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## On the non-existence of energy stable minimal cones

by

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**ABSTRACT.** — We show that there are no non-trivial (potential) energy stable minimal cones in  $\mathbb{R}^n \times \mathbb{R}^+$  with singularity at 0, if  $2 \leq n \leq 5$ . The sharpness of this result is demonstrated by proving that a certain six dimensional cone in  $\mathbb{R}^7$  is stable. Moreover, we extend all results to the more general  $\alpha$ -energy functional.

*Key words* : Stable cones.

**RÉSUMÉ.** — L'on démontre que, si  $2 \leq n \leq 5$ , il n'existe pas dans  $\mathbb{R}^n \times \mathbb{R}^+$  de cônes minimaux stables en énergie. Ce résultat est optimal, car l'on exhibe dans  $\mathbb{R}^7$  un cône de dimension 6 qui est stable. On étend également ces résultats à des fonctionnelles d'énergie plus générales.

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A well known result due to J. Simons [SJ] states that there are no non-trivial  $n$ -dimensional stable minimal cones in  $\mathbb{R}^{n+1}$  (with singularity at zero), provided  $n \leq 6$ . One of the crucial ingredients in his proof is an important identity for the Laplacian of the second fundamental form for minimal hypersurfaces. Using sharper estimates than had previously been

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realized, Schoen-Simon-Yau [SSY] gave a considerably simpler proof of Simons' result.

Simons also proved that the seven dimensional cone  $x_1^2 + \dots + x_4^2 = x_5^2 + \dots + x_8^2$  in  $\mathbb{R}^8$  is stable and, in fact, it was proved by Bombieri-De Giorgi-Giusti [BDG] that it even minimizes area in  $\mathbb{R}^8$ . This result dashes the hope for general interior regularity of codimension one solutions to the problem of least area in  $\mathbb{R}^8$ .

In two papers [D 1] and [D 2] the author has investigated the cones

$$C_n^\alpha := \left\{ x = (x_1, \dots, x_{n+1}); 0 \leq x_{n+1} \leq \sqrt{\frac{\alpha}{n-1}} [x_1^2 + \dots + x_n^2]^{1/2} \right\} \subset \mathbb{R}^{n+1}$$

which have boundaries of least  $\alpha$ -energy

$$\mathcal{E}_\alpha = \int x_{n+1}^\alpha |D\varphi_U| \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+,$$

provided one of the following conditions holds:

(i)  $\alpha + p \geq 6$ , where  $\alpha \geq 2$  and  $p := n - 1 \geq 2$ ,

or

(ii)  $\alpha + p \geq 7$ , for  $\alpha \geq 1$  and  $p \geq 1$ .

Here  $\mathbb{R}^+ = \{t \geq 0\}$ ,  $U \subset \mathbb{R}^n \times \mathbb{R}^+$ , and  $|D\varphi_U|$  is the  $n$ -dimensional Hausdorff measure restricted to the reduced boundary of  $U$ . Also a set  $C \subset \mathbb{R}^n \times \mathbb{R}^+$  with characteristic function  $\varphi_C$  has a boundary of least  $\alpha$ -energy in  $\mathbb{R}^n \times \mathbb{R}^+$ , if and only if for each  $g \in L_{loc}^1(\mathbb{R}^n \times \mathbb{R}^+)$  with compact support  $K \subset \mathbb{R}^n \times \mathbb{R}^+$  we have

$$\int_K x_{n+1}^\alpha |D\varphi_C| \leq \int_K x_{n+1}^\alpha |D(\varphi_C + g)|.$$

Furthermore it could be shown in [D 2] that the six dimensional boundary of the cone  $C_6^1$  does not minimize the energy  $\mathcal{E}_1$  in  $\mathbb{R}^6 \times \mathbb{R}^+$ . Similarly, the cone  $C_2^5$  does not minimize  $\mathcal{E}_5$  in  $\mathbb{R}^2 \times \mathbb{R}^+$ .

We wish to emphasize the physical relevance of the problem. Namely if we regard the boundary  $M = \partial U$  of  $U$  as a material surface of constant mass density, then  $\mathcal{E}_1$  corresponds to the potential energy of  $M$  under gravitational forces. Here we have of course assumed that the gravitational force acts in the  $-x_{n+1}$  direction. Therefore, we refer to  $\mathcal{E}_\alpha$  as the  $\alpha$ -energy, and, in particular, if  $\alpha = 1$  we shall simply omit the addition " $\alpha$ ".

