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A general approach to the existence of minimizers of one-dimensional non-coercive integrals of the calculus of variations

by

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ABSTRACT. — We present a general approach to get the existence of minimizers for a class of one-dimensional non-parametric integrals of the calculus of variations with non-coercive integrands. Motivated by the concrete applicative relevance of the problems, we extend the notion of solution to a class of locally absolutely continuous functions with generic boundary values. We extend by lower semi-continuity the functionals and we prove for them representation formulae. Assuming some structure conditions on the partial derivatives of the integrand, we obtain some $W^{1,\infty}_{loc}$ *a priori* estimates that we use as a main tool to get existence. The results are then applied to get existence theorems for the classical Fermat's problem and for recent optimal foraging models of behavioural ecology.

Key words : Calculus of variations, non-coercive integrals, optimal foraging.

RÉSUMÉ. — On présente une approche générale pour obtenir l'existence de minimiseurs pour une classe d'intégrales unidimensionnelles non paramétriques du calcul des variations avec intégrandes non coercitives. En

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vue des applications concrètes, on étend la notion de solution à une classe de fonctions absolument continues localement ayant des valeurs aux bords génériques. On étend par semi-continuité les fonctionnelles pour lesquelles on prouve des formules de représentation. Supposant quelques conditions de structure sur les dérivées partielles de l'intégrande, on obtient *a priori* des estimations $W^{1,\infty}_{\mathrm{loc}}$ que l'on utilise pour obtenir l'existence. Les résultats sont ensuite appliqués à des exemples classiques (notamment au problème de Fermat) et à des modèles récents d'approvisionnement optimal en écologie du comportement, pour lesquels on démontre des théorèmes d'existence.

1. INTRODUCTION

Most of the classical examples of problems in the calculus of variations are related to the minimization, in a given class of functions $v=v(x)$, of one-dimensional non-parametric integrals of the type

$$F(v)=\int_a^b f(x, v, v')\,dx. \tag{1.1}$$

Often the problems are *non-coercive*, in the sense that the function $f=f(x, s, \xi)$ grows (at most) linearly when $|\xi| \to +\infty$. We recall for example some classical problems (i) to (iii) and recent ones (iv), (v) with non-coercive integrand (see sections 5 and 6 for more details):

(*i*) *The brachistocrone problem.* First considered by J. Bernoulli in 1696, it is related to the minimization of the integral

$$F(v)=\int_a^b \sqrt{\frac{1+v'^2}{v}}\,dx \tag{1.2}$$

in the class of functions $v \in W^{1,1}(a, b)$ such that $v(a)=0$, $v(b)=B>0$ and $v(x) \geqq 0$ for $x \in [a, b]$.

(ii) *The surface of revolution of minimal area.* One is led to minimize the one-dimensional integral

$$F(v)=\int_a^b v\sqrt{1+v'^2}\,dx \tag{1.3}$$

in the class of functions $v \in W^{1,1}(a, b)$ such that $v(a)=A$, $v(b)=B$ and $v(x) \geqq 0$ for $x \in [a, b]$.

(iii) *Fermat's principle in geometrical optics.* The problem is to find the trajectory $v=v(x)$ of a ray of light propagating in an inhomogeneous medium, so that the time $F=F(v)$ required to travel from the point $(0, S_0)$ to the point $(1, S_1)$ is minimal. It consists then in the minimization of

$$F(v)=\int_0^1 a(x, v)\sqrt{1+v'^2}\,dx \tag{1.4}$$

in the class of functions $v\in W^{1,1}(0,1)$ such that $v(0)=S_0$, $v(1)=S_1$ and $v(x)\geqq 0$ for $x\in[0,1]$. The coefficient $a=a(x, s)$ is equal to the inverse of the velocity of light in the medium and is proportional to the index of refraction. Of course, the integrals (1.2) and (1.3) are particular cases of (1.4).

(iv) *Adiabatic model of the atmosphere.* This model, considered recently by Ball [6], predicts a finite height to the atmosphere. The problem consists in the minimization of

$$F(v)=\int_0^1 \left\{\frac{p_0}{(\gamma-1)\,v'^{\gamma-1}}+r_0\,g v\right\}dx \tag{1.5}$$

in the class of functions $v\in W^{1,1}(0, 1)$ such that $v(0)=0$, $v(1)=h$ and $v'(x)>0$ for a.e. $x\in[0,1]$. The positive constants p_0, r_0, $\gamma>1$ and h are gas constants and g the gravity constant.

(v) *Models in behavioural ecology.* This variational problem appears in models of optimal foraging theory and has been first considered by Arditi and Dacorogna [2]. The problem is related to the minimization of an integral of the type

$$F(v)=\int_0^1 \{\rho(x)e^{-v'}+h(x)(1+v)^p\}\,dx, \tag{1.6}$$

for $p>1$, in the class of functions $v\in W^{1,1}(0,1)$ such that $v(0)=0$, $v(1)=S>0$ and $v'(x)\geqq 0$ for a.e. $x\in[0,1]$. In this case the integrand is bounded (and therefore non-coercive) with respect to $v'\geqq 0$.

The classical problems (i) and (ii) have been extensively studied each by itself, by considering the special structure of each integrand f; one can *see* for example [12], [32], [33] (*see* also the recent paper [14] for the brachistocrone). Up to now, an existence theorem for Fermat's problem (iii) seems not known. The more recent problem (v), motivated by models in behavioural ecology ([2], [3], [4], [9], [20]), has been considered in [8], [10], [11] for some functions $\rho(x)$ and $h(x)$.

Up to now, a general theorem for the minimization of non-parametric non-coercive integrals seems not known. An approach that has been already used in the literature consists in proving that a related parametric problem has a solution that, in some cases, is also a minimizer of the original non-parametric integral. Problems of slow growth are treated with this method in chapter 14 of the book of Cesari [12]. However, with this method, it is not possible to handle most of the applications and many classical examples, as the integral (1.3) of the surface of revolution of minimal area, which is a particular but relevant example of our approach.

In this paper, we present a general approach to get the existence of minimizers for a class of one-dimensional non-parametric integrals of the calculus of variations. We shall show (theorems 4.1, 4.3 and 4.4) that it is possible not to assume that the problem is coercive, by assuming instead some structure conditions on the partial derivatives of f [*see* in particular (3.5) and (3.7)]. We then apply the general results to solve the classical Fermat's problem (iii) (*see* theorem 5.1) and to obtain a new existence theorem for some models in behavioural ecology including (v), under new assumptions relevant and natural in the specific application (theorem 6.1).

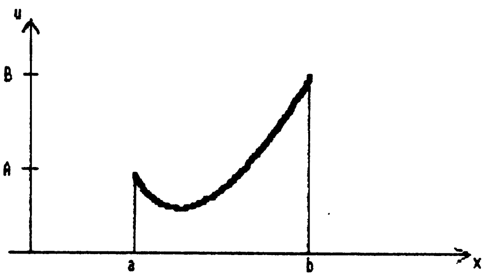
We first introduce a natural extension of the notion of solution. We consider the class of locally absolutely continuous functions v with generic boundary values $v(a)$ and $v(b)$ (and not necessarily the original prescribed values A and B) and we extend to this class the given integral functional $F(v)$ "by lower semicontinuity".

Under assumptions general enough to be verified in many applications including the above examples, we prove in section 2 that the extended functional $\bar{F}(v)$ has an integral representation. By using then the $W^{1,\infty}_{\mathrm{loc}}$ *a priori* estimates given in section 3, we propose in section 4 three existence theorems including the non-coercive case.

Let us rapidly describe what we obtain by considering again the example (ii) of surfaces of revolution of minimal area. It is well-known that the integral F in (1.3) has no minimum in the Sobolev class of functions $v \in W^{1,1}(a, b)$ such that $v(a) = A$ and $v(b) = B$, if $b - a$ is too large with respect to A and B. But geometrically and physically (and also analytically, by considering integrals in parametric form), there always exists a surface of revolution of minimal area whose shape u is either a catenary (*Fig.* 1), or equal to zero in the open interval $]a, b[$ (*Fig.* 2).

In the case of Figure 2, the surface of revolution is the union of two disks perpendicular to the x axis, of radius respectively A and B, positioned respectively at $x = a$ and $x = b$. The extended functional $\bar{F}$ at u turns out to be equal to the sum of the areas of the two disks, while ingenuously we could have taken as an extension $\bar{F} = F$ and obtain $\bar{F}(u) = 0$ (if we defined the extension $\bar{F}$ as F itself for all $v \in W^{1,1}_{\mathrm{loc}}$, we would then obtain that $u = 0$ is always the minimizer of $\bar{F}$ among functions with generic boundary values).

We consider again example (ii) together with some other classical examples [including the preceding ones (i) to (iv)] in section 5. In particular, with theorem 5.1 we obtain an existence result for Fermat's problem (iii) that, to our knowledge, is new. In section 6, we prove an existence theorem



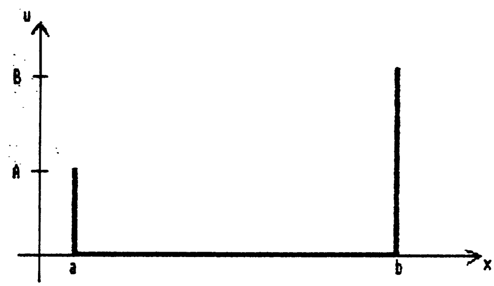


Fig. 1 Fig. 2

for the ecological problem (v) under assumptions that are new with respect to [11] and that are natural in the application.

From a mathematical point of view, the solutions we find are those of an extended (or relaxed) problem. But at least for the cases that we consider in sections 5 and 6, they have a concrete applicative (geometrical, physical or ecological) relevance in the sense that it is meaningless, from the point of view of the applications, to discriminate if they assume the boundary data or not.

