ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 5, nº 4 (1988), p. 323-345
<http://www.numdam.org/item?id=AIHPC_1988__5_4_323_0>

Ann. Inst. Henri Poincaré,

Vol. 5, n° 4, 1988, p. 323-345. *Analyse non linéaire*

Holes and obstacles (*)

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ABSTRACT. — In this paper we investigate the effect of a partial obstacle on a semilinear elliptic B.V.P. which has, in general, no solution.

We show that highly unstable solutions arise, a phenomena previously observed for the same equation in presence of holes in the domain.

RÉSUMÉ. — Dans cet article nous examinerons l'effet d'un obstacle partiel pour un problème semi-linéaire elliptique au bord qui, en général, n'a pas de solution.

(*) Partially supported by Ministero Pubblica Istruzione (Fondi 40%).

Classification A.M.S. : 35 A 15, 35 D 10, 35 J 20, 35 J 85, 58 E 35.

Nous montrons que des solutions hautement instables surgissent; il s'agit d'un phénomène précédemment observé pour le même genre d'équation, en présence de trous dans le domaine.

Mots clés : Inégalité variationnelle, croissance critique, concentration-compacité, minmax, point critique.

0. INTRODUCTION

In a remarkable series of papers J. M. Coron and A. Bahri have been giving a complete explanation of a phenomenon previously observed by Kazdan and Warner [9], i. e. the role of the geometry of the domain with respect to existence-non existence for non linear elliptic boundary value problems of the form

$$\left.\begin{aligned} -\Delta u = u^{2^*-1} \quad & \text{in } \Omega \subseteq \mathbb{R}^N, \text{ open and bounded} \\ u \in H_0^1(\Omega), \quad & u \geqq 0 \end{aligned}\right\} \tag{0.1}$$

It is well known that (0. 1) has only the trivial solution if Ω is starshaped. Conversely, A. Bahri and J. M. Coron showed, roughly speaking, that "holes" in Ω induce richer topology on the energy sublevels for (0. 1). This, in turn, is responsable of the existence of non trivial critical points for the energy associated to (0. 1).

In this paper we prove that a similar effect results by imposing a bilateral condition to (0. 1). More precisely, we are interested to the following free boundary problem:

Given $\psi \in H_0^1(\Omega) \cap C^0(\bar{\Omega})$, $\psi \geqq 0$ and a smooth closed subset $C \subset \Omega$, find $u \in H_0^1(\Omega) \cap C^0(\Omega)$ and a closed set $E \subseteq C$ such that

$$\begin{aligned} -\Delta u = |u|^{2^*-1} \quad & \text{in } \Omega \setminus E \\ u = \psi \quad & \text{in } E \\ u \leqq \psi \quad & \text{in } C \end{aligned} \tag{0.2}$$

In case $\psi = 0$, a solution to (0. 2) solves (0. 1) in $\Omega \setminus C$, and hence (0. 2) includes the study of (0. 1) for domains with "holes".

The paper is organized as follows.

In Section 1 we discuss the behaviour of P.S. sequences for the following variational inequality:

PROBLEM 1:

find $u \in \mathbf{K}$ such that

$$\int_\Omega \nabla u \nabla (v-u) \geqq \int_\Omega u^{2^*-1} (v-u), \qquad \forall v \in \mathbf{K}$$

where $\mathbf{K}$ is the closed convex set of functions $u \in H_0^1(\Omega)$ such that $u \geqq 0$ a. e. in Ω and $u \leqq \psi$ on C in the sense of H^1 (*see* [10], Definition 5.1, p. 35).

In Section 2 we give a variational principle for Problem 1 and prove, under additional hypothesis on C, the existence of non trivial critical points for the energy functional associated to Problem 1.

In the last section, we will prove a regularity result for Problem 1 which insures that every solution of Problem 1 solves the free boundary problem (0.2).

NOTATIONS. — We denote by $\|.\|$ the norm in the Sobolev space $H_0^1(\Omega)$, and for $p \geqq 1$, $|.|_p$ will denote the usual norm in $L^p = L^p(\Omega)$. If $u, w \in L^p$, we write $u \vee w = \text{Max}\{u, w\}$, $u \wedge w = \text{Min}\{u, w\}$.

All the inequalities between H^1 functions on the closed set C have to be regarded in the H^1 sense.

1. THE BEHAVIOUR OF P.S. SEQUENCES

DEFINITION 1.1. — $u_n \in H_0^1(\Omega)$ is called a *P.S.* sequence for Problem 1 if

(i) $u_n \in \mathbf{K}$;

(ii) $\operatorname{Sup}_n \left\{ \frac{1}{2} \int_\Omega |\nabla u_n|^2 - \frac{1}{2^*} \int_\Omega |u_n|^{2^*} \right\} < +\infty$;

(iii) $\exists z_n \in H_0^1(\Omega)$, $z_n \to 0$ in $H_0^1(\Omega)$ s. t.

$$\int_\Omega \nabla u_n \nabla (v-u_n) - \int_\Omega |u_n|^{2^*-1} (v-u_n) \geqq \int_\Omega \nabla z_n \nabla (v-u_n)$$

$$\forall v \in \mathrm{K}.$$

PROPOSITION 1.2. — *Every P.S. sequence is bounded in* $H_0^1(\Omega)$.

Proof. — Choosing $v_n = 2u_n - \min\{u_n, \psi\} = u_n + (u_n - \psi)^+$ in (iii), we get

$$\int |\nabla (u_n - \psi)^+|^2 + \int \nabla \psi \nabla (u_n - \psi)^+$$
$$\geq \int |u_n|^{2^*} - \int \psi^{2^*} - \int |u_n|^{2^*-1} \psi - \|(u_n - \psi)^+\|$$

for n large and hence, by Hölder inequality,

$$\|u_n\|^2 + c_1 \|u_n\| \geq |u_n|_{2^*}^{2^*} - c_2 |u_n|_{2^*}^{2^*-1} - c_3$$

Since by (ii) we have

$$|u_n|_{2^*}^{2^*} = \frac{2^*}{2} \|u_n\|^2 + O(1)$$

we readily get the boundeness of $\|u_n\|$. ■

Remark 1.3. — In view of Proposition 1.2, we will always assume in the sequel, that if u_n is a P.S. sequence then $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ for some $u \in \mathbf{K}$, and $\lim \int |\nabla u_n|^2$, $\lim \int |u_n|^{2^*}$ exist. Moreover, we can suppose that $|\nabla u_n|^2, |u_n|^{2^*}$ converge weakly in the sense of measures. ■

PROPOSITION 1.4. — *Let* $u_n \rightharpoonup u$ *be a P.S. sequence. Then u is a solution of Problem* 1.