In this paper we will employ the method of Schoen-Simon-Yau [SSY] to obtain a result on the non-existence of non-trivial  $\alpha$ -stable minimal cones in  $\mathbb{R}^n \times \mathbb{R}^+$  *i. e.*, cones which are stable with respect to the  $\alpha$ -energy  $\mathcal{E}_\alpha$ . We in fact prove (Theorem 2) that such a result holds true provided that

$$\alpha + p < 3 + \sqrt{8}, \quad p = n - 1.$$

On the other hand we show in Theorem 1 that the cones

$$x_{n+1} = + \sqrt{\frac{\alpha}{p}} \left[ x_1^2 + \dots + x_n^2 \right]^{1/2}$$

$$\alpha + p \geq 3 + \sqrt{8}.$$

Note that this in particular implies stability, if  $\alpha = 1, n = 6$ , or  $\alpha = 5, n = 2$ , but because of [D 2] the boundaries of the set  $C_6^1$  or  $C_2^5$  do not minimize the corresponding  $\alpha$ -energy in  $\mathbb{R}^n \times \mathbb{R}^+$ . In fact, we might even obtain a field of  $\alpha$ -stable and non-minimizing minimal cones, e.g. the two-dimensional cones  $x_3 = \sqrt{\alpha} [x_1^2 + x_2^2]^{1/2}$  in  $\mathbb{R}^3$  where  $2 + \sqrt{8} \leq \alpha \leq 5$ .

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### 1. NOTATIONS AND RESULTS

In this section we set up our terminology and, in particular, we give simple expressions for the first and second variation of the  $\alpha$ -energy. Finally, we formulate our main results.

Let  $M$  be an  $n$ -dimensional submanifold of class  $C^2$  contained in the open half-space  $\mathbb{R}^n + \mathring{\mathbb{R}}^+ \subset \mathbb{R}^{n+1}$ ,  $\mathring{\mathbb{R}}^+ = \{t > 0\}$ , and let  $U \subset \mathbb{R}^n \times \mathring{\mathbb{R}}^+$  be open with  $U \cap M \neq \emptyset$ ,  $(\text{clos } M - M) \cap U = \emptyset$ ,  $\mathcal{H}_n(M \cap K) < \infty$  for each compact set  $K \subset U$ ; here  $\mathcal{H}_t$ ,  $t \geq 0$ , denotes  $t$ -dimensional Hausdorff measure. We consider one parameter families  $\{\Phi_t\}$ ,  $-1 \leq t \leq 1$ , of diffeomorphisms from  $U$  into  $U$ , with the following properties:

$$\Phi(t, x) =: \Phi_t(x) \in C^2((-1, 1) \times U, U), \tag{1}$$

$$\Phi_0(x) = x \text{ for all } x \in U, \tag{2}$$

$$\Phi_t(x) = x \text{ for all } t \in (-1, 1)$$

$$\text{and all } x \in U - K \text{ for some compact set } K \subset U. \tag{3}$$

Put

$$X(x) := \frac{\partial \Phi}{\partial t}(t, x) \Big|_{t=0},$$

and

$$Z(x) := \frac{\partial^2 \Phi}{\partial t^2}(t, x) \Big|_{t=0},$$

to denote the initial velocity and acceleration vectors of  $\Phi_t$  respectively. Then, because of (3),  $X$  and  $Z$  have compact support  $K \subset U$ , and furthermore

$$\Phi_t(x) = x + tX(x) + \frac{t^2}{2}Z(x) + o(t^2).$$

Let  $M_t := \Phi_t(M)$  denote the image of  $M = M_0$  under  $\Phi_t$ ; then we are interested in the first and second variation of the  $\alpha$ -energy functional

$$\mathcal{E}_\alpha(M) = \int_M x_{n+1}^\alpha d\mathcal{H}_n,$$

where  $x = (x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}^+$ ,  $\alpha > 0$ , *i. e.* we want to compute

$$\delta \mathcal{E}_\alpha(M, X) = \frac{d}{dt} \int_{M_t} x_{n+1}^\alpha d\mathcal{H}_n \Big|_{t=0},$$

and

$$\delta^2 \mathcal{E}_\alpha(M, X, Z) = \frac{d^2}{dt^2} \int_{M_t} x_{n+1}^\alpha d\mathcal{H}_n \Big|_{t=0}.$$

Choose a local field of orthonormal frames  $\tau^1, \dots, \tau^n, \nu$  such that  $\tau^1, \dots, \tau^n \in T_x M$  are tangent to  $M$ . For a given vectorfield  $Y$  on  $M$  (not necessarily tangential) we denote by  $D_{\tau^i} Y$  the directional derivative of  $Y$  in the direction  $\tau^i$ . Also

$$\operatorname{div} Y = \sum_{i=1}^n (D_{\tau^i} Y) \tau^i$$

stands for the divergence on  $M$ , and

$$\nabla f = \sum_{i=1}^n (D_{\tau^i} f) \tau^i$$

denotes the gradient of the function  $f \in C^1(M, \mathbb{R})$  respectively. We shall also employ the symbol  $\Delta$  to denote the Laplacian on  $M$ , *i. e.*  $\Delta = \nabla_i \nabla_i$  where  $\nabla_i = D_{\tau^i}$ , and  $Y^\perp = Y - \sum_{j=1}^n (Y \cdot \tau^j) \tau^j$  stands for the normal part of  $Y$ .

