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A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps

by

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ABSTRACT. — We characterize the maps in $H^1(B^3, S^2)$ which can be approximated by smooth ones.

Key-words: Sobolev Spaces, density of smooth maps.

RÉSUMÉ. — Nous caractérisons les applications de $H^1(B^3, S^2)$ qui peuvent être approchées par des applications régulières.

I. INTRODUCTION

We consider maps from the unit open ball B^3 (in $\mathbb{R}^3$) to the unit sphere S^2 in $\mathbb{R}^3$, and the Sobolev space $H^1(B^3, S^2)$ defined by:

$$H^1(B^3, S^2) = \{ u \in H^1(B^3, \mathbb{R}^3), u(x) \in S^2 \text{ a. e.} \}.$$

In [SU], R. Schoen and K. Uhlenbeck have proved that smooth maps are not dense in the Sobolev space $H^1(B^3, S^2)$. They showed that the radial projection π from B^3 to S^2 defined by $\pi(x) = \dfrac{x}{|x|}$ (which is in $H^1(B^3, S^2)$) cannot be approximated by regular maps.

Given a map u in $H^1(B^3, S^2)$, we recall the definition of the vector field $D(u)$ introduced in [BCL]:

(1) $D(u) = (u.u_y \wedge u_z, u.u_z \wedge u_x, u.u_x \wedge u_y)$ $(D(u)$ is in $L^1(B^3, \mathbb{R}^3))$.

If u is regular except at most at a finite number of point singularities $a_1, a_2, \ldots, a_N$ in B^3 $\left(\text{that is } u \text{ is in } C^\infty\left(B^3 \setminus \bigcup_{i=1}^{N} a_i,\ S^2\right)\right)$ then

$$\text{(2)} \qquad \operatorname{div} D(u) = 4\pi \sum_{i=1}^{N} \deg(u, a_i)\delta_{a_i},$$

where deg (u, a_i) denotes the Brouwer degree of u restricted to any small sphere around a_i, and which we will call the degree of u at a_i.

For example, for the radial projection $\pi(x) = \dfrac{x}{|x|}$ which has only one singularity at the origin, of degree one, we have $\operatorname{div} D(\pi) = 4\pi\delta_0$.

The central result of this paper is a proof of a conjecture of H. Brezis:

THEOREM 1. — A map u in $H^1(B^3, S^2)$ can be approximated by smooth maps if and only if $\operatorname{div} D(u) = 0$.

The fact that this condition is a necessary one, is obvious. Indeed, let u_n be a sequence of maps in $C^\infty(B^3, S^2)$ converging for the H^1 norm to some map u in $H^1(B^3, S^2)$. Then we have $\operatorname{div} D(u_n) = 0$, and it is easy to verify that $D(u_n) \to D(u)$ in L^1, and hence $\operatorname{div} D(u) = 0$. The rest of the paper is devoted to the proof of the fact that this condition is sufficient.

We recall some usual notations:

For $\eta > 0$ and $x_0 \in \mathbb{R}^3$ we set $B^3(x_0; \eta) = \{x \in \mathbb{R}^3 / |x - x_0| < \eta\}$ and $B^3(\eta) = B^3(0; \eta)$.

For $r > 0$, we set $S_r^2 = \{x \in \mathbb{R}^3 / |x| = r\}$.

For u in $H^1(B^3, S^2)$, $E(u)$ represents the Dirichlet Energy integral $\int_{B^3} |\nabla u|^2 dx$ and $\|u\|$ denotes the usual H^1 norm, i. e.

$$\|u\|^2 = \int_{B^3} |\nabla u|^2 dx + \int_{B^3} |u|^2 dx.$$

If W is some domain in B^3, $E(u; W)$ represents the integral $\int_W |\nabla u|^2 dx$.

If Σ is a surface, $E(u; \Sigma)$ is the surface integral $E(u; \Sigma) = \int_\Sigma |\nabla u|^2 d\sigma$.

For $x = (x_1, x_2, x_3)$ in $\mathbb{R}^3$, we set $|||x||| = \underset{i \in \{1,2,3\}}{\operatorname{Max}} \{|x_i|\}$.

Approximation theorems are also a key tool in our proofs. For that purpose we introduce the subsets R_1 and R_2 of $H^1(B^3, S^2)$ defined in the following way:

Let R_1 be the subset of maps of $H^1(B^3, S^2)$ which are smooth except at most at a finite number of point singularities, that is

$$(3)\quad R_1 = \left\{ u \in H^1(B^3, S^2) / \exists (a_1, \ldots, a_N),\ a_i \in B^3,\quad u \in C^\infty\left(\bar{B}^3 \backslash \bigcup_{i=1}^{N} \{a_i\}, S^2\right) \right\}.$$

It is known (see Bethuel-Zheng [BZ]) that R_1 is dense in $H^1(B^3, S^2)$. We shall use a more precise result:

Let R_2 be the subset of maps u in R_1 $\left(\text{thus } u \text{ is in } C^\infty\left(\bar{B}^3 \backslash \bigcup_{j=1}^{N} \{a_j\}, S^2\right)\right.$ for some points $a_1, \ldots, a_N$) such that there is some rotation R_i such that $u(x) = \pm R_i \dfrac{x - a_i}{|x - a_i|}$ in some neighborhood of a_i (the sign + corresponds to singularities of degree $+1$ and the sign $-$ to the singularities of degree -1). In the Appendix (Lemma A1) we prove that R_2 is dense in $H^1(B^3, S^2)$.