Proof. — Choosing in (iii) $v_n = u_n + (u_n - u - \psi)^+$ we get, denoting $\vartheta_n = u_n - u$:

$$\int \nabla u_n \nabla (\vartheta_n - \psi)^+ \geq \int |u_n|^{2^*-1} (\vartheta_n - \psi)^+ + o(1),$$

i. e.

$$\int \nabla u_n \nabla \vartheta_n - \int \nabla u_n \nabla (\vartheta_n \wedge \psi) \geq \int |u_n|^{2^*-1} (\vartheta_n - \vartheta_n \wedge \psi) + o(1). \quad (1.1)$$

We claim that

$$\int \nabla u_n \nabla (\vartheta_n \wedge \psi) \to 0; \quad (1.2)$$

$$\int |u_n|^{2^*-1}(\vartheta_n \wedge \psi) \to 0. \tag{1.3}$$

From the claim it follows, using (1.1):

$$\lim \int |\nabla u_n|^2 - \int |\nabla u|^2 \geqq \lim \int |u_n|^{2^*} - \int u^{2^*}. \tag{1.4}$$

Since (iii) yields in the limit

$$\int \nabla u \nabla v - \int u^{2^*-1} v \geqq \lim \left(\int |\nabla u_n|^2 - \int |u_n|^{2^*} \right)$$

we see from (1.4) that u solves Problem 1.

It remains to prove (1.2) and (1.3). Since $\vartheta_n \wedge \psi \to 0$ a.e., (1.3) follows from Lebesgue's dominated convergence Theorem. Finally, setting $v = u + (\vartheta_n - \psi)^+$ in (iii), we get

$$\limsup \int \nabla u_n \nabla (\vartheta_n \wedge \psi) \leqq \lim \int |u_n|^{2^*-1} (\vartheta_n \wedge \psi) = 0. \tag{1.5}$$

On the other hand, since $\vartheta_n \wedge \psi \rightharpoonup 0$ in H_0^1, we have

$$\liminf \int \nabla u_n \nabla (\vartheta_n \wedge \psi) = \liminf \int \nabla \vartheta_n \nabla (\vartheta_n \wedge \psi). \tag{1.6}$$

But, denoted by χ_n the characteristic function of $\{ \vartheta_n \geqq \psi \}$, it results

$$\lim \left| \int (\nabla \vartheta_n \nabla \psi) \chi_n \right| \leqq \text{cost.} \lim \left(\int |\nabla \psi|^2 \chi_n \right)^{1/2} = 0$$

since $\chi_n \to 0$ almost for every x for which $\psi(x) > 0$. Thus (1.6) gives

$$\liminf \int \nabla u_n \nabla (\vartheta_n \wedge \psi) \geqq 0. \tag{1.7}$$

Hence, (1.7), (1.5) yield (1.2). ■

In view of the above Lemma, we will be concerned in the following with P.S. sequences which weakly converge to zero.

Remark 1.5. – Let $u_n \rightharpoonup 0$ be a P.S. sequence. Since (iii) implies, taking $v=0$, $\lim \int |\nabla u_n|^2 \leq \lim \int |u_n|^{2^*}$, from (1.4) we get

$$\lim \int |\nabla u_n|^2 = \lim \int |u_n|^{2^*}. \quad \blacksquare$$

Let us now introduce the energy functional:

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{2^*} \int_\Omega u^{2^*}$$

and

$$S = \operatorname{Inf} \left\{ \frac{\|u\|^2}{|u|_{2^*}^2} \,\middle|\, u \in H_0^1(\Omega), u \neq 0 \right\}.$$

The main result in this section is the following

THEOREM 1.6. – *Let $u_n \rightharpoonup 0$ be a P.S. sequence with $E(u_n) \to c \neq 0$. Then* $\lim E(u_n) = (k/N) S^{N/2}$ *for some* $k \in \mathbb{N}$.

One of the basic ingredients in the proof of Theorem 1.6 is a Lemma, essentially contained in P. L. Lions [12], concerning the local behaviour of weakly convergent sequences satisfying some kind of "reverse" inequalities.

LEMMA 1.7. – *Let $u_n \in L^{2^*}(\mathbb{R}^N)$, $\nabla u_n \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^N, \mathbb{R}^N)$. Let $U \subseteq \mathbb{R}^N$ be a given open set, and assume*

$$\lim \int |\nabla u_n|^2 \varphi^2 \leq \lim \int |u_n|^{2^*} \varphi^2, \qquad \forall \varphi \in C_0^\infty(U). \tag{1.8}$$

Then there is a (possibily empty) finite set of points $x_1, \ldots, x_m \in U$ such that

$$\liminf \int_K |\nabla u_n|^2 = 0, \qquad \forall \text{compact } K \subset U \setminus \{x_1, \ldots, x_m\}; \tag{1.9}$$

$$\liminf \int_{B_r(x_j)} u_n^{2^*} \geq S^{N/2}, \qquad \forall j = 1, \ldots, m \text{ and } r > 0 \text{ small.} \tag{1.10}$$

Proof. — We have to prove:

$$\exists x^0 \in \mathrm{U}:$$

$$\operatorname{liminf} \int_{\mathrm{B}_r(x^0)} |\nabla u_n|^2 > 0, \forall \Rightarrow \operatorname{liminf} \int_{\mathrm{B}_r(x^0)} u_n^{2^*} \geqq \mathrm{S}^{\mathrm{N}/2}, \forall > 0.$$

Let $\varphi \in \mathrm{C}_0^\infty(\mathrm{B}_r(x^0))$, $\varphi = 1$ in $\mathrm{B}_{r/2}(x^0)$. We have:

$$0 < \lim \int |\nabla (u_n \varphi)|^2 = \lim \int |\nabla u_n|^2 \varphi^2 \leqq \lim \int u_n^{2^*-2} (u_n \varphi)^2$$

$$\leqq \operatorname{liminf} \left(\int_{\mathrm{B}_r(x^0)} u_n^{2^*} \right)^{2/\mathrm{N}} \left(\int |u_n \varphi|^{2^*} \right)^{2/2^*}$$

$$\leqq \operatorname{liminf} \left(\int_{\mathrm{B}_r(x^0)} u_n^{2^*} \right)^{2/\mathrm{N}} \mathrm{S}^{-1} \int |\nabla (u_n \varphi)|^2$$

by (1.8), Hölder and Sobolev inequalities. Thus (1.10) readily follows. ■

Remark 1.8. — Let $u_n \rightharpoonup 0$ be a P.S. sequence. After extending u_n to be equal to zero outisde Ω, an application of Lemma 1.7, with $\mathrm{U} = \mathbb{R}^\mathrm{N}$, gives *there is a finite set of points,* $x_1, \ldots, x_m \in \overline{\Omega \setminus \mathrm{C}}$, *such that:*

$$u_n \to 0 \quad in\, \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^\mathrm{N} \setminus \{x_1, \ldots, x_m\}),$$

$$a_j := \lim \int_{\mathrm{B}_r(x_j)} |u_n|^{2^*} \geqq \mathrm{S}^{\mathrm{N}/2}, \qquad \forall j \text{ and } r > 0 \text{ small enough}$$

