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The Stefan problem in heterogeneous media

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ABSTRACT. — The nonlinear heat-transfer equation with a nonlinear boundary flux condition is investigated. The medium is assumed heterogeneous (*i. e.* the coefficients may depend on space variables) but piecewise homogeneous. The enthalpy formulation is employed. Existence of a weak solution is demonstrated, using an approximation by the Rothe method and a certain regularization of the contact conditions between the homogeneous subdomains. Phase transitions described as multiphase Stefan problems in some subdomains are also admitted, and a degeneration of the parabolic type of the equation is covered, too.

Key words : The Stefan problems, heterogeneous media, weak solution, existence, approximation.

RÉSUMÉ. — Dans cet article le problème de la conduction de chaleur dans un milieu hétérogène (avec des parts homogènes) est étudié. L'équation parabolique, formulée en version enthalpique, et la condition aux limites sont non linéaires. On prouve l'existence de la solution faible en utilisant la méthode de Rothe et une certaine régularisation de la condition du contact entre des subdomaines homogènes. Un problème multiphase de Stefan et une dégénération de la parabolicité sont aussi considérés.

Classification A.M.S. : 35 K 55, 80 A 20.

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1. INTRODUCTION AND FORMULATION OF THE PROBLEM

This paper deals with the nonlinear heat-transfer equation with spacial and temperature dependent coefficients:

$$c(x, \theta)\frac{\partial \theta}{\partial t} = \nabla (k(x, \theta)\nabla\theta) \quad \text{on } \mathrm{Q} = \Omega \times (0, \mathrm{T}), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz boundary Γ, $n \in \mathbb{N}$, $\theta = \theta(x, t)$ is an unknown temperature field, $x \in \Omega$, $t \in (0, \mathrm{T})$, $\mathrm{T} > 0$, $c = c(x, \theta)$ is the heat capacity multiplied by the mass density (depending on space and temperature), and $k = k(x, \theta)$ is the heat conductivity. On the boundary Γ we assume a nonlinear Newton condition:

$$k(x, \theta)\frac{\partial \theta}{\partial \nu} + g(x, t, \theta) = 0 \quad \text{on } \Sigma = \Gamma \times (0, \mathrm{T}), \tag{1.2}$$

where ν is the unit outward normal to Γ and $g: \Sigma \times \mathbb{R} \to \mathbb{R}$ is a density of the heat flux through the boundary Γ. We will pay our attention to the case when the properties of media depend discontinuously on $x \in \Omega$, confining ourselves to the case when the medium is piecewise homogeneous. For this reason we divide the domain Ω into m disjoint parts Ω_i, $i = 1, \ldots, m$, $m \in \mathbb{N}$, and suppose that the medium is homogeneous on each Ω_i:

$$c(x, \theta) = c_i(\theta) \quad \text{and} \quad k(x, \theta) = k_i(\theta) \quad \text{for } x \in \Omega_i. \tag{1.3}$$

Also we admit some phase transition of the Stefan type, which can be described by means of Dirac distributions in the coefficients c_i. E. g. an M-phase Stefan problem in i-th subdomain Ω_i is created by putting:

$$c_i(\theta) = c_i^0(\theta) + \sum_{l=1}^{\mathrm{M}-1} \mathrm{L}_{il}\,\delta(\theta - \theta_{il}),$$

where c_i^0 is a regular part of the distribution c_i, δ is the Dirac distribution with the support at $\theta = 0$, and θ_{il} and L_{il} is the temperature and the latent heat, respectively, of the transition between the l-th and $(l+1)$-th phases of the material that occupies the i-th subdomain.

Such problems arise very often in engineering applications and in nature too. Let us mention, e. g., the temperature field investigated in geophysics, where the temperature ranges many hundreds degrees (thus the material cannot be considered as linear) and the properties of different plates of the lithosphere differ from each other, hence the material is discontinuously heterogeneous. Another example is the temperature field within metal casting, when one of the subdomains Ω_i is occupied by molten metal while the other Ω_j contains sand. Of course, in both examples we neglected convection of the materials and many other physical events.

It can be said that, in fact, homogeneous or linear media are only a certain simplification of situations which may appear in nature. Also usual boundary conditions, which neglect the temperature field outside the domain Ω, should be understood as such a simplification. Thus the media that are simultaneously nonlinear and discontinuously heterogeneous are of great importance not only from the viewpoint of applications, but also from a theoretical viewpoint. It is interesting that in this case we are forced to use a technique that does not check the time derivative of the temperature and the quality of the temperature field thus obtained is worse than the quality that can be obtained (under some regularity assumptions) in the linear or "smoothly heterogeneous" case, see [6]; *cf.* also Remark 4.1 below. As a side effect, this technique covers also the case where some coefficient c_i is not greater than a positive constant, it means that the equation is not of a strongly parabolic type.

An efficient method to solve the nonlinear heat-transfer equation is based on the enthalpy and the Kirchhoff transformations, and there is a large, ever-growing amount of literature about this method; the references at the end of this paper represent only a short sample. The heterogeneous case has been already investigated by Niezgódka and Pawłow [6], but only for the case that the coefficients $c(., \theta)$ and $k(., \theta)$ are continuous (and even smooth). In the homogeneous case, the degeneration of the parabolic type of the equation has been investigated in [7].

The crucial difficulty in our problem can be outlined as follows: In the linear or the homogeneous case the spacial part of (1.1) represents a monotone [according to the scalar product of the space $L^2(\Omega)$] and potential operator with respect to the temperature or to the temperature after the Kirchhoff transformation, respectively. However, this operator is neither monotone nor potential in the nonlinear and heterogeneous case; *cf.* Remark 3.1 below. For the homogenous case and the Dirichlet boundary condition, a monotonicity according to the scalar product of the space $H^{-1}(\Omega)$ has been demonstrated in [4] Sec. II.3, but such monotonicity is preserved for our heterogeneous case with the Newton boundary condition only under somewhat restrictive conditions; see Remark 4.2 below. Also an m-akretivity in the space $L^1(\Omega)$ [it means that the corresponding Rothe operator is a contraction in $L^1(\Omega)$], which is valid in the homogeneous case (*see* E. Magenes *et al.* [5]), cannot be transferred to our nonlinear heterogeneous case.

In this paper we will demonstrate existence of a weak solution of our heterogeneous Stefan problem. We will employ an enthalpy formulation, a regularization of contact conditions that appear between the subdomains Ω_i and Ω_j, the Rothe method and the Schauder fixed-point theorem.

2. AN ENTHALPY FORMULATION OF THE PROBLEM

For every $i=1, \ldots, m$, let us consider the functions α_i, β_i: $\mathbb{R} \to \mathbb{R}$ such that

$$c_i(\theta)=\frac{d\alpha_i^{-1}}{d\theta} \quad \text{and} \quad k_i(\theta)=\frac{d\beta_i \circ \alpha_i^{-1}}{d\theta}. \tag{2.1}$$

Of course, in case of the Stefan problem when c_i contains some Dirac distribution, (2.1) is to be understood in the sense of distributions. Then (1.1) restricted to the i-th domain can be rewritten into the form:

$$\left.\begin{aligned}\theta(x, t)&=\alpha_i(w(x, t))\\ \frac{\partial w}{\partial t}&=\Delta\beta_i(w)\end{aligned}\right\} \quad \text{on } Q_i=\Omega_i\times(0, T). \tag{2.2}$$

where $w=w(x, t)$ is an unknown enthalpy, $\alpha_i(w)$ is the temperature, and $\beta_i(w)$ is the temperature after the Kirchhoff transformation, we will say briefly the Kirchhoff temperature ($\beta_i \circ \alpha_i^{-1}$ is the Kirchhoff transformation, α_i^{-1} is the so-called enthalpy transformation). In three-dimensional problems (*i. e.* $n=3$) the physical dimension of enthalpy, temperature, and the Kirchhoff temperature is $\mathrm{J}\,m^{-3}$, K, and $\mathrm{W}\,m^{-1}$, respectively. The boundary conditions (1.2) can be now rewritten as:

$$\frac{\partial}{\partial\nu}\beta_i(w)+g(x, t, \alpha_i(w))=0 \quad \text{on } \Sigma_i=\Gamma_i\times(0, T), \tag{2.3}$$

where $\Gamma_i=\Gamma\cap\bar{\Omega}_i$; $\bar{\Omega}_i$ denotes the closure of Ω_i in $\mathbb{R}^n$. As the solutions of (2.2) for $i=1, \ldots, m$ are to solve (1.2), we must impose still the so-called Hugeniot contact conditions on the boundaries $\Gamma_{ij}=\bar{\Omega}_i\cap\bar{\Omega}_j$ between the i-th and the j-th subdomains:

$$\left.\begin{aligned}&\frac{\partial}{\partial\nu}\beta_i(w)=\frac{\partial}{\partial\nu}\beta_j(w) &(2.4)\\ &\alpha_i(w)=\alpha_j(w) &(2.5)\end{aligned}\right\} \quad \text{on } \Sigma_{ij}=\Gamma_{ij}\times(0, T),$$

where ν is a normal to Γ_{ij} (its orientation is not imporatant here). The conditions (2.4) and (2.5) express the requirement for the heat flux and the temperature not to have jumps on Γ_{ij}. Of course, the system should be completed by an initial condition:

$$w(x, 0)=w^0(x) \quad \text{on } \Omega. \tag{2.6}$$

Note that instead of the unknown temperature in the problem (1.1)-(1.2) we have got an unknown enthalpy in the system (2.2)-(2.5). The formulation based on the enthalpy is more sensible especially in the case of the Stefan problem, where some α_i^{-1} may be a multivalued mapping. Let us remark that usually only the Kirchhoff temperature (and thus only a function β_i) appears in the enthalpy formulation. In the homogeneous case it is good enough, at least from the mathematical point of view, but in our heterogeneous case temperature and the Kirchhoff temperature must be distinguished from each other because they differ qualitively (the latter may have jumps on Γ_{ij}, while the former may not). For an example of the functions α_i and β_i in a case of a certain type of steel we refer to [8].