Let us conclude this introduction by recalling that in the extension from F to $\bar{\mathrm{F}}$, we follow a scheme that already exists in the literature. Since Lebesgue [22], it has been considered by De Giorgi, Giusti, Miranda ([19], [27]), by Serrin [30] and many others (for example [16], [18], [24], [28]; *see* also the relaxation procedure by Ekeland and Temam [17]). In particular Dal Maso [16] considers the extension of an integral functional to $\mathrm{BV}(\Omega)$, the space of functions of bounded variation, where Ω is an open set of $\mathscr{R}^n$, with $n \geqq 1$. Our extension in section 2 is related to his extension, but for $n=1$, our more general assumptions are satisfied in particular by problems like the above (i) to (v). Finally, let us mention that a similar extension procedure has been recently considered by Marcellini [26], to treat the phenomenon of cavitation in nonlinear elasticity.

2. REPRESENTATION FORMULAE FOR EXTENDED FUNCTIONALS

Let us consider non-coercive functionals of the type

$$\mathrm{F}(v)=\int_0^1 f(x, v, v')\,dx, \tag{2.1}$$

defined for functions $v:[0,1]\rightarrow\mathbb{R}$ belonging to a subset $\mathscr{W}_p$ of $\mathrm{W}^{1,p}(0,1)$, the usual Sobolev's space, for $p \geqq 1$.

For functions belonging to $\bar{\mathscr{W}}_p$, *i.e.* the closure of $\mathscr{W}_p$ in $W^{1,p}_{loc}(0, 1)$ with respect to its weak topology, let us define the weak lower semicontinuous extension $\bar{F}$ of F by:

(2.2) $\bar{F}(v)=\inf\{\liminf_k F(v_k) : (v_k)\subset\mathscr{W}_p,$

v_k *weakly converges to* v *in* $W^{1,p}_{loc}\}$, for every $v\in\bar{\mathscr{W}}_p$.

We are then concerned in this section with finding a representation formula of $\bar{F}$ for different choices of $\mathscr{W}_p$.

We first consider the following subspace $\mathscr{W}_p$ of functions satisfying boundary conditions:

$$\mathscr{W}_p=\{v\in W^{1,p}(0,1) : v(0)=S_0,\ v(1)=S_1\}, \tag{2.3}$$

where S_0, $S_1\in\mathbb{R}$, and we give the representation of $\bar{F}$ in theorem 2.1. In this case, $\bar{\mathscr{W}}_p=W^{1,p}_{loc}(0, 1)$ and we define the values at $x=0$ and at $x=1$ of a function $v\in\bar{\mathscr{W}}_p$ by:

$$\begin{aligned} v(0)&=\inf\{\liminf_{k\to+\infty} v(x_k) : x_k\to 0\},\\ v(1)&=\inf\{\liminf_{k\to+\infty} v(x_k) : x_k\to 1\}. \end{aligned} \tag{2.4}$$

In principle $v(0)$ and $v(1)$ could take infinite values, but, with the assumptions we shall make, we can restrict ourselves to the case where they are finite. We shall make the following assumptions:

(2.5) $f=f(x, s, \xi)$ is a Carathéodory function defined in $[0, 1]\times\mathbb{R}\times\mathbb{R}$, convex with respect to ξ;

(2.6) there exist a constant $K\geqq 0$, a convex function $h=h(\xi)$ and continuous functions $a=a(x, s)$, $b=b(x, s)$ such that, for every $(x, s, \xi)\in[0, 1]\times\mathbb{R}\times\mathbb{R}$:

(*a*) $a(x, s)h(\xi)-K\leqq f(x, s, \xi)\leqq a(x, s)h(\xi)+b(x, s)$,

(*b*) $|\xi|\leqq h(\xi)$,

(*c*) either $a=a(x, s)$ is bounded from below by a positive constant, or $a=a(s)$ is independent of x and is positive a. e. in $\mathbb{R}$.

We shall furthermore use the following notations:

$$A(x, y)=\int_0^y a(x, s)\,ds, \qquad \text{for} \quad (x, y)\in[0, 1]\times\mathbb{R}, \tag{2.7}$$

$$\tilde{h}_\pm=\lim_{\xi\to\pm\infty}\frac{h(\xi)}{|\xi|}. \tag{2.8 a}$$

Since h is convex, the limits in (2.8 *a*) exist in $\mathbb{R}\cup\{+\infty\}$; only for the sake of simplicity, we shall assume that:

$$\tilde{h}_+=\tilde{h}_-=\tilde{h}. \tag{2.8 b}$$

THEOREM 2.1 (*Representation in the unconstrained case*). – *Let* F, $\bar{F}$ *and* $\mathscr{W}_p$ *be defined respectively by* (2.1), (2.2) *and* (2.3), *for* $p\geqq 1$. *Under* *assumptions* (2.5), (2.6) *and* (2.8), *for every* $v \in \overline{\mathscr{W}}_p = W^{1,p}_{\text{loc}}(0,1)$, *the following representation formula holds:*

$$(2.9) \quad \overline{F}(v) = F(v) + \tilde{h}\{|A(0, v(0)) - A(0, S_0)| + |A(1, v(1)) - A(1, S_1)|\}.$$

Note that in (2.9), we write by abuse of notation $F(v)$ instead of

$$\int_0^1 f(x, v, v')\,dx, \text{ for } v \in \overline{\mathscr{W}}_p.$$

Remark 2.2. — Assumption (2.8 *b*) is not necessary in theorem 2.1. With a similar proof, without assuming that $\tilde{h}_+ = \tilde{h}_-$, we would get, instead of (2.9):

$$(2.10) \quad \overline{F}(v) = F(v) + \tilde{h}_+\{[A(0, v(0)) - A(0, S_0)]^+ + [A(1, v(1)) - A(1, S_1)]^+\} + \tilde{h}_-\{[A(0, v(0)) - A(0, S_0)]^- + [A(1, v(1)) - A(1, S_1)]^-\},$$

where $[t]^+$ and $[t]^-$ denote respectively the positive and negative parts of $t = [t]^+ - [t]^-$.

Remark 2.3. — Following the same proof of theorem 2.1, other growth conditions than (2.6) can be considered to obtain (2.9). Motivated by integrals of nonlinear elasticity (e. g. [5], [26]), we could replace (2.6 *a*) by:

$$(2.11) \quad a(x, s)h(\xi) - K \leqq f(x, s, \xi) \leqq a(x, s)h(\xi) + x^\alpha(1-x)^\beta b(x, s)(1 + |\xi|^q),$$

where α, β, $q \geqq 0$ and α, $\beta > q - 1$.

For some applications, we next consider the same representation problem for another subset $\mathscr{W}_p$ of $W^{1,p}(0,1)$, namely for functions satisfying boundary conditions and a supplementary monotonicity constraint:

$$(2.12) \quad \mathscr{W}_p = \{v \in W^{1,p}(0,1) : v(0) = S_0,\ v(1) = S_1,\ v'(x) \geqq 0 \text{ a. e.}\},$$

where $S_0 \leqq S_1$ and $p \geqq 1$. The weak closure of $\mathscr{W}_p$ in $W^{1,p}_{\text{loc}}$ is in this case

$$(2.13) \quad \overline{\mathscr{W}}_p = \{x \in W^{1,p}_{\text{loc}}(0,1) : v(0) \geqq S_0,\ v(1) \leqq S_1,\ v'(x) \geqq 0 \text{ a. e.}\}$$

and with the monotonicity constraint, the values at $x=0$ and at $x=1$ of $v \in \overline{\mathscr{W}}_p$ are defined naturally respectively as the infimum and the supremum of the values $v(x)$ for $x \in]0,1[$.

THEOREM 2.4 (*Representation for constrained problems*). — *Let* F, $\overline{F}$, *and* $\mathscr{W}_p$ *be defined respectively by* (2.1), (2.2) *and* (2.12), *for* $p \geqq 1$. *The representation formula* (2.9) *holds, under* (2.5), (2.8) *and under the following assumption:*

(2.14) *there exist a non-negative convex function* $h = h(\xi)$, *continuous functions* $a = a(x, s)$, $b_1 = b_1(x, s)$ *and* $b_2 = b_2(x, s)$ *with* $a \geqq 0$ such that, for every $x \in [0, 1]$, $s \in \mathbb{R}$, $\xi \geqq 0$:
$a(x, s)h(\xi) - b_1(x, s) \leqq f(x, s, \xi) \leqq a(x, s)h(\xi) + b_2(x, s)$.

Proof of theorem 2.1. – Let $v \in \bar{\mathcal{W}}_p$ be fixed with $p \geqq 1$. If $v \in \mathcal{W}_p$, the result $\bar{\mathrm{F}} = \mathrm{F}$ follows immediately from the weakly lower semicontinuity of F in $\mathrm{W}^{1,p}_{\mathrm{loc}}$. We shall separate the problem at $x=0$ and at $x=1$. For simplicity, let us assume that $v(1) = \mathrm{S}_1$ and let us consider the general case at $x=0$, the other cases being similar. We have then to show that:

$$\bar{\mathrm{F}}(v) = \mathrm{F}(v) + \tilde{h} \, | \mathrm{A}(0, v(0)) - \mathrm{A}(0, \mathrm{S}_0) |. \tag{2.15}$$

In a first step, we show the inequality $\geqq$ in (2.15). Let us fix any sequence $(v_k) \subset \mathcal{W}_p$, with v_k weakly converging to v in $\mathrm{W}^{1,p}_{\mathrm{loc}}$, such that, up to a subsequence still denoted by (v_k):

$$\lim_{k \to +\infty} v_k(x) = v(x), \quad \text{for every } x \in]0, 1[, \tag{2.16}$$

$$\liminf_{k \to +\infty} \mathrm{F}(v_k) = \lim_{k \to +\infty} \mathrm{F}(v_k).$$

