of  $\frac{v}{|v|}$ . By rotation, it is enough to define this space

when  $\frac{v}{|v|} = \pm e_1$  in which case we define  $\mathcal{H}_v^1$  to be  $\mathcal{H}^1(\mathbb{R}_+^2 \times \mathbb{R}_+^N)$  i.e. the

$H^1$  Hardy-Stein-Weiss space on the product domain  $\mathbb{R}_+^2 \times \mathbb{R}_+^N$ . In order to conclude the proof of (22) – and similarly for (23) – we use a two stages interpolation argument observing first that it is enough to show, using the Marcinkiewicz interpolation theorem and the easy case  $p=2$ , that the

(continuous) linear map  $(f \mapsto \overset{T_\varepsilon^1}{f_2})$  is of weak type  $(q, q)$  with an operator norm bounded by  $C\varepsilon^{1/q}$  for any  $q \in (1, 1)$ . Then, if  $f \in L_{x,v}^q$ , we are going to use, for each  $v$ , a Calderon-Zygmund decomposition of  $f(\cdot, v)$  in  $L^2(\mathbb{R}_x^N)$  and  $\mathcal{H}_v^1$  – as defined above –: indeed, by the results of [5], [18], for all  $\alpha > 0, t > 0$ , we can find  $f_1(\cdot, v) \in \mathcal{H}_v^1, f_2(\cdot, v) \in L^2(\mathbb{R}_x^N)$  such that  $f = f_1 + f_2$  and

$$\left. \begin{aligned} \|f_2(\cdot, v)\|_{L^2(\mathbb{R}_x^N)}^2 &\leq C(\alpha t)^{2-q} \|f(\cdot, v)\|_{L^q(\mathbb{R}_x^N)}^q, \\ \|f_1(\cdot, v)\|_{\mathcal{H}_v^1}^2 &\leq C(\alpha t)^{1-q} \|f(\cdot, v)\|_{L^q(\mathbb{R}_x^N)}^q \end{aligned} \right\} \tag{26}$$

where  $C \geq 0$  is independent of  $\alpha, t, v, f$ . Then, we argue as follows  $\text{meas} \{ |T_\varepsilon^1 f| > \alpha \}$

$$\begin{aligned} &\leq \text{meas} \left\{ |T_\varepsilon^1 f_2| \geq \frac{\alpha}{2} \right\} + \text{meas} \left\{ |T_\varepsilon^1 f_1| \geq \frac{\alpha}{2} \right\} \\ &\leq C \left\{ \frac{1}{\alpha^2} \|T_\varepsilon^1 f_2\|_{L_x^2}^2 + \frac{1}{\alpha} \|T_\varepsilon^1 f_1\|_{L_x^1} \right\} \\ &\leq C \left\{ \frac{\varepsilon}{\alpha^2} \|f_2\|_{L_{x,v}^2}^2 + \frac{1}{\alpha} \int_{\mathbb{R}^N} \|f_1\|_{\mathcal{H}_v^1} dv \right\} \end{aligned}$$

in view of the bounds already proven. Then, (26) yields

$$\text{meas} \{ |T_\varepsilon^1 f| > \alpha \} \leq C \alpha^{-q} \{ \varepsilon t^{2-q} + t^{1-q} \} \|f\|_{L_{\xi,v}^q}^q \leq (C a^{-1} \varepsilon^{1/q} \|f\|_{L_{\xi,v}^q})^q$$

where we choose  $t = \frac{1}{\varepsilon}$ , proving thus our claim.  $\square$

We may now state and prove a first extension of Theorem 1:

**THEOREM 2.** — *Let  $1 < p \leq 2$ , let  $G \in L^p(\mathbb{R}^N \times \mathbb{R}^N)$  and let  $f \in L^p(\mathbb{R}^N \times \mathbb{R}^N)$  satisfy (1) with  $g = (-\Delta_x + I)^{\nu/2} (\Delta_x + I)^{m/2} G$  where  $\tau \in [0, 1)$ ,  $m > 0$ . Then,  $\bar{f} \in B_2^{s,p}(\mathbb{R}^N)$  where  $s = (1 - \tau)(p')^{-1}(m + 1)^{-1}$ .*

*Remarks.* — 1. Analogues of remarks 1, 2, 4-6 following Theorem hold here.