Let us derive a weak formulation of the system (2.2)-(2.6). We assume:

$$\begin{aligned} &\Omega,\ \Omega_i \text{ are bounded Lipschitz domains, } i=1,\ \ldots,\ m, \\ &\Omega_i \cap \Omega_j = \varnothing \quad \text{for } i \neq j,\ \bigcup_{i=1}^{m} \bar{\Omega}_i = \bar{\Omega}. \end{aligned} \tag{2.7}$$

The standard norm and the scalar product of a Hilbert space $L^2(\Xi)$ will be denoted by $\|\,.\,\|_\Xi$ and $\langle\,.,\ .\,\rangle_\Xi$, respectively; e.g. for $\Xi = Q$, $\langle u,\ u\rangle_Q$ will mean $\int_Q u.v\,dx\,dt$. Let us denote by $H^1(Q)$ the usual Sobolev space of all functions $Q \to \mathbb{R}$ that belongs together with their first distributional derivatives to $L^2(Q)$. For every $z \in H^1(Q)$ we have by (2.2):

$$\sum_i \left\langle \frac{\partial w}{\partial t} - \Delta \beta_i(w),\ z \right\rangle_{Q_i} = 0,$$

where $\sum_i$ denotes the summation for $i=1,\ \ldots,\ m$. Now we use the integration per partes in time and Green's theorem in space. By (2.3), (2.4) and (2.6) we get

$$\begin{aligned} \sum_i \Bigg(\left\langle w,\ \frac{\partial z}{\partial t} \right\rangle_{Q_i} &- \langle \nabla \beta_i(w),\ \nabla z \rangle_{Q_i} - \langle g(\alpha_i(w)),\ z \rangle_{\Sigma_i} \\ &+ \langle w^0,\ z(.,\ 0) \rangle_{\Omega_i} + \langle w(.,\ T),\ z(.,\ T) \rangle_{\Omega_i} \Bigg) = 0, \end{aligned} \tag{2.8}$$

where the integrals over Σ_i and Ω_j are to be understood in the sense of traces. Let us note that the terms of the type $\left\langle \frac{\partial}{\partial \nu} \beta_i(w),\ z \right\rangle_{\Sigma_{ij}}$ do not occure in (2.8) because of (2.4) and the fact that the traces on Σ_{ij} of the functions $z|_{Q_j}$ and $z|_{Q_i}$ are the same since $z \in H^1(Q)$. We will use the following convention: $\alpha(w)$ denotes the function defined a.e. on Q by $\alpha(w)(x,\ t) = \alpha_i(w(x,\ t))$ if $x \in \Omega_i$. Analogously we define $\beta(w)$ a.e. on Q, $g(\alpha(w))$ a.e. on $\Sigma = \Gamma \times (0,\ T)$, etc. The integral identity (2.8) can be then rewritten as follows:

$$\begin{aligned} \left\langle w,\ \frac{\partial z}{\partial t} \right\rangle_Q &- \sum_i \langle \nabla \beta(w),\ \nabla z \rangle_{Q_i} - \langle g(\alpha(w)),\ z \rangle_\Sigma \\ &+ \langle w^0,\ z(.,\ 0) \rangle_\Omega = \langle w(.,\ T),\ z(.,\ T) \rangle_\Omega. \end{aligned}$$

Thus we come to the following definition of a weak solution of the system (2.2)-(2.6). Note that the condition (2.5), which has not been included into (2.8), is reflected in (2.9).

DEFINITION 2.1. — *A function* $w \in L^2(Q)$ *will be called a weak solution of the Stefan problem in a heterogeneous medium,* i. e. *of the problem* (2.2)-(2.6), *if*

$$\alpha(w) \in L^2(0, T; H^1(\Omega)), \tag{2.9}$$

and the following integral identity holds for every $z \in H^1(Q)$ *such that* $z(., T) = 0$ (*in the sense of traces on* $\Omega \times \{T\}$):

$$\left\langle w, \frac{\partial z}{\partial t} \right\rangle_Q - \sum_i \langle \nabla \beta(w), \nabla z \rangle_{Q_i} - \langle g(\alpha(w)), z \rangle_\Sigma + \langle w^0, z(., 0) \rangle_\Omega = 0. \tag{2.10}$$

It should be emphasized that the traces on Σ are understood in the following manner: due to (2.9), $\alpha(w)|_\Sigma$ belongs to the space $L^2(0, T; H^{1/2}(\Gamma))$, and thus it belongs to $L^2(\Sigma)$ as well. It is weaker than the usual understanding of traces (*i. e.* for $z \in H^1(Q)$, $z|_\Sigma \in H^{1/2}(\Sigma)$). As usual, $L^2(0, T; .)$ denotes the space of squared integrable (in the Bochner sense) functions of the interval $[0, T]$, and $H^{1/2}$ denotes the corresponding Sobolev-Slobodeckii space.

3. A REGULARIZATION AND AN APPROXIMATION OF THE PROBLEM

The Stefan problems are usually treated by means of a regularization of the functions α_i and β_i (then the regularization of α_i^{-1} is a singlevalued or even smooth function). We will not need such a regularization here, but what we need is a regularization of the contact condition (2.5) of the Dirichlet type by replacing it with a condition of the Newton type:

$$\rho \frac{\partial}{\partial \nu} \beta_i(w) = \alpha_j(w) - \alpha_i(w) \quad \text{on } \Sigma_{ij}, \tag{3.1}$$

where ν is the unit normal to Σ_{ij} oriented from Q_i to Q_j (now its orientation is important). Note that (3.1) is symmetric with respect to the indices i and j if (2.4) and the orientation of ν are taken into consideration. The parameter $\rho > 0$ represents a thermal resistivity of the surface Γ_{ij} (its physical dimension is $m^2 \mathrm{KW}^{-1}$ for three-dimensional problems). The original problem can be understood as a "limit" case of the problem (2.2), (2.3), (2.4), (2.6), (3.1) when the thermal resisitivity of the surfaces Γ_{ij} tends to zero. It should be pointed out that the problem with $\rho > 0$ has its own physical meaning. Hereafter we show, as a by-product, the existence of a weak solution for $\rho > 0$ and a convergence for $\rho \searrow 0$.

Let us define:

$$\mathscr{H}^1(\Omega)=\prod_i \mathrm{H}^1(\Omega_i), \qquad \mathscr{H}^1(\mathrm{Q})=\prod_i \mathrm{H}^1(\mathrm{Q}_i),$$

and endow them by the standard product norms, i. e. $\|(u_1, \ldots, u_m)\|_{\mathscr{H}^1(\Omega)}=(\Sigma_i \|u_i\|^2_{\mathrm{H}^1(\Omega_i)})^{1/2}$ and analogously for $\mathscr{H}^1(\mathrm{Q})$. We will consider $\mathrm{H}^1(\Omega)$ imbedded continuously into $\mathscr{H}^1(\Omega)$ by means of $u \mapsto (u_1, \ldots, u_m)$ with $u_i=u|_{\Omega_i}$. Similarly $\mathrm{H}^1(\mathrm{Q}) \subset \mathscr{H}^1(\mathrm{Q})$. For $u \in \mathscr{H}^1(\Omega)$ and $i \neq j$ let us denote by $\delta_{ij} u \in \mathrm{H}^{1/2}(\Gamma_{ij})$ the difference between the traces on Γ_{ij} of $u|_{\Omega_i}$ and $u|_{\Omega_j}$. It is evident that $u \in \mathscr{H}^1(\Omega)$ belongs to $\mathrm{H}^1(\Omega)$ if and only if $\delta_{ij} u=0$ for every $i \neq j$. By using the per-partes integration in time and Green's theorem in space, we get the following definition of a weak solution for $\rho > 0$:

DEFINITION 3.1. — *A function* $w_\rho \in \mathrm{L}^2(\mathrm{Q})$ *will be called a weak solution of the Stefan problem with thermal resistivity* $\rho > 0$, i. e. *the problem* (2.2), (2.3), (2.4), (2.6) *and* (3.1), *if*

$$\alpha(w_\rho) \in \mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega)) \tag{3.2}$$

and the following integral identity holds for every $z \in \mathscr{H}^1(\mathrm{Q})$ *such that* $z(., \mathrm{T})=0$ *(in the sense of traces on* $\Omega \times \{\mathrm{T}\}$*):*

$$\left\langle w_\rho, \frac{\partial z}{\partial t}\right\rangle_{\mathrm{Q}} - \sum_i \langle \nabla \beta(w_\rho), \nabla z \rangle_{\mathrm{Q}_i} - \langle g(\alpha(w_\rho)), z \rangle_\Sigma + \langle w^0, z(., 0) \rangle_\Omega - \frac{1}{\rho} \sum_{i>j} \langle \delta_{ij} \alpha(w), \delta_{ij} z \rangle_{\Sigma_{ij}}=0. \tag{3.3}$$

Now we employ the approximation by means of the well-known Rothe method, which is based on a time semi-discretization (hereinafter considered as equidistant, $\eta > 0$ will denote the lenght of a time step, T/η integer).