From the definition (2.4), there exists a sequence $(x_\nu) \subset]0, 1[$ with $x_\nu \to 0$ such that $\lim_{\nu \to +\infty} v(x_\nu) = v(0)$. Moreover, from (2.16), for every fixed ν, $\lim_{k \to +\infty} v_k(x_\nu) = v(x_\nu)$. There exists then k_ν such that $| v_{k_\nu}(x_\nu) - v(x_\nu) | < 1/\nu$. Therefore, we have that:

$$| v_{k_\nu}(x_\nu) - v(0) | < \frac{1}{\nu} + | v(x_\nu) - v(0) |,$$

and we hence get a sequence $(x_\nu) \subset]0, 1[$ and a subsequence (v_{k_ν}) of (v_k) that we denote by (v_ν) such that:

$$x_\nu \to 0 \quad \text{and} \quad v_\nu(x_\nu) \to v(0), \quad \text{as } \nu \to +\infty. \tag{2.17}$$

We then have

$$\begin{aligned} \liminf_k \mathrm{F}(v_k) &= \lim_\nu \mathrm{F}(v_\nu) \\ &\geqq \liminf_\nu \int_0^{x_\nu} f(x, v_\nu, v'_\nu)\, dx + \liminf_\nu \int_{x_\nu}^1 f(x, v_\nu, v'_\nu)\, dx. \end{aligned} \tag{2.18}$$

For fixed ν_0, the last integral term in (2.18) becomes greater than

$$\liminf_{\nu \geqq \nu_0} \int_{x_{\nu_0}}^1 f(x, v_\nu, v'_\nu)\, dx \geqq \int_{x_{\nu_0}}^1 f(x, v, v')\, dx, \tag{2.19}$$

by the weak lower semicontinuity, since $f(x, v, \cdot)$ is convex. Letting then $\nu_0 \to +\infty$, we get

$$\lim_{\nu_0 \to +\infty} \int_{x_{\nu_0}}^1 f(x, v, v')\, dx = \mathrm{F}(v). \tag{2.20}$$

With (2.6*a*) and the Jensen inequality in a generalized form (*see* e. g. [26]), the first integral in (2.18) becomes

$$(2.21)\quad \int_0^{x_\nu} f(x, v_\nu, v'_\nu)\,dx \geqq \int_0^{x_\nu} \{ a(x, v_\nu) h(v'_\nu) - \mathrm{K} \}\,dx \geqq \mathrm{V}_\nu \frac{h(t_\nu)}{t_\nu} - \mathrm{K}\, x_\nu,$$

where the following notations have been introduced:

$$(2.22)\qquad \mathrm{V}_\nu = \int_0^{x_\nu} a(x, v_\nu)\, v'_\nu\,dx, \qquad t_\nu = \frac{\mathrm{V}_\nu}{\displaystyle\int_0^{x_\nu} a(x, v_\nu)\,dx}.$$

Let us separate the two cases in (2.6*c*). If $a(x, s)$ is independent of x, we have, with (2.7), that $\mathrm{V}_\nu = \mathrm{A}(0, v_\nu(x_\nu)) - \mathrm{A}(0, v_\nu(0))$, and then, with (2.17) and since $v_\nu(0) = \mathrm{S}_0$, that

$$(2.23)\qquad \lim_{\nu \to +\infty} \mathrm{V}_\nu = \mathrm{A}(0, v(0)) - \mathrm{A}(0, \mathrm{S}_0).$$

With (2.6), we furthermore have that

$$(2.24)\quad |\mathrm{A}(0, v_\nu(x)) - \mathrm{A}(0, v_\nu(0))| \leqq \int_0^{x} a(0, v_\nu(t))\,|v'_\nu(t)|\,dt \leqq \mathrm{F}(v_\nu) + \mathrm{K}.$$

If $\lim_\nu \mathrm{F}(v_\nu) = +\infty$, the inequality $\geqq$ in (2.15) would be obvious. Let us suppose then that $\lim_\nu \mathrm{F}(_\nu)$ is finite. With (2.24), we get the uniform boundedness of (v_ν) in $\mathrm{L}^\infty(0, 1)$ with respect to ν, since A is strictly increasing.

If $a(x, s) \geqq c > 0$ depends on x and s, we have, with (2.6), that

$$(2.25)\quad c\,|v_\nu(x) - v_\nu(0)| \leqq c \int_0^1 |v'_\nu(x)|\,dx \leqq \int_0^1 a(x, v_\nu)\, h(v'_\nu)\,dx \leqq \mathrm{F}(v_\nu) + \mathrm{K}.$$

Thus in this case, (v_ν) is uniformly bounded in $\mathrm{W}^{1,1}(0, 1)$ and in $\mathrm{L}^\infty(0, 1)$. With the uniform continuity of a in $[0, 1] \times [-\sup_\nu \|v_\nu\|_{\mathrm{L}^\infty}, \sup_\nu \|v_\nu\|_{\mathrm{L}^\infty}]$, we have that for $\varepsilon > 0$ being fixed and ν sufficiently large:

$$(2.26)\quad |\mathrm{V}_\nu - \mathrm{A}(0, v_\nu(x_\nu)) + \mathrm{A}(0, \mathrm{S}_0)| = \left| \int_0^{x_\nu} \{ a(x, v_\nu(x)) - a(0, v_\nu(x)) \}\, v'_\nu(x)\,dx \right| \leqq \varepsilon \int_0^1 |v'_\nu(x)|\,dx.$$

We hence get (2.23) in that case too. If the limit in (2.23) is different from zero, since a is bounded in $[0, 1] \times [-\sup_v \| v_v \|_{L^\infty}, \sup_v \| v_v \|_{L^\infty}]$, we have that t_v in (2.22) goes either to $+\infty$ or to $-\infty$. Since t_v has the same sign as V_v,

$$V_v \frac{h(t_v)}{t_v} = | V_v | \frac{h(t_v)}{| t_v |}.$$

Hence, by (2.21),

$$\liminf_v \int_0^{x_v} f(x, v_v, v_v') \, dx \geqq \tilde{h} \, | A(0, v(0)) - A(0, S_0) |,$$

which concludes with (2.18) and (2.20) the first step of the proof.

In the following second step, we show the inequality $\leqq$ in (2.15). To do that, we consider a particular sequence in the definition (2.2) of $\bar{F}$ in the following way. From the definition (2.4) of $v(0)$, we know that there exists a sequence $(x_k) \subset]0, 1[$, with $x_k \to 0$, such that $v(x_k) \to v(0)$. We then consider the following sequence $(v_k) \subset \mathscr{W}_p$:

$$(2.27) \qquad v_k(x) = \begin{array}{ll} S_0 + \dfrac{v(x_k) - S_0}{x_k} x, & \text{for} \quad 0 \leqq x < x_k, \\ v(x), & \text{for} \quad x_k \leqq x \leqq 1. \end{array}$$

With this choice, obviously v_k weakly converges to v in $W^{1, p}_{\text{loc}}$, as $k \to \infty$. From the definition (2.2) of $\bar{F}$,

$$\bar{F}(v) \leqq \limsup_k F(v_k)$$
$$\leqq \limsup_k \int_0^{x_k} f(x, v_k, v_k') \, dx + \limsup_k \int_{x_k}^1 f(x, v, v') \, dx.$$

From (2.6):

$$\int_0^{x_k} f(x, v_k, v_k') \, dx \leqq \int_0^{x_k} \{ a(x, v_k) h(v_k') + b(x, v_k) \} \, dx.$$

But with the continuity of b,

$$\int_0^{x_k} b(x, v_k) \, \mathrm{dx} \to 0, \quad \text{as} \, k \to +\infty.$$

We hence obtain the result, following the same kind of computations as in the first step, for the particular sequence (v_k). This concludes the second step and hence the proof of the representation formula (2.9). □

Proof of theorem 2.4. – We can follow the beginning of the proof of theorem 2.1, up to (2.20). With (2.14), we should now replace (2.21) by:

$$(2.28)\quad \int_0^{x_\nu} f(x, v_\nu, v'_\nu)\,dx \geqq \int_0^{x_\nu} \{a(x, v_\nu)\,h(v'_\nu) - b_1(x, v_\nu)\}\,dx$$
$$\geqq V_\nu \frac{h(t_\nu)}{t_\nu} - \int_0^{x_\nu} b_1(x, v_\nu)\,dx,$$

where V_ν and t_ν are defined by (2.22). With the monotonicity constraint $v'_\nu \geqq 0$ a.e. in the definition (2.12) of $\mathscr{W}_p$, we get that

$$\int_0^1 |v'_\nu(x)|\,dx = v_\nu(1) - v_\nu(0) = S_1 - S_0,$$

i.e. the uniform boundedness of the sequence (v_ν) in $W^{1,1}$ with respect to ν. The last integral term in (2.28) being bounded since b_1 is continuous, we hence get (2.26) and then (2.23). We then conclude like in theorem 2.1. □

3. *A PRIORI* ESTIMATES

We are concerned in this section with proving *a priori* estimates for the solutions of the standard coercive variational problem

$$(3.1)\quad \inf\left\{ F(v) = \int_0^1 f(x, v, v')\,dx : v \in W^{1,p}(0, 1),\ v(0) = S_0,\ v(1) = S_1 \right\},$$

where S_0, $S_1 \in \mathbb{R}$, $p > 1$ and $f = f(x, s, \xi)$ is a non-negative function of class C^2 satisfying the following assumption: there exist $K \geqq 0$, $m > 0$ and an increasing function $M : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for every $(x, s, \xi) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$:

$$(3.2)\quad m|\xi|^p - K \leqq f(x, s, \xi) \leqq M(|s|)(1 + |\xi|^p),$$
$$(3.3)\quad |f_s(x, s, \xi)| \leqq M(|s|)(1 + |\xi|^p),$$
$$(3.4)\quad f_{\xi\xi}(x, s, \xi) > 0,$$

where $f_s = \dfrac{\partial}{\partial s} f$. Let us recall that under (3.2) and (3.4), problem (3.1) has minimizers in $W^{1,p}$ (of course, they also belong to $C^0([0, 1])$, by the imbedding theorem). We now state the first of the two *a priori* estimates' theorems that we shall use in the existence theorems of the next section.