LEMMA 1. — *Let  $M, \Phi_t: U \rightarrow U$  and*

$$\begin{aligned} X(x) &= (X_1(x), \dots, X_{n+1}(x)), \\ Z(x) &= (Z_1(x), \dots, Z_{n+1}(x)) \end{aligned}$$

*be defined as above. Then*

$$\delta \mathcal{E}_\alpha(M, X) = \int_M \{ x_{n+1}^\alpha \operatorname{div} X + \alpha x_{n+1}^{\alpha-1} X_{n+1} \} d\mathcal{H}_n \quad (4)$$

and

$$\delta^2 \mathcal{E}_\alpha(M, X, Z) = \int_M \left\{ \alpha(\alpha-1)x_{n+1}^{\alpha-2}X_{n+1}^2 + \alpha x_{n+1}^{\alpha-1}Z_{n+1} + 2\alpha x_{n+1}^{\alpha-1}X_{n+1} \operatorname{div} X + x_{n+1}^\alpha [\operatorname{div} Z + (\operatorname{div} X)^2 + \sum_{i=1}^n |(D_{\tau^i} X)^\perp|^2 - \sum_{i,j=1}^n (\tau^i D_{\tau^j} X)(\tau^j D_{\tau^i} X)] \right\} d\mathcal{H}_n. \quad (5)$$

*Proof.* — From the general area formula (see e. g. [SL], §8, or [FH] 3.2.20 Cor.), we infer that

$$\mathcal{E}_\alpha(\Phi_t(M \cap K)) = \int_{M \cap K} (\psi_t)_{n+1}^\alpha J\psi_t d\mathcal{H}_n,$$

where  $\psi_t = \Phi_t|_{M \cap U}$ ,  $J\psi_t$  denotes the Jacobian of  $\psi_t$  and  $(\psi_t)_{n+1}^\alpha$  is the  $(n+1)$ -th component of  $\psi_t$  to the power  $\alpha$ . The Jacobian  $J\psi_t$  may be computed as in [SL], p. 50,

$$J\psi_t = 1 + t \cdot \operatorname{div} X + \frac{t^2}{2} \left\{ \operatorname{div} Z + (\operatorname{div} X)^2 + \sum_{i=1}^n |(D_{\tau^i} X)^\perp|^2 - \sum_{i,j=1}^n (\tau^i D_{\tau^j} X)(\tau^j D_{\tau^i} X) \right\} + o(t^2).$$

Similarly we find

$$(\psi_t)_{n+1}^\alpha = x_{n+1}^\alpha + t\alpha x_{n+1}^{\alpha-1}X_{n+1} + \frac{t^2}{2} [\alpha(\alpha-1)x_{n+1}^{\alpha-2}X_{n+1}^2 + \alpha x_{n+1}^{\alpha-1}Z_{n+1}] + o(t^2).$$

Now the result follows immediately by computing the coefficients of  $t$  and  $\frac{t^2}{2}$  in the product  $(\psi_t)_{n+1}^\alpha \cdot J\psi_t$ .  $\square$

Lemma 1 motivates the following definition.

**DEFINITION 1.** — A  $C^1$ -submanifold  $M \subset \mathbb{R}^n \times \mathbb{R}^+$  is called  $\alpha$ -stationary in  $U \subset \mathbb{R}^n \times \mathbb{R}^+$ , if  $\mathcal{E}_\alpha(M \cap K) < \infty$  for all compact sets  $K \subset U$ , and

$$\int_M \{ x_{n+1}^\alpha \operatorname{div} X + \alpha x_{n+1}^{\alpha-1} X_{n+1} \} d\mathcal{H}_n = 0, \quad (6)$$

for all vector fields  $X \in C_c^1(U, \mathbb{R}^{n+1})$ .