2. We do not know if  $\bar{f} \in B_p^{s,p}(\mathbb{R}^N)$  or even  $H^{s,p}(\mathbb{R}^N)$ .

*Sketch of proof.* — We only explain the case when  $\tau = 0$ , the case when  $\tau > 0$  being easily adapted. We modify the decomposition (6) accordingly to the observations made above. We now write (with the same treatment of the case  $j = 0$  as above, treatment we do not detail below)

$$\hat{f}(\xi, v) = \sum_j \varphi_0 \left( 2^{\beta j} \frac{\xi \cdot v}{|\xi|} \right) \hat{f}_j(\xi, v) + \sum_j \varphi_1 \left( 2^{\beta j} \frac{\xi \cdot v}{|\xi|} \right) \hat{f}_j(\xi, v) \quad (27)$$

where  $\beta = \frac{1 - \tau}{1 + m}$ . Next, (8) becomes

$$\begin{aligned} \hat{f}(\xi, v) = \sum_j \varphi_0 \left( 2^{\beta j} \frac{\xi \cdot v}{|\xi|} \right) \hat{f}_j(\xi, v) \\ - i \sum_j \varphi_1 \left( 2^{\beta j} \frac{\xi \cdot v}{|\xi|} \right) \frac{1}{\xi \cdot v} \cdot (I - \Delta_v)^{m/2} \hat{G}_j(\xi, v). \end{aligned} \quad (28)$$

And integrating with respect to  $\bar{\Psi}(v) dv$ , we deduce

$$\begin{aligned} \bar{f}(\xi) = \mathcal{F}^{-1} \left\{ \sum_j \int_{\mathbb{R}^N} \varphi_0 \left( 2^{\beta j} \frac{\xi \cdot v}{|\xi|} \right) \hat{f}_j(\xi, v) \bar{\Psi}(v) dv \right. \\ \left. - i \sum_j \int_{\mathbb{R}^N} (I - \Delta_v)^{m/2} \left\{ \varphi_1 \left( 2^{\beta j} \frac{\xi \cdot v}{|\xi|} \right) \frac{1}{\xi \cdot v} \bar{\Psi}(v) \right\} G(\xi, v) \right\}, \end{aligned} \quad (29)$$

a decomposition that we may write again with obvious notations

$$\bar{f} = \sum_j A_j \Delta_j f - \sum_j B_j \Delta_j G.$$

Then, the proof of Lemma 1 adapts easily (if  $m$  is an even integer — and the general case follows from complex interpolation) and yields that  $A_j$  and  $B_j$  are bounded from  $L^1(\mathbb{R}^N \times \mathbb{R}^N)$  into  $L^1(\mathbb{R}^N)$ , uniformly in  $j$  — one just replaces at the end of the proof  $\frac{\theta_1(\eta_1)}{\eta_1}$  by  $\frac{\theta_1(\eta_1)}{\eta_1^{m+1}}$  ( $m \in \mathbb{N}$ ,  $m \geq 0$ ) which

is also in  $\mathcal{F}L^1(\mathbb{R})$  since  $\theta_1$  vanishes in a neighborhood of 0,  $\theta_1 \in L^\infty$  and  $\theta_2(\eta_1) \equiv 1$  for  $|\eta_1|$  large. A similar adaptation of Lemma 2 yields that  $A_j, B_j$  are bounded from  $L^2(\mathbb{R}^N \times \mathbb{R}^N)$  into  $L^2(\mathbb{R}^N)$ , with a norm bounded from above by  $C2^{-\beta j/2}$  for some positive constant  $C$  independent of  $j$ . We then conclude exactly as in the proof of Theorem 1.  $\square$

We may now consider a general extension of Theorems 1-2 namely the case when  $f \in L^p, g = (I - \Delta_x)^{\tau/2} (I - \Delta_v)^{m/2} G, G \in L^q$  with  $1 < p < \infty, 1 \leq q < \infty$ . Our result will require the introduction of Besov spaces modelled on Lorentz spaces instead of  $L^p$  spaces: if we replace in the definition of  $B_q^{s,p}$  spaces the  $L^p$  norm of the dyadic blocks by Lorentz norms  $L^{p,r}$ , we obtain a space we denote by  $B_{q,r}^{s,p}$  - see J. Peetre [21] for more details. We may now state the