THÉORÈME 3.1 (*First* a priori *estimates' theorem*). — *Let* $f\in C^2(]0,1[\times\mathbb{R}\times\mathbb{R})$ *satisfy* (3.2), (3.3) *and* (3.4). *Let us assume that*

(3.5) *the function* φ, *defined by*
$$\varphi(x,s,\xi)=f_s(x,s,\xi)-\xi f_{s\xi}(x,s,\xi)-f_{x\xi}(x,s,\xi),$$
has a definite sign (i.e. $\varphi\geqq 0$ *or* $\varphi\leqq 0$) *for every* $(x,s,\xi)\in]0,1[\times\mathbb{R}\times\mathbb{R}$.

Then, every minimizer u of (3.1) *satisfies the following estimate*

$$\|u'\|_{L^\infty(\delta,1-\delta)}\leqq\frac{2}{\delta}\|u\|_{L^\infty(0,1)},\quad \textit{for every } \delta\in\left]0,\frac{1}{2}\right[. \tag{3.6}$$

We state a second theorem in the case where φ has not necessarily a definite sign.

THEOREM 3.2 (*Second* a priori *estimates' theorem*). — *Let* $f\in C^3(]0,1[\times\mathbb{R}\times\mathbb{R})$ *satisfy* (3.2), (3.3) *and* (3.4). *Let us assume that for every* $\delta\in\left]0,\frac{1}{2}\right[$ *and for every* $r\geqq 0$,

(3.7) *there exists* $K_0=K_0(\delta,r)\geqq 0$ *such that for every* $(x,s,\xi)\in[\delta,1-\delta]\times[-r,r]\times\mathbb{R}$ *with* $|\xi|>K_0$, *if* $\varphi(x,s,\xi)=0$, *then* $\xi\psi(x,s,\xi)>0$, *where* φ *is defined by* (3.5) *and* ψ *is defined by*
$$\psi(x,s,\xi)=\varphi_x(x,s,\xi)+\xi\varphi_s(x,s,\xi).$$

Then, every minimizer u of (3.1) *satisfies the following estimate*

$$\|u'\|_{L^\infty(\delta,1-\delta)}\leqq\max\left\{K_0\left(\frac{\delta}{2},\|u\|_{L^\infty(0,1)}\right);\frac{4}{\delta}\|u\|_{L^\infty(0,1)}\right\},\quad \textit{for every } \delta\in\left]0,\frac{1}{2}\right[. \tag{3.8}$$

We shall use theorems 3.1 and 3.2 in section 5 for classical examples and in section 6 for models in behavioural ecology.

Remark 3.3. — Assumption (3.7) is satisfied if, for example, φ is different from zero for $|\xi|$ sufficiently large.

Proof of theorem 3.1. — Let $u\in W^{1,p}$ be a minimizer of (3.1). The convexity of $f(x,s,\xi)$ with respect to ξ and the condition (3.2) ensures (*see* lemma 2.1 in [25]) that

$$|f_\xi(x,s,\xi)|\leqq C M(|s|)(1+|\xi|^{p-1}),$$

for every $(x,s,\xi)\in[0,1]\times\mathbb{R}\times\mathbb{R}$ and for some $C>0$. Then, with (3.3), u solves the Euler equation of F in the weak form (this is classical, *see* e.g.

theorem 1.10.1 in [28] or section 3.4.2 in [15]):

$$\int_0^1 \{ f_\xi(x, u, u') v'(x) + f_s(x, u, u') v(x) \} \, dx = 0,$$
$$\text{for every } v \in W_0^{1,p}(0, 1).$$

But since $f \in C^2$, then $u \in C^2$ and u solves the Euler equation of F in the strong form (*see* [28] again):

$$\frac{d}{dx}\{ f_\xi(x, u, u') \} = f_s(x, u, u'), \quad \text{for every } x \in]0, 1[,$$

which can be written, since $u, f \in C^2$:

$$f_{\xi\xi}(x, u, u') u''(x) = \varphi(x, u, u'), \quad \text{for every } x \in]0, 1[, \tag{3.9}$$

where φ is defined by (3.5). With (3.4) and (3.5), u is either convex or concave in $[0, 1]$. Let us assume that $\varphi \geqq 0$ and that u is convex in $[0, 1]$, the case $\varphi \leqq 0$ and u concave being similar. Using the well-known monotonicity of the differential quotients of a convex function, we have that for $0 \leqq y < x < z \leqq 1$,

$$\frac{u(y) - u(x)}{y - x} \leqq u'(x) \leqq \frac{u(z) - u(x)}{z - x}. \tag{3.10}$$

Choosing $y = 0$ and $z = 1$ in (3.10), we get for $x \in]\delta, 1 - \delta[$, where $\delta \in \left]0, \frac{1}{2}\right[$:

$$|u'(x)| \leqq \max\left\{ \frac{|u(x) - u(0)|}{x}; \frac{|u(1) - u(x)|}{1 - x} \right\} \leqq \frac{2}{\delta} \| u \|_{L^\infty(0, 1)}. \tag{3.11}$$

We hence get (3.6), which concludes the proof of theorem 3.1. □

Proof of theorem 3.2. – Let $u \in W^{1,p}$ be a minimizer of (3.1) and let r in (3.7) be equal to $\| u \|_{L^\infty(0, 1)}$. The following lemma 3.4 characterizes the concavity or convexity properties of u that we need to prove theorem 3.2:

LEMMA 3.4. – *Let $u \in W^{1,p}$ be a minimizer of* (3.1). *With the assumptions and the notations of theorem* 3.2, *for every $\delta \in \left]0, \frac{1}{2}\right[$ fixed, there exist x_1, x_2 with $\delta \leqq x_1 \leqq x_2 \leqq 1 - \delta$ such that:*

(I) $|u'(x)| \leqq K_0$, *for every* $x \in]x_1, x_2[$,

(II) $|u'(x)| > K_0$, *for every* $x \in]\delta, x_1[\cup]x_2, 1 - \delta[$.

Moreover,

(III) *if* $u'(x) > K_0$ [*resp.* $u'(x) < -K_0$] *in* $]\delta, x_1[$, *then* u *is concave* (*resp. convex*) *in* $]\delta, x_1[$,

(IV) *if* $u'(x) > K_0$ [*resp.* $u'(x) < -K_0$] *in* $]x_2, 1-\delta[$, *then* u *is convex* (*resp. concave*) *in* $]x_2, 1-\delta[$.

We shall first achieve the proof of theorem 3.2 and then we shall prove lemma 3.4. Let us use the statement of lemma 3.4 with δ replaced by $\delta/2$ and $K_0 = K_0(\delta/2, r)$. If $\delta \geqq \max\{x_1; 1-x_2\}$, then $]\delta, 1-\delta[\subseteqq]x_1, x_2[$; thus, with assertion (I) of lemma 3.4, there is nothing more to prove, since $|u'(x)| \leqq K_0$, for every $x \in]\delta, 1-\delta[$. If $\delta < \max\{x_1; 1-x_2\}$, then it remains to be proved the pointwise *a priori* estimate (3.8) on the set $]\delta, x_1[\bigcup]x_2, 1-\delta[$. Let us assume, for example, that $\delta < x_1$ and let us prove the estimate in the interval $]\delta, x_1[$ (the case $1-\delta > x_2$ being similar). With assertion (II) of lemma 3.4, $u'(x)$ is either larger than K_0 or smaller than $-K_0$ in $]\delta/2, x_1[$, since u' is continuous. If for example $u'(x) < -K_0$ in $]\delta/2, x_1[$, then with (III) of lemma 3.4, u is convex in $]\delta/2, x_1[$ and thus, like in the proof of theorem 3.1, from the left handside of (3.10) with $y = \delta/2$, we have

$$\frac{u(x) - u(\delta/2)}{x - \delta/2} \leqq u'(x) \leqq -K_0, \qquad \text{for} \quad x \in]\delta, x_1[,$$

and hence (3.8), since $x > \delta$, which concludes the proof of theorem 3.2. $\square$

Proof of lemma 3.4. – Like in the proof of theorem 3.1, since now $f \in C^3$, then $u \in C^3$; differentiating (3.9) we get:

$$(3.12) \quad f_{\xi\xi}(x, u, u')\, u'''(x) = \psi(x, u, u') + \left\{ \varphi_\xi(x, u, u') - \frac{d}{dx} f_{\xi\xi}(x, u, u') \right\} u''(x),$$

for every $x \in]0, 1[$, where φ and ψ are respectively defined by (3.5) and (3.7).

Let $\delta \in \left]0, \frac{1}{2}\right[$ be fixed. Let us define the following subset $\mathscr{Y}$ of $]\delta, 1-\delta[$:

$$\mathscr{Y} = \{ x \in]\delta, 1-\delta[: |u'(x)| > K_0 \},$$

where K_0 is given by (3.7). If $\mathscr{Y}$ is empty, the lemma is proved by choosing $x_1 = \delta$ and $x_2 = 1-\delta$. Suppose then that $\mathscr{Y}$ is not empty and define the subset $\mathscr{X}$ of $\mathscr{Y}$:

$$\mathscr{X} = \{ x \in]\delta, 1-\delta[: |u'(x)| > K_0,\ u''(x) = 0 \}.$$

If $\mathscr{X}$ is not empty, consider $x \in \mathscr{X}$. From (3.9), $\varphi(x, u(x), u'(x)) = 0$ and with (3.7), $u'(x)\psi(x, u(x), u'(x)) > 0$. From (3.4) and (3.12), $u'(x)$, $\psi(x, u(x), u'(x))$ and $u'''(x)$ have the same sign. Hence if $x \in \mathscr{X}$, one of the two following assertions is verified:

(i) $u'(x) > K_0$, $u''(x) = 0$, $u'''(x) > 0$ and then u' has a strict local minimum at x,

(ii) $u'(\mathrm{x}) < -\mathrm{K}_0$, $u''(x)=0$, $u'''(x)<0$ and then u' has a strict local maximum at x.