(for some subsequence). In fact (1.8) is easily checked, taking $v = (1-\varphi) u_n$, $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^\mathrm{N})$, $0 \leqq \varphi \leqq 1$, in (iii). ■

Remark 1.9. — Let $u_n \rightharpoonup 0$ be a P.S. sequence. If $u_n \not\to 0$ in $\mathrm{H}_0^1(\Omega)$, necessarily

$$\lim \mathrm{E}(u_n) \geqq (1/\mathrm{N})\, \mathrm{S}^{\mathrm{N}/2}. \tag{1.11}$$

In fact, by the previous Remark, $\lim \int |u_n|^{2^*} \geqq \mathrm{S}^{\mathrm{N}/2}$. On the other hand, by Remark 1.5, we have $\lim \mathrm{E}(u_n) = (1/\mathrm{N}) \lim \int |u_n|^{2^*}$, and (1.11) follows. ■

Proof of Theorem 1.6. – By Remark 1.5, it amounts to prove

$$\lim \int |u_n|^{2^*} = k\,S^{N/2} \quad \text{for some } k \in \mathbb{N}. \tag{1.12}$$

In view of Remark 1.8, we can assume there is a finite set of "concentration points" $x_1, \ldots, x_m$ in $\overline{\Omega \setminus C}$, such that

$$\int_K |\nabla u_n|^2 \to 0 \quad \text{if } K \subseteq \bar{\Omega} \setminus \{x_1, \ldots, x_m\}, \text{ K compact,}$$

$$\lim \int_{B_r(x^0)} |u_n|^{2^*} \geqq S^{N/2}, \qquad \forall j = 1, \ldots, m \text{ and } r > 0.$$

In order to prove (1.12) we will use an iteration procedure, which, at each step, reduces the energy by exactly $S^{N/2}$. This will be done "blowing" each singularity x_j. In what follows, we will use quite the same arguments as in Brezis [3] (*see* also [4], [13]).

To perform the "blowing up" technique, let $\delta \in]0, S^{N/2}[$ be given and let $\varepsilon_n > 0$ be such that

$$\delta \leqq \operatorname*{Sup}_{x' \in \overline{B_\rho(x^0)}} \int_{B_{\varepsilon_n}(x')} |u_n|^{2^*} \leqq S^{N/2} - \delta. \tag{1.13}$$

Here x^0 denotes any of the "concentration points" x_j, and ρ is chosen in order $B_{2\rho}(x^0)$ contains only x^0 as a concentration point.

Now, let $x_n \in \overline{B_\rho}(x^0)$ be such that

$$\int_{B_{\varepsilon_n}(x_n)} |u_n|^{2^*} = \operatorname*{Sup}_{x' \in \overline{B_\rho(x^0)}} \int_{B_{\varepsilon_n}(x')} |u_n|^{2^*}. \tag{1.14}$$

Notice that $\varepsilon_n \to 0$. In fact, if (for a subsequence) $\varepsilon_n \geqq \varepsilon^0 > 0$, by (1.13) we get

$$S^{N/2} - \delta \geqq \int_{B_{\varepsilon_n}(x^0)} |u_n|^{2^*} \geqq \int_{B_{\varepsilon^0}(x^0)} |u_n|^{2^*}$$

while

$$\lim \int_{B_{\varepsilon^0}(x^0)} |u_n|^{2^*} \geqq S^{N/2}$$

by assumption. Also, $x_n \to x^0$. In fact, $x_n \to y$ implies

$$\delta \leqq \int_{B_{\varepsilon_n}(x_n)} |u_n|^{2^*} \leqq \int_{B_r(y)} |u_n|^{2^*}$$

for any given $r > 0$, provided n is sufficiently large. But, if r is small, $\lim \int_{B_r(y)} |u_n|^{2^*} = 0$ if $y \neq x^0$ again by assumption.

Now, define

$$\tilde{u}_n(x) = \varepsilon_n^{N/2^*} u_n(x_n + \varepsilon_n x).$$

Remark that $u_n = 0$ outside

$$\Omega_n := \frac{\Omega - x_n}{\varepsilon_n}.$$

Since $\int_{\mathbb{R}^N} |\nabla \tilde{u}_n|^2 = \int_\Omega |\nabla u_n|^2$ and $\int_{\mathbb{R}^N} |\tilde{u}_n|^{2^*} = \int_\Omega |u_n|^{2^*}$ we can assume there is ω, with $\int_{\mathbb{R}^N} |\nabla \omega|^2 < +\infty$, such that

$$\nabla u_n \rightharpoonup \nabla \omega \quad \text{weakly in } L^2(\mathbb{R}^N, \mathbb{R}^N),$$

and

$$u_n \rightharpoonup \omega \quad \text{weakly in } L^{2^*}(\mathbb{R}^N).$$

Finally, let us set $U := \{ z \in \mathbb{R}^N \mid x_n + \varepsilon_n z \in \Omega \setminus C, \ \forall n \text{ large} \}$. Notice that

$$U = \mathbb{R}^N \qquad \text{if} \quad (1/\varepsilon_n)\,\mathrm{dist}(x_n, (\Omega \setminus C)^{\complement}) \to +\infty,$$

while $U=\varnothing$ iff

$$x_n \notin \Omega \quad \text{and} \quad (1/\varepsilon_n)\operatorname{dist}(x_n, \partial\Omega) \to +\infty$$

or

$$x_n \in C \quad \text{and} \quad (1/\varepsilon_n)\operatorname{dist}(x_n, \partial C) \to +\infty.$$

In case $x^0 \in \partial\Omega$ and $(1/\varepsilon_n)$ dist $(x_n, \partial\Omega) \to l < \infty$, or $x^0 \in \partial C$ and $(1/\varepsilon_n)\operatorname{dist}(x_n, \partial C) \to l < \infty$ clearly U is an half space.