**LEMMA 2.** — Suppose  $M$  is  $\alpha$ -stationary in  $U$  and of class  $C^2$ . Then the mean curvature  $H$  of  $M$  with respect to the unit normal  $\nu = (\nu_1, \dots, \nu_{n+1})$  is given by

$$H(x) = \alpha x_{n+1}^{-1} \nu_{n+1} \quad \text{for all } x \in M \cap U.$$

*Proof.* — Take some arbitrary function  $\xi \in C_c^1(M, \mathbb{R})$  with compact support in  $U$  and put  $X = \xi \cdot v$ . Then we infer from (6)

$$\begin{aligned} 0 &= \int_M \{ x_{n+1}^\alpha \operatorname{div}(\xi \cdot v) + \alpha x_{n+1}^{\alpha-1} \xi \cdot v_{n+1} \} d\mathcal{H}_n \\ &= \int_M \{ \operatorname{div}(x_{n+1}^\alpha \xi \cdot v) + \alpha x_{n+1}^{\alpha-1} \xi \cdot v_{n+1} \} d\mathcal{H}_n, \\ &= - \int_M \{ x_{n+1}^\alpha \xi \cdot v \cdot \underline{H} - \alpha x_{n+1}^{\alpha-1} \xi \cdot v_{n+1} \} d\mathcal{H}_n, \end{aligned}$$

where  $\underline{H} = vH$  is the mean curvature vector of  $M$ . The lemma follows by applying the fundamental lemma in the calculus of variations.  $\square$

We take again the special variation  $X = \xi \cdot v$ ,  $\xi \in C_c^1(M, \mathbb{R})$  and find successively,

$$\begin{aligned} \operatorname{div} X &= -X \cdot \underline{H} = -\alpha v_{n+1} x_{n+1}^{\alpha-1} \xi \\ \sum_{i=1}^n |(D_i X)^\perp|^2 &= \sum_{i=1}^n |v D_i \xi|^2 = |\nabla \xi|^2, \end{aligned}$$

and

$$\sum_{i, j=1}^n (\tau^i D_j X) (\tau^j D_i X) = \xi^2 |A|^2,$$

where  $|A|$  denotes the length of the second fundamental form  $A = h_{ij} \tau^i \otimes \tau^j$ , *i. e.*

$$|A|^2 = \sum_{i, j=1}^n h_{ij}^2.$$

Thus we have proved

LEMMA 3. — Suppose  $M \subset \mathbb{R}^n \times \mathbb{R}^+$  is a submanifold of class  $C^2$  which is  $\alpha$ -stationary in  $U \subset \mathbb{R}^n \times \mathbb{R}^+$ ,  $(\operatorname{clos} M - M) \cap U = \emptyset$ . If  $X = \xi \cdot v$  for some function  $\xi \in C_c^1(M, \mathbb{R})$  with compact support in  $U$ , then the second variation is given by

$$\delta^2 \mathcal{E}(M, \xi) = \int_M \{ x_{n+1}^\alpha |\nabla \xi|^2 - \alpha x_{n+1}^{\alpha-2} v_{n+1}^2 \xi^2 - x_{n+1}^\alpha |A|^2 \xi^2 \} d\mathcal{H}_n. \quad \square$$

Hence it is reasonable to define stability as follows.

DEFINITION 2. — Suppose  $M \subset \mathbb{R}^n \times \mathbb{R}^+$  is a  $n$ -dimensional submanifold of class  $C^2$  which is  $\alpha$ -stationary in  $U \subset \mathbb{R}^n \times \mathbb{R}^+$ ,  $(\operatorname{clos} M - M) \cap U = \emptyset$ . Then  $M$  is called  $\alpha$ -stable in  $U$ , if

$$\int_M \{ x_{n+1}^\alpha |\nabla \xi|^2 - \alpha x_{n+1}^{\alpha-2} v_{n+1}^2 \xi^2 - x_{n+1}^\alpha |A|^2 \xi^2 \} d\mathcal{H}_n \geq 0 \quad (7)$$

for each  $\xi \in C_c^1(M, \mathbb{R})$  with compact support in  $U$ . In particular, if  $\mathcal{C} = \text{clos } M$  is a cone in  $\mathbb{R}^n \times \mathbb{R}^+$  with singularity at  $\{0\}$ , and if  $M = \mathcal{C} - \{0\} \subset \mathbb{R}^n \times \mathbb{R}^+$  is  $\alpha$ -stationary in  $\mathbb{R}^n \times \mathbb{R}^+$ , then  $\mathcal{C}$  is called  $\alpha$ -stable if (7) holds for all  $\xi \in C_c^1(M, \mathbb{R})$ .

Put  $c_n^\alpha(y) = \sqrt{\frac{\alpha}{p}} [y_1^2 + \dots + y_n^2]^{1/2}$ ,  $y \in \mathbb{R}^n$ ,  $\alpha > 0$ ,  $p = n - 1$  and define the cones

$$\mathcal{C}_n^\alpha = \{ (y, c_n^\alpha(y)) : y \in \mathbb{R}^n \},$$

then we have

**THEOREM 1.** — *The cones  $\mathcal{C}_n^\alpha$  are  $\alpha$ -stable, if  $\alpha + p \geq 3 + \sqrt{8}$ .*

Observe that the critical number  $3 + \sqrt{8}$  also enters the discussion of the ordinary differential system [11] in [D 1]. Here, it appears as a necessary, though not sufficient condition for the construction of a minimal foliation about the cone  $\mathcal{C}_n^\alpha$ .