Suppose that $\mathscr{Y} =]\delta, 1-\delta[$ and let us analyze only the case of a point $x \in \mathscr{X}$ satisfying (i) [therefore $u'(x) > \mathrm{K}_0$], the case (ii) being similar. The point x must then be a strict global minimum of u', since, if it were a strict local (but not global) minimum, it would imply the existence elsewhere of an interior local maximum. For the same reason, x is the unique minimum point of u' in $]\delta, 1-\delta[$. Thus, we can choose $x_1 = x_2 = x$ and u' is decreasing in $]\delta, x_1[$ (*i.*. u is concave) and u' is increasing in $]x_2, 1-\delta[$ (*i.e.* u is convex) and the lemma is satisfied.

Suppose now that $\mathscr{Y} \neq]\delta, 1-\delta[$. The set of points $x \in]\delta, 1-\delta[$ such that $|u'(x)| \leqq \mathrm{K}_0$ is an interval, since, if not, there would exist either a local maximum point x of u', with $u'(x) > \mathrm{K}_0$, or a local minimum point x of u', with $u'(x) < -\mathrm{K}_0$, which is in contradiction with (i) or (ii). Let us define

$$x_1 = \inf \{x \in [\delta, 1-\delta] : |u'(x)| \leqq \mathrm{K}_0\}$$

and

$$x_2 = \sup \{x \in [\delta, 1-\delta] : |u'(x)| \leqq \mathrm{K}_0\}.$$

In this case, $\mathscr{Y} =]\delta, x_1[\cup]x_2, 1-\delta[$ and (I) and (II) of the lemma are satisfied. Since $\mathscr{X}$ is empty, then u'' has a definite sign; if $]\delta, x_1[$ is not empty, then u' is decreasing [if $u'(x_1) = \mathrm{K}_0$] or increasing [if $u'(x_1) = -\mathrm{K}_0$] in $]\delta, x_1[$ and (III) and (IV) are satisfied. This concludes the proof of the lemma, since a similar analysis holds for $]x_2, 1-\delta[$. □

Remark 3.5. – In the proof of lemma 3.4, in particular in (3.12), it is sufficient to assume that the third derivative $f_{\xi\xi\xi}$ exists almost everywhere and is locally bounded in $\mathbb{R}$, instead of being continuous. We shall use this remark in the proof of theorem 6.1.

4. EXISTENCE THEOREMS

In theorem 4.1, we shall assume the following growth conditions on f, stronger than (2.6):

(4.1) there exist L, $\mathrm{K} \geqq 0$, $p \geqq 1$, a convex function $h = h(\xi)$, and continuous functions $a = a(x, s)$, $b = b(x, s)$ such that, for every $(x, s, \xi) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$:

(*a*) $a(x, s)\, h(\xi) - \mathrm{K} \leqq f(x, s, \xi) \leqq a(x, s)\, h(\xi) + b(x, s)$,

(*b*) $|\xi| \leqq h(\xi) \leqq \mathrm{L}(1 + |\xi|^p)$,

(*c*) either $a = a(x, s)$ is bounded from below by a positive constant, or $a = a(s)$ is independent of x and is positive a.e. in $\mathbb{R}$.

THEOREM 4.1 (*Existence in the unconstrained case*). — *Let* $f=f(x, s, \xi)$ *be a convex function with respect to* ξ, *satisfying* (4.1) *and* (3.3). *Let us assume that either* $f\in C^2$ *and* (3.5) *holds or* $f\in C^3$ *and* (3.7) *holds. Then the variational problem related to the function* $\bar{F}$ *in* (2.9):

$$\text{(4.2)} \qquad \min \left\{ \bar{F}(v) : v \in W^{1,p}_{\text{loc}}(0, 1) \right\}$$

admits a solution, which belongs to $W^{1,\infty}_{\text{loc}}(0, 1)$.

Remark 4.2. — Instead of (4.1 *b*) [and also of (2.6 *b*)], we could assume that there exist exponents $p \geqq r \geqq 1$ such that

$$\text{(4.3)} \qquad |\xi|^r \leqq h(\xi) \leqq L(1+|\xi|^p).$$

In this case, we should simply change in the proof (4.12) [and similarly (2.24)] with

$$\text{(4.4)} \qquad |B(u_\varepsilon(x))|^r \leqq \int_0^x a(u_\varepsilon(t))\,|u'_\varepsilon(t)|^r\,dt,$$

where $B(s)=\int_{S_0}^{s} [a(t)]^{1/r}\,dt$; (4.4) is a consequence of Hölder inequality.

A similar result holds under the constraint $v(x)\geqq 0$ for $x\in[0, 1]$:

THEOREM 4.3 (*Existence for the constrained problem* $v\geqq 0$). — *Let* $f=f(x, s, \xi)$ *be defined in* $[0, 1]\times[0, +\infty[\times\mathbb{R}$ *and let it satisfy the same assumptions of theorem* 4.1 *for every* $x\in[0, 1]$, $\xi\in\mathbb{R}$, *but only for* $s\geqq 0$. *if*

$$\text{(4.5)} \qquad f(x, s, \xi)=f(x, s-\xi), \quad \textit{for every } (x, s, \xi)\in[0, 1]\times[0, +\infty[\times\mathbb{R},$$

then the variational problem related to the functional $\bar{F}$ *in* (2.9), *with* S_0, $S_1\geqq 0$:

$$\text{(4.6)} \qquad \min \left\{ \bar{F}(v) : v\in W^{1,p}_{\text{loc}}(0, 1),\ v(x)\geqq 0 \textit{ for } x\in[0, 1] \right\}$$

admits a solution, which belongs to $W^{1,\infty}_{\text{loc}}(0, 1)$.

We state a third existence theorem with the constraint $v'(x)\geqq 0$ a.e. in $[0, 1]$ under the following growth conditions of f, more general than (4.1):

(4.7) there exist L, $K\geqq 0$, $p\geqq 1$, a convex function $h=h(\xi)$, and continuous functions $a=a(x, s)$, $b=b(x, s)$ such that, for every $(x, s, \xi)\in[0, 1]\times\mathbb{R}\times\mathbb{R}$:

(*a*) $a(x, s)\,h(\xi)-K\leqq f(x, s, \xi)\leqq a(x, s)\,h(\xi)+b(x, s)$,

(*b*) $0\leqq h(\xi)\leqq L(1+|\xi|^p)$,

(*c*) $a(x, s)\geqq 0$.

THEOREM 4.4 (*Existence for the constrained problem* $v'\geqq 0$). — *Let* $f=f(x, s, \xi)$ *be a convex function with respect to* ξ, *satisfying* (4.7) *and* (3.3). *Let us assume either that* $f\in C^2$ *and* (3.5) *holds or* $f\in C^3$ and (3.7) *holds. Then the variational problem related to the functional* $\bar{F}$ *in* (2.9), *with* $S_0 \leqq S_1$,

$$\text{(4.8)}\quad \min\left\{\bar{F}(v): v\in W^{1,p}_{loc}(0,1),\ v(0)\geqq S_0,\ v(1)\leqq S_1,\ v'(x)\geqq 0 \textit{ a.e. in } [0,1]\right\}$$

admits a solution $u\in W^{1,\infty}_{loc}(0,1)$ *which satisfies the estimate* [$K_0=K_0(\delta/2, r)$ *is given by* (3.7) *with* $r>\max\{|S_0|; |S_1|\}$]

$$\text{(4.9)}\quad \|u'\|_{L^\infty(\delta,1-\delta)}\leqq \max\left\{K_0; \frac{4}{\delta}|S_0|; \frac{4}{\delta}|S_1|\right\},\quad \textit{for every } \delta\in\left]0,\frac{1}{2}\right[.$$

Remark 4.5. — Instead of the constraint $v'(x)\geqq 0$, one could consider $v'(x)\geqq a$ or $v'(x)\leqq b$ for some $a, b\in\mathbb{R}$ and apply theorem 4.4 after change of variable. Of course, the existence theorem in the case of a constraint $a\leqq v'(x)\leqq b$ is trivial.

Remark 4.6 (*Example of non-existence*). — The Weierstrass functional

$$F(v)=\int_{-1}^{1} x^2 v'^2\,dx$$

has no minimizer in the class ($A<B$ given):

$$\mathscr{W}_2=\left\{v\in W^{1,2}(-1,1): v(-1)=A,\ v(1)=B,\ v'(x)\geqq 0 \text{ a.e. in } [-1,1]\right\}.$$

This is well-known and can be proved by observing that $F(v)>0$ for every $v\in\mathscr{W}_2$ and that $F(v_k)\to 0$ as $k\to+\infty$, where v_k is constantly equal to A in $[-1,0]$, is equal to B in $[1/k,1]$ and whose derivative is equal to $k(B-A)$ in $]0,1/k[$. Since $\tilde{h}=+\infty$, then $\bar{F}=F$ if the boundary values are satisfied and $\bar{F}=+\infty$ if not. Then, neither the variational problem

$$\min\left\{\bar{F}(v): v\in W^{1,2}_{loc}(-1,1),\ v(-1)\geqq A, v(1)\leqq B,\ v'(x)\geqq 0 \text{ a.e. in } [-1,1]\right\}$$

has a solution. Note that the main assumptions of theorem 4.4 [either (3.5) or (3.7)] are not satisfied, since $\varphi=-4x\xi$ and $\psi=-4\xi$.