Let us remark that $\omega=0$ a.e. in $\bar{U}^C$. In fact, if $z \notin \bar{U}$, $B_r(z) \cap \bar{U}=\varnothing$ then, either $B_{r\varepsilon_n}(x_n+\varepsilon_n z) \subset \bar{\Omega}^C$ or $B_{r\varepsilon_n}(x_n+\varepsilon_n z) \subset \mathring{C}$. In both cases:

$$\lim_n \int_{B_r(z)} |\tilde{u}_n|^{2^*} = \lim_n \int_{B_{\varepsilon_n r}(x_n+\varepsilon_n z)} |u_n|^{2^*} = 0.$$

Using Lemma 1.7 we can exclude the case $\omega \equiv 0$ in $\mathbb{R}^N$. In fact, since, as one can easily check in this case, $\tilde{u}_n$ satisfies (1.8) while (1.10) cannot be satisfied, in view of (1.13), at any point, an application of Lemma 1.7 yields $\tilde{u}_n \to 0$ in $H^1_{loc}(\mathbb{R}^N)$, contraddicting the inequality on the left in (1.13).

The first consequence is that $U \neq \varnothing$; thus, either $U=\mathbb{R}^N$ or U is an half space. Later we will rule out the second alternative.

We are now in position to prove (1.12). Il will require a few steps:

Step 1. – $\tilde{u}_n \to \omega$ in $H^1_{loc}(U)$;

Step 2. – $-\Delta\omega=\omega^{2^*-1}$ in U, $\omega \in H^1_0(U)$, $\omega>0$ and hence $U=\mathbb{R}^N$;

Step 3. – $\lim \int |u_n-\hat{\omega}_n|^{2^*} = \lim \int ((u_n-\hat{\omega}_n)^+)^{2^*}$ where

$$\hat{\omega}_n(x)=\varepsilon_n^{-N/2^*}\,\omega\left(\frac{x-x_n}{\varepsilon_n}\right);$$

Step 4. – $u_{1,n}(x):=(u_n-\hat{\omega}_n)^+$ is a P.S. sequence;

Step 5. – Proof of (1.12) concluded.

Proof of Step 1. – In order to apply Lemma 1.7 to $\tilde{\eta}_n:=\tilde{u}_n-\omega$, let us fix $\varphi \in C^\infty_0(U)$, $0 \leqq \varphi \leqq 1$. Notice that

$$v_n:=u_n+\varphi\left(\frac{x-x_n}{\varepsilon_n}\right)\left(\varepsilon_n^{-N/2^*}\,\omega\left(\frac{x-x_n}{\varepsilon_n}\right)-u_n\right)$$

is admissible for (iii) in Definition 1.1, and $(v_n)_n$ is uniformly bounded in $H_0^1(\Omega)$; hence

$$\lim \int_{\mathbb{R}^N} \nabla \tilde{u}_n \nabla (\varphi \tilde{\eta}_n) \leqq \lim \int_{\mathbb{R}^N} |\tilde{u}_n|^{2^*-1} \tilde{\eta}_n \varphi.$$

Using a Lemma by Brezis and Lieb [5], one can verify that

$$\lim \int_{\mathbb{R}^N} \nabla \tilde{u}_n \nabla (\varphi \tilde{\eta}_n) = \lim \int_{\mathbb{R}^N} |\nabla \tilde{\eta}_n|^{2^*} \varphi$$

and

$$\lim \int_{\mathbb{R}^N} |\tilde{u}_n|^{2^*-1} \tilde{\eta}_n \varphi = \lim \int_{\mathbb{R}^N} |\tilde{\eta}_n|^{2^*} \varphi.$$

Thus Lemma 1.7 applies to get $\tilde{\eta}_n \to 0$ in $H^1_{loc}(U)$, since the inequality

$$\lim \int_{B_r(x)} |\tilde{u}_n - \omega|^{2^*} \geqq S^{N/2}$$

cannot be satisfied for any $x \in U$, in view of (1.13) and the obvious inequality:

$$\lim \int_{B_r(x)} \tilde{u}_n^{2^*} \geqq \lim \int_{B_r(x)} |\tilde{u}_n - \omega|^{2^*}.$$

Proof of Step 2. — Standard arguments in variational inequalities insure it is enough to prove

$$\int_{B_r(z)} \nabla \omega \nabla (\xi - \omega) \geqq \int_{B_r(z)} \omega^{2^*-1} (\xi - \omega) \qquad \forall \xi \in H_0^1(B_r(z)) + \omega|_{B_r(z)}, \quad \xi \geqq 0 \tag{1.15}$$

for every $r > 0$, $z \in U$ for which $B_{2r}(z) \subset U$. Thus, given ξ, we extend it outside $B_r(z)$, setting $\xi = \omega$. Now, given $\vartheta \in C_0^\infty(\mathbb{R}^N)$, $0 \leqq \vartheta \leqq 1$, $\vartheta = 1$ on $B_r(z)$, $\vartheta = 0$ outside $B_{2r}(z)$, we see that

$$v_n := u_n + \vartheta\left(\frac{x - x_n}{\varepsilon_n}\right)\left(\varepsilon_n^{-N/2^*} \xi\left(\frac{x - x_n}{\varepsilon_n}\right) - u_n\right)$$

is admissible for (iii) in Definition 1.1, and we obtain

$$\int_{\mathbb{R}^N} \nabla \tilde{u}_n \nabla (\xi - \tilde{u}_n)\, \vartheta \geqq o(1) + \int_{\mathbb{R}^N} \tilde{u}_n^{2^*-1} (\xi - \tilde{u}_n)\, \vartheta.$$

By $H^1_{loc}(U)$ convergence we can pass to the limit, getting

$$\int_{\mathbb{R}^N} \nabla \omega \nabla (\xi - \omega)\, \vartheta \geqq \int_{\mathbb{R}^N} \omega^{2^*-1} (\xi - \omega)\, \vartheta$$

i.e. (1.15), because $\xi - \omega = 0$ outside $B_r(z)$ and $\vartheta = 1$ on $B_r(z)$.

Furthermore, since $\omega = 0$ outside U, clearly $\omega \in H^1_0(U)$. Since $\omega \not\equiv 0$, as we have noticed before, by Pohozaev identify this implies U cannot be an half space, and hence $U = \mathbb{R}^N$.