**THEOREM 2.** — *Suppose  $\mathcal{C} \subset \mathbb{R}^n \times \mathbb{R}^+$  is an  $\alpha$ -stable  $n$ -dimensional cone with singularity at  $\{0\}$ . If  $\alpha + p < 3 + \sqrt{8}$  then  $\mathcal{C}$  is a hyperplane  $\mathcal{P}$ . Furthermore,  $\mathcal{P}$  must be perpendicular to the plane  $\{x_{n+1} = 0\}$ .*

**COROLLARY.** — *In particular, if  $2 \leq n \leq 5$  there are no non-trivial (potential-) energy stable cones in  $\mathbb{R}^n \times \mathbb{R}^+$  with singularity at  $\{0\}$ .*

## 2. PROOFS

Let  $\xi \in C_c^1(\mathcal{C}_n^\alpha - \{0\}, \mathbb{R})$  be arbitrary and put  $X(x) = x \cdot |x|^{-2} \xi^2$  for  $x \in \mathbb{R}^n \times \mathbb{R}^+$  where  $|x|^2 = (x_1^2 + \dots + x_{n+1}^2)$ . A standard calculation yields (see [SL], § 17)

$$\begin{aligned} \text{div } X = \sum_{i=1}^n (D_{\tau^i} X) \tau^i &= 2 |x|^{-2} (x \nabla \xi) \xi \\ &\quad + (n-2) \xi^2 |x|^{-2} + 2 |x|^{-2} \xi^2 |(D|x|)^+|^2. \end{aligned}$$

Since  $\mathcal{C}_n^\alpha - \{0\}$  is  $\alpha$ -stationary in  $\mathbb{R}^n \times \mathbb{R}^+$ , we conclude from (6) that

$$\int_{\mathcal{C}_n^\alpha - \{0\}} x_{n+1}^\alpha \{ 2 |x|^{-2} (x \nabla \xi) \xi + (n-2 + \alpha) |x|^{-2} \xi^2 \} d\mathcal{H}_n \leq 0.$$

We apply Schwarz inequality and obtain

$$\left( \frac{n-2+\alpha}{2} \right)^2 \int_{\mathcal{C}_n^\alpha - \{0\}} x_{n+1}^\alpha |x|^{-2} \xi^2 d\mathcal{H}_n \leq \int_{\mathcal{C}_n^\alpha - \{0\}} x_{n+1}^\alpha |\nabla \xi|^2 d\mathcal{H}_n.$$



Therefore  $\mathcal{C}_n^\alpha$  is  $\alpha$ -stable, if

$$\left(\frac{n-2+\alpha}{2}\right)^2 \geq |x|^2 |A|^2 + \alpha x_{n+1}^{-2} |x|^2 v_{n+1}^2. \tag{8}$$

An elementary calculation shows that for the cone  $\mathcal{C}_n^\alpha$  the length of the second fundamental form is given by

$$|A|^2 = \frac{\alpha p}{\alpha + p} r^{-2} = \alpha |x|^{-2} \quad \text{for all } x \in \mathcal{C}_n^\alpha - \{0\},$$

where we have put  $r^2 = (x_1^2 + \dots + x_n^2)$ . Then along  $\mathcal{C}_n^\alpha$ ,  $x_{n+1} = \sqrt{\frac{\alpha}{p}} r$  and we infer from (8) that  $\mathcal{C}_n^\alpha$  is stable, if

$$\begin{aligned} \left(\frac{n-2+\alpha}{2}\right)^2 &\geq \alpha + \alpha x_{n+1}^{-2} |x|^2 v_{n+1}^2 \\ &= \alpha + \frac{\alpha p}{\alpha + p} \left[1 + \frac{r^2}{x_{n+1}^2}\right] = \alpha + p. \end{aligned}$$

This is true in case that  $\alpha + p \geq 3 + \sqrt{8}$ . Theorem 1 follows.