DEFINITION 3.2. — *A function* $w_{\rho\eta} \in \mathrm{L}^2(\mathrm{Q})$ *will be called a Rothe approximate solution of the Stefan problem with thermal resisitivity* $\rho > 0$ *if* $w_{\rho\eta}(x, t)=w^k_{\rho\eta}(x)$ *for a. a.* $x \in \Omega$ *and* $(k-1)\eta < t < k\eta$, $k=1, \ldots, \mathrm{T}/\eta$ *(thus* $w_{\rho\eta}$ *is picewise constant in time), where* $w^k_{\rho\eta} \in \mathrm{L}^2(\Omega)$ *are functions such that* $\alpha(w^k_{\rho\eta}) \in \mathscr{H}^1(\Omega)$ *and the following recursive integral identity holds for every* $v \in \mathscr{H}^1(\Omega)$ *and all* $k=1, \ldots, \mathrm{T}/\eta$:

$$\eta \sum_i \langle \nabla \beta(w^k_{\rho\eta}), \nabla v \rangle_{\Omega_i} + \langle w^k_{\rho\eta}, v \rangle_\Omega + \int_{k\eta-\eta}^{k\eta} \langle g(., t, \alpha(w^k_{\rho\eta})), v \rangle_\Gamma \, dt + \frac{\eta}{\rho} \sum_{i>j} \langle \delta_{ij} \alpha(w^k_{\rho\eta}), \delta_{ij} v \rangle_{\Gamma_{ij}} = \langle w^{k-1}_{\rho\eta}, v \rangle_\Omega, \tag{3.4}$$

$$w^0_{\rho\eta}=w^0.$$

Remark 3.1. — Now we can outline what is the crucial difficulty in the nonlinear problem in a heterogeneous medium. Put $w_{\rho\eta}^k = w_a$ and $w_{\rho\eta}^k = w_b$ and subtract (3.4) for these two choices. We thus get the terms

$$\eta \sum_i \langle \nabla(\beta(w_a) - \beta(w_b)), \nabla v \rangle_{\Omega_i}$$

and

$$\frac{\eta}{\rho} \sum_{i>j} \langle \delta_{ij}(\alpha(w_a) - \alpha(w_b)), \delta_{ij} v \rangle_{\Gamma_{ij}}.$$

Putting $v = \alpha(w_a) - \alpha(w_b)$, the second term is nonnegative, but the first term may have generally an arbitrary sign even if β_i are nondecreaing. For the choice $v = \beta(w_a) - \beta(w_b)$ the situation is converse, it means that the second term does not preserve any sign. In no case we get any monotonicity, and that is why usual methods for homogeneous meadia fail in our heterogeneous case. It causes also serious difficulties concerning uniqueness of the weak solution, its continuous dependence on the data, and efficient numerical methods. It is quite surprising because the "discontinuously heterogeneous" nonlinear heat-transfer problem is physically reasonable. Thus it is a challenge to develop new techniques for this problem.

4. EXISTENCE AND CONVERGENCE RESULTS

We impose the following assumptions on the data α_i, β_i, g, and w^0. We assume that there are real constants $\theta_{\min} \leqq \theta_{\max}$, $0 < k_{\min} \leqq k_{\max}$, $g_{\max} > 0$ such that:

(4.1) $\forall i = 1, \ldots, m: \alpha_i, \beta_i: \mathbb{R} \to \mathbb{R}$ are continuous, nondecreasing functions such that $\beta_i \circ \alpha_i^{-1}$ is an absolutely continuous (single-valued) function, (2.1) holds in the sense of distributions or a. e., respectively, $c_i(\theta) \geqq 0$, $k_{\max} \geqq k_i(\theta) \geqq k_{\min}$ for a. a. $\theta \in \mathbb{R}$, and $\lim_{|w| \to \infty} |\alpha_i(w)| = \infty$,

(4.2) $g: \Sigma \times \mathbb{R} \to \mathbb{R}$ is measurable, $|g(x, t, \theta)| \leqq g_{\max}$ for a. a. $(x, t, \theta) \in \Sigma \times [\theta_{\min}, \theta_{\max}]$, $g(x, t, .): \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing for a. a. $(x, t) \in \Sigma$, and $g(., ., \theta_{\min}) \leqq 0$ and $g(., ., \theta_{\max}) \geqq 0$ a. e. on Σ,

(4.3) $$w^0 \in L^\infty(\Omega) \text{ and } \theta_{\min} \leqq \alpha(w^0) \leqq \theta_{\max} \text{ a. e. on } \Omega.$$

Note that, if the Stefan problem actually appears in the i-th subdomains (hence c_i contains a Dirac distribution), α_i^{-1} is a multivalued mapping because α_i is then constant on an interval with a lenght equal just to the latent heat of the phase change in question. Of course, since $\beta_i \circ \alpha_i^{-1}$ should be a singlevalued mapping, β_i must be constant on this interval too; *cf.* the example in [8].

Note also that, since c_i may approach zero, the equation need not be of a strong parabolic type. To avoid some technical difficulties, we have not admitted c_i to be zero on an interval of a positive lenght [in such a case α_i and β_i would not be singlevalued mappings as assumed in (4.1)], but the results below could be extended to this case too.

Now we are going to prove existence of some approximate solution $w_{\rho\eta}$ by means of the Schauder fixed-point theorem.

PROPOSITION 4.1. — *Under the assumptions* (2.7), (4.1)-(4.3), *for every* $\rho > 0$ *and* $\eta > 0$ *there exists at least one Rothe approximate solution* $w_{\rho\eta}$ *due to Definition* 3.2. *Moreover,*

$$\theta_{\min} \leqq \alpha(w_{\rho\eta}) \leqq \theta_{\max} \quad a.e.\ on\ Q. \tag{4.4}$$

Proof. — We are to prove that, for every $w_{\rho\eta}^{k-1} \in L^2(\Omega)$ such that $\theta_{\min} \leqq \alpha(w_{\rho\eta}^{k-1}) \leqq \theta_{\max}$ there exists at least one $w_{\rho\eta}^k \in L^2(\Omega)$ satisfying (3.4) for every $v \in \mathscr{H}^1(\Omega)$ and such that $\theta_{\min} \leqq \alpha(w_{\rho\eta}^k) \leqq \theta_{\max}$. Now we pull the subdomains Ω_i apart, considering the following m problems: for $\theta_{ij} \in L^2(\Gamma_{ij})$ find $w_i \in L^2(\Omega_i)$ such that

$$\begin{aligned} \eta \langle \nabla \beta(w_i), \nabla v \rangle_{\Omega_i} &+ \langle w_i, v \rangle_{\Omega_i} \\ &+ \int_{k\eta-\eta}^{k\eta} \langle g(.,t,\alpha(w_i)), v \rangle_{\Gamma_i} dt + \frac{\eta}{\rho} \sum_{j \neq i} \langle \alpha(w_i), v \rangle_{\Gamma_{ij}} \\ &= \langle w_{\rho\eta}^{k-1}, v \rangle_{\Omega_i} + \frac{\eta}{\rho} \sum_{j \neq i} \langle \theta_{ij}, v \rangle_{\Gamma_{ij}} \end{aligned} \tag{4.5}$$

holds for every $v \in H^1(\Omega_i)$; $i = 1, \ldots, m$. It is evident that, if

$$\theta_{ij} = \alpha(w_j)\big|_{\Gamma_{ij}} \quad \text{for every } i, j, i \neq j, \tag{4.6}$$

then $w_{\rho\eta}^k$, defined a. e. on Ω by $w_{\rho\eta}^k(x) = w_i(x)$ for $x \in \Omega_i$, would satisfy (3.4).

Putting $y_i = \beta_i(w_i)$, we can rewrite (4.5) into the form of a variational inclusion (realize that, by (4.1), β_i^{-1} is a multivalued mapping with a maximal monotone graph):

$$\begin{aligned} \eta \langle \nabla y_i, \nabla v \rangle_{\Omega_i} &+ \langle \beta_i^{-1}(y_i), v \rangle_{\Omega_i} \\ &+ \int_{k\eta-\eta}^{k\eta} \langle \bar{g}(.,t,y_i), v \rangle_{\Gamma_i} dt \\ &+ \frac{\eta}{\rho} \sum_{j \neq i} \langle \alpha_i \circ \beta_i^{-1}(y_i), v \rangle_{\Gamma_{ij}} \\ &\ni \langle w_{\rho\eta}^{k-1}, v \rangle_{\Omega_i} + \frac{\eta}{\rho} \sum_{j \neq i} \langle \theta_{ij}, v \rangle_{\Gamma_{ij}}, \end{aligned} \tag{4.7}$$

where $\bar{g}(.,.,y) = g(.,.,\alpha \circ \beta^{-1}(y))$. The inclusion (4.7) is nothing else than the usual necessary conditions for $y_i \in H^1(\Omega_i)$ to be a minimizer of the convex, coercive (not necessarily differentiable) functional J_i: $H^1(\Omega_i) \to \mathbb{R}$ defined by:

$$(4.8)\quad J_i(y_i) = \frac{\eta}{2}\|\nabla y_i\|^2_{\Omega_i} + \int_{\Omega_i} B(y_i)\,dx + \int_{k\eta-\eta}^{k\eta}\int_{\Gamma_i} G(x, t, y_i)\,d\sigma\,dt + \frac{\eta}{\rho}\sum_{j \neq i}\int_{\Gamma_{ij}} A(y_i)\,d\sigma - \langle w^{k-1}_{\rho\eta}, y_i\rangle_{\Omega_i} - \frac{\eta}{\rho}\sum_{j \neq i}\langle \theta_{ij}, y_i\rangle_{\Gamma_{ij}},$$

where

$$B(y) = \int_0^y \beta_i^{-1}(\zeta)\,d\zeta, \qquad G(.,\,.,\,y) = \int_0^y \bar{g}(.,\,.,\,\zeta)\,d\zeta,$$

and

$$A(y) = \int_0^y \alpha_i \circ \beta_i^{-1}(\zeta)\,d\zeta.$$

By the monotonicity of β_i, $\alpha_i \circ \beta_i^{-1}$ and g, the functions A, B and G, are convex with respect to y, hence J_i is actually a convex functional on $H^1(\Omega_i)$. It is evident that (4.8) is equivalent to the fact that 0 belongs to the subdifferential ∂J_i at y_i. Without lost of generality we may and will suppose that there is some j, $j \neq i$, such that Γ_{ij} has a positive $(n-1)$-dimensional measure. Roughly speaking, it means that every Ω_i is "connected" with some other Ω_j. If it would not be true and some Ω_i would be isolated, we could divide such Ω_i into two subdomains connected with each other. By this assumption and (4.1), the first and the forth terms in (4.8) ensure a uniform convexity of J_i with respect to the norm of $H^1(\Omega_i)$, from which follows particularly a coercivity of J_i on $H^1(\Omega_i)$. Suppose, for a moment, that g has been changed outside the interval $[\theta_{\min}, \theta_{\max}]$ in order to have a linear growth. It ensures the continuous dependence of the third term in (4.8) on y_i. The other terms are continuous evidently, therefore the whole functional J_i is continuous, and thanks to its convexity it is also weakly lower semi continuous. Existence of its minimizer y_i then results from reflexivity of $H^1(\Omega_i)$. Taking some w_i such that $\beta_i(w_i) = y_i$, we obtain a solution of (4.5). Moreover, the uniform convexity ensures uniqueness of the minimizer y_i of J_i and the (even uniform) continuity of the mapping $\theta_{ij} \mapsto y_i$ from $H^{-1/2}(\Gamma_{ij})$ to $H^1(\Omega_i)$.

Now we will prove that the mapping $\theta_{ij} \mapsto \alpha_i(w_i)|_{\Gamma_{ij}}$ is completely continuous from $L^2(\Gamma_{ij})$ to itself. Let $\{\theta^l_{ij}\}_{l \in \mathbb{N}}$ be a sequence in $L^2(\Gamma_{ij})$ converging weakly to θ_{ij}. As the imbedding of $L^2(\Gamma_{ij})$ into $H^{-1/2}(\Gamma_{ij})$ is compact, this sequence converges strongly in $H^{-1/2}(\Gamma_{ij})$, and therefore $y^l_i \to y_i$ strongly in $H^1(\Omega_i)$; y_i and y^l_i denote the unique solutions of (4.7) corresponding to θ_{ij} and θ_{ij}^l, respectively. As the sequence $\{\alpha_i \circ \beta_i^{-1}(y_i^l)\}_{l \in \mathbb{N}}$ is bounded in $H^1(\Omega_i)$, we can take a subsequence (denote it by the same indices, for simplicity) such that $\alpha_i \circ \beta_i^{-1}(y_i^l) = \alpha_i(w_i^l) \to \theta_i$ weakly in $H^1(\Omega_i)$. Therefore $\alpha_i(w_i^l) \to \theta_i$ strongly in $L^2(\Omega_i)$. In view of the continuity of the Nemytskii operator induced by $\alpha_i \circ \beta_i^{-1}$ (note that, thanks to (4.1), $\alpha_i \circ \beta_i^{-1}: \mathbb{R} \to \mathbb{R}$ is continuous with a linear growth), we can see that $\alpha_i(w_i^l) \to \alpha_i \circ \beta_i^{-1}(y_i) = \alpha_i(w_i)$ in $L^2(\Omega_i)$, which shows that $\theta_i = \alpha_i(w_i)$ and the whole sequence $\{\alpha_i(w_i^l)\}_{l \in \mathbb{N}}$ converges weakly in $H^1(\Omega_i)$ to θ_i. Thus the respective traces on Γ_{ij} converge strongly in $L^2(\Gamma_{ij})$.

Let us define the mapping $\Phi: \prod_{i \neq j} L^2(\Gamma_{ij}) \to \prod_{i \neq j} L^2(\Gamma_{ij})$ by prescribing $q = (\theta_{ij})_{i \neq j} \mapsto \Phi(q)$ with $[\Phi(q)]_{ji} = \alpha_i(w_i)|_{\Gamma_{ij}}$ where w_i is a solution of (4.5). We have already shown that Φ is completely continuous. We have assumed that $\theta_{\min} \leqq \alpha(w_{\rho\eta}^{k-1}) \leqq \theta_{\max}$. By the maximum principle, which holds for the nonlinear heat-transfer equation as a consequence of the monotonicity of α_i, β_i and g, and by the assumption (4.2) it can be proved that Φ maps the set $\mathcal{M} = \{q = (\theta_{ij}) \in \prod_{i \neq j} L^2(\Gamma_{ij});\ \forall i, j, i \neq j: \theta_{\min} \leqq \theta_{ij} \leqq \theta_{\max}\}$ into itself.

As $\mathcal{M}$ is convex, closed and bounded in the reflexive separable space $\prod_{i \neq j} L^2(\Gamma_{ij})$ and Φ is completely continuous, we can see that $\Phi(\mathcal{M})$ is strongly compact, and therefore we can use the Schauder fixed point theorem to get some $q = (\theta_{ij})_{i \neq j} \in \mathcal{M}$ such that $q = \Phi(q)$. From the definition of Φ it is evident that (4.6) is satisfied, and then $w_{\rho\eta}^k$ constructed as described above solves (3.4). Starting with (4.3), we can verify recursively that $\theta_{\min} \leqq \alpha(w_{\rho\eta}^k) \leqq \theta_{\max}$, and thus also (4.4). The existence of some Rothe approximate solution $w_{\rho\eta}$ by Definition 3.2 has been proved. ■

Now we are to state and prove some *a priori* estimates. Let us denote by $w_{\rho\eta}^I \in C^0(0, T; L^2(\Omega))$ the linear interpolation of $w_{\rho\eta}$ over the intervals $[k\eta - \eta, k\eta]$, that means $w_{\rho\eta}^I(x, .)$ is linear on each of these intervals, and $w_{\rho\eta}^I(., k\eta) = w_{\rho\eta}^k$ for $k = 0, \ldots, T/\eta$. For $f \in L^2(Q)$ let us put

$$\| f \|_{L^2(0, T; H^1(\Omega))^*} = \sup \{\langle f, z \rangle_Q;\ \| z \|_{L^2(0, T; H^1(\Omega))} \leqq 1\},$$

which makes $L^2(Q)$ imbedded continuously into the dual space to $L^2(0, T; H^1(\Omega))$ endowed with the standard dual norm. Also we will use the known fact that $L^2(0, T; H^1(\Omega))^*$ is isometrically isomorphic to $L^2(0, T; H^1(\Omega)^*)$, where $H^1(\Omega)^*$ is the dual space to $H^1(\Omega)$; for $f \in L^2(\Omega)$ we put

$$\| f \|_{H^1(\Omega)^*} = \sup \{\langle f, v \rangle_\Omega;\ \| v \|_{H^1(\Omega)} \leqq 1\},$$

thus $L^2(\Omega)$ is imbedded continuously to $H^1(\Omega)^*$. The space $L^2(0, T; \mathcal{H}^1(\Omega)^*)$ is treated analogously.

Proposition 4.2. — *Let* (2.7), (4.1)-(4.3) *be valid, and* $w_{\rho\eta}$ *be a Rothe approximate solution due to Definition* 3.2. *Then there are some constants*

$C_0, \ldots, C_4$, *independent of* ρ, $\eta > 0$, *such that:*

$$\| w_{\rho\eta} \|_{L^\infty(Q)} \leqq C_0, \tag{4.9}$$

$$\| \alpha(w_{\rho\eta}) \|_{L^2(0,T;\mathscr{H}^1(\Omega))} \leqq C_1, \tag{4.10}$$

$$\| \delta_{ij} \alpha(w_{\rho\eta}) \|_{\Sigma_{ij}} \leqq C_2 \sqrt{\rho} \quad \text{for all } i \neq j, \tag{4.11}$$

$$\left\| \frac{\partial}{\partial t} w^{\mathrm{I}}_{\rho\eta} \right\|_{L^2(0,T;H^1(\Omega)^*)} \leqq C_3, \tag{4.12}$$

$$\left\| \frac{\partial}{\partial t} w^{\mathrm{I}}_{\rho\eta} \right\|_{L^2(0,T;\mathscr{H}^1(\Omega)^*)} \leqq C_3 + C_4/\sqrt{\rho}. \tag{4.13}$$

Proof. – The estimate (4.9) follows immediately from (4.4) and from the coercivity of all α_i; *see* (4.1).