Finally, let us recall that ω is uniquely determined (up to translations and changes of scale) and satisfies

$$\int |\nabla \omega|^{2^*} = \int \omega^{2^*} = S^{N/2}.$$

Proof of Step 3. — It is enough to observe that

$$\int_\Omega |u_n - \hat{\omega}_n|^{2^*} = \int_{\mathbb{R}^N} |\tilde{u}_n - \omega|^{2^*} = \int_{\mathbb{R}^N} [(\tilde{u}_n - \omega)^+]^{2^*} + \int_{\mathbb{R}^N} [(\omega - \tilde{u}_n)^+]^{2^*}.$$

Since $0 \leqq (\omega - \tilde{u}_n)^+ \leqq \omega$ and $\omega - \tilde{u}_n \to 0$ a.e. in $\mathbb{R}^N$, the claim follows by Lebesgue Theorem.

Proof of Step 4. — First of all, remark that $u_{1,n} \in \mathbf{K}$. Let $\eta_n = u_n - \hat{\omega}_n$, so that $u_{1,n} = \eta_n \vee 0$. We will prove later that:

$$\lim \int_\Omega \nabla \eta_n \nabla (\varphi - \eta_n) \geqq \lim \int_\Omega |\eta_n|^{2^*-2} \eta_n (\varphi - \eta_n) \tag{1.16}$$

uniformly for φ on bounded subsets of $\mathbf{K}$. Choosing $\varphi = \eta_n \vee 0$ in (1.16) and setting $\tilde{\eta}_n = \tilde{u}_n - \omega$ we get

$$\lim \int_\Omega \nabla \eta_n \nabla (\eta_n \wedge 0) \leqq \lim \int_{\mathbb{R}^N} |\tilde{\eta}_n|^{2^*-2} \tilde{\eta}_n (\tilde{\eta}_n \wedge 0) = 0$$

by Lebesgue Theorem, since $-\omega \leqq \tilde{\eta}_n \wedge 0 \leqq 0$. Thus we can replace η_n by $\eta_n \vee 0 = u_{1,n}$ in (1.16) and this completes the proof of Step 4.

Inequality (1.16) follows by Step 2, since

$$\int_\Omega \nabla \eta_n \nabla(\varphi-\eta_n)$$

$$= \int_\Omega \nabla u_n \nabla(\varphi-u_n) + \int_{\mathbb{R}^N} \nabla \tilde{u}_n \nabla \omega - \int_{\mathbb{R}^N} \nabla \omega \nabla(\tilde{\varphi}_n - \tilde{\eta}_n)$$

$$\geqq \int_\Omega u_n^{2^*-1}(\varphi-u_n) + \int_{\mathbb{R}^N} |\nabla \omega|^2 - \int_{\mathbb{R}^N} \omega^{2^*-1}(\tilde{\varphi}_n-\tilde{\eta}_n) + o(1)$$

$$= \int_\Omega u_n^{2^*-1}(\varphi-\eta_n) - \int_{\mathbb{R}^N} \tilde{u}_n^{2^*-1}\omega + \int_{\mathbb{R}^N} \omega^{2^*} - \int_\Omega \hat{\omega}_n^{2^*-1}(\varphi-\eta_n) + o(1)$$

$$= \int_\Omega (u_n^{2^*-1} - \hat{\omega}_n^{2^*-1})(\varphi-\eta_n) + o(1)$$

where the $o(1)$ are uniform on φ and, as usual,

$$\tilde{\varphi}_n(x) = \varepsilon_n^{N/2^*} \varphi(\varepsilon_n x + x_n).$$

Since $\varphi-\eta_n$ is uniformly bounded in $L^{2^*}(\Omega)$ for φ on bounded subsets of $H_0^1(\Omega)$, it is enough to prove:

$$z_n := (\tilde{u}_n^{2^*-1} - \omega^{2^*-1}) - |\tilde{u}_n - \omega|^{2^*-2}(\tilde{u}_n - \omega) \to 0 \quad \text{in } L^{2N/(N+2)}(\mathbb{R}^N).$$

First remark that $\tilde{u}_n \leqq \omega \Rightarrow |z_n| \leqq \omega^{2^*-1}$ and hence, by Lebesgue Theorem,

$$\lim \int |z_n|^{2N/(N+2)} = \lim \int_{\{\tilde{u}_n \geqq \omega\}} |z_n|^{2N/(N+2)}.$$

Since

$$\int_{\{\tilde{u}_n \geqq \omega\}} |z_n|^{2N/(N+2)} = \int_{\mathbb{R}^N} [(\tilde{\eta}_n^+ + \omega)^{2^*-1} - \omega^{2^*-1} - (\tilde{\eta}_n^+)^{2^*-1}]^{2N/(N+2)}$$

and $\tilde{\eta}_n = \tilde{u}_n - \omega \rightharpoonup 0$, the conclusion follows from

LEMMA 1.10. — *Let $f_n \in L^q$, $q \geqq 2$, be an L^q-uniformly bounded sequence, and suppose $f_n \to 0$ a. e., $f_n \geqq 0$. Then for every $f \in L^q$, $f \geqq 0$ it results*

$$g_n := (f_n + f)^{q-1} - f^{q-1} - (f_n)^{q-1} \to 0 \quad \text{in } L^{q/(q-1)}.$$

Proof. – First remark that $g_n \geqq 0$ a. e. Hence, using a Lemma by Brezis and Lieb [5] we immediately get

$$\int |f_n|^q + \int |f|^q = \int |f_n + f|^q + o(1)$$

$$= \int [g_n + f^{q-1} + (f_n)^{q-1}]^{q/(q-1)} + o(1)$$

$$\geqq \int |g_n|^{q/(q-1)} + \int |f_n|^q + \int |f|^q + o(1). \quad \blacksquare$$

Proof of Step 5. – By Step 3, we get

$$\lim \int_\Omega |u_{1,n}|^{2^*} = \lim \int_\Omega |u_n - \hat{\omega}_n|^{2^*} = \lim \int_{\mathbb{R}^N} |\tilde{u}_n|^{2^*} - S^{N/2}.$$

The last equality follows by Brezis-Lieb Lemma and Step 2. Thus $\lim \int_\Omega |u_n|^{2^*} = \lim \int_\Omega |u_{1,n}|^{2^*} + S^{N/2}$. In view of Step 4, the same argument can be iterated k times, if $k\, S^{N/2} \leqq \lim \int |u_n|^{2^*} < (k+1)\, S^{N/2}$, obtaining, for the kth iterate $u_{k,n}$, the equality

$$\lim \int |u_{k,n}|^{2^*} = \lim \int |u_n|^{2^*} - k\, S^{N/2}. \tag{1.17}$$

This implies $\lim \int |u_{k,n}|^{2^*} < S^{N/2}$. Thus $u_{k,n}$ is a P.S. sequence satisfying

$$\lim E(u_{k,n}) = (1/N) \lim \int |u_{k,n}|^{2^*} < (1/N)\, S^{N/2}.$$

An application of Remark 1 .9 yields $u_{k,n} \to 0$ in $H_0^1(\Omega)$ and (1.12) follows from (1.17). ■