**THEOREM 3.** - Let  $1 \leq q < \infty$ , let  $G \in L^q(\mathbb{R}^N \times \mathbb{R}^N)$ , let  $1 < p < \infty$  and let  $f \in L^p(\mathbb{R}^N \times \mathbb{R}^N)$  satisfy (1) with  $g = (I - \Delta_x)^{\tau/2} (I - \Delta_v)^{m/2} G$  where  $\tau \in [0, 1), m \geq 0$ . Then,

$$\begin{aligned} \bar{f} \in B_{t,r}^{s,r} \quad \text{where} \quad s &= (1 - \tau)\alpha, \quad \alpha = \frac{1}{p} \left( m + \frac{1}{p} + \frac{1}{q} \right)^{-1}, \\ \frac{1}{r} &= \frac{\alpha}{q} + \frac{(1 - \alpha)}{p}, \quad \bar{p} = \max(p, p'), \\ q &= \min(q, q'), \quad \frac{1}{t} = \frac{1 - \alpha}{\max(p, 2)} + \frac{\alpha}{\max(q, 2)}. \end{aligned}$$

*Remarks.* - 1. Remarks analogous to those made after Theorems 1 and 2 also hold here.

2. Of course,  $\bar{f} \in L^p$ . Notice also that Theorem 3 yields that  $\bar{f} \in B_{\infty}^{s,r}$ .

3. All these results can be localized.

4. If we are interested in the  $L^1_{loc}$  compactness of a sequence  $\bar{f}_n$  of averages, in view of the proof of Theorem 3 (and of Remark 3 above), it is enough to assume that  $\bar{f}_n$  is bounded in  $L^p_{loc}$  for some  $p > 1$  and that  $\{(I - \Delta_x)^{-1/2} (I - \Delta_v)^{-m/2} g_n\}$  is compact in  $L^1_{loc}$  for some  $m \geq 0$ .

*Sketch of proof.* - We only explain the case when  $\tau = m = 0$ , the general case following (as in Theorem 2) by easy adaptations. Hence, we assume that  $g = G \in L^q$ . Then, let  $\lambda > 0$ , we replace (6) by

$$\begin{aligned} f &= \sum_j (\hat{f}_j^\lambda + \hat{g}_j^\lambda), \quad \hat{f}_j^\lambda = \varphi_0 \left( \lambda \frac{\xi \cdot v}{|\xi|} \right) \hat{f}_j, \\ \hat{g}_j^\lambda &= (i(\xi \cdot v))^{-1} \varphi_1 \left( \lambda \frac{\xi \cdot v}{|\xi|} \right) \hat{g}_j \end{aligned} \tag{30}$$

which leads to

$$\bar{f} = \sum_j \bar{f}_j, \quad \bar{f}_j = A_j^\lambda \Delta_j f - i B_j^\lambda \Delta_j g \tag{31}$$

Then, the proofs of Lemma 1 and 2 imply

$$\|A_j^\lambda \Delta_j f\|_{L^p} \leq C \lambda^{-1/\bar{p}} \|\Delta_j f\|_{L^p} \tag{32}$$

$$\|B_j^\lambda \Delta_j g\|_{L^q} \leq C \lambda^{-1/q} 2^{-j} \|\Delta_j g\|_{L^q} \tag{33}$$

and we may conclude easily.  $\square$

Another possible extension – which can be connected with the extensions described in Theorems 2-3 or in the remarks following these results – consists in assuming that  $f, g \in L^q(\mathbb{R}_v^N; L^p(\mathbb{R}_x^N))$  with  $1 < q \leq p$ :

**THEOREM 5.** – *Let  $1 < q \leq p$ , let  $f \in L^q(\mathbb{R}_v^N; L^p(\mathbb{R}_x^N))$  satisfy (1) with  $g \in L^q(\mathbb{R}_v^N; L^p(\mathbb{R}_x^N))$ . Then,  $f \in B_t^{s,p}$  where  $s = \frac{1}{q}$ ,  $t = \max(p, 2) \frac{q'}{p}$ .*

*Sketch of proof.* – Theorem 5 is proven in the same way than Theorem 1. We just observe that we have

$$\|A_j \Delta_j f\|_{L^p(\mathbb{R}^N)} \leq C 2^{-j/\bar{q}} \|\Delta_j f\|_{L^q(\mathbb{R}_v^N; L^p(\mathbb{R}_x^N))} \tag{34}$$

and

$$\|B_j \Delta_j g\|_{L^p(\mathbb{R}^N)} \leq C 2^{-j/\bar{q}} \|\Delta_j g\|_{L^q(\mathbb{R}_v^N; L^p(\mathbb{R}_x^N))}. \tag{35}$$