We may put $v = \alpha(w^k_{\rho\eta})$ into (3.4) and sum (3.4) for $k = 1, \ldots, T/\eta$. We obtain:

$$\eta \sum_{k=1}^{T/\eta} \left(\sum_i \langle \nabla \beta(w^k_{\rho\eta}), \nabla \alpha(w^k_{\rho\eta}) \rangle_{\Omega_i} + \frac{1}{\rho} \sum_{i>j} \| \delta_{ij} \alpha(w^k_{\rho\eta}) \|^2_{\Gamma_{ij}} \right)$$
$$= \sum_{k=1}^{T/\eta} \Big(\langle w^{k-1}_{\rho\eta} - w^k_{\rho\eta}, \alpha(w^k_{\rho\eta}) \rangle_\Omega$$
$$- \int_{k\eta-\eta}^{k\eta} \langle g(.,t,\alpha(w^k_{\rho\eta})), \alpha(w^k_{\rho\eta}) \rangle_\Gamma \, dt \Big)$$
$$= \left\langle \frac{\partial}{\partial t} w^{\mathrm{I}}_{\rho\eta}, \alpha(w^{\mathrm{I}}_{\rho\eta}) - \alpha(w_{\rho\eta}) \right\rangle_Q - \left\langle \frac{\partial}{\partial t} w^{\mathrm{I}}_{\rho\eta}, \alpha(w^{\mathrm{I}}_{\rho\eta}) \right\rangle_Q$$
$$- \langle g(\alpha(w_{\rho\eta})), \alpha(w_{\rho\eta}) \rangle_\Sigma = I_1 + I_2 + I_3.$$

In view of (4.1), the left-hand side can be estimated from below by

$$k_{\min} \Sigma_i \| \nabla \alpha(w_{\rho\eta}) \|^2_{Q_i} + \frac{1}{\rho} \Sigma_{i>j} \| \delta_{ij} \alpha(w_{\rho\eta}) \|^2_{\Sigma_{ij}}.$$

Let us estimate from above the right-hand side. As α_i are nondecreasing, the term I_1 is non-positive and can be replaced by zero. Denoting by A_i a primitive function to α_i, we can rewrite the term I_2:

$$I_2 = -\sum_i \left\langle \frac{\partial}{\partial t} A_i(w^{\mathrm{I}}_{\rho\eta}), 1 \right\rangle_{Q_i} = \sum_i \langle A_i(w^0) - A_i(w^{T/\eta}_{\rho\eta}), 1 \rangle_{\Omega_i}.$$

Thanks to (4.9) we get immediately an estimate for I_2. By using (4.2) and (4.4), we obtain an estimate for I_3, namely $|I_3| \leqq \mathrm{meas}_n \Sigma \max \{ |\theta_{\min}|, |\theta_{\max}| \} g_{\max}$, where meas_n denotes the n-dimensional Lebegue measure. It completes the proof of (4.10) and (4.11).

Now we go on to (4.12). First, for every $v \in \mathscr{H}^1(\Omega)$, we can rewrite (3.4) into the form:

$$(4.14)\quad \langle (w^k_{\rho\eta} - w^{k-1}_{\rho\eta})/\eta,\, v \rangle_\Omega = -\sum_i \langle \nabla \beta(w^k_{\rho\eta}),\, \nabla v \rangle_{\Omega_i} - \frac{1}{\eta}\int_{k\eta-\eta}^{k\eta} \langle g(.,\, t,\, \alpha(w^k_{\rho\eta})),\, v \rangle_\Gamma\, dt - \frac{1}{\rho}\sum_{i>j} \langle \delta_{ij}\alpha(w^k_{\rho\eta}),\, \delta_{ij} v \rangle_{\Gamma_{ij}}.$$

By integration for $t \in [0, T]$ we obtain for every $z \in \mathscr{H}^1(Q)$:

$$\left| \left\langle \frac{\partial}{\partial t} w^I_{\rho\eta},\, z \right\rangle_Q \right| \leqq \sum_i (\| \nabla \beta(w_{\rho\eta}) \|^2_{Q_i} + \| \nabla z \|^2_{Q_i}) + \| g(\alpha(w_{\rho\eta})) \|^2_\Sigma + \| z \|^2_\Sigma + \sqrt{\frac{1}{\rho}} \sum_{i>j} \left(\frac{1}{\rho} \| \delta_{ij}\alpha(w_{\rho\eta}) \|^2_{\Sigma_{ij}} + \| \delta_{ij} z \|^2_{\Sigma_{ij}} \right)$$

$$\leqq k^2_{\max} \| \alpha(w_{\rho\eta}) \|^2_{L^2(0,\, T;\, \mathscr{H}^1(\Omega))} + \| z \|^2_{L^2(0,\, T;\, \mathscr{H}^1(\Omega))} + \text{meas}_n\, \Sigma\, g^2_{\max} + \sqrt{\frac{1}{\rho}} \sum_{i>j} C^2_2 + \left(C^2 + \sqrt{\frac{1}{\rho}} \sum_{i>j} C^2_{ij} \right) \| z \|^2_{L^2(0,\, T;\, \mathscr{H}^1(\Omega))}.$$

We have used the estimate (4.11) and the inequalities: $\| v \|_\Gamma \leqq C \| v \|_{\mathscr{H}^1(\Omega)}$ and $\| \delta_{ij} v \|_{\Gamma_{ij}} \leqq C_{ij} \| v \|_{\mathscr{H}^1(\Omega)}$, which hold for all $i \neq j$ and every $v \in \mathscr{H}^1(\Omega)$. It yields the estimate (4.13):

$$\left\| \frac{\partial}{\partial t} w^I_{\rho\eta} \right\|_{L^2(0,\, T;\, \mathscr{H}^1(\Omega)^*)} \leqq k^2_{\max} C^2_1 + 1 + \text{meas}_n\, \Sigma\, g^2_{\max} + C^2 + \sqrt{\frac{1}{\rho}} \sum_{i>j} (C^2_2 + C^2_{ij}).$$

If $v \in H^1(\Omega)$ in (4.14), $\delta_{ij} v = 0$ and therefore the last term in (4.14) vanishes. Thus we obtain the estimate (4.12):

$$\left\| \frac{\partial}{\partial t} w^I_{\rho\eta} \right\|_{L^2(0,\, T;\, H^1(\Omega)^*)} \leqq k^2_{\max} C^2_1 + 1 + \text{meas}_n\, \Sigma'\, g^2_{\max} + C^2. \quad \blacksquare$$

We will need still an additional assumption on g:

$$(4.15)\quad g(x,\, t,\, \theta) = \sum_{l=1}^{L} g_l(x,\, t,\, \theta), \qquad L \in \mathbb{N}, \quad \text{and} \quad \forall l = 1, \ldots, L$$

$\exists \sigma_l : [\theta_{\min}, \theta_{\max}] \to \mathbb{R}$ increasing and Lipschitz continuous such that $g_l(x, t, \sigma_l^{-1}(.)) : \mathbb{R} \to \mathbb{R}$ is linear for a. a. $(x, t) \in \Sigma$.

Let us remark that this condition does not seem to be a considerable restriction for g. E. g. it admits the boundary conditions of the Stefan-Boltzmann type: let $\theta_{\min} \geqq 0$ and

$$g(x,\, t,\, \theta) = a(x)(\theta - u(x,\, t)) + b(x)(\theta^4 - u(x,\, t)^4),$$

where $a(x) \geqq 0$, $b(x) \geqq 0$, and $u(x, t)$ is the outside temperature. Then (4.15) is satisfied with

$$\mathrm{L}=2, \quad g_1(x, t, \theta)=a(x)(\theta-u(x, t)), \quad \sigma_1(\theta)=\theta,$$
$$g_2(x, t, \theta)=b(x)(\theta^4-u(x, t)^4), \quad \sigma_2(\theta)=\theta^4.$$

THEOREM 4.1. — *Let* (2.7), (4.1)-(4.3) *and* (4.15) *be valid.*

(*i*) *Let* $\rho>0$ *is fixed, and* $\{w_{\rho\eta}\}_{\eta>0}$ *be a sequence of Rothe approximate solutions by Definition* 3.2 *with* $\eta \searrow 0$. *Then there is a subsequence (denote it, for simplicity, by the same indices) such that:*

$$\alpha(w_{\rho\eta}) \to \alpha(w_\rho) \quad \textit{weakly in } \mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega)),$$
$$w^{\mathrm{I}}_{\rho\eta} \to w_\rho \quad \textit{strongly in } \mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega)^*),$$

and every w_ρ *thus obtained is a weak solution of the Stefan problem with thermal resistivity* $\rho>0$ *due to Definition* 3.1.

(*ii*) *Let* $\{w_{\rho\eta}\}_{\rho, \eta>0}$ *be a sequence of Rothe approximate solutions by Definition* 3.2 *with* $\rho \searrow 0$, $\eta \searrow 0$, *and* $\eta^2/\rho \to 0$. *Then there is a subsequence (again denoted by the same indices) such that*

$$\alpha(w_{\rho\eta}) \to \alpha(w) \quad \textit{weakly in } \mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega)),$$
$$w^{\mathrm{I}}_{\rho\eta} \to w \quad \textit{strongly in } \mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega)^*),$$

and every w *thus obtained is a weak solution of the Stefan problem in heterogeneous medium by Definition* 2.1.