*Proof of Theorem 2.* — In the following we shall always assume that  $M = \mathcal{C} - \{0\}$  is an  $\alpha$ -stable cone in  $\mathbb{R}^n \times \mathbb{R}^+$ , so that in particular (7) holds true. Replacing  $\xi$  by  $|A| \xi$  in (7) we get

$$\begin{aligned} &\int_M \{x_{n+1}^\alpha |A|^4 \xi^2 + \alpha x_{n+1}^{\alpha-2} v_{n+1}^2 |A|^2 \xi^2\} d\mathcal{H}_n \\ &\leq \int_M x_{n+1}^\alpha \{ |A|^2 |\nabla \xi|^2 + |\nabla |A||^2 \xi^2 + 2 \xi |A| (\nabla \xi \nabla |A|) \} d\mathcal{H}_n. \tag{9} \end{aligned}$$

Now

$$\begin{aligned} 2 \int_M x_{n+1}^\alpha |A| \xi (\nabla \xi \nabla |A|) d\mathcal{H}_n &= \int_M x_{n+1}^\alpha (\nabla \xi^2) \nabla \left(\frac{1}{2} |A|^2\right) d\mathcal{H}_n \\ &= - \int_M x_{n+1}^\alpha \xi^2 \Delta \left(\frac{1}{2} |A|^2\right) d\mathcal{H}_n - \int_M \xi^2 (\nabla x_{n+1}^\alpha) \left(\nabla \frac{1}{2} |A|^2\right) d\mathcal{H}_n. \tag{10} \end{aligned}$$

In order to conclude further we need a sharp estimate for the Laplacian of  $|A|^2$ . This will be provided by the following

LEMMA 4 ([SSY], [SL, appendix B]). — *If  $M = \mathcal{C} - \{0\}$  is a cone, then*

$$-\frac{1}{2} \Delta |A|^2 \leq |A|^4 - 2 |x|^{-2} |A|^2 - |\nabla |A||^2 - h_{ij} H_{,ij} - H h_{mi} h_{mj} h_{ij} \tag{1}$$

(1) The summation convention is used freely here!

Here  $H_{,ij}$  denote the second covariant derivatives of the mean curvature  $H$  with respect to  $\tau^i$  and  $\tau^j$ , and, as above,  $h_{ij}$  are the coefficients of  $A$ .

*Proof of Lemma 4.* – B.8 Lemma and B.9 Lemma in [SL] yield the relations

$$\Delta \left( \frac{1}{2} |A|^2 \right) = \sum_{i,j,k} h_{ij,k}^2 - |A|^4 + h_{ij} H_{,ij} + H h_{mi} h_{mj} h_{ij},$$

here  $H = h_{kk} = \text{trace } A$  and  $h_{ij,k}$  denotes the covariant derivative of  $A$  with respect to  $\tau^k$ ; also

$$\sum_{i,j,k} h_{ij,k}^2 - |\nabla |A||^2 \geq 2 |x|^{-2} |A|^2 \quad \text{for all } x \in M.$$

Both relations imply Lemma 4.  $\square$

From (9), (10) and Lemma 4 we conclude that

$$\begin{aligned} \int_M \xi^2 \left\{ 2 x_{n+1}^\alpha |x|^{-2} |A|^2 + \alpha x_{n+1}^{\alpha-2} |A|^2 v_{n+1}^2 \right. \\ \left. + \nabla (x_{n+1}^\alpha) \nabla \left( \frac{1}{2} |A|^2 \right) + x_{n+1}^\alpha h_{ij} H_{,ij} + x_{n+1}^\alpha H h_{mi} h_{mj} h_{ij} \right\} d\mathcal{H}_n \\ \leq \int_M x_{n+1}^\alpha |A|^2 |\nabla \xi|^2 d\mathcal{H}_n. \quad (11) \end{aligned}$$

Relation (11) will be of crucial importance in what follows.

To begin, select an orthonormal frame  $\tau^1, \dots, \tau^n \in T_x M$  so that  $\tau^n = \frac{x}{|x|}$  and  $\tau^1, \dots, \tau^n$  are constant along the ray through  $x$ . Also we can assume that  $\tau_{n+1}^1 = \tau_{n+1}^2 = \dots = \tau_{n+1}^{n-1} = 0$ . Then  $h_{in} = h_{ni} = 0$  for  $i \in \{1, \dots, n\}$  and, since  $h_{ij}(\lambda x) = \lambda^{-1} h_{ij}(x)$ ,  $\lambda > 0$ , we have  $h_{ij,n} = -|x|^{-1} h_{ij}$ .

We first compute the expression

$$\begin{aligned} (\nabla x_{n+1}^\alpha) \left( \nabla \frac{1}{2} |A|^2 \right) &= \alpha x_{n+1}^{\alpha-1} (D_{\tau^k} x_{n+1}) \left( D_{\tau^k} \left( \frac{1}{2} |A|^2 \right) \right) \\ &= \alpha x_{n+1}^{\alpha-1} h_{ij} h_{ij,k} \tau_{n+1}^k = -\alpha x_{n+1}^\alpha |x|^{-2} |A|^2, \quad (12) \end{aligned}$$