Proof of theorem 4.1. — We shall add a coercive ε-term of f to apply one of the *a priori* estimates' theorems of section 3. Let us fix $\varepsilon\in]0,1]$ and let us define, for $(x,s,\xi)\in[0,1]\times\mathbb{R}\times\mathbb{R}$:

$$\text{(4.10)}\quad f^\varepsilon(x,s,\xi)=f(x,s,\xi)+\varepsilon(1+\xi^2)^{q/2},$$

where $q=p$ if $p>1$ or q is any number strictly greater than one, if $p=1$. We have that

$$\text{(4.11)}\quad \varepsilon|\xi|^q-K\leqq f^\varepsilon(x,s,\xi)\leqq M(|s|)(1+|\xi|^q),$$

for every $(x,s,\xi)\in[0,1]\times\mathbb{R}\times\mathbb{R}$, where

$$M(|s|)=\max\left\{a(x,t)L+|b(x,t)|+2^{q-1}: x\in[0,1],\ |t|\leqq|s|\right\}$$

is increasing. Since furthermore $f^{\varepsilon}_{\xi\xi}>0$, the following functional

$$F^{\varepsilon}(v)=\int_0^2 f^{\varepsilon}(x, v, v')\,dx$$

admits a minimizer $u_{\varepsilon}\in\mathscr{W}_q$ [$\mathscr{W}_q$ is defined by (2.3)]. Let $A(x,y)$ be defined by (2.7). As in (2.24), with (4.1), we obtain, for $x\in]0, 1[$:

$$(4.12)\quad |A(0, u_{\varepsilon}(x))-A(0, S_0)| \leqq \int_0^x a(0, u_{\varepsilon}(t))\,|u'_{\varepsilon}(t)|\,dt\leqq F(u_{\varepsilon})+K,$$

in the case where $a=a(s)$ is independent of x or an expression similar to (2.25) if $a=a(x, s)$. $F(u_{\varepsilon})$ is bounded uniformly in ε (since $\varepsilon\leqq 1$ and u_{ε} minimizes F^{ε}) and with (4.1), $A(x,\cdot)$ is strictly increasing. Hence with (4.12), the sequence (u_{ε}) is bounded in $L^{\infty}(0, 1)$ uniformly with respect to ε: there exists $C_1>0$ such that $\|u_{\varepsilon}\|_{L^{\infty}(0, 1)}\leqq C_1$, for every $\varepsilon\in]0, 1]$.

Since the definitions of φ in (3.5) and of ψ in (3.7) are the same for f and f^{ε}, with (4.11), we can apply one of the two *a priori* estimates' theorems 3.1 or 3.2 to f^{ε}. With (4.12), we get that for $\delta\in\left]0, \frac{1}{2}\right[$ fixed,

$$(4.13)\quad \begin{cases} \text{there exists } C_2>0 \text{ such that } \|u'_{\varepsilon}\|_{L^{\infty}(\delta, 1-\delta)}\leqq C_2, \\ \qquad\qquad \text{for every } \varepsilon\in]0, 1], \end{cases}$$

where $C_2=C_2(K_0, C_1, \delta)$. Then (u_{ε}) is weakly* relatively compact in $W^{1,\infty}_{\text{loc}}(0, 1)$: there exists $u\in W^{1,\infty}_{\text{loc}}$ such that, up to a subsequence if necessary, u_{ε} weakly* converges to u in $W^{1,\infty}_{\text{loc}}$, as $\varepsilon\to 0$.

Let $v\in\mathscr{W}_q$. By the definition (2.2) of $\bar{F}$ and since u_{ε} minimizes F^{ε} in $\mathscr{W}_q$, we have that:

$$(4.14)\quad \bar{F}(u)\leqq\liminf_{\varepsilon\to 0} F(u_{\varepsilon})\leqq\liminf_{\varepsilon\to 0} F^{\varepsilon} u_{\varepsilon})\leqq\liminf_{\varepsilon\to 0} F^{\varepsilon}(v)=F(v).$$

Let us now consider any $w\in\mathscr{W}_p$. Since $W^{1,q}$ is dense in $W^{1,p}$ (recall that $q\geqq p$), there exists a sequence $(v_k)\subset\mathscr{W}_q$ with $v_k\to w$ in $W^{1,p}$ and since F is strongly continuous in $W^{1,p}$ by (4.1 *b*), then $F(v_k)\to F(w)$. Thus, by (4.14), $\bar{F}(u)\leqq F(w)$. Let us finally consider any $v\in\bar{\mathscr{W}}_p=W^{1,p}_{\text{loc}}(0, 1)$. By the definition (2.2) of $\bar{F}$, there exists a sequence $(v_k)\subset\mathscr{W}_p$ that weakly converges to v in $W^{1,p}_{\text{loc}}$ such that $F(v_k)\to\bar{F}(v)$. We hence get that $\bar{F}(u)\leqq\bar{F}(v)$, for $v\in W^{1,p}_{\text{loc}}$, which concludes the proof of theorem 4.1. $\square$

Proof of theorem 4.3. – We extend f for $s<0$ by defining

$$(4.15)\quad f(x, s, \xi)=f(x, -s, \xi),\quad \text{for every } (x, s, \xi)\in[0, 1]\times\mathbb{R}^-\times\mathbb{R}.$$

Let us note that f is a Carathéodory function in $[0, 1]\times\mathbb{R}\times\mathbb{R}$ (in general only continuous with respect to s at $s=0$). We define $f^{\varepsilon}(x, s, \xi)$ as in (4.10) and, like in the proof of theorem 4.1, we denote by u_{ε} a solution of the variational problem

$$(4.16)\qquad \min\left\{\int_0^1 f^\varepsilon(x, v, v')\,dx : v\in W^{1,q}(0, 1),\ v(0)=S_0,\ v(1)=S_1\right\}.$$

By (4.5), we have that $f^\varepsilon(x, s, \xi)=f^\varepsilon(x, s, -\xi)$ and by (4.15), we have also that $f^\varepsilon(x, s, \xi)=f^\varepsilon(x, -s, \xi)$. It follows that

$$f^\varepsilon(x, |v|, |v'|)=f^\varepsilon(x, v, v'),\qquad \text{for}\quad v\in W^{1,q}(0, 1)$$

and thus, since $S_0, S_1\geqq 0$, $|u_\varepsilon|$ is a minimizer of (4.16) too. By changing u_ε with $|u_\varepsilon|$ if necessary, we have that $u_\varepsilon(x)\geqq 0$ for every $x\in[0, 1]$. Repeating the argument of theorem 3.1 (or 3.2), since the integrand f^ε is smooth in $[0, 1]\times[0, \infty[\times\mathbb{R}$, we can show that u_ε is a smooth function and we can then conclude as in theorem 4.1. $\square$

Proof of theorem 4.4. — Like in the proof of theorem 4.1, we shall add a coercive ε-term to f to apply one of the *a priori* estimates' theorems. With a penalization type method (*see* e.g. [23]), we shall furthermore handle the constraint $v'\geqq 0$ by adding a k-term. Let us fix $\varepsilon\in]0, 1]$ and $k\geqq 0$. Let us then define, for $(x, s, \xi)\in[0, 1]\times\mathbb{R}\times\mathbb{R}$:

$$(4.17)\qquad f^{\varepsilon, k}(x, s, \xi)=f(x, s, \xi)+\varepsilon(1+\xi^2)^{q/2}+k(\xi^-)^q,$$

where $q=\max\{p; 4\}$ and $\xi^-=-\min\{\xi; 0\}$ denotes the negative part of ξ. Since $q\geqq 4$, then $f^{\varepsilon, k}$ is either of class C^2 (if $f\in C^2$) or of class C^3 (if $f\in C^3$). Since $f(x, s, \cdot)$ is convex, $f^{\varepsilon, k}_{\xi\xi}>0$ and

$$(4.18)\qquad \varepsilon|\xi|^q-K\leqq f^{\varepsilon, k}(x, s, \xi)\leqq M(|s|)(1+|\xi|^q),$$

for every $(x, s, \xi)\in[0, 1]\times\mathbb{R}\times\mathbb{R}$, where

$$M(|s|)=\max\{a(x, t)L+|b(x, t)|+2^{q-1}+k : x\in[0, 1],\ |t|\leqq|s|\}.$$

Then, the functional

$$F^{\varepsilon, k}(v)=\int_0^1 f^{\varepsilon, k}(x, v, v')\,dx$$

admits a minimizer $u_{\varepsilon, k}$ belonging to $\mathscr{W}_q$ [$\mathscr{W}_q$ is defined by (2.3)]. In particular, $F^{\varepsilon, k}(u_{\varepsilon, k})$ is uniformly bounded with respect to ε and k. Indeed, for v affine belonging to $\mathscr{W}_q$ (*i.e.* $v'=S_1-S_0$), we have that:

$$F^{\varepsilon, k}(u_{\varepsilon, k})\leqq F^{\varepsilon, k}(v)\leqq F(v)+\int_0^1\{1+v'^2\}^{q/2}\,dx,$$

since $[v']^-\equiv 0$ and $\varepsilon\leqq 1$. If we denote by C the right hand side, we then have that

$$(4.19)\qquad F^{\varepsilon, k}(u_{\varepsilon, k})\leqq C,\quad \text{for every } \varepsilon\in]0, 1] \text{ and } k\geqq 0.$$

From (4.18), for every fixed $\varepsilon\in]0, 1]$, the sequence $(u_{\varepsilon, k})$ is then bounded in $W^{1,q}(0, 1)$ uniformly with respect to $k>0$. There exists then

$u_\varepsilon \in W^{1,q}(0, 1)$, such that, up to a subsequence if necessary,

(4.20) $u_{\varepsilon, k}$ *converges to* u_ε *weakly in* $W^{1,q}(0, 1)$ *and strongly in* $L^\infty(0, 1)$, *as* $k \to \infty$.

With (4.7) and (4.17), we have that:

$$k \int_0^1 \{[u'_{\varepsilon, k}(x)]^-\}^q \, dx - K \leqq F^{\varepsilon, k}(u_{\varepsilon, k}).$$

Thus, with (4.19), we have that:

$$\int_0^1 \{[u'_{\varepsilon, k}(x)]^-\}^q \, dx \leqq \frac{C+K}{k}, \quad \text{for every } k>0. \tag{4.21}$$

We obtain that $u'_\varepsilon(x) \geqq 0$ a.*e*. in $]0, 1[$, since, by (4.21), it is the weak limit of the sequence of non-negative functions:

$[u'_{\varepsilon, k}]^+ = u'_{\varepsilon, k} + [u'_{\varepsilon, k}]^-$, that *weakly converges to* u'_ε *in* $L^q(0, 1)$, *as* $k \to \infty$.