In order to conclude, we observe that on one hand

$$\|\Delta_j \varphi\|_{L^\infty(\mathbb{Z}, L^1(\mathbb{R}_v^N; L^p(\mathbb{R}_x^N)))} \leq \|\varphi\|_{L^1(\mathbb{R}_v^N; L^p(\mathbb{R}_x^N))}$$

and on the other hand, as we remarked in the proof of Theorem 1, we have with  $\sigma = \max(p, 2)$

$$\|\Delta_j \varphi\|_{L^\sigma(\mathbb{Z}, L^p(\mathbb{R}_v^N; L^p(\mathbb{R}_x^N)))} \leq C \|\varphi\|_{L^p(\mathbb{R}_v^N; L^p(\mathbb{R}_x^N))} \tag{36}$$

for all  $\varphi$ . Therefore,  $\|\Delta_j \varphi\|_{L^q(\mathbb{R}_v^N; L^p(\mathbb{R}_x^N))} \in l^t(\mathbb{Z})$  by interpolation where  $\frac{1}{t} = \frac{1-\theta}{\sigma}$ ,  $\theta + \frac{1-\theta}{p} = \frac{1}{q}$ .  $\square$

### III. THE TIME-DEPENDENT CASE AND APPLICATIONS

It is quite clear that the method of proof introduced in section II can be applied to more general operators  $(a(v) \cdot \nabla_x)$  than  $(v \cdot \nabla_x)$ . Of special interest is, of course, the “relativistic streaming” operator  $(v(1 + |v|^2)^{-1/2} \cdot \nabla_x)$  for which the same type of results hold – with exactly the same gain of regularity. Also, exactly as in [15], one can replace the space of velocities  $(v \in \mathbb{R}^N)$  and the measure with respect to which one averages [namely  $\psi(v) dv$ ] by more general ones. In particular, one can do so in order to accommodate in the same setting the case of equations (1) and (2). We will not do so here and we just present (without proofs) the analogues of Theorems 2 and 3 for the case of (2).

**THEOREM 5.** — Let  $p \in (1, \infty)$ , let  $G \in L^p(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R})$  and let  $f \in L^p(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R})$  satisfy (2) with  $g = (I - \Delta_x)^{\tau/2} (I - \Delta_v)^{m/2} G$ ,  $\tau \in [0, 1)$ ,  $m \geq 0$ . Let  $\bar{\Psi} \in \mathcal{D}(\mathbb{R}_v^N)$ , we denote by  $f(x, t) = \int_{\mathbb{R}^N} f(x, v, t) \bar{\Psi}(v) dv$ . Then,  $\bar{f} \in B_{t, \infty}^{s, p}(\mathbb{R}^N \times \mathbb{R})$  where  $t = \max(p, 2)$ ,  $s = (1 - \tau) \bar{p}^{-1} (m + 1)^{-1}$ ,  $\bar{p} = \max(p, p')$ .

**THEOREM 6.** — Let  $1 \leq q < \infty$ , let  $G \in L^q(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R})$ , let  $1 < p < \infty$  and let  $f \in L^p(\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R})$  satisfy (2) with  $g = (I - \Delta_x)^{\tau/2} (I - \Delta_v)^{m/2} G$ ,  $\tau \in [0, 1)$ ,  $m \geq 0$ . Then,  $\bar{f} \in B_{t, \infty}^{s, r}(\mathbb{R}^N \times \mathbb{R}^N)$  where  $s = (1 - \tau) \alpha$ ,  $\alpha = \frac{1}{p} \left( m + \frac{1}{p} + \frac{1}{q} \right)^{-1}$ ,  $\frac{1}{r} = \frac{\alpha}{q} + \frac{1 - \alpha}{p}$ ,  $\bar{p} = \max(p, p')$ ,  $\underline{q} = \min(q, q')$ ,  $\frac{1}{t} = \frac{1 - \alpha}{\max(p, 2)} + \frac{\alpha}{\max(q, 2)}$ .

*Remark.* — All the remarks made after Theorems 1-4 adapt easily here.