Let u be a map of R_1 such that the degree of u restricted to ∂B^3 is zero and moreover the degree of u at each singularity is $+1$ or -1. We recall the definition of the length of a minimal connection of u, which is introduced in [BCL] (part II, p. 654). Let A be the set of the point singularities of u; we may divide A in two subsets A^+ and A^-, A^+ (resp. A^-) being the set of singularities of degree $+1$ (resp. -1). If we set $p = \# A^+ = \# A^- = \dfrac{\# A}{2}$ we may write:

$$A^+ = \{P_i / 1 \le i \le p\}, \qquad A^- = \{N_i / 1 \le i \le p\}.$$

The definition of the length of a minimal connection is given by:

$$(4)\quad L(u) = \min \left\{ \sum_{i=1}^{p} |P_i - N_{\sigma(i)}|,\ \sigma \text{ is a permutation of } \{1, \ldots, p\} \right\}.$$

This means that we take all possible pairings of points of A^+ with points of A^-, we sum the distances between the paired points, and finally the length of a minimal connection is equal to the minimum of all these sums. If we don't assume that the degree of u at the singularities is $+1$ or -1, the points are counted according to their degree.

We recall a result of [BCL], namely that the length of a minimal connection is also given by:

$$(5)\qquad L(u) = \operatorname{Max} \left\{ \int \xi \, d\mu;\ \xi \in \operatorname{Lip}(B^3) \quad \text{and} \quad |\nabla \xi|_\infty \le 1 \right\}$$

where
$$d\mu = \sum_{i=1}^{p} \delta_{P_i} - \sum_{i=1}^{p} \delta_{N_i} = \frac{1}{4\pi} \operatorname{div} D(u).$$

Thus

$$(6)\quad L(u) = \frac{1}{4\pi} \operatorname{Max} \left\{ \sum_{i=1}^{p} (\xi(P_i) - \xi(N_i));\ \xi \in \operatorname{Lip}(B^3);\ |\nabla\xi|_\infty \leq 1 \right\}.$$

In § II, we prove Theorem 1 assuming u is smooth in some neighborhood of the boundary. The proof is divided in the following steps:

First (§ II.1), we show that deg $(u_{|\partial B^3})$ is equal to zero, and we show that for every sequence $u_n \in R_2$, approximating u in H^1 and such that $u_n = u$ on the boundary, we have $L_n = L(u_n)$ (the length of a minimal connection of u_n) goes to zero when $n \to +\infty$.

In § II.2 we present a basic construction for « removing » a pair of singularities P and N, of degree +1 and −1 respectively. This is the main tool of the paper.

In § II.3 we use the previous construction to prove Theorem 2 below, which is a result concerning the approximation of maps of R_2 by smooth maps.

THEOREM 2. — Let v be in R_1 such that the degree of v restricted to ∂B^3 is zero. Then:

$$(7)\qquad \operatorname{Inf}\{E(v-\varphi)\,|\,\varphi \in C^\infty(B^3, S^2),\ \ \varphi = v \text{ on } \partial B^3\} \leq 8\pi L(v).$$

More precisely, there is a sequence of maps v_m in $C^\infty(B^3, S^2)$ such that $v_m = v$ on ∂B^3,

$$\operatorname{meas}\{x \in B^3/v_m(x) \neq v(x)\} \to 0, \qquad \lim_{n\to+\infty} E(v_m) \leq E(v) + 8\pi L(v)$$

and v_m converges weakly to v in H^1.

In § II.4 combining Lemma 1 and Theorem 2 we complete the proof of Theorem 1 when u is smooth near the boundary.

In § III, we first prove Theorem 1 in the general case.

As a by product of our methods, we also obtain the following Theorem.

THEOREM 3. — $C^\infty(B^3, S^2)$ is dense (and in fact sequentially dense) for the weak topology in $H^1(B^3, S^2)$.

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II. PROOF OF THEOREM 1, WHEN u IS SMOOTH NEAR THE BOUNDARY

We assume in this section that u is smooth in some open neighborhood of the boundary ∂B^3, u is in $H^1(B^3, S^2)$, and $\operatorname{div} D(u) = 0$.

II.1. A lemma concerning the convergence of the length of a minimal connection.

Before stating the lemma, we first remark that:

$$\deg(u_{|\partial B^3}) = 0. \tag{8}$$

To prove this claim, we may for example apply [BCL] (Theorem B1, p. 691). Since u is smooth on $B^3/B^3(r)$ for some $r < 1$ large enough, we have for every Lipschitz map $\xi \in \operatorname{Lip}(B^3)$ with compact support in B^3 and $\xi \equiv 1$ on B_r^3:

$$-\int_{B^3\setminus B_r^3} D\cdot\nabla\xi = 4\pi\deg(u_{|\partial B^3}). \tag{9}$$

Since $\nabla\xi = 0$ on B_r^3 this implies:

$$-\int_{B^3} D\cdot\nabla\xi = 4\pi\deg(u_{|\partial B^3}).$$

Since $\xi = 0$ on ∂B^3 it follows that:

$$4\pi\deg(u_{|\partial B^3}) = -\int D\cdot\nabla\xi = \int_{B^3}(\operatorname{div} D)\cdot\xi = 0.$$

This completes the proof of the claim.

LEMMA 1. — Let u be as above, and let u_n be a sequence in R_2 converging to u in $H^1(B^3, S^2)$ and such that u_n restricted to ∂B^3 is equal to u restricted to ∂B^3 (the existence of such a sequence u_n for every u as above, is proved in Lemma A.1 of the Appendix). Then $L_n = L(u_n)$ (which can be defined since $\deg(u_{n|\partial B^3}) = 0$) goes to zero when n goes to $+\infty$.

Proof of lemma 1. — We use the expression of the length of a minimal connection given by equality (5): for every $n \in \mathbb{N}$ there is some Lipschitz map $\xi_n \in \operatorname{Lip}(B^3)$ such that $|\nabla\xi_n|_\infty \leq 1$ and:

$L_n = \frac{1}{4\pi}\int_{B^3}(\operatorname{div} D_n)\xi_n d\sigma$ where we have set $D_n = D(u_n)$ and $D = D(u)$.