Remark 1.11. – From the results in this Section it follows that if $u_n \rightharpoonup u$ is a P.S. sequence (u non necessarily zero) and $E(u_n) \to c$, then

$$E(u_n) = E(u) + (k/N)\, S^{N/2} + o(1) \quad \text{for some } k \in \mathbb{N} \cup \{0\}. \tag{1.18}$$

Also, $k=0$ if and only if $u_n \to u$ strongly. In order to prove (1.18) consider the sequence $\vartheta_n := u_n - u$ and use Proposition 1.1 to verify

$$\lim \int \nabla \vartheta_n \nabla (\varphi - \vartheta_n) \geqq \lim \int |\vartheta_n|^{2^*-2} \vartheta_n (\varphi - \vartheta_n) \tag{1.19}$$

uniformly with respect to φ on bounded subsets of $\mathbf{K}$. From (1.19) follows, choosing $v = \vartheta_n \vee 0$ as test function,

$$\lim \int |\nabla (\vartheta_n \wedge 0)|^2 \leqq \lim \int |\vartheta_n \wedge 0|^{2^*} = 0 \tag{1.20}$$

by Lebesgue theorem, since $-u \leqq \vartheta_n \wedge 0 \leqq 0$. Thus, we can replace ϑ_n by $\vartheta_n \vee 0$ in (1.19), and this proves that $(\vartheta_n \vee 0)_n$ is a P.S. sequence for Problem 1. Thus Theorem 1.6 implies that $E(\vartheta_n \vee 0) = (k/N)\ S^{N/2}$ for some $k \in \mathbb{N} \cup \{0\}$, with $k=0$ iff $\vartheta_n \vee 0 \to 0$ i.e., by (1.20), $\vartheta_n \to 0$. Now, from (1.20) and Taylor's expansion formula we easily get (1.18).

In particular, this result implies that the energy functional

$$f(u) := \begin{cases} E(u) & \text{if } u \in \mathbf{K} \\ +\infty & \text{otherwise in } H_0^1(\Omega) \end{cases}$$

verifies P. S. condition (in the sense of Szulkin [14]) at every energy level except for those of the form $E(u) + (k/N)\, S^{N/2}$, where u is a solution to Problem 1 and $k \geqq 1$ an integer. ■

2. THE EXISTENCE THEOREM

In this Section we will use a Min-Max principle in order to get the existence of a non trivial solution to Problem 1. More precisely, we prove that if $u \equiv 0$ is the only solution to Problem 1 with energy less than $(1/N)\ S^{N/2}$ and the set $\Omega \diagdown C$ verifies a geometrical assumption (as in Coron [8]), then there exists a critical point of "saddle type" for the functional f with energy in $](1/N)\, S^{N/2}, (2/N)\, S^{N/2}[$. Notice that by Remark 1.11, under this hypothesis f verifies P.S. condition in this interval.

In order to prove our existence theorem, we will construct, following Coron [8], a continuous map g^0 defined on an $N+1$-dimensional cylinder

Z with values in $\mathbb{K}$, such that

$$c^0 := \operatorname*{Sup}_{\partial Z} f(g^0) \geqq \frac{1}{N} S^{N/2}.$$

Then, we define

$$\Sigma := \{ g \in C^0(Z, H_0^1) \mid g_{|\partial Z} = g^0{}_{|\partial Z} \},$$
$$c := \operatorname*{Inf}_{\Sigma} \operatorname*{Sup}_{Z} f(g)$$

and prove that

$$c^0 < c < (2/N)\, S^{N/2}.$$

Since f verifies P.S. condition in a neighbourhood of c, an application of the deformation Lemma by Szulkin [14] for functionals of the form C^1 + convex-proper-lower semicontinuous gives the existence of a critical point at the level c, and this will complete the proof of the following

THEOREM 2.1. — *If* Ω, C *verify: there exist* $x^0 \in \mathbb{R}^N$ *and* $R_2 > R_1 > 0$ *such that*

$$\{ x \in \mathbb{R}^N \mid R_1 \leqq |x - x^0| \leqq R_2 \} \subset \Omega \setminus C$$
$$\{ x \in \mathbb{R}^N \mid |x - x^0| \leqq R_1 \} \not\subset \overline{\Omega \setminus C}$$

and R_2/R_1 *is large enough, then Problem* 1 *has a non trivial solution.*

Proof. — First of all we remark as in [8] that we can suppose

$$x^0 = 0, \qquad R_1 = \alpha^{-1}, \qquad R_2 = \alpha$$

for some $\alpha > 1$, so that the hypothesis "R_2/R_1 large enough" in Theorem 2.1 means "α large enough".

For the construction of the map g^0 we will use the functions

$$\Gamma: \ H_0^1(\Omega) \to \mathbb{R}, \qquad \Gamma(u) = \int |\nabla u|^2 - \int |u|^{2^*}$$

$$F: \ H_0^1(\Omega) \to \mathbb{R}^N, \qquad F(u) = S^{-N/2} \int x \, |\nabla u|^2 \, dx.$$

As an immediate consequence of the Concentration-Compactness Lemma by P.L. Lions [12], we get the following

LEMMA 2.2. – *For every neighbourhood* V *of* $\Omega\setminus C$ *there exist some* $\varepsilon>0$ *s. t.*

$$u\neq 0, \qquad \Gamma(u)=0,$$
$$f(u)\leqq(1/N)\,S^{N/2}+2\varepsilon \quad\Rightarrow\quad F(u)\in V.$$

Now, fix a point $a^0\notin\overline{\Omega\setminus C}$, $|a^0|<\alpha^{-1}$, a compact neighbourhood V of $\Omega\setminus C$ such that $a^0\notin V$, and correspondingly fix $\varepsilon>0$ as in Lemma 2.2, in such a way that

$$\sigma+\xi\neq a^0 \qquad \text{if} \quad |\sigma|=1, \quad |\xi|\leqq\varepsilon.$$