Proof. — Let us take a test function z for (3.3), it means $z \in \mathscr{H}^1(\mathrm{Q})$ and $z(., \mathrm{T})=0$. As $\mathrm{C}^1(0, \mathrm{T}; \mathscr{H}^1(\Omega))$ is dense in $\mathscr{H}^1(\mathrm{Q})$, we may and will suppose that $z \in \mathrm{C}^1(0, \mathrm{T}; \mathscr{H}^1(\Omega))$, that means $t \mapsto z(., t)$ is a continuously differentiable mapping from $[0, \mathrm{T}]$ to $\mathscr{H}^1(\Omega)$. Put $v=v^{k-1}=z(., k\eta-\eta)$ into (3.4). Then sum (3.4) for $k=1, \ldots, \mathrm{T}/\eta$ and use the discrete analogy of integration per partes:

$$\sum_{k=1}^{\mathrm{T}/\eta} (w^k-w^{k-1})\, v^{k-1} = w^{\mathrm{T}/\eta} v^{\mathrm{T}/\eta} - w^0 v^0 - \sum_{k=1}^{\mathrm{T}/\eta} w^k (v^k-v^{k-1}).$$

Realizing that $v^{\mathrm{T}/\eta}=0$, we get:

$$\begin{aligned}(4.16)\quad \sum_{k=1}^{\mathrm{T}/\eta} \Big(\eta \sum_i \langle \nabla \beta(w^k_{\rho\eta}), \nabla z(., k\eta-\eta)\rangle_{\Omega_i} &- \langle w^k_{\rho\eta}, z(., k\eta)-z(., k\eta-\eta)\rangle_\Omega \\ &+ \int_{k\eta-\eta}^{k\eta} \langle g(., t, \alpha(w^k_{\rho\eta})), z(., k\eta-\eta)\rangle_\Gamma\, dt \\ + \frac{\eta}{\rho} \sum_{i>j} \langle \delta_{ij} \alpha(w^k_{\rho\eta}), \delta_{ij} z(., k\eta-\eta)\rangle_{\Gamma_{ij}} dt\Big) &- \langle w^0, z(., 0)\rangle_\Omega = 0.\end{aligned}$$

Denote by $z^{\mathrm{I}}_\eta \in \mathrm{C}^0(0, \mathrm{T}; \mathscr{H}^1(\Omega))$ the linear interpolation of z over the intervals $[k\eta-\eta, k\eta]$, and by $z_\eta \in \mathrm{L}^\infty(0, \mathrm{T}; \mathscr{H}^1(\Omega))$ the piecewise constant (regarding time) function defined by $z_\eta(., t)=z(., k\eta-\eta)$ for

$k\eta-\eta<t<k\eta$, $k=1, \ldots, T/\eta$. We can rewrite (4.16) as follows:

$$\sum_i \langle \nabla \beta(w_{\rho\eta}), \nabla z_\eta \rangle_{Q_i} - \left\langle w_{\rho\eta}, \frac{\partial}{\partial t} z^I_\eta \right\rangle_Q + \langle g(\alpha(w_{\rho\eta})), z_\eta \rangle_\Sigma + \frac{1}{\rho} \sum_{i>j} \langle \delta_{ij}\alpha(w_{\rho\eta}), \delta_{ij} z_\eta \rangle_{\Sigma_{ij}} - \langle w^0, z(.,0)\rangle_\Omega = 0.$$

Comming back to the original test function z, we get:

$$(4.17)\quad \sum_i \langle \nabla \beta(w_{\rho\eta}), \nabla z \rangle_{Q_i} - \left\langle w_{\rho\eta}, \frac{\partial z}{\partial t} \right\rangle_Q + \langle g(\alpha(w_{\rho\eta})), z \rangle_\Sigma + \sum_{i>j} \langle \delta_{ij}\alpha(w_{\rho\eta}), \delta_{ij} z \rangle_{\Sigma_{ij}} - \langle w^0, z(.,0)\rangle_\Omega = R_{\rho\eta},$$

where $R_{\rho\eta}$ denotes the remainder:

$$(4.18)\quad R_{\rho\eta} = \sum_i \langle \nabla \beta(w_{\rho\eta}), \nabla (z-z_\eta) \rangle_{Q_i} + \left\langle w_{\rho\eta}, \frac{\partial}{\partial t} (z^I_\eta - z) \right\rangle_Q + \langle g(\alpha(w_{\rho\eta})), z-z_\eta \rangle_\Sigma + \frac{1}{\rho} \sum_{i>j} \langle \delta_{ij}\alpha(w_{\rho\eta}), \delta_{ij}(z-z_\eta) \rangle_{\Sigma_{ij}} = R^{(1)}_{\rho\eta} + R^{(2)}_{\rho\eta} + R^{(3)}_{\rho\eta} + R^{(4)}_{\rho\eta}.$$

The term $R^{(1)}_{\rho\eta}$ tends to zero provided $\eta \searrow 0$ because of the estimate $\|\nabla \beta(w_{\rho\eta})\|_{Q_i} \leqq k_{\max} C_1$ [*cf.* (4.1) and (4.10)] and because $\|\nabla(z-z_\eta)\|_{Q_i} = o(\eta)$ for $\eta \searrow 0$ since $z \in C^1(0, T; \mathscr{H}^1(\Omega))$. Also the term $R^{(2)}_{\rho\eta}$ converges to zero if $\eta \searrow 0$ because of (4.9) and because $\frac{\partial}{\partial t} z^I_\eta \to \frac{\partial z}{\partial t}$ in $L^2(Q)$ as a consequence of the assumed smoothness of z. Using the linearity and continuity of the trace operator, we can see that $z|_\Sigma \in C^1(0, T; L^2(\Gamma))$ and $z|_{\Sigma_{ij}} \in C^1(0, T; L^2(\Gamma_{ij}))$. It yields the estimates: $|R^{(3)}_{\rho\eta}| = o(\eta)$ and $|R^{(4)}_{\rho\eta}| = o(\eta/\sqrt{\rho})$. The last estimate uses (4.11). Summarizing, we have got:

$$(4.19)\quad R_{\rho\eta} \to 0 \text{ provided } \eta \searrow 0 \text{ and } \rho > 0 \text{ is fixed.}$$

In view of the *a priori* estimates (4.9), (4.10), (4.12) we can choose a subsequence of $\{w_{\rho\eta}\}_{\eta>0}$ with $\eta \searrow 0$ (indexed, for simplicity, by the same indices) such that

$$(4.20)\quad w^I_{\rho\eta} \to w_\rho \quad \text{weakly in } H^1(0, T; H^1(\Omega)^*) \cap L^2(Q),$$

$$(4.21)\quad \beta(w_{\rho\eta}) \to y_\rho \quad \text{weakly in } L^2(0, T; \mathscr{H}^1(\Omega)),$$

$$(4.22)\quad \alpha(w_{\rho\eta}) \to \theta_\rho \quad \text{weakly in } L^2(0, T; \mathscr{H}^1(\Omega)),$$

$$(4.23)\quad g(\alpha(w_{\rho\eta})) \to \gamma_\rho \quad \text{weakly in } L^2(\Sigma),$$

$$(4.24)\quad \sigma_l(\alpha(w_{\rho\eta})) \to s_{l\rho} \quad \text{weakly in } L^2(0, T; \mathscr{H}^1(\Omega)) \text{ for all } l=1, \ldots, L,$$

and

$$(4.25)\quad w_{\rho\eta} \to w'_\rho \quad \text{weakly in } L^2(Q).$$

The convergence (4.24) results from (4.10) and from the Lipschitz continuity of σ_l assumed in (4.15).

The following assertion is well known (for more general case we refer to J.-L. Lions [4], Chap. I, Theorem 5.1): If B_0, B, B_1 are Hilbert spaces such that $B_0 \subset\subset B \subset B_1$, then

$$L^2(0, T; B_0) \cap H^1(0, T; B_1) \subset\subset L^2(0, T; B),$$

where $\subset$ and $\subset\subset$ denote the continuous and the compact imbeddings, respectively. We employ this assertion by the following manner: since $H^1(\Omega) \subset \mathscr{H}^1(\Omega) \subset\subset L^2(\Omega)$, we have $L^2(\Omega) \subset\subset \mathscr{H}^1(\Omega)^* \subset H^1(\Omega)^*$ if identifying $L^2(\Omega)$ with its own dual, and we can put $B_0 = L^2(\Omega)$, $B = \mathscr{H}^1(\Omega)^*$, $B_1 = H^1(\Omega)^*$. Then (4.20) results in

$$(4.26)\qquad w^{\mathrm{I}}_{\rho\eta} \to w_\rho \quad \text{strongly in } L^2(0, T; \mathscr{H}^1(\Omega)^*).$$

By (4.19), (4.21), (4.22), (4.23) and (4.25), we can pass to the limit in (4.17), which yields the identity:

$$(4.27)\quad \sum_i \langle \nabla y_\rho, \nabla z \rangle_{Q_i} - \left\langle w'_\rho, \frac{\partial z}{\partial t} \right\rangle_Q + \langle \gamma_\rho, z \rangle_\Sigma + \frac{1}{\rho} \sum_{i>j} \langle \delta_{ij} \theta_\rho, \delta_{ij} z \rangle_{\Sigma_{ij}} - \langle w^0, z(., 0) \rangle_\Omega = 0.$$

Of course, we have used the fact that the operator $\theta \mapsto \delta_{ij}\theta$ is continuous from $\mathscr{H}^1(\Omega)$ to $L^2(\Gamma_{ij})$, and thus also from $L^2(0, T; \mathscr{H}^1(\Omega))$ to $L^2(\Sigma_{ij})$, and, being linear, this operator remains continuous with respect to the weak topologies, as well.