Integrating by parts, we obtain:

$$(10) \qquad 4\pi L_n = \int_{\partial B^3} (D_n \cdot n)\xi_n d\sigma - \int_{B^3} D_n \cdot \nabla \xi_n dx.$$

Since $D_n \cdot n$ depend only on the value of u_n restricted to the boundary (more precisely, $D_n \cdot n = u_n \cdot u_{n_x} \wedge u_{n_y} = \varphi \cdot \varphi_x \wedge \varphi_y$ on the boundary, where x, y are orthonormal coordinates on ∂B^3), we have:

$$(11) \qquad D_n \cdot n = D \cdot n \quad \text{on} \quad \partial B^3.$$

On the other hand, since div D = 0 in B^3, we may write, using an integration by parts:

$$(12) \qquad 0 = \int (\operatorname{div} D) \cdot \xi_n d\sigma = \int_{\partial B^3} (D \cdot n)\xi_n d\sigma - \int D \cdot \nabla \xi_n dx.$$

Thus combining (10), (11) and (12) we obtain

$$4\pi L_n = - \int_{B^3} (D_n - D) \nabla \xi_n dx + \int_{\partial B^3} (D_n \cdot n - D \cdot n)\xi d\sigma = \int_{B^3} (D_n - D) \nabla \xi d\sigma.$$

Since $| \nabla \xi_n |_\infty \leq 1$, we find:

$$4\pi | L_n | \leq \int_{B^3} | D_n - D | dx.$$

Since D_n converges strongly to D in L^1, we see that $| L_n | \to 0$ when n goes to $+\infty$. This completes the proof of Lemma 1. Before completing the proof of Theorem 1 when u is smooth near the boundary, we shall prove Theorem 2, and for this purpose we present next a basic construction for « removing » a pair of singularities P and N of degree $+1$ and -1 respectively. This construction will be used and adapted in a forthcoming paper [Be1].

II.2. The basic construction for « removing » a pair of singularities.

Let W be some open domain in $\mathbb{R}^3$. We consider a map v in $H^1(W, S^2)$ such that v has only two point singularities P and N of degree $+1$ and -1 respectively, that is v is in $C^\infty(W \setminus \{ P, N \}, S^2)$. We assume furthermore that the segment [PN] is included in W. We are going to show how to « remove » the two singularities; more precisely we are going to modify v only in a small neighborhood of [PN], in such a way that the new map is smooth in this neighborhood, and the new energy is not increased too much, namely by $8\pi | P - N |$. This is the content of the following lemma:

LEMMA 2. — Let v be as above. There is a sequence of smooth maps $v_m \in C^\infty(W, S^2)$, which coincide with v outside some small neighborhood K_m of [PN] such that:

$$\text{(13)} \qquad \begin{aligned} &\text{meas} \quad K_m \to 0 \quad \text{when} \quad m \to +\infty; \\ &\lim_{m \to +\infty} E(v_m; K_m) = 8\pi \,|\, P - N \,|. \end{aligned}$$

Proof of Lemma 2. — Without loss of generality we may assume in addition that there are some rotations R_+ and R_-, and some $r_0 > 0$ small such that:

$$\text{(14)} \qquad \begin{aligned} v(x) &= R_+\left(\frac{x - P}{|x - P|}\right) \quad \text{on} \quad B^3(P; r_0); \\ v(x) &= - R_-\left(\frac{x - N}{|x - N|}\right) \quad \text{on} \quad B^3(N; r_0). \end{aligned}$$

Indeed, applying the proof of Lemma A1, we may approximate v by maps satisfying (14), which differ from u only on a small neighborhood of the singularities (and the estimates below do not depend on the approximation). We set $d = |P - N|$. We choose normal coordinates such that $P = (0, 0, 0)$ and $N = (0, 0, d)$. We let r be such that $0 < r < \frac{r_0}{2}$. Since v is smooth on $W \backslash (B^3(P, r) \cup B^3(N; r))$ there is some constant $d(r)$ such that $|\nabla v|_\infty \leq d(r)$ on $W \backslash (B^3(P; r) \cup B^3(N; r))$.

For $m \in \mathbb{N}^*$ large enough, we set $a_m = \dfrac{d}{2(m-1)}$ and we consider the set K_m defined by $K_m = [-a_m, a_m]^2 \times [-a_m, d + a_m]$. For m large enough K_m is in W. We are going to construct a map $v'_m \in H^1(W, S^2)$ such that $v'_m = v$ on $W \backslash K_m$, and such that v_m is continuous on K_m except at a finite number of point singularities of degree zero. On the other hand any such map can be strongly approximated by smooth maps on K_m, having the same boundary value (see the proof of Theorem 5 in [BZ] or [Be1]).

We divide K_m in m 3-dimensional cubes $C_{m,j}$ (which in fact are translates of $[-a_m, a_m]^3$) defined by:

$$C_{m,j} = [-a_m, a_m]^2 \times [(-1+j)a_m; (1+j)a_m] \quad \text{for} \quad j = 0 \quad \text{to} \quad m-1.$$

For the cubes $C_{m,j}$ which do not intersect $B(P; r) \cup B(N; r)$ we have $|\nabla v|_\infty \leq d(r)$ on $C_{m,j}$ and thus

$$\text{(15)} \qquad E(v; \partial C_{m,j}) \leq 24 d(r)^2 a_m^2.$$

For the cubes $C_{m,j}$ which intersect $B^3(P; r) \cup B^3(N; r)$, we have the relations (14) holding for v on these cubes, and thus it is easy to verify that $\operatorname{Sup}\{|\nabla v(x)|_\infty,\ u \in \partial C_{m,j}\} \leq \frac{1}{a_m}$, which leads to the inequality

$$E(v; \partial C_{m,j}) \leq 24\,. \tag{16}$$

Since we have at most $T_m(r) = \frac{r}{2a_m} + 2$ cubes $C_{m,j}$ which intersect $B(P; r) \cup B(N; r)$, combining (16) and (15), we obtain:

$$\sum_{j=0}^{m-1} E(v; \partial C_{m,j}) \leq 24T_m(r) + m[24d(r)^2 a_m^2] \leq C_1 \frac{r}{a_m} + C_2 d(r^2) a_m^2 m\,. \tag{17}$$