Let us now conclude with a brief sketch of possible applications to kinetic models (like Vlasov-Maxwell, Boltzmann systems...): let  $f^n$  be founded in  $L^\infty(0, \infty; L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N, 1 + |v|^2))$ ,  $f^n \geq 0$ , satisfy in renormalized sense the following equation

$$\frac{\partial f^n}{\partial t} + v \cdot \nabla_x f^n + F^n \cdot \nabla_v f^n = C^n \quad \text{in } \mathbb{R}_x^N \times \mathbb{R}_v^N \times (0, \infty) \quad (37)$$

where we assume

$$\left. \begin{aligned} & \operatorname{div}_v (F^n) = 0, \\ & F^n \text{ is bounded in } L^1_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)) \\ & C^n = C^n_+ - C^n_-, \quad C^n_\pm \geq 0, \\ & C^n_\pm 1_{(f^n \leq M)} \text{ is bounded in } L^1_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)) \end{aligned} \right\} \quad (38)$$

$$\text{for all } M \in (0, \infty). \quad (39)$$

With these conditions, the notion of renormalized solutions, as introduced in [7], [8], consists in imposing that, for all  $\beta \in C^1([0, \infty), [0, \infty))$  such that  $\beta'(t) = 0$  if  $t$  is large, the following holds

$$\begin{aligned} \frac{\partial}{\partial t} \{ \beta(f^n) \} + v \cdot \nabla_x \{ \beta(f^n) \} + \operatorname{div}_v \{ F^n \beta(f^n) \} \\ = \beta'(f^n) C^n \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)) \end{aligned} \quad (40)$$

observing that  $F^n \beta(f^n)$ ,  $C^n \beta'(f^n) \in L^1_{\text{loc}}$  because of (38), (39). Notice, by the way, that, at least formally, the bound on  $C^n_- 1_{(f^n \leq M)}$  implies the bound on  $C^n_+ 1_{(f^n \leq M)}$  as it is easily seen upon integrating (40). Exactly as in [7], we see that we can localize (40) in an arbitrary set of the form  $B_R \times B_R \times (\varepsilon, T)$  for all  $R, T < \infty$ ,  $\varepsilon > 0$ . Therefore, we deduce from our regularity results that, for each  $\psi \in \mathcal{D}(\mathbb{R}^N)$   $\left( \int \beta(f^n)(x, v, t) \psi(x) dv \right)_n$  is



relatively compact in  $L^1(B_R \times (\varepsilon, T))$  for all  $R, T < \infty, \varepsilon > 0$ . In view of the bounds we assumed on  $f^n$ , we deduce easily that

$$\int_{\mathbb{R}^N} \beta(f^n) \psi \, dv \text{ is relatively compact in } L^q(0, T; L^1(B_R)), \quad (41)$$

for all  $1 \leq q < \infty, R, T < \infty, \psi \in L^1 + L^\infty, \beta \in C([0, \infty), [0, \infty)), \frac{\beta(t)}{\beta} \xrightarrow{t \rightarrow \infty} 0$ .

And if we know that  $f^n$  is weakly compact in  $L^1_{loc}$ , then we may even take  $\beta$  such that  $\beta(t)(1+t)^{-1}$  is bounded and thus, in particular,  $\beta(t) \equiv t$ . This line of arguments shows that the conclusion (41) is valid for general Boltzmann-type models (including Vlasov-Maxwell-Boltzmann systems, or systems, or models for inelastic collisions...) and whenever we have the additional information of the weak compactness in  $L^1_{loc}$  of  $f^n$  then all the convergence analysis made in [7] on averages or on  $Q^\pm(f^n, f^n)$  applies. So, this seems to be the unique role of entropy in the results of [7]. In particular, this shows that this part of the results of [7] is now available for Vlasov-Maxwell-Boltzmann systems.

The above arguments also show that the main stability result proved in [9] on sequences of smooth solutions of Vlasov-Maxwell systems is still valid for sequences of weak solutions since we have in that situation

$$\frac{\partial f^n}{\partial t} + v \cdot \nabla_x f^n = \text{div}_v(F^n f^n) \text{ in } \mathcal{D}'(\mathbb{R}_x^N + \mathbb{R}_v^N \times (0, \infty))$$

where  $f^n$  is bounded in  $L^\infty(0, \infty; L^2_{x,v} \cap L^1_{x,v}(1+|v|^2))$ ,  $\frac{F^n}{1+|v|}$  is bounded in  $L^\infty((0, \infty) \times \mathbb{R}_v^N; L^2(\mathbb{R}_x^N))$ . Then, our results immediately yield that  $\int f^n \psi(v) \, dv$  is relatively compact in  $L^q(0, T; L^p(B_R))$  (for all  $R, T < \infty, 1 \leq q < \infty, 1 \leq p < 2$  and for all  $\psi$  such that  $\frac{\psi}{(1+|v|)^\sigma} \in L^1 + L^\infty, \sigma \in [0, 2)$ ). This then allows to follow *mutatis mutandis* the arguments of [9].

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