Since $u'_\varepsilon(x) \geqq 0$ a.e. we also have an L^∞ bound for u_ε, uniform with respect to $\varepsilon \in]0, 1]$:

$$S_0 = u_\varepsilon(0) \leqq u_\varepsilon(x) \leqq u_\varepsilon(1) = S_1, \qquad \text{for} \quad x \in]0, 1[.$$

We can now use one of the *a priori* estimates' theorems 3.1 or 3.2 for $u_{\varepsilon, k}$ [we consider here explicitly the case of theorem 3.2 and the estimate (3.8)]; with (4.20), we get:

$$\begin{aligned} \| u'_\varepsilon \|_{L^\infty(\delta, 1-\delta)} &\leqq \liminf_{k \to \infty} \| u'_{\varepsilon, k} \|_{L^\infty(\delta, 1-\delta)} \\ &\leqq \max \left\{ K_0; \frac{4}{\delta} \| u_\varepsilon \|_{L^\infty(0, 1)} \right\} \leqq \max \left\{ K_0; \frac{4}{\delta} |S_0|; \right. \\ &\qquad \left. \frac{4}{\delta} |S_1| \right\}, \qquad \text{for} \quad \delta \in \left] 0, \frac{1}{2} \right[. \end{aligned}$$

We can then conclude in a similar way as in the proof of theorem 4.1 from (4.13) □

5. APPLICATIONS TO CLASSICAL EXAMPLES

In this section, we apply the existence results of section 4 to the classical examples (i) to (iii) and then to the example (iv) described in the introduction.

THEOREM 5.1 (*Appliation to classical examples*). – *Let*

$$F(v) = \int_0^1 a(x, v) \sqrt{1 + v'^2} \, dx, \tag{5.1}$$

where $a \in C^2(]0, 1[\times[0, +\infty[) \cap C^0([0, 1]\times[0, +\infty[)$ *satifies at least one of the following conditions:*

(5.2) $a = a(s) > 0$, *for* $s > 0$, *independent of* x, *is monotone;*

(5.3) $a = a(s) \geqq 0$, *independent of* x, *is such that, if* $a'(\bar{s}) = 0$ *for some* $\bar{s} \geqq 0$, *then* $a''(\bar{s}) > 0$;

(5.4) $a = a(x) \geqq C > 0$, *independent of* s, *is such that, if* $a'(\bar{x}) = 0$ *for some* $\bar{x} \in [0, 1]$, *then* $a''(\bar{x}) < 0$;

(5.5) $a = a(x, s) \geqq C > 0$ *and* $a_x(x, s) \neq 0$ *for* $(x, s) \in]0, 1[\times[0, +\infty[$.

Then, the extended functional $\bar{F}$ *is given by* (2.9) *with* $\tilde{h} = 1$ *and the variational problem*

$$(5.7) \qquad \min\{\bar{F}(v) : v \in W^{1,1}_{\mathrm{loc}}(0, 1),\ v(x) \geqq 0 \text{ for } x \in [0, 1]\}$$

has a solution which belongs to $W^{1,\infty}_{\mathrm{loc}}(0, 1)$.

Theorem 5.1 contains the functionals described in the introduction (ii) and (iii), *i.e.* the surface of minimal revolution area ($a(s) = s$) and Fermat's principle [since $a = a(x, s) \geqq 1/c$, where c is the velocity of light in the vacuum].

Remark 5.2 (*Application to the brachistocrone problem*). – To treat the brachistocrone problem (i), we need a further approximation argument. In fact, if $a(s) = s^{-1/2}$, we can consider, for every $k \in \mathbb{N}$, a function $a_k(s)$ that is equal to $a(s)$ for $s \geqq 1/k$, and that is extended as an affine $C^1(\mathbb{R})$ function for $s < 1/k$. Then, we repeat the proof of theorem 4.1 in this case. We consider first a minimizer $u_{\varepsilon,k} \in W^{1,2}(0, 1)$, with $u_{\varepsilon,k}(0) = 0$, $u_{\varepsilon,k}(1) = S_1$, of the functional

$$F^{\varepsilon,k}(v) = \int_0^1 a_k(v)\sqrt{1 + v'^2}\, dx + \varepsilon \int_0^1 v'^2\, dx.$$

By posing $A_k(y) = \int_0^y a_k(s)\, ds$ (note that A_k is increasing with respect to k), as in (4.12), we obtain

$$A_1(u_{\varepsilon,k}(x)) \leqq A_k(u_{\varepsilon,k}(x)) \leqq F^{\varepsilon,k}(u_{\varepsilon,k}), \qquad \text{for} \quad k \geqq 1.$$

To get an L^∞ bound for $u_{\varepsilon,k}$ (uniform with respect to ε and k), since $A_1(s)$ is strictly increasing, we have to show that $F^{\varepsilon,k}(u_{\varepsilon,k})$ is bounded. By posing $v(x) = S_1 x$, we have [since $a_k(s) \leqq a(s)$ and $\varepsilon \leqq 1$]

$$F^{\varepsilon,k}(u_{\varepsilon,k}) \leqq F^{\varepsilon,k}(v) \leqq \int_0^1 \{a(v)\sqrt{1 + v'^2} + v'^2\}\, dx = \text{const.},$$

since $a(v(x)) = (S_1 x)^{-1/2}$ is an $L^1(0, 1)$ function. As in (4.13), we then get an estimate (uniform with respect to ε and k)

$$\|u'_{\varepsilon,k}\|_{L^\infty(\delta, 1-\delta)} \leqq C = C(\delta), \quad \text{for every } \delta \in \left]0, \frac{1}{2}\right[$$

We can then proceed in the same way as in theorem 4.1.

Remark 5.3. — Similarly to theorem 5.1, we can also treat some other integrands f. For example, we can assume that $f(x, s, \xi)=a(s)h(\xi)$ is of class C^3, where $h(\xi)$ is convex and $a>0$ a.e. in $\mathbb{R}$, that there exist $p \geqq 1$, $L>0$ such that $|\xi| \leqq h(\xi) \leqq L(1+|\xi|^p)$, that $h(\xi)-\xi h'(\xi)>0$ (resp. <0) for every $\xi \in \mathbb{R}$ and that if $a'(\bar{s})=0$ for some $\bar{s} \in \mathbb{R}$, then $a''(\bar{s})>0$ (resp. <0). Then, the representation formula (2.10) [or (2.9) if $\tilde{h}_+=\tilde{h}_-$] holds for $\bar{F}$ and problem (4.2) admits a solution which belongs it $W^{1,\infty}_{\text{loc}}$.

Proof of theorem 5.1. — For $f(x, s, \xi)=a(x, s)\sqrt{1+\xi^2}$, φ and ψ defined by (3.5) and (3.7) are

$$\varphi(x, s, \xi)=(1+\xi^2)^{-1/2}\{a_s(x, s)-\xi a_x(x, s)\},$$
$$\psi(x, s, \xi)=(1+\xi^2)^{-1/2}\{(1-\xi^2)a_{xs}(x, s)+\xi[a_{ss}(x, s)-a_{xx}(x, s)]\}.$$

Theorem 5.1 is then obtained as a corollary of theorems 2.1 and 4.3. In fact, under (5.2) φ has a definite sign and (3.5) holds; under either (5.3) or (5.4), if $\varphi=0$, then $\xi\psi>0$ and (3.7) holds. Finally, under (5.5), for every $\delta \in \left]0, \frac{1}{2}\right[$ and $r \geqq 0$, there exist C_1, $C_2>0$ such that $|a_s(x, s)| \leqq C_1$ and $|a_x(x, s)| \geqq C_2$, for $(x, s) \in [\delta, 1-\delta] \times [0, r]$; this implies that $\varphi \neq 0$ for $|\xi|$ large and again (3.7) is satisfied (*see* remark 3.3) □

We mention at last the application of theorem 2.4 and 4.4 to a recent problem appearing in an adiabatic model prodicting a finite height h_0 to the atmosphere:

$$(5.8) \quad \inf\left\{F(v)=\int_0^1 \left\{\frac{p_0}{(\gamma-1)v'^{\gamma-1}}+r_0 g v\right\} dx : v \in W^{1,1}(0, 1),\right.$$
$$\left. v(0)=0,\ v(1)=h,\ v'(x)>0\right\},$$

where the positive constants p_0, r_0, $\gamma>1$, g and h stand respectively for the atmosphere's pressure and density in the reference configuration, the adiabatic and gravity constants and the height of the atmosphere. It has been shown by Ball in [6] that (5.8) has a solution (satisfying the boundary conditions) for $h \leqq h_0$ and a generalised solution [verifying $v(0)=0$ and $\lim_{x \to 1} v(x)=h_0$] for $h>h_0$. Applying our results to (5.8) [as in remark 5.2, we must first approximate $h(\xi)=\xi^{1-\gamma}$ by the increasing sequence $h_k(\xi)$ of convex functions which are equal to $h(\xi)$ for $\xi \geqq 1/k$ and which are extended as $C^2(\mathbb{R})$ functions for $s<1/k$], since $\tilde{h}=0$ and $\varphi(x, s, \xi) \equiv r_0 g>0$, we obtain by theorem 2.4 that $\bar{F}=F$ and by theorem 4.4 that there exists a solution v which belongs to $W^{1,\infty}_{\text{loc}}$ for the problem (4.8) associated to (5.8). In this case, by the convexity of $v(x)$, it is easy to show that $v(0)=0$.

6. APPLICATION IN BEHAVIOURAL ECOLOGY

A fundamental question appearing in behavioural ecology (*see* e. g. [7], [8], [20]), in particular in the study of behaviour of animals while foraging (*i. e.* search and acquisition of food), is the following. An animal is going each day around in its habitat to find food. Imagine that the food resource is renewed each day with the same distribution and that some regions of the habitat are more risky to exploit, for example because of the presence of predators. Assuming that the animal has learnt the food and risk distributions, what is the optimal way to exploit the habitat in order to balance the needs of maximizing the food gained and of minimizing the risks incurred?