In order to complete the proof of Lemma 2, we shall use the following standard technique (see e. g. [BC], Theorem 2, part C), which is stated in the following lemma:

LEMMA 3. — Let $\mu > 0$, $\epsilon > 0$ and $d \in \mathbb{Z}$ be given, and $C_\mu = [-\mu, \mu]^3$. Let φ be a smooth map from ∂C_μ to S^2 having degree d_0. Then there is some $0 < \alpha_0 < \mu$, depending only on $|\nabla v|_\infty$ and ϵ such that for every $0 < \alpha < \alpha_0$, there is some smooth map $\bar{\varphi}$ from ∂C_μ to S^2 having the following properties:

$$\begin{cases} \bar{\varphi} \ \text{has degree } d \text{ on } \ \partial C_\mu\,; \\ \bar{\varphi} = \varphi \quad \text{on} \quad \partial C_\mu \backslash B^2(0, \alpha) \times \{\mu\}\,; \\ E(\bar{\varphi}; B^2(0, \alpha) \times \{\mu\}) \leq 8\pi\,|d - d_0| + \epsilon\,. \end{cases} \tag{18}$$

Proof of Lemma 2 completed. — As a first step, we are going to define a smooth map v'_m on $\bigcup_{j=0}^{m-1} \partial C_{m,j}$, such that $v'_m = v$ on ∂K_m and such that the degree of v'_m restricted to each $\partial C_{m,j}$ is zero (afterwards, we will extend v'_m inside each cube $C_{m,j}$).

DEFINITION of v_m on $\bigcup_{j=0}^{m-1} \partial C_{m,j}$. — Let $\epsilon > 0$ be small. We first apply lemma 3 to $C_{m,0}$, $\varphi = v$ restricted to $\partial C_{m,0}$ (which has degree $+1$), $d = 0$, and $\alpha = \operatorname{Min}(\epsilon a_m, \alpha_0)$. Lemma 3 gives us a map $\bar{\varphi}$ from $\partial C_{m,0}$ to S^2, satisfying (18). On $\partial C_{m,0}$ we define v'_m by

$$v'_m = \bar{\varphi} \quad \text{on} \quad \partial C_{m,0}\,.$$

Thus v'_m has the following properties on $\partial C_{m,0}$:

$$(19)\qquad \begin{cases} -v'_m \text{ has degree zero on } \partial C_{m,0} \\ -v'_m = v \quad \text{on} \quad \partial C_{m,0}\backslash B^2(0,\alpha)\times\{+a_m\} \\ -E(v'_m; B^2(0,\alpha)\times\{a_m\}) \le 8\pi+\epsilon \end{cases}$$

Hence, v'_m is equal to v on $\partial C_{m,0} \cap \partial K_m$. We now consider the next cube $C_{m,1} = [-a_m, a_m]^2 \times [+a_m, 3a_m]$ and the smooth map $\tilde{v}_m$ from $\partial C_{m,1}$ to S^2 defined by

$$(20)\qquad \begin{cases} \tilde{v}_m = v'_m \text{ on } \partial C_{m,0} \cap \partial C_{m,1} \text{ that is on the face } [-a_m, a_m]\times\{a_m\} \\ \tilde{v}_m = v \quad \text{elsewhere i. e. on} \quad \partial C_{m,1}\backslash[-a_m, a_m]^2\times\{a_m\}. \end{cases}$$

It is easy to see that the degree of $\tilde{v}_m$, on $\partial C_{m,1}$ is $+1$.

We apply once more Lemma 3 to $\tilde{v}_m$, $\partial C_{m,1}$ and $d = 0$. Lemma 3 provides us a new map from $\partial C_{m,1}$ to S^2 satisfying (18). We take v'_m equal to this new map. Note that this definition of v'_m on $\partial C_{m,1}$ is compatible with the previous definition of v'_m on $\partial C_{m,0}$. Moreover v'_m has degree zero on $\partial C_{m,1}$ and $v'_m = v$ on $\partial C_{m,1} \cap \partial K_m$. Repeating this argument m times, we define a smooth map v'_m on $\bigcup_{j=0}^{m-1} \partial C_{m,j}$ such that $v'_m = v$ on ∂K_m and such that the degree of v'_m restricted to each cube $C_{m,j}$ is zero.

<u>DEFINITION of v'_m on $K_m = \bigcup_{j=0}^{m-1} C_{m,j}$.</u> — For each cube $C_{m,j}$ we extend v'_m defined on $\partial C_{m,j}$ to $C_{m,j}$ in the following way:

$$(21)\qquad v'_m(x) = v'_m\left(\frac{x-x_j}{|||x-x_j|||} + x_j\right)$$

on $C_{m,j}$ where x_j is the barycenter of $C_{m,j}$.

It is easy to see that $v'_m = v$ on ∂K_m, that v'_m is in $H^1(K_m, S^2)$ continuous except at the points x_j, where the degree of v'_m is zero, and for every small open neighborhood of the points x_j, Lipschitz outside this neighborhood. If we estimate the energy of v'_m on $\partial C_{m,j}$ easy calculations, combining (18), (20), and (21) show that, for $j = 0$ to $m - 1$:

$$E(v'_m; \partial C_{m,j}) \le E(v; \partial C_{m,j}) + 2(8\pi + \epsilon)$$

and

$$(22)\qquad E(v'_m; C_{m,j}) \le C_3 a_m E(v; \partial C_{m,j}) + 2a_m(8\pi+\epsilon)K(\epsilon)$$

where $K(\epsilon)$ is a constant depending only on ϵ which goes to 1 when ϵ goes to zero. Adding all these inequalities for $j = 0$ to $m - 1$ we obtain

$$(23)\qquad E(v'_m; K_m) \le C_3 a_m \sum_{j=0}^{m-1} E(v; \partial C_{m,j}) + 2ma_m^3(8\pi+\epsilon)K(\epsilon).$$