and then

$$\begin{aligned}
 \frac{1}{\alpha} H_{,ij} &= \frac{1}{\alpha} \nabla_i \nabla_j H = \nabla_i \nabla_j \left( \frac{v_{n+1}}{x_{n+1}} \right) \\
 &= \nabla_i \left\{ -x_{n+1}^{-2} (\nabla_j x_{n+1}) v_{n+1} + x_{n+1}^{-1} \nabla_j v_{n+1} \right\} \\
 &= 2 x_{n+1}^{-3} \nabla_i x_{n+1} \nabla_j x_{n+1} v_{n+1} - x_{n+1}^{-2} (\nabla_i \nabla_j x_{n+1}) v_{n+1} \\
 &\quad - x_{n+1}^{-2} \nabla_j x_{n+1} \nabla_i v_{n+1} \\
 &\quad - x_{n+1}^{-2} \nabla_i x_{n+1} \nabla_j v_{n+1} + x_{n+1}^{-1} \nabla_i \nabla_j v_{n+1} \\
 &= 2 x_{n+1}^{-3} \tau_{n+1}^i \tau_{n+1}^j v_{n+1} \\
 &\quad - x_{n+1}^{-2} \nabla_i \tau_{n+1}^j v_{n+1} - x_{n+1}^{-2} \tau_{n+1}^j \nabla_i v_{n+1} \\
 &\quad - x_{n+1}^{-2} \tau_{n+1}^i \nabla_j v_{n+1} + x_{n+1}^{-1} \nabla_i \nabla_j v_{n+1}.
 \end{aligned}$$

By virtue of

$$\nabla_i v = -h_{il} \tau^l \quad \text{and} \quad \nabla_i \tau^j = h_{ij} v$$

we obtain

$$\begin{aligned}
 \frac{1}{\alpha} H_{,ij} &= 2 x_{n+1}^{-3} \tau_{n+1}^i \tau_{n+1}^j v_{n+1} \\
 &\quad - x_{n+1}^{-2} h_{ij} v_{n+1}^2 + x_{n+1}^{-2} \tau_{n+1}^j h_{il} \tau_{n+1}^l \\
 &\quad + x_{n+1}^{-2} \tau_{n+1}^i h_{jl} \tau_{n+1}^l - x_{n+1}^{-1} \nabla_i [h_{jl} \tau_{n+1}^l].
 \end{aligned}$$

Using the Codazzi equations we conclude

$$\begin{aligned}
 \nabla_i [h_{jl} \tau_{n+1}^l] &= h_{j,l} \tau_{n+1}^l + h_{jl} \nabla_i \tau_{n+1}^l \\
 &= h_{ij, l} \tau_{n+1}^l + h_{jl} h_{il} v_{n+1},
 \end{aligned}$$

whence

$$\begin{aligned}
 \frac{1}{\alpha} h_{ij} H_{,ij} &= 2 x_{n+1}^{-3} \tau_{n+1}^i \tau_{n+1}^j h_{ij} v_{n+1} - x_{n+1}^{-2} |A|^2 v_{n+1}^2 \\
 &\quad + x_{n+1}^{-2} h_{ij} h_{il} \tau_{n+1}^j \tau_{n+1}^l + x_{n+1}^{-2} h_{ij} h_{jl} \tau_{n+1}^i \tau_{n+1}^l \\
 &\quad - x_{n+1}^{-1} h_{ij} h_{ij, l} \tau_{n+1}^l - x_{n+1}^{-1} h_{ij} h_{jl} h_{il} v_{n+1}.
 \end{aligned}$$

Thus

$$\frac{1}{\alpha} h_{ij} H_{,ij} = -x_{n+1}^{-2} |A|^2 v_{n+1}^2 + |x|^{-2} |A|^2 - x_{n+1}^{-1} h_{ij} h_{jl} h_{il} v_{n+1},$$

and finally

$$h_{ij} H_{,ij} = -\alpha x_{n+1}^{-2} |A|^2 v_{n+1}^2 + \alpha |x|^{-2} |A|^2 - H h_{ij} h_{jl} h_{il}. \quad (13)$$

(12), (13), and (11) yield the relation

$$2 \int_M x_{n+1}^\alpha |x|^{-2} |A|^2 \xi^2 d\mathcal{H}_n \leq \int_M x_{n+1}^\alpha |A|^2 |\nabla \xi|^2 d\mathcal{H}_n \tag{14}$$

for all  $\xi \in C_c^1(M, \mathbb{R})$ .