Let ω be the unique positive and radially symmetric (around the origin) solution of

$$-\Delta\omega=\omega^{2^*-1} \quad \text{on } \mathbb{R}^N, \qquad \int_{\mathbb{R}^N}|\nabla\omega|^2<+\infty \tag{2.1}$$

and let

$$\omega_t^\sigma=(1-t)^{-N/2^*}\,\omega\left(\frac{x-t\sigma}{1-t}\right)$$

for $t\in[0,1[$, $\sigma\in\partial B^N$, where $B^N=\{\xi\in\mathbb{R}^N\,|\,|\xi|\leqq 1\}$. Then, ω_t^σ solves (2.1) and for every σ, t it results

$$\int_{\mathbb{R}^N}|\nabla\omega_t^\sigma|^2=\int_{\mathbb{R}^N}(\omega_t^\sigma)^{2^*}=S^{N/2}.$$

If α is large, we can find, as in [8], a cut-off function $\varphi\in C_0^\infty(\Omega)$ with support in $\Omega\setminus C$, such that $0\leqq\varphi\leqq 1$, $\varphi\equiv 1$ on a neighbourhood of ∂B^N, and such that the functions:

$$v_t^\sigma:=\frac{\|\varphi\omega_t^\sigma\|^{N/2^*}}{|\varphi\omega_t^\sigma|_{2^*}^{N/2}}(\varphi\omega_t^\sigma), \qquad v_t^\sigma\in\mathbf{K}$$

verify:

$$f(v_t^\sigma)\leqq(2/N)\,S^{N/2}-\varepsilon, \qquad \forall\,\sigma\in\partial B^N, \quad \forall\,t\in[0,1[, \tag{2.2}$$
$$f(v_{t^0}^\sigma)\leqq(1/N)\,S^{N/2}+\varepsilon, \qquad \forall\,\sigma\in\partial B^N, \tag{2.3}$$
$$|F(v_{t^0}^\sigma)-\sigma|<\varepsilon. \tag{2.4}$$

for t^0 large enough. Remark that since $\Gamma(v_t^\sigma)=0$, $\forall \sigma$, $\forall t$, from (2.2), (2.3) it follows

$$\frac{1}{N} S^{N/2} < \operatorname*{Max}_{\mu>0} f(\mu v_t^\sigma) = f(v_t^\sigma) = \frac{1}{N} \| v_t^\sigma \|^2 \leqq \frac{2}{N} S^{N/2} - \varepsilon, \qquad \forall \sigma, \quad \forall t,$$

and

$$\frac{1}{N} S^{N/2} < \operatorname*{Max}_{\mu>0} f(\mu v_{t^0}^\sigma) = f(v_{t^0}^\sigma) \leqq \frac{1}{N} S^{N/2} + \varepsilon, \qquad \forall \sigma.$$

Moreover, if $\lambda > 1$ is big enough then

$$f(\lambda v_t^\sigma) < 0, \qquad \forall \sigma \in \partial B^N, \quad \forall t \in [0, 1[.$$

Now we can define our "boundary data" $g^0 : Z := [0, 1] \times B^N \to \mathbf{K}$ by setting:

$$g^0(s, t\sigma) := \lambda s v_{tt^0}^\sigma$$

for $s \in [0, 1]$, $t \in [0, 1]$, $\sigma \in \partial B^N$. Remark that g^0 is well-defined and continuous on Z since for $t=0$ v_0^σ does not depend on σ. By observations above, we have

$$c^0 = \operatorname*{Sup}_{\partial Z} f(g^0) \leqq \frac{1}{N} S^{N/2} + \varepsilon \qquad \text{and} \qquad \operatorname*{Sup}_{Z} f(g^0) \leqq \frac{2}{N} S^{N/2}.$$

Thus, to conclude the proof of the Theorem is enough to verify:

$$\operatorname*{Sup}_{Z} f(g) \geqq \frac{1}{N} S^{N/2} + 2\varepsilon$$

for every

$$g \in C^0(Z, H_0^1), \qquad g_{|\partial Z} = g^0{}_{|\partial Z}.$$

Suppose by contradiction that there exists a $g \in C^0(Z, \mathbf{K})$ such that $g = g^0$ on ∂Z,

$$f(g(s, \xi)) \leqq (1/N) S^{N/2} + 2\varepsilon, \qquad \forall (s, \xi) \in Z \tag{2.5}$$

and consider the map

$$\begin{cases} G: \ Z \to \mathbb{R}^{N+1}, \\ G(s, \xi) = (s, F(g(s, \xi))). \end{cases}$$

We claim that

$$\deg(G, Z, (\lambda^{-1}, a^0)) = 1$$

since the map

$$\begin{cases} H: \quad [0,1] \times Z \to \mathbb{R}^{N+1} \\ H(t; s, \xi) := t\, G(s, \xi) + (1-t)(s, \xi) = (s, t\, F(g(s, \xi)) + (1-t)\, \xi) \end{cases}$$

is an admissible homotopy between G and Id_Z. In fact if $H(t; s, \xi) = (\lambda^{-1}, a^0)$ then necessarily $s = \lambda^{-1}$ and $\xi \notin \partial B^N$ since $\forall\, \sigma \in \partial B^N$

$$t\, F(g(\lambda^{-1}, \sigma)) + (1-t)\, \sigma = t\,(F(v_{t_0}^{\sigma}) - \sigma) + \sigma \neq a^0$$

because of (2.4).

Let us define the sets:

$$Z^+ = \{ (s, \xi) \in Z \mid \Gamma(g(s, \xi)) > 0 \} \cup \{ (0, \xi) \mid \xi \in B^N \}$$
$$Z^- = \{ (s, \xi) \in Z \mid \Gamma(g(s, \xi)) < 0 \}$$
$$Z^0 = \{ (s, \xi) \in Z \mid \Gamma(g(s, \xi)) = 0, s > 0 \}.$$

Notice that Z^+ is open in Z and Z^0 is closed in Z since $\Gamma(u) > 0$ if $u \neq 0$ and $\|u\|$ is small. Moreover

$$\begin{aligned} &(s, \xi) \in Z^+ \quad \text{for } (s, \xi) \in \partial Z, \qquad 0 \leqq s < \lambda^{-1} \\ &(\lambda^{-1}, \xi) \in Z^0 \quad \text{for } \xi \in \partial B^N. \\ &(s, \xi) \in Z^- \quad \text{for } (s, \xi) \in \partial Z, \qquad \lambda^{-1} < s \leqq 1 \end{aligned} \tag{2.6}$$

By Lemma 2.2 and (2.5) we have that $F(g(Z^0)) \subset V$ and in particular

$$F(g(s, \xi)) \neq a^0, \qquad \forall (s, \xi) \in Z^0. \tag{2.7}$$

Hence, by excision property we have

$$1 = \deg(G, Z, (\lambda^{-1}, a^0)) = \deg(G, Z^+, (\lambda^{-1}, a^0)) + \deg(G, Z^-, (\lambda^{-1}, a^0))$$

while on the other hand we shall prove that

$$\deg(G, Z^+, (\lambda^{-1}, a^0)) = 0; \tag{2.8}$$

$$\deg(G, Z^-, (\lambda^{-1}, a^0)) = 0 \tag{2.9}$$

getting in this way a contradiction which proves Theorem 2.1.