Using relation (17) we find

$$\text{(24)} \qquad E(v'_m; K_m) \leq C_3 r + C_2 d^2(r) m a_m^3 + \frac{2m}{2(m-1)} d(8\pi + \epsilon) K(\epsilon).$$

If we let m go to $+\infty$, ϵ to zero we see that

$$\lim_{m \to +\infty} E(v'_m; K_m) \leq C_4 r + 8\pi d.$$

Now we only have to let r go to zero, to see that if we take some convenient subsequent we have:

$$\text{(25)} \qquad \lim_{m \to +\infty} E(v'_m; K_m) \leq 8\pi d.$$

Since $v'_m = v$ on ∂K_m, we may extend v'_m to W by:

$$v'_m = v \quad \text{on} \quad W \backslash K_m.$$

Since v'_m has only point singularities of degree zero, using the proof of [BZ], Theorem 5 or [Be1], Lemma 1, we see that v'_m can be strongly approximated by smooth maps, equal to v'_m and thus to v outside K_m. This completes the proof of Lemma 2.

REMARK. — If in the assumptions of Lemma 2 we do not assume that the segment [PN] is contained in W, the conclusion still holds, except that we have to use the geodesic distance within W, between P and N instead of $|P - N|$.

II.3. **Proof of Theorem 2.**

Let v be in R_1 such that $\deg(v_{|\partial B^3}) = 0$. The minimal connection gives us a pairing of the point singularities of v, (P_1, N_1), (P_2, N_2), ... (P_p, N_p) where for $i = 1$ to p the degree of v at P_i is $+1$, the degree of v at N_i is -1. By the definition (4) of $L(v)$ we have:

$$\text{(32)} \qquad L(v) = \sum_{i=1}^{p} |P_i - N_i|.$$

Given a pair of point singularities (P_i, N_i) we may assume without loss of generality tha v has no other singularities on the segment $[P_i, N_i]$. Indeed since we consider only approximating sequences in R_1, we may always slightly change our approximating maps in such a way that this holds. Hence, for $i = 1$ to p let W_i be an open neighborhood of the segment $[P_i N_i]$ in B^3 such that v has no other singularities in W_i than P_i and N_i. Since v is in R_1, v satisfies the hypothesis of Lemma 2 on W_i. Thus we may apply lemma 2 to v on W_i, for $i = 1$ to p. This lemma gives us the existence of a sequence of smooth meas v_m and of some small neighborhoods $K_{m,i}$ of $[P_iN_i]$ in W_i such that

$$\text{meas} \bigcup_{i=1}^{p} K_{m,i} \to 0 \quad \text{when} \quad m \to +\infty. \tag{33}$$

$$\lim_{m \to +\infty} E(v_m, K_{m,i}) = 8\pi \,|\, P_i - N_i \,| \tag{35}$$

$$v_m = v \quad \text{on} \quad B^3 \setminus \bigcup_{i=1}^{p} K_{m,i}. \tag{35}$$

Adding (34) for $i = 1$ to p we obtain

$$\lim_{m \to +\infty} E(v_m; K_{m,i}) = 8\pi \sum_{i=1}^{p} |\, P_i - N_i \,| = 8\pi L(v) \tag{36}$$

(33) and (35) imply that v_m tends to v almost everywhere. Since $E(v_m)$ is bounded (by (36)), v_m tends weakly to v. Thus

$$\lim_{m \to +\infty} E(v_m - v) = \lim_{m \to +\infty} E(v_m) - E(v) = 8\pi L(v). \tag{37}$$

This implies inequality (7) and completes the proof of Theorem 2.

II.4. Proof of Theorem 1 completed in the case u is smooth near the boundary.

Let u be smooth near the boundary, such that div $D(u) = 0$. Let u_m be a sequence of maps in R_2 converging strongly to u in $H^1(B^3, S^2)$ and such that $u_m = u$ on the boundary. By Lemma A.1 of the Appendix we know that such a sequence exists. By Lemma 1 we know that $L(u_m)$ goes to zero when $m \to +\infty$. Theorem 2 shows that there is a sequence of smooth maps v_m from B^3 to S^2, which are equal to u on ∂B^3, such that

$$E(v_m - u_m) \leq 8\pi L(u_m) + \frac{1}{m}.$$

Thus

$$E(u - v_m) \leq \frac{1}{m} + 8\pi L(u_m) + E(u - u_m). \tag{38}$$

Since $L(u_m)$ and $E(u - u_m)$ go to zero when m goes to $+\infty$ (38) shows that $E(u - v_m)$ and thus $\| u - v_m \|$ go to zero when m goes to $+\infty$. This proves that u is in the strong closure of $C^\infty(B^3, S^2)$ in $H^1(B^3, S^2)$ and completes the proof of Theorem 1 when u is smooth near the boundary.

III. PROOF OF THEOREM 1 IN THE GENERAL CASE.

In this section we assume only that $u \in H^1(B^3, S^2)$ and $\operatorname{div} D(u) = 0$. Let u_n be a sequence of maps in R_2 approximating u strongly in $H^1(B^3, S^2)$. Since we have no regularity assumption on u near the boundary, it may happen that $\deg(u_n, \partial B^3)$ is different from zero. Thus we cannot define the length of a minimal connection of u_n as it was done in section II, using formulas (4), (5) and (6). To overcome this difficulty, we need a slightly different definition of the length of a minimal connection which is introduced in [BCL], part II, Example 3, p. 655: let v be in R_2 such that $\deg(v_{|\partial B^3}) = d$ (possibly different from zero). We may pair each singularity in B^3 either to another singularity with the opposite degree in B^3 or to a fictitious point singularity on the boundary with the opposite degree: that means that we allow connections to the boundary. Pairing all singularities in this way, we take the configuration that minimizes the sum of the distances between the paired singularities (real and fictitious) obtained by this method, and we denote by $\tilde{L}(v)$ this minimum, the length of the minimal connection when we allow connections with the boundary. Even if $d = 0$, $\tilde{L}(v)$ may be different from $L(v)$, in fact $\tilde{L}(v) \leq L(v)$. As for the functional L, $\tilde{L}(v)$ can be defined using the $D(v)$ vector field in the following way:

$$(39) \quad \tilde{L}(v) = \operatorname{Max}\left\{ \int_{B^3} \xi d\mu ; \xi \in \operatorname{Lip}(B^3), \xi = 0 \text{ on } \partial B^3, \text{ and } |\nabla \xi|_\infty \leq 1 \right\}$$

where $d\mu = \dfrac{1}{4\pi} \operatorname{div} D(v)$.

Using $\tilde{L}$ instead of L, and the relation (39), the method of section II, where we made the assumption that u is smooth near the boundary, can be carried over to the general case. The proof of Theorem 1 in the general case is thus divided in the following steps.

First (§ III.1) we prove Lemma 1 *bis,* which is similar to Lemma 1.

LEMMA 1 *bis.* — Let u be in $H^1(B^3, S^2)$, such that $\operatorname{div} D(u) = 0$, and let u_n be a sequence in R_2 tending strongly to u in $H^1(B^3, S^2)$. Then $\tilde{L}(u_n)$ goes to zero when n goes to $+\infty$.

Then, we adapt our basic construction for « removing singularities » to the case where a singularity is connected to the boundary. More precisely we prove in § III.2 the following lemma:

LEMMA 2 *bis.* — Let W be some open domain in R^3 such that ∂W is smooth. Let $v \in H^1(\bar{W}, S^2)$ be smooth except at a point singularity P of degree $+1$ or -1. Let N be a point on ∂W such that $|P - N| = d(P, \partial W)$.

Then there is some sequence of maps v_m in $C^\infty(W, S^2)$, which coincide with v outside some small neighborhood K_m of [PN] in W such that:

$$\text{meas } K_m \to 0 \quad \text{when} \quad m \to +\infty$$
$$\lim E(v_m; K_m) = 8\pi \,|\, P - N \,| = 8\pi d(P, \partial W).$$

In § III.3 using Lemma *bis*, we give the following easy modification of Theorem 2:

THEOREM 2 *bis*. — Let v be in R_2. Then we have:

$$\text{Inf}\,\{E(v - \varphi)/\varphi \in C^\infty(B^3, S^2)\} \leq 8\pi \tilde{L}(v). \tag{40}$$

Moreover there is a sequence of maps $v_m \in C^\infty(B^3, S^2)$ tending weakly to v in $H^1(B^3, S^2)$ such that mes $\{x \in B^3\ v_m(x) \neq v(x)\} \to 0$ when $m \to +\infty$, and $\lim_{n \to +\infty} E(v_m) = E(v) + 8\pi L(v)$.

In § III.4 combining lemma 1 *bis* with Theorem 2 *bis* we complete the proof of Theorem 1 in the general case.

In § III.5 we give a proof to Theorem 3, concerning weak density of smooth maps in $H^1(B^3, S^2)$.

III.1. **Proof of Lemma 1 *bis*.**

Applying relation (39) to u_n we see that there is some map ξ_n in Lip (B^3), such that $\xi_n = 0$ on ∂B^3, $|\,\nabla \xi_n\,|_\infty \leq 1$ and

$$4\pi \tilde{L}(u_n) = \int_{B^3} \text{div } D_n \xi_n dx \quad \text{where} \quad D_n = D(u_n). \tag{41}$$

Integrating (41) by parts, and using the fact that $\xi_n = 0$ on ∂B^3 we find

$$4\pi \tilde{L}(u_n) = \int_{B^3} \text{div } D_n \xi_n dx = \int_{B^3} (D_n - D) \nabla \xi_n + \int_{B^3} D \cdot \nabla \xi_n. \tag{42}$$

Using the convergence of D_n to D in $L^1(B^3)$ and the fact that div D $= 0$ we find that $\tilde{L}(v_n)$ goes to zero when n tends to $+\infty$. This completes the proof of Lemma 1 *bis*.

III.2. **Proof of Lemma 2 *bis*.**

We assume for instance that the degree of the singularity P is $+1$. Moreover for simplicity, we may assume that ∂W is flat in some neighborhood of N and that ∂W is orthogonal there to [PN] (the general case is technically more involved but the method remains essentially the same).

We may choose orthonormal coordinates such that $N = (0, 0, 0)$ and $P=(0, 0, d)$ where $d=|P-N|$ and $\partial W \cap B^3(N, r_1)=B^2(0; r_1) \times \{0\}$, for some r_1 small enough. For $m \in \mathbb{N}$, large enough we consider the cube $C_m = \left[-\frac{1}{m}, \frac{1}{m}\right]^2 \times \left[0, \frac{2}{m}\right]$ and the restriction of v to ∂C_m, which has degree zero. Applying Lemma 3 to ∂C_m, v and $d = -1$, we see that there is some smooth map $\tilde{v}_m$ from ∂C_m to S^2 such that:

$$\begin{cases} -\tilde{v}_m \text{ has degree } -1 \text{ on } \partial C_m \\ -\tilde{v}_m = v \text{ on } \partial C_m \Big\backslash \left[-\frac{1}{m}, \frac{1}{m}\right] \times \{0\} \\ \quad \left(\text{note that } \left[\frac{1}{m}, \frac{1}{m}\right] \times \{0\} \subset \partial W\right); \\ \quad E\left(\tilde{v}_m; \left[-\frac{1}{m}, \frac{1}{m}\right] \times \{0\}\right) \le \\ \quad \le E\left(\tilde{v}_m; \left[-\frac{1}{m}, \frac{1}{m}\right]^2 \times \{0\}\right) + 8\pi + \frac{1}{m}. \end{cases}$$