If  $\xi$  does not have compact support in  $M = \mathcal{C} - \{0\}$  then (14) continues to hold, if only

$$\int_M x_{n+1}^\alpha |x|^{-2} |A|^2 \xi^2 d\mathcal{H}_n < \infty. \tag{15}$$

In fact, replace  $\xi$  by  $\xi \cdot \gamma_\varepsilon$  where  $\gamma_\varepsilon$  is a suitable cut off function with

$$\gamma_\varepsilon = \begin{cases} 1 & \text{for } |x| \in (\varepsilon, \varepsilon^{-1}) \\ 0 & \text{for } |x| < \frac{\varepsilon}{2} \text{ or } |x| > 2\varepsilon^{-1} \end{cases}$$

and  $0 \leq \gamma_\varepsilon \leq 1$ ,  $|\nabla \gamma_\varepsilon(x)| \leq 3|x|^{-1}$  in all of  $\mathbb{R}^n \times \mathbb{R}^+$ . Then  $\xi \cdot \gamma_\varepsilon$  is admissible in (14) and the assertion follows by letting  $\varepsilon \rightarrow 0^+$  and using (15).

Note that (15) is satisfied, if

$$\int_M |x|^{\alpha-2} |A|^2 \xi^2 d\mathcal{H}_n < \infty. \tag{16}$$

From the coarea formula we infer that

$$\int_M \varphi(x) d\mathcal{H}_n(x) = \int_0^\infty r^{n-1} \int_\Sigma \varphi(r\omega) d\mathcal{H}_{n-1} dr \tag{17}$$

for all non-negative  $\varphi \in C^0(M)$ , where  $\Sigma = M \cap S^n$ , and  $S^n \subset \mathbb{R}^{n+1}$  denotes the unit  $n$ -sphere. Also, since  $M$  is a cone, we find

$$|A(x)|^2 = |x|^{-2} |A(x/|x|)|^2 \text{ for all } x \in M.$$

Hence, we readily infer from (17) and (16) that

$$\xi = |x|^{1+\varepsilon-\alpha} \cdot |x|_1^{1+\alpha-(n/2)-2\varepsilon},$$

where

$$|x|_1 = \max(1, |x|),$$

is admissible in (14), if  $\varepsilon > \frac{\alpha}{2}$  (where we have of course assumed that  $n \geq 2$ ).

Furthermore we find

$$|\nabla \xi|^2 \leq \begin{cases} (1+\varepsilon-\alpha)^2 |x|^{2\varepsilon-2\alpha} & \text{in } M \cap B_1(0), \quad B_1(0) = \{|x| < 1\}, \\ \left(2 - \frac{n}{2} - \varepsilon\right)^2 |x|^{2-n-2\varepsilon} & \text{in } (\mathbb{R}^{n+1} - B_1(0)) \cap M \end{cases}$$

and (14) implies

$$\begin{aligned}
 & 2 \int_{\mathbf{M} \cap \mathbf{B}_1} x_{n+1}^\alpha |A|^2 |x|^{2\varepsilon-2\alpha} d\mathcal{H}_n \\
 & \quad + 2 \int_{\mathbf{M} \cap (\mathbb{R}^{n+1} - \mathbf{B}_1)} x_{n+1}^\alpha |A|^2 |x|^{2-n-2\varepsilon} d\mathcal{H}_n \\
 & \leq (1+\varepsilon-\alpha)^2 \int_{\mathbf{M} \cap \mathbf{B}_1} x_{n+1}^\alpha |A|^2 |x|^{2\varepsilon-2\alpha} d\mathcal{H}_n \\
 & \quad + \left(2 - \frac{n}{2} - \varepsilon\right)^2 \int_{\mathbf{M} \cap (\mathbb{R}^{n+1} - \mathbf{B}_1)} x_{n+1}^\alpha |A|^2 |x|^{2-n-2\varepsilon} d\mathcal{H}_n.
 \end{aligned}$$

We would like to choose  $n, \varepsilon, \alpha$  so that

$$\varepsilon > \frac{\alpha}{2}, \quad (1+\varepsilon-\alpha)^2 < 2 \quad \text{and} \quad \left(\frac{n}{2} + \varepsilon - 2\right)^2 < 2. \quad (18)$$

(18) is equivalent to

$$-1 - \sqrt{2} + \alpha < \varepsilon < \sqrt{2} + \alpha - 1 \quad \text{and} \quad \frac{\alpha}{2} < \varepsilon < 2 + \sqrt{2} - \frac{n}{2}. \quad (19)$$

If  $\alpha + n < 4 + 2\sqrt{2}$  then a suitable choice of  $\varepsilon$  is

$$\varepsilon = \frac{\alpha}{2} + \delta,$$

where

$$\delta = N^{-1} \left[ 2 + \sqrt{2} - \frac{n}{2} - \frac{\alpha}{2} \right] > 0$$

with  $N \in \mathbb{N}$  large. Thus we conclude that  $|A|^2 \equiv 0$  i.e.  $\mathbf{M}$  is a hyperplane  $\mathcal{P}$ . Because of  $0 = \mathbf{H} = \alpha \frac{v_{n+1}}{x_{n+1}}$  we must have  $v_{n+1} = 0$  as required.

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