In the past few years, optimal foraging theory has developed to answer such theoretical questions (*see* reviews in [21], [29]). There have been many attempts to formalize the above problem and partial answer (without presence of predators) has been given by the well-known "patch model" of Charnov [13] for some particular type of food distribution, by Andersson [1] for uniform food distributions or by many others (*see* references in [8], [20]). More recently, Arditi and Dacorogna ([2], [3], [4]) and Botteron and Arditi [9] have proposed models formalizing that question, generalizing the results of Charnov [13] to arbitrary food distributions and introducing in [9] arbitrary risk densities. The mathematical problem in [9] turns out to be the minimization of a non-coerciive functional of the calculus of variations:

$$\inf \left\{ \mathrm{F}(v) = \int_0^1 (\rho(x)\,\Phi_0(v') + h(x)\,\mathrm{G}(v))\,dx : v \in \mathscr{W}_1 \right\}, \tag{6.1}$$

[a relevant example is $\Phi_0(\xi) = e^{-\xi}$ and $\mathrm{G}(s) = (1+s)^p$, for $p \geqq 1$] where for $\mathrm{S} \geqq 0$,

$$\mathscr{W}_1 = \{ v \in \mathrm{W}^{1,1}(0, 1) : v(0) = 0,\ v(1) = \mathrm{S},\ v'(x) \geqq 0 \text{ a. e.} \}. \tag{6.2}$$

We quickly recall the meaning of the notations used in (6.1) and (6.2) (*see* e. g. [8] for more details). The animal is described by its schedule $v = v(x)$ (*i. e.* time against position), for $x \in [0, 1]$. The interval [0, 1] represents a one-dimensional habitat or a closed curve in a two-dimensional domain with $x = 0$ and $x = 1$ corresponding to the central place (*i. e.* nest or cache). The animal covers its habitat during the "foraging period" S ($v(0) = 0$, $v(1) = \mathrm{S}$) with an upper bound on its velocity [equivalent after change of variable to $v'(x) \geqq 0$ a. e.]. The food distribution in the habitat is arbitrary (*i. e.* neither necessarily "patchy" as in [13] nor uniform as in [1]). It is described by a given food density $\rho = \rho(x)$. The function $h = h(x)$ is related to a given risk density $c = c(x)$ [more precisely, $h(x) = -c'(x)$]. The "foraging presence" $v'(x)$ represents the time (in some convenient unit) during which the animal consumes the resource available at point x. The function Φ_0 results from the dynamics of food acquisition; typically, the local renewal rate of food resource is assumed to be slow, so that as the animal stays in the same place, the rate at which it acquires food drops (e. g. $\Phi_0(\xi)=e^{-\xi}$ with a Lotka-Volterra functional response, *see* [2]). The term $\rho(x)\Phi_0(v')$ in (6.1) represents the density of food remaining at point x after the passage of the animal and the term $h(x)\mathrm{G}(v)$ the density of risk cost. Food gains and risk costs are accounted for in common units of fitness. Fitness (*see* e. g. [7], [8], [31]) is a measure of the survival and reproductive success of the animal. In the evolutionary approach, a behaviour is called "optimal" if it maximizes fitness. This maximization is equivalent in these models to (6.1).

Mathematically, this problem has been solved in [2], [3] for the first models in bounded and unbounded habitats without introduction of risk [*i.e.* $\mathrm{G}\equiv 0$ in (6.1)], then in [10] for a generalised version in bounded habitat. Risk costs have been introduced in [9] and (6.1) has been solved by showing the sufficiency of the necessary conditions given by the Euler equation for the simple case $\mathrm{G}(v)=v$. This last resolution has been extended in [11] to more general G but with some restrictions on ρ and h (more precisely, ρ increasing and h strictly positive). With rearrangement techniques and without considering the Euler equation, solutions have been shown to exist for small value of S (*i.e.* $\mathrm{S}\leqq\mathrm{S}_c$). For large values of S (*i.e.* $\mathrm{S}>\mathrm{S}_c$), it has been shown that (6.1) has no solution (*i.e.* satisfying the prescribed boundary values) but has a solution in the sense described here [satisfying $v(0)=0$, $\lim_{x\to 1} v(x)=\mathrm{S}_c$].

However, in the point of view of the relevance of the application, it seemed interesting to handle the more general case where $h(x)$ is not restricted to be strictly positive [corresponding to a density of risk $c(x)$ strictly decreasing in [0, 1], since $h(x)=-c'(x)$], but where $h(x)$ can vary from negative to positive values [corresponding to the more realistic situation of a risk $c(x)$ increasing with distance to the central place].

Obtained as a corollary of the results of the previous sections, the following theorem 6.1 handles this more realistic situation and gives a new existence result under assumptions less restrictive than in [11] and more relevant for the specific application (*see* remark 6.2).

THEOREM 6.1 (*Application to models in behavioural ecology*). — *Let* F, $\bar{\mathrm{F}}$ *and* $\mathscr{W}_1$ *be defined respectvely by* (6.1), (2.2) *and* (6.2), *where*

$$f(x, s, \xi)=\rho(x)\,\Phi_0(\xi)+h(x)\,\mathrm{G}(s)$$

is of class C^3. *Assume that* $\Phi_0:\ \mathbb{R}^+\to\mathbb{R}^+$ *is convex and* $\Phi_0'(\xi)<0$, *for* $\xi\geqq 0$, $\mathrm{G}:[0,\mathrm{S}]\to[0,+\infty[$, $\mathrm{G}'(s)>0$ *for* $s\in[0,\mathrm{S}]$, $\rho\geqq 0$; *assume also that* $h(x)\rho'(x)\geqq 0$ *and that* $h'(\bar{x})>0$ *if* $h(\bar{x})=\rho'(\bar{x})=0$. *Then*, $\bar{\mathrm{F}}=\mathrm{F}$ *and it has* *a minimizer* (*belonging to* $W^{1,\infty}_{loc}$ *in*

$$\mathscr{W}_1 = \{ v \in W^{1,1}_{loc}(0, 1) : v(0) \geqq 0, \quad v(1) \leqq S, v'(x) \geqq 0 \ a.e. \}.$$

Remark 6.2. — The case where ρ if increasing and h is strictly positive, as in the resolution of Botteron and Dacorogna [11], is contained in theorem 6.1. The assumptions required in theorem 6.1 are satisfied in many relevant cases. For example, in the situation of an animal crossing a closed curve in a two-dimensional domain with $x=0$ and $x=1$ corresponding to the central place, with risk density c increasing with distance to the central place (this assumption is very natural, *see* [9] and the references quoted there) with maximum risk at a point $x_0 \in]0, 1[$ [*i.e.* $h(x_0) = -c'(x_0) = 0$, $h'(x_0) = -c''(x_0) > 0$, and therefore without a definite sign for $h(x)$] and with food density ρ decreasing with distance to the central place, *i.e.* decreasing in $[0, x_0]$ and with food density ρ decreasing with distance to the central place, *i.e.* decreasing in $[0, x_0]$ and increasing in $[x_0, 1]$.

Proof of theorem 6.1. — We first extend G from $[0, S]$ to $\mathbb{R}$ so that it is a positive bounded function of class C^2 with $G'(s) \geqq c = c(r) > 0$ for $s \in [-r, r]$. Then, since Φ_0 is convex and bounded in $[0, +\infty[$,

$$\lim_{\xi \to +\infty} \frac{\Phi_0(\xi)}{\xi} = \lim_{\xi \to +\infty} \Phi_0'(\xi) = 0.$$

Let us extend Φ_0 for $\xi < 0$ by the Taylor polynomial:

$$\Phi_0(\xi) = \Phi_0(0)\xi + \frac{1}{2}\Phi_0''(0)\xi^2, \qquad \text{for} \quad \xi < 0.$$

Then, Φ_0 is a convex function of class $C^2(\mathbb{R})$ satisfying (4.7) *b*) with $p=2$. Let us also extend f for $\xi < 0$:

$$f(x, s, \xi) = \rho(x)\Phi_0(\xi) + x\xi^4 + h(x)G(s), \qquad \text{if} \quad \xi < 0.$$

Then, f is of class $C^2([0, 1] \times \mathbb{R} \times \mathbb{R})$ and satisfies the assumptions of theorem 2.4. Hence, $\bar{F}(v) = F(v)$, for every $v \in \mathscr{W}_p$. The function φ defined by (3.5) is

$$(6.3) \quad \varphi(x, s, \xi) = \begin{cases} h(x)G'(s) - \rho'(x)\Phi_0'(\xi), & \text{if} \quad \xi \geqq 0, \\ h(x)G'(s) - \rho'(x)\Phi_0'(\xi) - 4\xi^3, & \text{if} \quad \xi < 0 \end{cases}.$$

For ξ negative, (3.7) is satisfied, since $\varphi > 0$ for $|\xi|$ sufficiently large (*see* remark 3.3). For ξ positive, the function ψ defined by (3.7) is

$$\psi(x, s, \xi) = h'(x)G'(s) - \rho''(x)\Phi_0'(\xi) + \xi h(x)G''(s), \qquad \text{if} \quad \xi \geqq 0.$$

Let us show that the condition (3.7) is also satisfied for $\xi \geqq 0$. In fact, if $\varphi(x, s, \xi) = 0$, since the two terms $h(x)G'(s)$ and $\rho'(x)\Phi_0'(\xi)$ in (6.3) have the same sign, then necessarily $h(x)=\rho'(x)=0$. Let us denote by $m=m(\delta)$ the positive minimum of $h'(x)$ on the compact set $\{x\in[\delta, 1-\delta]:h(x)=\rho'(x)=0\}$ (note that this set contains at most one point). Then, $\psi(x, s, \xi)\geqq m(\delta)c(r)-\|\rho''\|_{L^\infty}|\Phi'_0(\xi)|$. Since $\Phi'_0(\xi)\to 0$ as $\xi\to+\infty$, there exists $K_0=K_0(\delta, r)$ such that $\psi(x, s, \xi)>0$ for $\xi>K_0$. □

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