We extend $\tilde{v}_m$ to C_m by :

$$\tilde{v}_m(x) = \tilde{v}_m\left(\frac{x - N_m}{|||x - N_m|||} + N_m\right) \text{ on } C_m \text{ where } N_m \text{ is the barycenter of } C_m.$$

On $\partial C_m \cap W$ we have thus:

$$\tilde{v}_m = v.$$

We extend $\tilde{v}_m$ to W by $v_m = v$ on $W \backslash C_m$. It is easy to see that $\tilde{v}_m$ is in $H^1(B^3, S^2)$, and the same calculations as in Lemma 2 show that

$$\dot{E}(\tilde{v}_m; C_m) \le \frac{8\pi}{m} + \frac{K}{m^2}$$

where K depends only on $|\nabla v|_\infty$ on some neighborhood of N in W. Since $\tilde{v}_m$ is continuous outside two point singularities P and N_m of degree $+1$ and -1 respectively, we may now apply Lemma 2 to $\tilde{v}_m$ and W and this completes the proof of Lemma 2 *bis*.

III.3. Proof of Theorem 2 *bis*.

The proof is the same as the proof of Theorem 2 : the minimal connection gives us a pairing of the point singularities, some of them being paired with other singularities with the opposite degree, others being connected to the boundary. The first ones are « removed » using Lemma 2. The other ones are removed using Lemma 2 *bis*.

III.4. **Proof of Theorem 1 in the general case.**

The proof is the same as the proof of Theorem 1, in the case u is smooth near the boundary; the only modification is to replace L by $\tilde{L}$, and to use Lemma 1 *bis* instead of Lemma 1, Theorem 2 *bis* instead of Theorem 2.

III.5. **Proof of Theorem 3.**

Let u be in $H^1(B^3, S^2)$. Let u_n be a sequence of maps in R_2 approximating strongly u. For $n \in \mathbb{N}^*$ let v_n be a map in $C^\infty(B^3, S^2)$ given by Theorem 2 *bis* applied to u_n such that

$$\text{mes}\{x \in B^3 / v_n(x) = u_n(x)\} \leq \frac{1}{n}$$

and

$$E(v_n) \leq E(u_n) + 8\pi\tilde{L}(u_n) + \frac{1}{n}. \tag{43}$$

One of the main results of [BCL] (Theorem 1.1 example 3) is that:

$$8\pi\tilde{L}(u_n) \leq E(u_n). \tag{44}$$

Combining (43) and (44) we see that:

$$E(v_n) \leq 2E(u_n) + \frac{1}{n} \leq 3E(u) \quad \text{for } n \text{ large enough}. \tag{45}$$

Thus $E(v_n)$ is bounded in $H^1(B^3, S^2)$. Passing to a subsequence if necessary, v_n converges weakly to some map u'. Since v_n converges a. e. to u, we have in fact $u' = u$. This completes the proof of Theorem 3.

REMARK 2. — Extension of the above results to the spaces $W^{1,p}(B^n, S^{n-1})$, $n \geq 3$.

1) Let $p = n - 1$. Let u be in $W^{1,p}(B^n, S^{n-1})$. Following ([BCL], Appendix B), we associate to u the vector-field $D \in L^1(B^n, \mathbb{R}^n)$, generalizing the definition of the case $n = 3$, with components D_j as follows:

$$D_j = \det\left(\frac{\partial u}{\partial x_1}, \dots \frac{\partial u}{\partial x_{j-1}}, u, \frac{\partial u}{\partial x_{j+1}}, \dots \frac{\partial u}{\partial x_n}\right).$$

Then, for this vector-field, Theorem 1 remains true: u in $W^{1,n-1}(B^n, S^{n-1})$ can be approximated by smooth maps in $C^\infty(B^n, S^{n-1})$ if and only if $\text{div}\, D = 0$. This result holds because all the technical tools involved are also valid if $n \geq 3$. Lemmas 1, 2 and 3 remain true as well as Theorem 3.

2) *The case $p < n-1$:* For $p < n-1$, we know by [BZ], Theorem 1, that smooth maps are dense in $W^{1,p}(B^n, S^{n-1})$. In this situation the energy for removing two singularities is arbitrarily small, as a simple scaling argument shows. This means that for any $\epsilon > 0$, we may construct a basic dipole of length L given, such that its energy is less than ϵ. Using the tools of Theorem 1, this leads to a new proof of the density result of [BZ], Theorem 1 in this special case.

In another direction, we may obtain specific density results (such density results are proved and used by Helein [H]). For example, we know, using approximation theorems such as those of Coron and Gulliver [CG], Theorem 3.2, or [BZ], Theorem 4 *bis,* that given a smooth boundary value ξ on ∂B^n, and a map u in $W^{1,p}(B^n, S^{n-1})$, of u in R_2 (where the definition of R_2 (where the definition of R_2 is generalized to the Sobolev spaces $W^{1,p}(B^n, S^{n-1})$ of u in $W^{1,p}$ such that u_n restricted to ∂B^n is ξ. If $\deg \xi = d (\neq 0)$, then u_n has at least $|d|$ point singularities. The previous methods allow us to eliminate all the singularities, except for $|d|$ of them, all of which have degree sign d. Moreover, we may force these singularities to be located at fixed points; for example, we may approximate u by maps u_n smooth except at zero, where they have degree d. For doing this we only have to create a basic dipole, as small as we wish, with the adequate singularity at the given point (for the approximating maps in R_2), and to eliminate all the other singularities by the method of Lemma 2.