It is obvious that the part (i) of the theorem will be proved if one shows that

$$(4.28)\qquad \theta_\rho = \alpha(w_\rho) \quad \text{a. e. on } Q,$$

and that w_ρ is a weak solution by Definition 3.1. Comparing (4.27) with (3.3), we observe that we are to show, besides (4.28), also

$$(4.29)\qquad w_\rho = w'_\rho \quad \text{a. e. on } Q,$$

$$(4.30)\qquad y_\rho = \beta(w_\rho) \quad \text{a. e. on } Q,$$

and

$$(4.31)\qquad \gamma_\rho = g(\theta_\rho) \quad \text{a. e. on } \Sigma.$$

Let us start with (4.28). Take some $w_0 \in L^2(Q)$ and a sequence $\{w_\varepsilon\}_{\varepsilon>0}$ such that $w_\varepsilon \to w_0$ strongly in $L^2(Q)$ for $\varepsilon \searrow 0$ and $\alpha(w_\varepsilon) \in L^2(0, T; \mathscr{H}^1(\Omega))$ for all $\varepsilon > 0$. Such a sequence can be obtained from w_0 by the mollifier technique. Now suppose that $\varepsilon > 0$ is fixed. From (4.21) and (4.26) it follows that

$$\langle\!\langle \alpha(w_{\rho\eta}) - \alpha(w_\varepsilon), w^{\mathrm{I}}_{\rho\eta} - w_\varepsilon \rangle\!\rangle \quad \text{converge to} \quad \langle\!\langle \theta_\rho - \alpha(w_\varepsilon), w_\rho - w_\varepsilon \rangle\!\rangle$$

for $\eta \searrow 0$, where $\langle\!\langle ., . \rangle\!\rangle$ denotes the duality pairing between

$L^2(0, T; \mathscr{H}^1(\Omega))$ and $L^2(0, T; \mathscr{H}^1(\Omega))^*$. Since $L^2(Q)$ has been imbedded into $L^2(0, T; \mathscr{H}^1(\Omega))^*$, we can even replace $\langle\!\langle .\,, . \rangle\!\rangle$ by $\langle .\,, . \rangle_Q$ because $w_\varepsilon, w_\rho, w^I_{\rho\eta} \in L^2(Q)$. This trick concerning dual spaces has been already used by A. Visintin [9], Sec. 2.

Let us emphasize that $\langle \alpha(w_{\rho\eta}) - \alpha(w_\varepsilon), w^I_{\rho\eta} - w_\varepsilon \rangle_Q$ need not be non-negative. However, it is obvious that

$$\langle \alpha(w_{\rho\eta}) - \alpha(w_\varepsilon), w^I_{\rho\eta} - w_\varepsilon \rangle_Q = \langle \alpha(w_{\rho\eta}) - \alpha(w_\varepsilon), w_{\rho\eta} - w_\varepsilon \rangle_Q + \langle \alpha(w_{\rho\eta}) - \alpha(w_\varepsilon), w^I_{\rho\eta} - w_{\rho\eta} \rangle_Q = I_1 + I_2,$$

where $I_1 \geqq 0$ because of the monotonicity of all α_i, $i = 1, \ldots, m$. As to the term I_2, by means of (4.10) we can estimate:

$$|I_2| \leqq (C_1 + \|\alpha(w_\varepsilon)\|_{L^2(0, T; \mathscr{H}^1(\Omega))}) \cdot \|w^I_{\rho\eta} - w_{\rho\eta}\|_{L_2(0, T; \mathscr{H}^1(\Omega)^*)},$$

where

$$\|w^I_{\rho\eta} - w_{\rho\eta}\|^2_{L^2(0, T; \mathscr{H}^1(\Omega)^*)} = \sum_{k=1}^{T/\eta} \int_{k\eta-\eta}^{k\eta} \left\| (w^k_{\rho\eta} - w^{k-1}_{\rho\eta}) \frac{t - k\eta + \eta}{\eta} \right\|^2_{\mathscr{H}^1(\Omega)^*} dt$$

$$= \frac{\eta}{3} \sum_{k=1}^{T/\eta} \|w^k_{\rho\eta} - w^{k-1}_{\rho\eta}\|^2_{\mathscr{H}^1(\Omega)^*} = \frac{\eta^2}{3} \left\| \frac{\partial}{\partial t} w^I_{\rho\eta} \right\|^2_{L^2(0, T; \mathscr{H}^1(\Omega)^*)}.$$

Using the *a priori* estimate (4.13), we obtain:

$$(4.32) \qquad |I_2| \leqq (C_1 + \|\alpha(w_\varepsilon)\|_{L^2(0, T; \mathscr{H}^1(\Omega))}) \cdot (C_3\eta + C_4\eta/\sqrt{\rho})/\sqrt{3}.$$

Thus $I_2 \to 0$ for $\eta \searrow 0$, and $\varepsilon, \rho > 0$ fixed. Passing with $I_1 + I_2$ to the limit for $\eta \searrow 0$, we get:

$$(4.33) \qquad \langle \theta_\rho - \alpha(w_\varepsilon), w_\rho - w_\varepsilon \rangle_Q \geqq 0.$$

Now we can pass to the limit with $\varepsilon \searrow 0$. Since $w_\varepsilon \to w_0$ strongly in $L^2(Q)$ and the Nemytskii operator $w \mapsto \alpha(w)$ is continuous in $L^2(Q)$, we can see that $\alpha(w_\varepsilon) \to \alpha(w_0)$ strongly in $L^2(Q)$. From (4.33) we thus obtain:

$$(4.34) \qquad \langle \theta_\rho - \alpha(w_0), w_\rho - w_0 \rangle_Q \geqq 0.$$

As (4.34) is valid for every $w_0 \in L^2(Q)$, we come to (4.28) by the usual Minty trick.

By the similar way we come to (4.30), using (4.21) and the estimate $\|\beta(w_{\rho\eta})\|_{L^2(0, T; \mathscr{H}^1(\Omega))} \leqq k_{\max} C_1$ instead of (4.22) and the estimate (4.10).

Now we are going to (4.29). As $L^2(\Omega)$ is continuously imbedded into $L^2(0, T; H^1(\Omega)^*)$, from (4.20) and (4.25) we can see that $w^I_{\rho\eta} \to w_\rho$ and $w_{\rho\eta} \to w'_\rho$ weakly in $L^2(0, T; H^1(\Omega)^*)$ as well. Using (4.12) we can obtain by the same manner as above the estimate

$$\|w^I_{\rho\eta} - w_{\rho\eta}\|_{L^2(0, T; H^1(\Omega)^*)} \leqq \eta C_3/\sqrt{3},$$

from which (4.29) immediately follows.

It remains to prove (4.31). Similarly as above, we can show, using (4.24) and monotonicity of the functions $\sigma_l \circ \alpha_i$, that for all $l=1, \ldots, L$: $s_{l\rho}=\sigma_l(\alpha(w_\rho))$ a.e. on Q. In view of (4.28) we can see that $s_{l\rho}=\sigma_l(\theta_\rho)$ a.e. on Q. Particularly the traces on Σ of $s_{l\rho}$ and $\sigma_l(\theta_\rho)$ are the same. Let us put $\bar{g}_l(x, t, s)=g_l(x, t, \sigma_l^{-1}(s))$. From (4.24) we get the weak convergence of the respective traces in $L^2(0, T; H^{1/2}(\Gamma))$, hence in $L^2(\Sigma)$ too. As $\bar{g}_l(x, t, .)$ is linear [*see* (4.15)] and the Nemytskii operator $s \mapsto \bar{g}_l(s)$ is continuous in $L^2(\Sigma)$, this operator is also weakly continuous in $L^2(\Sigma)$. It yields that $\bar{g}_l(\sigma_l(\alpha(w_{\rho\eta}))) \to \bar{g}_l(s_{l\rho})$ weakly in $L^2(\Sigma)$. Since $g(\alpha(w_{\rho\eta}))=\Sigma_l \bar{g}_l(\sigma_l(\alpha(w_{\rho\eta})))$ and $g(\theta_\rho)=\Sigma_l \bar{g}_l(\sigma_l(\theta_\rho))=\Sigma_l \bar{g}_l(s_{l\rho})$, we have got that $g(\alpha(w_{\rho\eta})) \to g(\theta_\rho)$. Comparing it with (4.23), we obtain just (4.31). The proof of the part (i) is just completed.