Proof of (2.8). — Fix $R>\lambda^{-1}$ such that $y\in\mathbb{R}^{N+1}$, $|y|\geqq R \Rightarrow y\notin G(Z)$, and consider the path

$$p:\quad [0,1]\to\mathbb{R}^{N+1},\qquad p(t)=(tR+(1-t)\lambda^{-1}, a^0).$$

We claim that $p(t)\notin G(\partial Z^+)$ for every t. Suppose this is not the case; then there exist $t\in[0,1]$ and $(s,\xi)\in\partial Z^+$ such that

$$(tR+(1-t)\lambda^{-1}, a^0)=(s, F(g(s,\xi))).$$

We first deduce that $s\geqq\lambda^{-1}$; on the other hand, from $F(g(s,\xi))=a^0$ and (2.7) it follows that $(s,\xi)\notin Z^0$. Since $\partial Z^+\subset\partial Z\cup Z^0$ we conclude that the only possibility is: $\xi\in\partial Z$ and $(s,\xi)\in Z^+$ which implies, together with (2.6), $s<\lambda^{-1}$, in contrast with $s\geqq\lambda^{-1}$.

Since $p(.)$ is admissible, we have that $\deg(G, Z^+, p(t))$ does not depend on t, and hence

$$\deg(G, Z^+, (\lambda^{-1}, a^0))=\deg(G, Z^+, (R, a^0))=0$$

since $(R, a^0)\notin G(Z)$.

Formula (2.9) can be proved in the same way, observing that the path

$$q:\quad [0,1]\to\mathbb{R}^{N+1},\qquad q(t)=(-tR+(1-t)\lambda^{-1}, a^0)$$

is admissible for the degree, and thus

$$\deg(G, Z^-, (\lambda^{-1}, a^0))=\deg(G, Z^-, (-R, a^0))=0.\quad\blacksquare$$

3. A REGULARITY REMARK

Before stating our regularity result, we point out some properties of solutions to Problem 1.

PROPOSITION 3 1. — *If u solves Problem* 1, *then*

$$\int\nabla u\nabla(v-u)\geqq\int u^{2^*-1}(v-u),\qquad\forall v\in H_0^1(\Omega),\quad v\leqq\psi \text{ on } C.\quad (3.1)$$

Proof. — Let us set $f:=u^{2^*-1}\in L^{2N/(N+2)}(\Omega)$, and let w be the unique solution of:

$$\begin{gathered} w\in H_0^1(\Omega), \qquad w\leqq\psi \text{ on } C \\ \int\nabla w\nabla(v-w)\geqq\int f(v-w), \qquad \forall v\in H_0^1,\quad v\leqq\psi \text{ on } C. \end{gathered} \tag{3.2}$$

In order to prove that $w=u$, we observe first of all that $w\geqq 0$ in Ω (i. e. $w\in\mathbf{K}$); in fact, choosing $v=w\vee 0$ as test function in (3.2) we get

$$\int|\nabla(w\wedge 0)|^2=\int\nabla w\nabla(w\wedge 0)\leqq\int f(w\wedge 0)$$

and hence $w\wedge 0=0$, since $f\geqq 0$ a. e. in Ω.

Thus Proposition 3.1 follows from uniqueness for the linear variational inequality:

$$u\in\mathbf{K},\qquad \int\nabla u\nabla(v-u)\geqq\int f(v-u),\qquad \forall v\in\mathbf{K}.\quad \blacksquare$$

From Proposition 3.1 it follows immediately that u is a weak solution of the equation

$$-\Delta u=u^{2^*-1}\quad \text{in } \Omega\setminus C. \tag{3.3}$$

We are now in position to state and prove our regularity result:

THEOREM 3.2. — *If* $\psi\in C^0(\bar\Omega)\cap H^1(\Omega)$ *and* u *solves Problem* 1, *then* u *is continuous in* Ω.

Proof. — We first prove that $u\in L^\infty(\Omega)$. Let u be the unique solution of:

$$-\Delta\mathrm{u}=0\quad \text{in } \Omega\setminus C,\qquad \mathrm{u}=u\quad \text{on } \partial(\Omega\setminus C).$$

From (3.3) it follows that the function $z:=u-\mathrm{u}$ solves

$$\text{(3.4)}\qquad \left\{\begin{aligned} -\Delta z&=a(x)z+g\quad \text{in } \Omega\setminus C\\ u&\in H_0^1(\Omega\setminus C)\end{aligned}\right.$$

where $a:=u^{2^*-2}\in L^{N/2}$, $g:=u^{2^*-2}\mathrm{u}\in L^{N/2}$, since $\mathrm{u}\in L^\infty$ by the maximum principle. The boundness of u is a consequence of the following Lemma, which is essentially contained in [6] (*see* also [7], Lemma 1.5):

LEMMA 3.3. — *Suppose* $a \in L^{N/2}$, $g \in L^q$ *with* $q \geqq N/2$ *and* z *solves* (3.4). *Then* $z \in L^\tau$, $\forall \tau < \infty$.

Applying Lemma 3.3 we easily get $u \in L^\infty(\Omega \setminus C)$ and finally, since $0 \leqq u \leqq \psi$ in C, we can conclude that $u \in L^\infty(\Omega)$.

We now set $f := u^{2^*-1} \in L^\infty(\Omega)$, $w := h - u$, where h solves

$$-\Delta h = f \quad \text{in}\, \Omega, \qquad h = 0 \quad \text{on}\, \partial\Omega.$$

Using Proposition 3.1 it is easy to verify that w is the unique solution of the linear variational inequality:

$$w \in H_0^1(\Omega), \qquad w \geqq h - \psi \quad \text{on}\, C$$

$$\int \nabla w \nabla (v - w) \geqq 0, \qquad \forall v \in H_0^1(\Omega), \quad v \geqq h - \psi \,\text{on}\, C.$$

Since $h - \psi$ is continuous on $\bar{\Omega}$, an application of a Theorem by Lewy-Stampacchia ([11], Part II) gives the continuity of w, and the theorem is proved. ■

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(*Manuscrit reçu le* 4 *juin* 1987.)