APPENDIX

We will consider more generally maps from $B^n = \{x \in \mathbb{R}^n / |x| < 1\}$ to $S^{n-1} = \partial B^n$, and the Sobolev spaces $W^{1,p}(B^n, S^{n-1})$, for $n-1 \leq p < n$, defined by:

$$W^{1,p}(B^n, S^{n-1}) = \{u \in W^{1,p}(B^n, \mathbb{R}^n)/u(x) \in S^{n-1} \text{ a. e.}\}.$$

We set $E(u) = \int_{B^n} |\nabla u|^p dx$, for u in $W^{1,p}(B^n, S^{n-1})$. The definitions of R_1 and R_2 are extended to $W^{1,p}(B^n, S^{n-1})$. Then we have the following lemma.

LEMMA A1. — *i)* R_2 is dense in $W^{1,p}(B^n, S^{n-1})$ for $n - 1 \leq p < n$.
ii) If u is in $W^{1,p}(B^n, S^{n-1})$ $(n - 1 \leq p < n)$ such that u restricted to the boundary is smooth, then there is a sequence of maps u_n in R_2 such that u_n converges strongly to u in $W^{1,p}$ and u_n coincides with u on the boundary.

Proof of Lemma A1 *i*). — By [BZ] we know that R_1 is dense in $W^{1,p}(B^n, S^{n-1})$. Thus we only have to prove that a given map u in R_1 can be strongly approximated by maps in R_2. Let $a_1, \ldots, a_l$ be the point singularities of u. Since the problem is local (as later considerations will show) we may assume that u has only one singularity centered at zero, that is $u \in C^\infty(B^n \backslash \{0\}, S^{n-1})$ (the general case is treated in a similar way, considering each singularity separately). The proof of Lemma A1 *i*) is divided in two steps. First we prove that u can be strongly approximated by maps $\bar{u}_m$ in $\cap C^0(B^n \backslash \{0\}, S^{n-1})$ such that:

- $\bar{u}_m = u$ on $B^n \backslash B^n(0; r_m)$ where $r_m > 0$ goes to zero when $m \to +\infty$;

$$u_m(u) = u_m\left(\frac{x}{|x|} r_m\right) \quad \text{on} \quad B^n(0, r_m).$$

In a second step we prove that $\bar{u}_m$ can be strongly approximated by maps u_m in R_2 having only one point singularity of zero, which are equal to u on $B^n \backslash B^n(0, r_m)$.

First step. — By Fubini's Theorem, there is some $r_m \in \left[\frac{1}{2m}, \frac{1}{m}\right]$ such that:

$$\text{(A1)} \qquad E(u; S^{n-1}_{r_m}) \leq 2m\, E\left(u; B^n\left(0, \frac{1}{m}\right)\right).$$

We set $u_m = u$ on $B^n \backslash B^n(0, r_m)$,

$$u_m = u\left(\frac{x}{|x|} r_m\right) \quad \text{on} \quad B^n(0, r_m).$$

Using relation (A1) we see that

$$\text{(A2)} \qquad E(u_m; B^n(0, r_m)) \leq 2E\left(u; B^n\left(0; \frac{1}{m}\right)\right).$$

Since $E\left(u;\ B^n\left(0;\ \frac{1}{m}\right)\right)$ goes to zero when m tends to $+\infty$, $\| u_m - u \|$ goes to zero when $m \to +\infty$. This completes the first step on the proof.

Second step. — In order to complete the proof of Lemma A1 we need the following result, the proof of which we will give after the completion of the proof of Lemma A1:

LEMMA A2. — Let v be a map in R_1 having only one point singularity at zero, and such that there is some $0 < r_0 < 1$ such that

$$\text{(A3)} \qquad v(x) = v\left(\frac{x}{|x|} r_0\right) \quad \text{on} \quad B^n(0; r_0).$$

Then v can be strongly approximated by maps v_m in R_2 having only one point singularity at 0, and which coincide with v outside some small neighborhood of 0.

Proof of Lemma A1 *completed.* — Since u_m verifies relation A3 we may apply Lemma A2 to u_m. This completes the proof of Lemma A1 *i*).

Assertion *ii*) follows from the corresponding result in [BZ] (Theorem 3 *bis*) and the above method.

We give next the proof of Lemma A2.

Proof of Lemma A2. — Let φ be the smooth map from S^{n-1} to S^{n-1} defined by:

$$\varphi(x) = v(xr_0) \quad \text{for} \quad x \in S^{n-1}(|x| = 1).$$

We assume, for instance, that the degree of the singularity 0 is $+1$. Then the degree of φ is also clearly $+1$, and φ is homotopic to every rotation R in $SO(n)$, restricted to S^{n-1}. This implies that there is some Lipschitz map Φ from $B^n \backslash B^n\left(0;\frac{1}{2}\right)$ to S^{n-1} such that $|\nabla\Phi|_\infty$ is bounded and:

$$\Phi = \varphi \quad \text{and} \quad \partial B^n$$

$$\Phi = R\left(\frac{x}{|x|}\right) \quad \text{on} \quad \partial B^n\left(0;\frac{1}{2}\right).$$

On $B^n\left(0;\frac{1}{2}\right)$ we extend Φ by:

$$\Phi(x) = R\left(\frac{x}{|x|}\right) \quad \text{on} \quad B^n\left(0;\frac{1}{2}\right).$$

Then clearly $E(\Phi; B^n)$ is bounded, Φ is continuous on $B^n\backslash\{0\}$, 0 is a singularity of Φ of degree +1. Now we define an approximating map v_m of v by:

$$v_m = v \quad \text{on} \quad B^n\backslash B^n\left(0;\frac{1}{m}\right)$$

$$v_m = \Phi(mx) \quad \text{on} \quad B^n\left(0;\frac{1}{m}\right).$$

Then and easy computation shows that

$$E\left(v_m; B^n\left(0;\frac{1}{m}\right)\right) \leq \frac{1}{m} E(\Phi; B^n).$$

Since $E(\Phi; B^n)$ is bounded (A4) then implies that $\| v_m - v \|$ goes to zero when $m \to +\infty$. This completes the proof of Lemma A2.

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