The part (ii) requires only slight modifications in the preceeding proof. First, instead of $z \in C^1(0, T; \mathscr{H}^1(\Omega))$ we can suppose $z \in C^1(0, T; H^1(\Omega))$, hence $\delta_{ij} z=0$, and also $\delta_{ij} z_\eta=0$, and thus the respective terms in (4.16), (4.17), (4.18), and (4.27) vanish. As now both ρ and η tend to zero, we choose a subsequence of $\{w_{\rho\eta}\}_{\rho, \eta > 0}$ such that $w^I_{\rho\eta} \to w$ weakly in $H^1(0, T; H^1(\Omega)^*) \cap L^2(\Omega)$, and analogously we modify also (4.21)-(4.25), the respective limits being denoted without the subscript ρ. To prove that $R^{(4)}_{\rho\eta} \to 0$ for $\rho \searrow 0$ and $\eta \searrow 0$, we employ the assumption $\eta^2/\rho \to 0$ (recall that we have shown $|R^{(4)}_{\rho\eta}|=o(\eta/\sqrt{\rho})$). This assumption is employed also for (4.32) to prove the modification of (4.33), it means $\langle \theta-\alpha(w_\varepsilon), w-w_\varepsilon \rangle_Q \geqq 0$. In this way we get the integral identity (2.10). Since $\rho \searrow 0$, from the *a priori* estimate (4.11) we get additionally that $\delta_{ij}\alpha(w)=0$ for all $i \neq j$, which proves (2.9). Thus w obtained in the above manner is actually a weak solution of the Stefan problem due to Definition 2.1. ■

We join Theorem 4.1 with Proposition 4.1, which gives immediately the following existence result:

COROLLARY 4.1. — *Under the assumptions* (2.7), (4.1)-(4.3), *and* (4.15), *there exists at least one weak solution w of the Stefan problem in the heterogeneous medium due to Definition* 2.1 *and, for every $\rho > 0$, at least one weak solution w_ρ of the Stefan problem with thermal resistivity ρ due to Definition* 3.1.

Assuming uniqueness, we can obtain also the following convergence result:

COROLLARY 4.2. — *Let* (2.7), (4.1)-(4.3), (4.15) *be valid and, for every $\rho > 0$, the weak solution due to Definition* 3.1 *be unique. Then there is a subsequence (denoted by the same indices) of $\{w_\rho\}_{\rho > 0}$ with $\rho \searrow 0$ such that:*

$$\alpha(w_\rho) \to \alpha(w) \quad \textit{weakly in } L^2(0, T; \mathscr{H}^1(\Omega))$$

and

$$w_\rho \to w \quad \textit{strongly in } \mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega)^*),$$

and every w thus obtained is a weak solution of the Stefan problem in the heterogeneous medium due to Definition 2.1. *If, in addition, the solution w due to Definition* 2.1 *is unique, then even the whole sequence* $\{w_\rho\}_{\rho>0}$ *converge to it.*

A sketch of the proof. – As the *a priori* estimates (4.9)-(4.13) do not depend on η, we can obtain the corresponding *a priori* estimates also for w_ρ by passing with η to zero. Thus we can see that there is a subsequence with $\rho \searrow 0$ such that $\alpha(w_\rho) \to \theta$ weakly in $\mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega))$ and $w_\rho \to w$ strongly in $\mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega)^*)$. Similarly as above we can prove that $\theta = \alpha(w)$. As $\mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega))$ is a separable Hilbert space, the weak topology on its bounded subsets is metrizable; let us denote by d some metric inducing on bounded subsets the weak topology. Thus

$$d(\alpha(w_\rho), \alpha(w)) + \| w_\rho - w \|_{\mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega)^*)} \to 0 \qquad \text{for} \quad \rho \searrow 0.$$

Since w_ρ is assumed to be a unique weak solution by Definition 3.1, by Theorem 4.1 (i) we can choose $\eta = \eta(\rho) > 0$ such that:

$$d(\alpha(w_{\rho\eta(\rho)}), \alpha(w_\rho)) + \| w_{\rho\eta(\rho)} - w_\rho \|_{\mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega)^*)} \leqq \rho.$$

In addition we can suppose $\eta(\rho) \leqq \rho^{1/2+\varepsilon}$ with some $\varepsilon > 0$. Therefore $\alpha(w_{\rho\eta(\rho)}) \to \alpha(w)$ weakly in $\mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega))$ and $w_{\rho\eta(\rho)} \to w$ strongly in $\mathrm{L}^2(0, \mathrm{T}; \mathscr{H}^1(\Omega)^*)$. As $\eta(\rho)^2/\rho \to 0$, all the assumptions of Theorem 4.1 (ii) are fulfilled, and we can see that w is a weak solution by Definition 2.1. ■

We have met the affair of uniqueness of the weak solution. Unfortunately, in general the uniqueness remains as an open problem, though it does not seem that there is any actual reason for the weak solution not to be unique. The original technique by Kamenomostkaya [3] to prove uniqueness has been used in [6], but only on assumptions that Γ is smooth and the medium is "smoothly heterogenous".

Remark 4.1. – The problem is considerably simplified by the additional assumption:

$$\beta_i(w) = k_i^{(1)} \alpha_i(w) + k_i^{(0)} \tag{4.35}$$

for all $w \in \mathbb{R}$, $i = 1, \ldots, m$, and some $k_i^{(0)}, k_i^{(1)} \in \mathbb{R}$. In other words, (4.35) means that the thermal conductivity $k_i(\theta)$ does not depend on the temperature θ, being equal just to the constant k_i^1. Then the approximate solution $w_{\rho\eta}$ due to Definition 3.2 can be obtained simply by minimizing (successively for $k = 1, \ldots, \mathrm{T}/\eta$) of strictly convex functionals of $\alpha(w_{\rho\eta}^k)$ over $\mathscr{H}^1(\Omega)$, without using the Schauder fixed-point theorem. Thus we get also uniqueness of $w_{\rho\eta}$. Also it can be demonstrated by modification of the method [5] that the corresponding Rothe operator is a contraction in $L^1(\Omega)$, like in the homogeneous case. Besides, passing to the limit in the boundary term in the proof of Theorem 4.1 can be performed as proposed by A. Visintin [9] without the condition (4.15). Under some regularity assumptions on the data g and w^0 and under the assumption $c_i(\theta) \geqq c_{\min} > 0$, we can even use techniques checking the time derivative of the temperature; *see*, e.g., [6], [8] and [9], Sec. 3. Thus the weak convergence in $L^2(0, T; \mathscr{H}^1(\Omega))$ in Theorem 4.1 and Corollary 4.2 can be replaced by the weak convergence in $\mathscr{H}^1(Q)$, and the proof of Theorem 4.1 can be considerably shortened.

Remark 4.2. — A particularly simple case appears when, in addition to (4.35), the boundary heat flux is linear, having the form:

$$g(x, t, \theta) = g_1(x).\theta + g_0(x, t) \qquad \text{with} \quad g_1(x) > 0 \text{ on } \Gamma. \tag{4.36}$$

Then we can follow the technique proposed in [4] Sec. II.3 for the case of a homogeneous medium and the Dirichlet boundary condition. In our case it can be demonstrated that the operator $\mathscr{A}: \mathscr{D}(\mathscr{A}) \to H^1(\Omega)^*$, where $\mathscr{D}(\mathscr{A}) = \{w \in L^2(\Omega);\ \alpha(w) \in H^1(\Omega)\}$, defined by

$$\mathscr{A}(w)(v) = \sum_i \langle \nabla \beta(w), \nabla v \rangle_{\Omega_i} + \langle g_1 \alpha(w), v \rangle_\Gamma$$

is monotone with respect to the scalar product $\langle\!\langle\ .,\ .\ \rangle\!\rangle$ in $H^1(\Omega)^*$ defined by $\langle\!\langle \zeta, \xi \rangle\!\rangle = \zeta(A^{-1}(\xi))$ with $A: H^1(\Omega) \to H^1(\Omega)^*$ defined by

$$A(\theta)(v) = \sum_i k_i^{(1)} \langle \nabla \theta, \nabla v \rangle_{\Omega_i} + \langle g_1 \theta, v \rangle_\Gamma.$$

Analogously, for $\rho > 0$ we define the operator $\mathscr{A}_\rho: \mathscr{D}(\mathscr{A}_\rho) \to \mathscr{H}^1(\Omega)^*$, where $\mathscr{D}(\mathscr{A}_\rho) = \{w \in L^2(\Omega);\ \alpha(w) \in \mathscr{H}^1(\Omega)\}$, by

$$\mathscr{A}_\rho(w)(v) = \sum_i \langle \nabla \beta(w), \nabla v \rangle_{\Omega_i} + \langle g_1 \alpha(w), v \rangle_\Gamma + \frac{1}{\rho} \sum_{i>j} \langle \delta_{ij} \alpha(w), \delta_{ij} v \rangle_{\Gamma_{ij}}.$$

This operator is monotone with respect to the scalar product on $\mathscr{H}^1(\Omega)^*$ defined like previously but, instead of A, by means of the operator $A_\rho: \mathscr{H}^1(\Omega) \to \mathscr{H}^1(\Omega)^*$ defined by

$$A_\rho(\theta)(v) = \sum_i k_i^{(1)} \langle \nabla \theta, \nabla v \rangle_\Omega + \langle g_1 \theta, v \rangle_\Gamma + \frac{1}{\rho} \sum_{i>j} \langle \delta_{ij} \theta, \delta_{ij} v \rangle_{\Gamma_{ij}}.$$

Our problems can be rewritten now as

$$\partial w/\partial t = \mathscr{A}(w) + \mathscr{G}(t) \qquad \text{and} \qquad \partial w_\rho/\partial t = \mathscr{A}_\rho(w_\rho) + \mathscr{G}(t)$$

with $\mathscr{G}(t) \in \mathscr{H}^1(\Omega)^*$ defined by $\mathscr{G}(t)(v) = \langle g_0(.,t), v \rangle_\Gamma$. The monotonicity of the operators $\mathscr{A}$ and $\mathscr{A}_\rho$ enables to use standard techniques (*see* [4]), particularly it yields uniqueness for both w and w_ρ; *cf.* also [8], Remark 4.5.

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